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Investigation of the Asymptotic and Monotonic Properties of Solutions of Functional Differential Equations with Multiple Delays and a Damping Term

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Abstract. Our objective in this work is to establish new asymptotic and monotonic properties of second-order differential equations involving multiple delay terms and a damping component. These properties are instrumental in advancing the understanding of the qualitative behavior of their solutions. This study employs a refined comparison technique utilizing first-order differential equations. The resulting criteria broaden and enhance previously established findings in the literature. A natural progression and forthcoming challenge of this research is examining the oscillatory behavior and asymptotic characteristics of solutions for higher-order neutral delay differential equations, where the interplay between delay and neutral components becomes progressively intricate.

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Key Words and Phrases: Differential equations, oscillatory characteristics, deviating argument, damping term

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1. Introduction

Second-order differential equations (DEs) are used in engineering as well as science to examine a variety of events. Particularly, second-order nonlinear equations play a significant role in this field. The nonlinear terms in these equations grow more quickly than the linear terms when the dependent variable rises. There are many applications for these equations, such as population dynamics models, nonlinear oscillatory systems, and mechanical systems with nonlinear damping [1–4]. Since second-order nonlinear differential equations display a wide range of phenomena, they are an interesting topic for applied mathematical study (see [5–8]).

The neutral delay differential equation (NDDE) is a type of delay differential equation (DDE) in which the highest-order derivative of the unknown function occurs both in

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delayed and non-delayed forms. This class of equations commonly appears in the dynamic modeling of various real-world systems, including pharmacokinetics, frequency-modulated processes, bursting rhythm phenomena in biological and medical sciences, and optimal control models within economic theory [9, 10]. In particular, second-order NDDE hold significant interest in fields such as robotics, where they are employed in the development of bipedal locomotion systems, and in biology, where they contribute to understanding the mechanisms underlying human postural balance [11, 12].

Within the present study, we analyze the oscillatory characteristics of solutions resulting from the following second-order NDDEs with multiple delays and a damping term:

$$\left(r\left(s\right)\left(z'\left(s\right)\right)^{\kappa}\right)' + h\left(s\right)\left(z'\left(s\right)\right)^{\kappa} + \sum_{i=1}^{n} q_i\left(s\right) x^{\kappa}\left(\eta_i\left(s\right)\right) = 0, \quad s \geq s_0, \tag{1}$$

in which

$$z\left(s\right) := x\left(s\right) + p\left(s\right) x\left(\tau\left(s\right)\right).$$

Our investigation proceeds under the following hypotheses:

(H$_1$) $\kappa \geq 1$ is a fraction formed by dividing odd positive integers, and $n \in \mathbb{Z}^+$;

(H$_2$) $r \in \mathbf{C}^1\left([s_0, \infty), \mathbb{R}^+\right)$, and

$$\int_{s_0}^{\infty} \frac{1}{r^{1/\kappa}\left(u\right)} \exp\left(-\frac{1}{\kappa}\int^{u} \frac{h\left(\rho\right)}{r\left(\rho\right)} \mathrm{d}\rho\right) \mathrm{d}u = \infty; \tag{2}$$

(H$_3$) $p,\ h,\ q_i \in \mathbf{C}\left([s_0, \infty), [0, \infty)\right)$, for $i = 1, 2, ..., n$, $0 \leq p(s) \leq p_0 < 1$, where p_0 is a constant, and q_i does not vanish identically on any half-line of the form $[t_*, \infty), t_* \geq t_0$;

(H$_4$) $\tau, \eta_i \in \mathbf{C}([s_0, \infty), \mathbb{R})$, $\tau(s) \leq s$, $\eta_i(s) < s$, $\lim_{s\to\infty} \tau\left(s\right) = \infty$, and $\lim_{s\to\infty} \eta_i\left(s\right) = \infty$, for $i = 1, 2, ..., n$.

Based on a solution to (1), we consider a real-valued function $x \in \mathbf{C}\left([s_*, \infty), \mathbb{R}\right)$, $s_* \geq s_0$, where z, $r\left(z'\right)^{\kappa} \in \mathbf{C}^1\left([s_*, \infty), \mathbb{R}\right)$, and x fulfills (1) on $[s_*, \infty)$. Our focus is confined to solutions x to (1) that meet $\sup\left\{|x\left(s\right)| : s \geq s_1\right\} > 0$ for all $s_1 \geq s_*$. A solution x to (1) is called oscillatory if it possesses zeros that are arbitrarily large; otherwise, it is referred to as nonoscillatory.

Condition (2) is called the canonical condition. It has a fundamental effect on the study because it ensures that there are no positive solutions with corresponding decreasing functions.

Remark 1. *All the relationships and inequalities in the study are eventually satisfied, i.e. for $t \geq t_*$ where $t_* \geq t_0$.*

2. Literature review

Since NDDE appears in the modeling of many systems in our lives, see [13–15]. The oscillation of DEs has been a major focus of interest among researchers. For example, [16, 17] was interested in studying first-order NDDEs but [18–21] was interested in the second-order equations. Conversely, the even/nth order equations were studied in works [22–25].

In 1978, Brands [26] proved that, in the case of bounded delays, the solutions of

$$x''(s)+q(s)\,x(s-\eta)=0,$$

are oscillatory if the solutions of $x''(s)+q(s)\,x(s)=0$ are also oscillatory. Then, based on the well-known results obtained by Ladas et al. [27] for the oscillation of the first-order DDE, Wei [28] provided the famous oscillation criteria

$$\limsup_{s\to\infty}\int_{\eta(s)}^{s}\eta(\rho)\,q(\rho)\,\mathrm{d}\rho>1$$

or

$$\liminf_{s\to\infty}\int_{\eta(s)}^{s}\eta(\rho)\,q(\rho)\,\mathrm{d}\rho>\frac{1}{\mathrm{e}},$$

for the second order DDE

$$x''(s)+q(s)\,x(\eta(s))=0.$$

Then Koplatadze et al. [29] improved these criteria to the form

$$\limsup_{s\to\infty}\int_{\eta(s)}^{s}q(\rho)\left[\eta(\rho)+\int_{s_1}^{\eta(\rho)}\nu\eta(\nu)\,q(\nu)\,\mathrm{d}\nu\right]\mathrm{d}\rho>1,$$

for $\eta'\geq 0$, or

$$\liminf_{s\to\infty}\int_{\eta(s)}^{s}q(\rho)\left[\eta(\rho)+\int_{s_1}^{\eta(\rho)}\nu\eta(\nu)\,q(\nu)\,\mathrm{d}\nu\right]\mathrm{d}\rho>\frac{1}{\mathrm{e}}.$$

Investigating the oscillation behavior by using the Riccati technique was a very useful tool in literature, since this technique aims to diminish the order of the DE to the first-order DE and allows us to benefit from the wide range of oscillation research regarding these first-order ones. Dzurina and Stavroulakis [30] used the Riccati technique to provide the oscillation criteria

$$\int^{\infty}R^{\kappa}(\eta(\rho))\,q(\rho)-\frac{\kappa\eta'(\rho)}{4L_1R(\eta(\rho))\,r^{1/\kappa}(\eta(\rho))}\mathrm{d}\rho=\infty, \tag{3}$$

where $\kappa\geq 1$, $\eta'>0$, and $0<L_1<1$, for the half linear DDE

$$\left(r(s)\left(x'(s)\right)^{\kappa}\right)'+q(s)\,x^{\kappa}(\eta(s))=0. \tag{4}$$

After that, Sun and Meng [31] developed (3) to

$$\int^{\infty} R^{\kappa}\left(\eta\left(\rho\right)\right) q\left(\rho\right) - \left(\frac{\kappa}{\kappa+1}\right)^{\kappa+1} \frac{\eta'\left(\rho\right)}{R\left(\eta\left(\rho\right)\right) r^{1/\kappa}\left(\eta\left(\rho\right)\right)} \mathrm{d}\rho = \infty.$$

One may find some oscillation results for the noncanonical case of (4), i.e.,

$$\int_{s_0}^{\infty} r^{-1/\kappa}\left(\rho\right) \mathrm{d}\rho < \infty,$$

in the works of Chatzarakis et al. [32, 33]. Wong [34] found the conditions that guarantee the oscillation of solutions to the NDDE:

$$\left(x\left(s\right) + px\left(s-\tau\right)\right)'' + q\left(s\right) f\left(x\left(s-\tau\right)\right) = 0,$$

where the delays and the neutral coefficient τ, η are constants. Xu and Meng [35] generalized the previous criteria to the NDDE

$$\left(r\left(s\right)\left(\left[x\left(s\right) + p\left(s\right) x\left(\tau\left(s\right)\right)\right]'\right)^{\kappa}\right)' + q\left(s\right) x^{\kappa}\left(\eta\left(s\right)\right) = 0. \tag{5}$$

In 2018, Grace et al. [36] established criteria that are sufficient for the solutions of (5) to oscillate when $0 \leq p < 1$, and showed that the oscillation can be ensured provided that at least one of the next conditions is satisfied

$$\limsup_{s\to\infty} \int_{\eta(s)}^{s} Q\left(\rho\right) \tilde{R}^{\kappa}\left(\eta\left(\rho\right)\right) \mathrm{d}\rho > 1,$$

for $\eta' > 0$, or

$$\liminf_{s\to\infty} \int_{\eta(s)}^{s} Q\left(\rho\right) \tilde{R}^{\kappa}\left(\eta\left(\rho\right)\right) \mathrm{d}\rho > \frac{1}{\mathrm{e}}.$$

On the other hand, Baculikova and Džurina in [37, 38] investigated the second-order Emden–Fowler NDDE (1). Employing the comparison theorem in order to formulate adequate criteria for the oscillation of the solutions to (1) when $\kappa = \beta = 1$, $0 \leq p < \infty$, and

$$\int_{s_0}^{\infty} r^{-1/\kappa}\left(\rho\right) \mathrm{d}\rho = \infty.$$

Bohner et al. [39] considered the noncanonical case of (1) for $\kappa = \beta$ and $0 \leq p < 1$. Then, Baculikova and Dzurina [40] extended the previous results for $\kappa \geq \beta$ and $\eta, \tau\left(s\right) > 0$ of (1). Liu et al. in [41] studied (1) by applying the Riccati technique under the conditions $\kappa \geq \beta$, $r'\left(s\right) > 0$, and $\eta'\left(s\right) > 0$, they provided new oscillation criteria for (1) for both canonical and noncanonical cases.

3. Asymptotic and Monotonic Properties

For convenience, we define the following notations:

$$\varrho(s) := \exp\left(\int^{s} \frac{h(\rho)}{r(\rho)} \mathrm{d}\rho\right),$$

$$\eta(s) := \min\{\eta_i(s),\ i = 1, 2, ..., n\},$$

$$Q(s) := \varrho(s) \sum_{i=1}^{n} q_i(s) (1 - p(\eta_i(s)))^{\kappa},$$

$$R(s) := \int_{s_1}^{s} \frac{1}{\varrho^{1/\kappa}(\rho) r^{1/\kappa}(\rho)} \mathrm{d}\rho,$$

The following lemmas provide some asymptotic and monotonic properties of the corresponding function to positive solutions. They also provide the necessary relationships for proving oscillation theorems.

Lemma 1. *Grant that x is a positive solution to (1). Hence, eventually,*

$$z(s) > 0, \quad z'(s) > 0, \quad (r(s)(z'(s))^{\kappa})' \leq 0, \tag{6}$$

and

$$\left[\varrho(s) r(s) (z'(s))^{\kappa}\right]' + \varrho(s) \sum_{i=1}^{n} q_i(s) x^{\kappa}(\eta_i(s)) \leq 0. \tag{7}$$

Proof. Presume that x is a positive solution to (1). Equation (1) transforms into

$$\begin{aligned} \frac{1}{\varrho(s)} \left[\varrho(s) r(s) (z'(s))^{\kappa}\right]' &= (r(s)(z'(s))^{\kappa})' + h(s)(z'(s))^{\kappa} \\ &= -\sum_{i=1}^{n} q_i(s) x^{\kappa}(\eta_i(s)), \end{aligned}$$

or

$$\left[\varrho(s) r(s) (z'(s))^{\kappa}\right]' \leq -\varrho(s) \sum_{i=1}^{n} q_i(s) x^{\kappa}(\eta_i(s)) \leq 0.$$

So, $r(z')^{\kappa}$ is a non-increasing function, and either $z'(s) > 0$ or $z'(s) < 0$, eventually.

Grant that $z'(s) < 0$ for $s \geq s_1$. Then, there is a $K_1 > 0$ such that

$$\varrho(s) r(s) (z'(s))^{\kappa} \leq -K_1,$$

and thus,

$$z'(s) \leq -K_1^{1/\kappa} \frac{1}{\varrho^{1/\kappa}(s) r^{1/\kappa}(s)}. \tag{8}$$

Integrating (8) over $[s_1, s]$, we arrive at

$$z(s) - z(s_1) \leq -K_1^{1/\kappa} \int_{s_1}^{s} \frac{1}{\varrho^{1/\kappa}(\rho)\, r^{1/\kappa}(\rho)} \mathrm{d}\rho.$$

Hence, using (2), we obtain $\lim_{s\to\infty} z(s) = -\infty$, a contradiction. So, $z'(s) > 0$, eventually.

Lemma 2. *Presume that x is a positive solution to (1). Therefore the following relations hold eventually:*
(P_1) $x(s) \geq (1 - p(s))\, z(s)$;
(P_2) $[\varrho(s)\, r(s)\, (z'(s))^{\kappa}]' + Q(s)\, z^{\kappa}(\eta(s)) \leq 0$;
(P_3) $z(s) \geq \varrho^{1/\kappa}(s)\, r^{1/\kappa}(s)\, z'(s)\, R(s)$;
(P_4) $z(s)/R(s)$ *is non-increasing function.*

Proof. Grant that x is a positive solution to (1). Based on the definition of z, we get

$$x(s) \geq z(s) - p(s)\, z(\tau(s)) \geq (1 - p(s))\, z(s).$$

Combined this with (7), we obtain

$$\begin{aligned}\left[\varrho(s)\, r(s) \left(z'(s)\right)^{\kappa}\right]' &\leq -\varrho(s) \sum_{i=1}^{n} q_i(s) \left(1 - p(\eta_i(s))\right)^{\kappa} z^{\kappa}(\eta_i(s)) \\ &\leq -z^{\kappa}(\eta(s))\, \varrho(s) \sum_{i=1}^{n} q_i(s) \left(1 - p(\eta_i(s))\right)^{\kappa} \\ &= -Q(s)\, z^{\kappa}(\eta(s)).\end{aligned}$$

Now, we have

$$z(s) \geq \int_{s_1}^{s} \frac{1}{\varrho^{1/\kappa}(\rho)\, r^{1/\kappa}(\rho)} \left[\varrho^{1/\kappa}(\rho)\, r^{1/\kappa}(\rho)\, z'(\rho)\right] \mathrm{d}\rho \geq \varrho^{1/\kappa}(s)\, r^{1/\kappa}(s)\, z'(s)\, R(s),$$

and hence

$$\left(\frac{z(s)}{R(s)}\right)' = \frac{R(s)\, z'(s) - \varrho^{-1/\kappa}(s)\, r^{-1/\kappa}(s)\, z(s)}{R^2(s)} \leq 0.$$

This completes the proof.

Lemma 3. *Grant that x is a positive solution to (1), provide*

$$\int_{s_1}^{\infty} R^{\kappa}(\eta(\rho))\, Q(\rho)\, \mathrm{d}\rho = \infty. \tag{9}$$

Then

$$\lim_{s\to\infty} \frac{z(s)}{R(s)} = 0.$$

Proof. Let's say that x is a positive solution to (1). From (P_3), (P_4), we get

$$\frac{z(\eta(s))}{R(\eta(s))} \geq \frac{z(s)}{R(s)} \geq \varrho^{1/\kappa}(s)\, r^{1/\kappa}(s)\, z'(s)\,.$$

Using (P_2), we have

$$\begin{aligned} -Q(s)\, z^{\kappa}(\eta(s)) &\geq \left[\varrho(s)\, r(s)\left(z'(s)\right)^{\kappa}\right]' \\ &= \left[\left(\varrho^{1/\kappa}(s)\, r^{1/\kappa}(s)\, z'(s)\right)^{\kappa}\right]' \\ &= \kappa\left(\varrho^{1/\kappa}(s)\, r^{1/\kappa}(s)\, z'(s)\right)^{\kappa-1}\left[\varrho^{1/\kappa}(s)\, r^{1/\kappa}(s)\, z'(s)\right]' \\ &\geq \kappa\left(\frac{z(\eta(s))}{R(\eta(s))}\right)^{\kappa-1}\left[\varrho^{1/\kappa}(s)\, r^{1/\kappa}(s)\, z'(s)\right]', \end{aligned}$$

and so

$$\begin{aligned} \left[\varrho^{1/\kappa}(s)\, r^{1/\kappa}(s)\, z'(s)\right]' &\leq -\frac{1}{\kappa}\frac{z^{1-\kappa}(\eta(s))}{R^{1-\kappa}(\eta(s))}Q(s)\, z^{\kappa}(\eta(s)) \\ &= -\frac{1}{\kappa}R^{\kappa-1}(\eta(s))\, Q(s)\, z(\eta(s))\,. \end{aligned} \tag{10}$$

Next, we see that

$$\begin{aligned} \left[\varrho^{1/\kappa}(s)\, r^{1/\kappa}(s)\, R^2(s)\left(\frac{z(s)}{R(s)}\right)'\right]' &= \left[R(s)\,\varrho^{1/\kappa}(s)\, r^{1/\kappa}(s)\, z'(s) - z(s)\right]' \\ &= R(s)\left[\varrho^{1/\kappa}(s)\, r^{1/\kappa}(s)\, z'(s)\right]' \\ &\leq -\frac{1}{\kappa}R(s)\, R^{\kappa-1}(\eta(s))\, Q(s)\, z(\eta(s))\,. \end{aligned} \tag{11}$$

Integration of (11) for s from s_1 to s yields

$$\varrho^{1/\kappa}(s)\, r^{1/\kappa}(s)\, R^2(s)\left(\frac{z(s)}{R(s)}\right)' \leq -L - \frac{1}{\kappa}\int_{s_1}^{s} R(\rho)\, R^{\kappa-1}(\eta(\rho))\, Q(\rho)\, z(\eta(\rho))\, \mathrm{d}\rho \tag{12}$$

where

$$L := -\varrho^{1/\kappa}(s_1)\, r^{1/\kappa}(s_1)\, R^2(s_1)\left.\left(\frac{z(s)}{R(s)}\right)'\right|_{s=s_1} > 0.$$

Let $w := z/R$. From (6) and (12), w is positive and decreasing. So,

$$\lim_{s\to\infty} w(s) = w_0 \geq 0.$$

Assume the contrary that $w_0 > 0$. Hence, (12) becomes

$$w'(s) \leq -\frac{1}{\kappa}\frac{1}{\varrho^{1/\kappa}(s)\, r^{1/\kappa}(s)\, R^2(s)}\int_{s_1}^{s} R(\rho)\, R^{\kappa-1}(\eta(\rho))\, Q(\rho)\, z(\eta(\rho))\, \mathrm{d}\rho. \tag{13}$$

Integration of (13) over $[s_1, \infty)$ gets

$$:= z(\eta(\rho))$$

$$\begin{aligned}
w_0 - w(s_1) &\leq -\frac{1}{\kappa}\int_{s_1}^{\infty}\frac{1}{\varrho^{1/\kappa}(u)\,r^{1/\kappa}(u)\,R^2(u)}\int_{s_1}^{u}R(\rho)\,R^{\kappa-1}(\eta(\rho))\,Q(\rho)\,z(\eta(\rho))\,\mathrm{d}\rho\mathrm{d}u \\
&= -\frac{1}{\kappa}\int_{s_1}^{\infty}\frac{1}{\varrho^{1/\kappa}(u)\,r^{1/\kappa}(u)\,R^2(u)}\int_{s_1}^{u}R(\rho)\,R^{\kappa}(\eta(\rho))\,Q(\rho)\,w(\eta(\rho))\,\mathrm{d}\rho\mathrm{d}u \\
&\leq -\frac{w_0}{\kappa}\int_{s_1}^{\infty}\frac{1}{\varrho^{1/\kappa}(u)\,r^{1/\kappa}(u)\,R^2(u)}\int_{s_1}^{u}R(\rho)\,R^{\kappa}(\eta(\rho))\,Q(\rho)\,\mathrm{d}\rho\mathrm{d}u \\
&= -\frac{w_0}{\kappa}\int_{s_1}^{\infty}R^{\kappa}(\eta(\rho))\,Q(\rho)\,\mathrm{d}\rho,
\end{aligned}$$

which contradicts (9), and thus $w_0 = 0$.

4. Oscillation Theorems

The following theorem uses a comparison approach with a first-order equation to study the oscillatory behavior of solutions to equation (1).

Theorem 1. *Assume that (9) holds. If the DDE*

$$w'(s) + \left(\frac{1}{\kappa\varrho^{1/\kappa}(s)\,r^{1/\kappa}(s)\,R^2(s)}\int_{s_0}^{s}R(\rho)\,R^{\kappa}(\eta(\rho))\,Q(\rho)\,\mathrm{d}\rho\right)w(\eta(s)) = 0 \tag{14}$$

is oscillatory, then equation (1) oscillates.

Proof. Assume the contrary that x is a positive solution to (1). Since in the demonstration of Lemma 3, we have that $w = z/R$ satisfies (12). Then,

$$\begin{aligned}
w'(s) &\leq \frac{-1}{\varrho^{1/\kappa}(s)\,r^{1/\kappa}(s)\,R^2(s)}\left[L + \frac{1}{\kappa}\int_{s_1}^{s}R(\rho)\,R^{\kappa}(\eta(\rho))\,Q(\rho)\,w(\eta(\rho))\,\mathrm{d}\rho\right] \\
&\leq \frac{-1}{\varrho^{1/\kappa}(s)\,r^{1/\kappa}(s)\,R^2(s)}\left[L + \frac{1}{\kappa}w(\eta(s))\int_{s_1}^{s}R(\rho)\,R^{\kappa}(\eta(\rho))\,Q(\rho)\,\mathrm{d}\rho\right] \\
&= \frac{-1}{\varrho^{1/\kappa}(s)\,r^{1/\kappa}(s)\,R^2(s)}\left[L - \frac{1}{\kappa}w(\eta(s))\int_{s_0}^{s_1}R(\rho)\,R^{\kappa}(\eta(\rho))\,Q(\rho)\,\mathrm{d}\rho\right. \\
&\quad \left. + \frac{1}{\kappa}w(\eta(s))\int_{s_0}^{s}R(\rho)\,R^{\kappa}(\eta(\rho))\,Q(\rho)\,\mathrm{d}\rho\right].
\end{aligned} \tag{15}$$

Since $w \to 0$ as $s \to \infty$, there is a $s_2 \geq s_1$ such that

$$L - \frac{1}{\kappa}w(\eta(s))\int_{s_0}^{s_1}R(\rho)\,R^{\kappa}(\eta(\rho))\,Q(\rho)\,\mathrm{d}\rho > 0 \text{ for } s \geq s_2. \tag{16}$$

Combining (15) and (16), we obtain

$$w'(s)+\left(\frac{1}{\kappa\varrho^{1/\kappa}(s)\,r^{1/\kappa}(s)\,R^{2}(s)}\int_{s_0}^{s}R(\rho)\,R^{\kappa}(\eta(\rho))\,Q(\rho)\,\mathrm{d}\rho\right)w(\eta(s))\leq 0. \tag{17}$$

Now, we have w is a positive solution to (17). According to Theorem 1 in [42], equation (14) holds as well a positive solution, a contradiction.

Using the oscillation criteria of solutions of first-order differential equations, which are known in the literature, we present the following corollary:

Corollary 1. *Assume that (9) holds. If*

$$\liminf_{s\to\infty}\int_{\eta(s)}^{s}\frac{1}{\varrho^{1/\kappa}(u)\,r^{1/\kappa}(u)\,R^{2}(u)}\int_{s_0}^{u}R(\rho)\,R^{\kappa}(\eta(\rho))\,Q(\rho)\,\mathrm{d}\rho\mathrm{d}u>\frac{\kappa}{\mathrm{e}}, \tag{18}$$

hence equation (1) oscillates.

Proof. Using Theorem 2 in [43], condition (18) implies oscillation of equation (14). Thus, from Theorem 1, equation (1) oscillates.

Using a different approach, we can establish another comparison criterion by relating the oscillation of the solutions of equation (1) to a first-order equation, as in the following theorem:

Theorem 2. *Equation (1) oscillates if condition (9) holds and the DDE*

$$H'(s)+\frac{1}{\kappa}R^{\kappa}(\eta(s))\,Q(s)\,H(\eta(s))\leq 0 \tag{19}$$

is oscillatory.

Proof. Assume the contrary that x is a positive solution to (1). As in the demonstration of Lemma 3, we arrive at (10). Using (P_3), (10) reduces to

$$\left[\varrho^{1/\kappa}(s)\,r^{1/\kappa}(s)\,z'(s)\right]'\leq-\frac{1}{\kappa}R^{\kappa}(\eta(s))\,Q(s)\,\varrho^{1/\kappa}(\eta(s))\,r^{1/\kappa}(\eta(s))\,z'(\eta(s))\,.$$

Then, $H:=\varrho^{1/\kappa}r^{1/\kappa}z'$ is a positive solution to

$$H'(s)+\frac{1}{\kappa}R^{\kappa}(\eta(s))\,Q(s)\,H(\eta(s))\leq 0.$$

According to Theorem 1 in [42], equation (19) holds as well a positive solution, a contradiction.

Corollary 2. *If*

$$\liminf_{s\to\infty}\int_{\eta(s)}^{s}R^{\kappa}(\eta(\rho))\,Q(\rho)>\frac{\kappa}{\mathrm{e}}, \tag{20}$$

then equation (1) oscillates.

Proof. First, we note that (9) is necessary for the validity of (20). Using Theorem 2 in [43], condition (20) implies oscillation of equation (19). Thus, from Theorem 2, equation (1) oscillates.

Example 1. *Consider the NDDE*

$$\left(x(s)+\frac{1}{2}x(\lambda s)\right)''+\frac{\ell}{s}\left(x(s)+\frac{1}{2}x(\lambda s)\right)'+\sum_{i=1}^{n}\frac{k_i}{s^2}x(\mu_i s)=0, \tag{21}$$

where $\ell\in[0,1)$, $\lambda,\mu_i\in(0,1]$, *and* $k_i>0$ *for* $i=1,2,...,n$. *Then,* $\varrho(s)=s^{\ell}$,

$$R(s)=\frac{1}{1-\ell}s^{1-\ell},$$

and

$$Q(s)=\frac{1}{2}s^{\ell}\sum_{i=1}^{n}\frac{k_i}{s^2}.$$

Assume that

$$\mu_0:=\min\{\mu_i,\ i=1,2,...,n\},$$

and

$$K:=\sum_{i=1}^{n}k_i.$$

Then, using Corollary 2, equation (21) oscillates if

$$\frac{1}{1-\ell}\mu_0^{1-\ell}K\ln\frac{1}{\mu_0}>\frac{2}{\mathrm{e}}.$$

5. Numerical Solutions

In this section, we include figures that show the numerical solution of the following example

Example 2. *Consider the NDDE*

$$\frac{d}{ds}\left((s+1)\frac{d}{ds}\left[x(s)+0.4\,x(s-1)\right]\right)$$
$$+\frac{1}{s+1}\frac{d}{ds}\left[x(s)+0.4\,x(s-1)\right]+2\,x(s-2)+2\,x(s-3)=0 \tag{22}$$

with history $x(s)=\cos(s)$ *for* $s\le 3$.

We solve equation (22) numerically by transforming it into a first-order system based on defining $x_1(s)=x(s)$, $x_2(s)=x_1'(s)$ *in the following form*

$$x_1'(s) \;=\; x_2(s),$$

$$x_2'(s) = -0.4\,x_2'(s-1) - \frac{s+2}{(s+1)^2}\left(x_2\left(s\right) + 0.4\,x\left(s-1\right)\right) - \frac{2}{s+1}\left(x_1\left(s-2\right) + x_1\left(s-3\right)\right). \tag{23}$$

We then use Matlab 2024 to find the numerical solutions of the transformed system (23).

The numerical solution of the given equation (22) is shown in Figure 1. The oscillatory behavior is present and can be seen in the solutions. The numerical solution shown in this figure is consistent with the analytical investigations discussed in the previous section.

Figure 1 shows the numerical solution of the given equation (22), whereas Figure 2 illustrates the numerical solution of the same equation in the absence of the damping term. The effect of damping can be clearly seen when comparing the two figures.

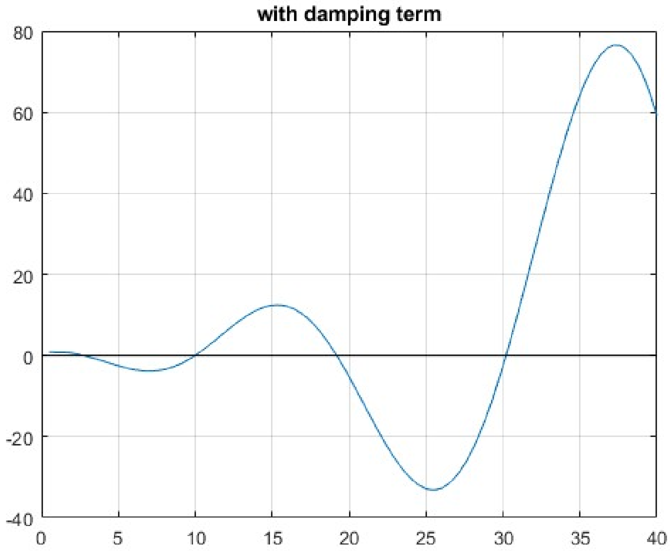

Figure 1: Numerical solution of equation (22)

6. Conclusion

Analyzing the monotonic, asymptotic, and oscillatory characteristics of solutions to DEs is fundamental for understanding the behavior of the models described by these equations. As it is shown in our findings, we derive some new monotonic and asymptotic properties for positive solutions to NDEs with multiple delays and a damping term (1). We then used these properties to derive sufficient criteria for the oscillation of all solutions of the equation under consideration. We also applied our outcomes to the special case in Example 21. The improved methodology employed in the analysis and investigation of the oscillatory behavior of equations with numerous delays and a middle term is what

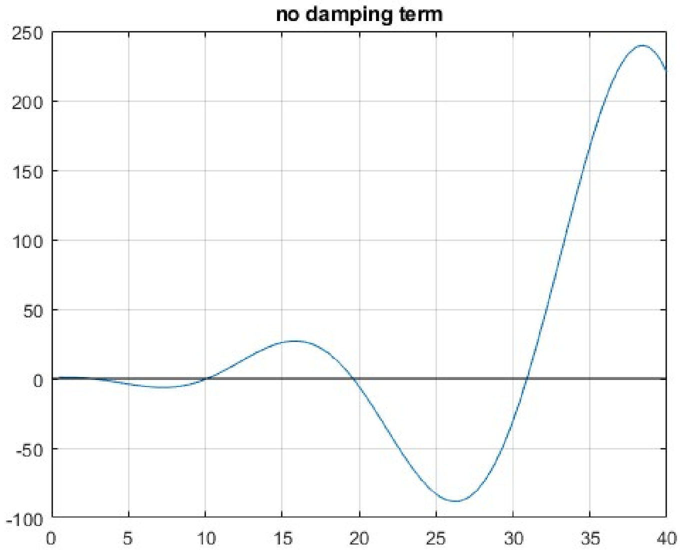

Figure 2: Numerical solution of equation (22) without damping

makes our results distinctive. A natural progression and forthcoming challenge of this research is examining the oscillatory behavior and asymptotic characteristics of solutions for higher-order neutral delay differential equations, where the interplay between delay and neutral components becomes progressively intricate.

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Conflict of interest

The authors declare there is no conflicts of interest.

Data availability statement

No data resulting from this study.

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