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The Energy of Cameron-Walker Graphs

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Abstract. The Cameron-Walker graphs are the graphs for which their matching number equals their induced matching number. In this paper, we study lower bounds for the graph energy in terms of induced matching numbers for the general graphs with equal matching and induced matching number. Moreover, we study the lower bounds of the graph energy of Cameron-Walker graphs.

2020 Mathematics Subject Classifications: 05C50, 05C70, 05C35

Key Words and Phrases: Cameron-Walker graph, matching number, graph energy, lower bound

1. Introduction

The energy of a graph is a measure that directly connects to Hückel Theory and is defined as the sum of the absolute values of the eigenvalues of its adjacency matrix. Let G be a simple, undirected graph with vertex set $V(G)$ and edge set $E(G)$. A *matching* in G is a subset $M \subseteq E(G)$ such that for any $e_1, e_2 \in M$, we have $e_1 \cap e_2 = \emptyset$. The *matching number*, denoted by $\mathrm{m}(G)$, is the maximum size of a matching in G. A matching M in a graph G is called an *induced matching* if, for any distinct $e_1, e_2 \in M$, there exists no $e \in E(G)$ such that $e \cap e_1 \neq \emptyset$ and $e \cap e_2 \neq \emptyset$. The *induced matching number* of G, denoted by $\mathrm{im}(G)$, is the maximum size of an induced matching in G. The adjacency matrix $A(G)$ of a graph $G = (V(G), E(G))$ is a symmetric matrix with entries 0 and 1. The energy of a graph G with $|V(G)| = n$ is defined as

$$\varepsilon(G) = \sum_{i=1}^{n} |\lambda_i|$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of the adjacency matrix of G.

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The *matching polynomial* of a graph G is given by

$$\mu(G) = \mu(G, \lambda) = \sum_{k \geq 0} (-1)^k m(G, k) \lambda^{n-2k}$$

where $m(G, k)$ is the number of k-matchings in G. The *matching energy* (ME) of G is defined as the sum of the absolute values of the zeros of its matching polynomial. This topic has been extensively studied [1–5].

A fundamental question in spectral graph theory concerns lower bounds on the energy of a graph in terms of its matching number. In [6], Wong et al. proved that

$$\varepsilon(G) \geq 2\mathrm{im}(G)$$

for any graph G. Moreover, they established the refined bound

$$\varepsilon(G) \geq 2\mathrm{im}(G) + \frac{\sqrt{5}}{5} c_1(G),$$

where $c_1(G)$ denotes the number of disjoint odd cycles in G. Later, in [7], Ashraf improved this result by showing that

$$\varepsilon(G) \geq 2\mathrm{im}(G) + c_0(G),$$

where $c_0(G)$ represents the number of disjoint odd cycles of length at least 5.

In this paper, we extend these results to the class of graphs known as Cameron-Walker graphs in which the matching number equals the induced matching number. In Section 3, for a given constant induced matching number, we determine the graph with minimum energy using matching energy. In Section 4, we establish new lower bounds on the energy of graphs where the matching and induced matching numbers are equal.

2. Preliminaries

This chapter is devoted to the definitions and previous results that will be used in the rest of the paper.

Definition 1. *Let G be a simple undirected graph of order n. If $m_k(G)$ is the number of k-matchings of G, $k \in \{0, 1, 2, ..., \lfloor \frac{n}{2} \rfloor\}$, then the matching energy of G is*

$$\mathrm{ME}(G) = \frac{2}{\pi} \int_0^\infty \frac{1}{x^2} \ln\Big[\sum_{k \geq 0} m_k(G) x^{2k}\Big] dx.$$

By the monotonicity of the logarithm, one can define a quasi-order relation "$\succeq$" as the following: Let G_1 and G_2 be two graphs. Then $G_1 \succeq G_2 \Longleftrightarrow m_k(G_1) \geq m_k(G_2)$ for all k.

The following theorem can provide the matching energy of a graph.

Theorem 1. *([1], Theorem 1) Let G be a simple graph and let $\sigma_1, \sigma_2, ..., \sigma_n$ be the zeros of the matching polynomial of G. Then,*

$$\mathrm{ME}(G) = \sum_{i=1}^{n} |\sigma_i|.$$

If $e = uv$ is an edge in G, then the equality $m(G; k) = m(G-e, k)+m(G-u-v; k-1)$ holds. So $m(G; k)$ increases whenever an edge is added to the graph. Hence, we can give the following result.

Theorem 2. *([1], Theorem 3) Let G be a graph and e be its edge. If $G-e$ is obtained by removing e from G and keeping all the vertices of G remaining, then $\mathrm{ME}(G-e) < \mathrm{ME}(G)$.*

The following theorem states that energy and matching energy coincide for trees.

Theorem 3. *([1], Theorem 2) If a graph G has no cycle, then its matching energy and energy are equal.*

In [8], Arizmendi et al. studied the vertex energy in a graph energy. The following theorem gives the lower bound of a vertex energy in terms of its degree.

Theorem 4. *([8], Theorem 3.3) Let G be a connected graph with at least one edge. Then for all $v_i \in V(G)$*

$$\varepsilon_G(v_i) \geq \frac{d_i}{\Delta}.$$

Equality holds if and only if G is isomorphic to the complete bipartite graph $K_{d,d}$.

3. Graph parameters and energy

In this section, we study the graphs with minimal energy with induced matching number $\mathrm{im}(G)$. Moreover, we provide lower bounds on graph energy with respect to the induced matching number.

Lemma 1. *Let G be any graph and H be its induced subgraph. Then $\varepsilon(H) \leq \varepsilon(G)$.*

Proof. Since H is an induced subgraph of G, then adjacency matrix of G is

$$A(G) = \begin{bmatrix} A(H) & X \\ X^T & A(G \backslash H) \end{bmatrix}$$

which implies that $A(H)$ is the principal submatrix of $A(G)$. Thus, the eigenvalues of $A(H)$ are less than those of $A(G)$. Therefore, $\varepsilon(H) < \varepsilon(G)$. If $G = H$, then $A(G\backslash H) = \mathbf{0}$, thus eigenvalues of $A(H)$ and $A(G)$ are same and $\varepsilon(H) = \varepsilon(G)$.

Next, we define a new graph called rake-graph RG_n over n vertices, and we show that this graph has the minimum energy among the graphs with induced matching number n.

Definition 2. *Let $n \geq 2$ be an integer. A graph RG_n, called a* rake graph*, is a graph on $2n+1$ vertices constructed as follows: take n disjoint copies of the complete graph K_2, and connect each of their vertices to a single additional vertex. Formally,*

$$V(RG_n) = \{v_0\} \cup \bigcup_{i=1}^{n} \{u_i, v_i\},$$

$$E(RG_n) = \bigcup_{i=1}^{n} \{(u_i, v_i), (v_0, v_i)\}.$$

Equivalently, the rake graph can be viewed as n disjoint edges (i.e., K_2 components), each of which is attached to a single central vertex v_0 via one of its endpoints.

It is easy to verify that both the matching number and the induced matching number of RG_n are equal to n.

Example 1. *Figure 1 depicts the graph RG_4. The matching number and the induced matching number of this graph are 4. This graph has minimal energy among the connected graphs with induced matching number 4 (See Theorem 5).*

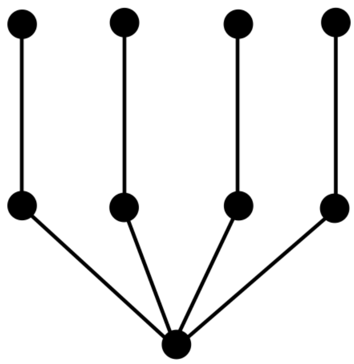

Figure 1: Graph RG_4.

If $\text{im}(G) = 1$, then it is clear that K_2 is the graph with minimum energy among the graphs with $\text{im}(G) = 1$. In the following theorem, we study the graphs with minimum energy among those with a constant induced matching number $\text{im}(G) > 2$.

Theorem 5. *Let G be a connected graph with induced matching number $\text{im}(G) > 2$. Then the energy and the matching energy of G are minimized by the graph $RG_{\text{im}(G)}$.*

Proof. In the view of Theorem 2, the graph G must be a tree since it has a minimal matching energy. By Theorem 3, it is enough to show that this graph has minimal matching energy with the given induced matching number. Let M be the set of induced matchings. Then $G[M]$ forms a disjoint union of graphs K_2 with the number of such graphs equal to $\mathrm{im}(G)$. Since M is a set of induced matchings, this implies that for each $e_i \in M$, there exists at least one edge e_i' adjacent to e_i with $e_i' \notin M$. Moreover, e_i' is adjacent to only e_i, otherwise e_i' would be a common edge of two edges in M and a contradiction. By minimality, we can consider one of the vertices of the edges $e_i = u_i v_i$ of M is a pendant vertex, say u_i.

Since $m(G;k) = m(G - e_i; k) + m(G - u_i - v_i; k-1)$ for all $e_i = u_i v_i \in M$, then the subgraph after removing edges $e_i \in M$, we have a cut-vertex left for minimality. Since the matching energy of a tree coincides with its energy, the graph $RG_{\mathrm{im}(G)}$ has the minimum energy among the graphs with induced matching number $\mathrm{im}(G) \geq 2$.

Theorem 6. *Let $G = (V, E)$ be a graph with induced matching number $\mathrm{im}(G)$. Then the followings hold:*

(1) For any graph G, $\varepsilon(G) \geq 2\mathrm{im}(G)$, and equality holds if and only if G is isomorphic to the disjoint union of graphs K_2.

(2) If G is connected and $\mathrm{im}(G) \geq 2$, then $\varepsilon(G) \geq \varepsilon(RG_{\mathrm{im}(G)})$.

Proof. (1) Assume that $M \subseteq E$ be an induced matching set. The induced subgraph on $G[M]$ forms a disjoint union of graphs K_2. Since each K_2 contributes 2 to the $\varepsilon(G)$, then by Lemma 1, we get $\varepsilon(G) \geq 2\mathrm{im}(G)$. If we assume G is isomorphic to the disjoint union of graphs K_2, i.e., $G \cong nK_2$, this implies that $\mathrm{im}(G) = n$. Since $\varepsilon(G) = 2n$, then equality holds.

(2) Assume that G is a connected graph. Since the minimal energy graph with induced matching number $\mathrm{im}(G) \geq 2$ is $RG_{\mathrm{im}(G)}$ by Theorem 5, then it is clear that $\varepsilon(G) \geq \varepsilon(RG_{\mathrm{im}(G)})$ for any connected graph G.

4. Energy of graphs with equal matching and induced matching number

Cameron and Walker gave a characterization of undirected connected finite simple graphs that satisfy $\mathrm{m}(G) = \mathrm{im}(G)$ in [9]. These graphs are known as Cameron-Walker graphs and have previously been studied from a commutative algebra perspective, particularly in relation to edge ideals and their algebraic invariants [10]. The authors gave a slightly modified definition of these graphs. Next, we provide the definitions of this graph in the light of [10].

Definition 3. *A finite simple and connected graph G satisfies $\mathrm{m}(G) = \mathrm{im}(G)$ if G is one of the following:*

- *(i) G is a star graph,*
- *(ii) G is a star triangle,*
- *(iii) G is a finite graph consisting of a connected supporting bipartite graph with vertex partition $U \sqcup V$ such that there is at least one leaf edge attached to each vertex $u \in U$ and that there may be some pendant triangles connected to each vertex $v \in V$.*

Definition 4. *A finite connected simple graph G is called a Cameron–Walker graph if* $\mathrm{im}(G) = \mathrm{m}(G)$ *and if G is neither a star nor a star triangle.*

The following proposition provides a lower bound for graphs that include an induced subgraph with $\mathrm{im}(G) = \mathrm{m}(G)$. *These graphs do not belong to the class of Cameron-Walker graphs.*

Proposition 1. *Let G be a graph containing an induced subgraph consisting of n pendant triangles sharing a common vertex. Then the energy of G satisfies $\varepsilon(G) \geq 2n + 1$.*

Proof. Let G' be the induced subgraph of G consisting of n pendant triangles sharing a common vertex. The adjacency matrix $A(G')$ of G' is $\begin{bmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1}^T & Q \end{bmatrix}$. Since $\mathrm{rank}(Q) = 2n$ and $\mathbf{1}$ is $1 \times 2n$ matrix with all 1's, it is clear that $\mathrm{rank}(A(G')) = 2n+1$. Hence $\varepsilon(G') \geq 2n+1$ by Lemma 2 in [11]. Thus, by Lemma 1, we get $\varepsilon(G) \geq 2n + 1$.

In the next proposition, we give a lower bound for the energy of a Cameron-Walker graph in terms of its matching number and maximum degree.

Proposition 2. *The graph $RG_{\mathrm{im}(G)}$ is the minimal energy Cameron-Walker graph with (induced) matching number* $\mathrm{im}(G) \geq 2$.

Proof. The graph $RG_{\mathrm{im}(G)}$ is a Cameron-Walker graph with supporting bipartite graph $K_{1,\mathrm{im}(G)}$. And the number of leaves is $\mathrm{im}(G)$, which implies (induced) matching number is $\mathrm{im}(G)$. Thus, by Theorem 5, $RG_{\mathrm{im}(G)}$ is a Cameron-Walker graph of minimal energy.

Proposition 3. *Let G be a Cameron-Walker graph. Then the followings hold:*

(1) If G is a C_3-free graph with m disjoint leaf edges, then $\varepsilon(G) \geq \varepsilon(RG_m)$.

(2) If G has m disjoint leaf edges and n disjoint pendant triangles, then $\varepsilon(G) \geq 2m+4n$.

(3) If G has m disjoint leaf edges and n pendant triangles, then $\varepsilon(G) \geq 2m + 2n + 1$.

Proof. (1) A Cameron-Walker graph is a connected graph. Since G is C_3-free and possesses m leaves, then there are no pendant triangles and the (induced) matching number of G is m. Hence, by Proposition 2, we conclude that $\varepsilon(G) \geq \varepsilon(RG_m)$.

(2) Let L be the set of disjoint leaf edges and let T be the set of disjoint pendant triangles. In a Cameron-Walker graph, it is clear that $L \cap T = \emptyset$. Since each leaf edge has energy equal to 2 and each pendant triangle has energy equal to 4, by Lemma 1 it follows that $\varepsilon(G) \geq 2m + 4n$.

(3) Now, without loss of generality, we assume that all the pendant triangles of G have a common vertex. Therefore, the induced subgraph on n pendant triangles has energy at least $2n + 1$ by Proposition 1. Since each leaf edge contributes 2 to the energy, then we get that $\varepsilon(G) \geq 2m + 2n + 1$.

Example 2. *The following graph G is a Cameron-Walker graph with 4 leaf edges and 2 pendant triangles. The energy $\varepsilon(G) \cong 19.55$ and (induced) matching number is 6.*

In the following theorem, we give a lower bound for a Cameron-Walker graph in terms of its matching number and number of triangles.

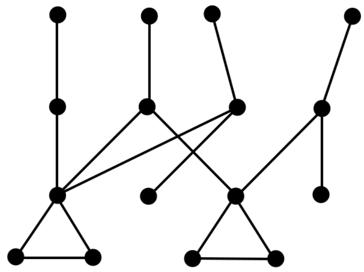

Figure 2: A Cameron-Walker Graph with 2 pendant triangles

Theorem 7. *If G is a Cameron-Walker graph with maximum degree Δ, then*

$$\varepsilon(G) \geq 2\mathrm{m}(G) + \frac{3}{\Delta} C_3(G),$$

where $m(G)$ is the matching number and $C_3(G)$ is the number of disjoint pendant triangles in G.

Proof. Let M be the maximum matching set of G. And let $C = \{C_1, C_2, ..., C_r\}$ be the set of disjoint triangles of G. Since G is a Cameron-Walker graph, the set M consists of disjoint pendant edges and edges of triangles with end vertices that have degree 2. Without loss of generality, let us assume that all the pendant triangles are disjoint in G. Let the sets $V(G)$ and $V(G[M])$ be the vertex sets of G and the induced graph on M, respectively. Thus, $\varepsilon(G) \geq \varepsilon(G[M]) + \varepsilon(V(G)\backslash G[M])$ by Lemma 1. It is clear that $\varepsilon(G[M]) = 2\mathrm{m}(G)$. And from Theorem 4, for any $v_i \in V(G)\backslash V(G[M])$ we have $\varepsilon(v_i) \geq \frac{3}{\Delta}$ since the degree of a vertex in $V(G)\backslash V(G[M])$ has degree at least 3. Thus, it follows that $\varepsilon(G) \geq 2\mathrm{m}(G) + \frac{3}{\Delta}C_3(G)$.

5. Conclusion

We investigated the graph energy of a special class of graphs—those for which the matching number equals the induced matching number, with particular focus on Cameron-Walker graphs. Using spectral graph-theoretic techniques, we established new lower bounds for the energy in terms of the induced matching number and provided comparisons with known bounds in the literature. Our results demonstrate that the structural constraints of Cameron-Walker graphs yield meaningful spectral restrictions, thereby influencing their energy in predictable ways. The Cameron-Walker graphs can have unique energy properties due to their controlled structure. Further studies can explore their other spectral properties.

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References

[1] I. Gutman and S. Wagner. The matching energy of a graph. *Discrete Appl. Math.*, 160:2177–2187, 2012.

[2] S. K. Ghezelahmad. Lower bounds on matching energy of graphs. *Discrete Appl. Math.*, 307:153–159, 2022.

[3] L. Zou and H.-H. Li. On matching energy of bicyclic graphs. *Int. J. Graph Theory Appl.*, 1(2):97–110, 2015.

[4] X. Chen, X. Li, and H. Lian. The matching energy of random graphs. *Discrete Appl. Math.*, 193:102–109, 2015.

[5] S. Ji, H. Ma, and G. Ma. The matching energy of graphs with given edge connectivity. *J. Inequal. Appl.*, 2015:415, 2015.

[6] D. Wong, X. Wang, and R. Chu. Lower bounds of graph energy in terms of matching number. *Linear Algebra Appl.*, 549:276–286, 2018.

[7] F. Ashraf. Energy, matching number and odd cycles of graphs. *Linear Algebra Appl.*, 577:159–167, 2019.

[8] O. Arizmendi, J. F. Hidalgo, and O. Juarez-Romero. Energy of a vertex. *Linear Algebra Appl.*, 557:464–495, 2018.

[9] K. Cameron and T. Walker. The graphs with maximum induced matching and maximum matching the same size. *Discrete Math.*, 299:49–55, 2005.

[10] T. Hibi, A. Higashitani, K. Kimura, and A. B. O'Keefe. Algebraic study on cameron-walker graphs. *J. Algebra*, 422:257–269, 2015.

[11] S. Akbari, E. Ghorbani, and S. Zare. Some relations between rank, chromatic number and energy of graphs. *Discrete Math.*, 309:601–605, 2009.