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Novel Fractional Hermite–Hadamard and Product-Type Inequalities via Raina Function and Preinvex Mappings with Entropy Applications

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Abstract. In this paper, we employ the Atangana-Baleanu fractional integral operator to develop new versions of Hermite–Hadamard and Pachpatte-type integral inequalities within the framework of generalized convexity involving Raina's function. By this approach, we derive a novel fractional integral identity associated with Raina's functions. Furthermore, leveraging Young's inequality, the power mean inequality, and Hölder's inequality, we establish several new extensions of Hermite–Hadamard-type inequalities via the Atangana–Baleanu fractional operator. Our results significantly improve upon existing findings, both in terms of generality and special cases. To validate our results, we provide remarks that recover various earlier inequalities. Additionally, we present applications related to entropy measures that demonstrate the practical utility of our main findings.

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1. Introduction

Convex analysis offers a strong mathematical foundation for researching and resolving issues in a wide range of fields. Convexity is a widely developed concept that is fundamentally introduced in the widely acclaimed book Inequalities by G. Pólya, G.H. Hardy, and J.E. Littlewood [1]. This foundational work, which focused solely on the study of inequalities, swiftly established itself as a standard reference for mathematicians and is still a must-read for anyone interested in learning more about this fascinating subject. Engineering [2], finance [3], economics [4], and optimization [5] are just a few of the fields in which the theory of convex functions finds extensive use. Furthermore, convex analysis serves as the foundation for the creation of strong numerical techniques, which offer the fundamental resources required to successfully explore and resolve challenging mathematical issues.

Mathematical inequalities offers a key basis for comprehending the perform of functions under integration, leading to crucial applications in both theoretical and applied mathematics. Fractional integrals inequalities involving convex functions, which establish connections between the integrals of convex functions and their values at particular points, are an effective tool in mathematical analysis. They are used in a number of areas, such as optimization, information theory, and probability theory. For numerical methods, such as the trapezoidal rule [6], Simpson's rule [7], and others, these inequalities are essential, particularly when estimating the error bounds.

Growing exponentially in popularity, fractional calculus allows one to define fractional derivatives and fractional integrals in several ways. Notably, the first concept of fractional calculus was put forth by Leibniz and L'Hospital in 1695. Particularly in view of the flaws of traditional calculus, the roots and ideas of fractional calculus have lately attracted great attention. Fractional calculus is the study of fractional order integrals and derivatives together with their applications in real and complex domains. Fractional integral inequalities allow us to determine the exact stability and uniqueness of fractional differential equations. Almost every nonlinear discipline or area of study in the modern world is impacted by fractional methods and tools. The subject fractional calculus has many applications in control systems [8], transform theory [9], nanotechnology [10], modeling [11, 12], fluid flow [13], mathematical biology [14], epidemiology [15], optimal control [16, 17] and physics [18]. The aforementioned widespread viewpoints and significance made the discussion of fractional operators intriguing to readers and academics. When defining and assessing statistical issues and formulas that resemble quadratures, this theory is helpful. An important development in fractional calculus is the Atangana-Baleanu fractional integral operator (ABFIO), which provides improved modeling capabilities for complex systems. For other relevant inequalities involving different fractional operators, we refer the reader to the following sources [19–23]

Because of their strong applications in analysis, differential equations, and the applied sciences, fractional convex integral inequalities have attracted a lot of attention lately. By using generalized proportional fractional integral operators defined with respect to another function, Rashid et al. [24] introduced new estimates of integral inequalities and offered a versatile framework for extending classical results like Hermite–Hadamard inequalities. Similarly, a larger class of functions could be analyzed under fractional integral operators thanks to the development of some mean-type fractional integral inequalities by Samraiz et al. [25] using various convexity concepts. However, Akın [26] extended the use of convex integral inequalities in dynamic systems and brought continuous and discrete fractional analysis together by establishing fractional maximal delta integral inequalities on time scales. To gain greater control over memory effects, Mohammed et al. [27] derived generalized Hermite–Hadamard inequalities using tempering fractional integrals. In the context of weighted inequalities, Rahman et al. [28] investigated weighted fractional integral inequalities for Chebyshev functionals, offering useful resources for weighting structures and monotonicity in fractional calculus. By developing generalized reverse Minkowski inequalities using conformable fractional integrals, Rashid et al. [29] expanded on classical inequalities and provided insight into fractional variational problems. Rahman et al. [30] recently introduced hattaf fractional operators, which are a more general type of fractional operator. They also make it possible to use new types of integral inequalities in many different fields. Tunç et al. [31] enhanced the resources for fractional integral analysis by proposing novel generalized fractional integral operators that resulted in new Hermite–Hadamard and Ostrowski-type inequalities. Sahoo et al. [32] presented generalized exponential-type convex functions and employed them to formulate novel Ostrowski-type fractional integral inequalities, thereby encompassing a broader spectrum of convexity behavior. Kashuri et al. [33] provided a unifying approach and illustrated the flexibility of the fractional integral framework by constructing integral inequalities using general fractional operators. Through their work with fractional calculus involving general analytic kernels, Mohammed and Fernandez [34] expanded on this idea, allowing for a highly flexible approach to solving integral inequalities suitable for a range of applications. Finally, Rahman et al. [35] studied fractional integral inequalities for monotone weighted Chebyshev functionals, which are crucial for fractional analysis that involves monotonicity and weighted conditions. For additional results on inequalities derived via alternative fractional operators, the reader is referred to the following literature [36–40].

The primary contribution of this study lies in the introduction of a novel class of generalized convex mappings, developed primarily based on Condition A, which is formally defined below. This newly established class incorporates the Raina function in a manner that, to the best of our knowledge, has not previously been explored in the context of the related inequalities. Moreover, while prior work on inequalities associated with generalized convexity has predominantly utilized classical integrals, Riemann–Liouville fractional integrals, and Caputo–Fabrizio operators, our approach leverages the Atangana–Baleanu fractional operators to generalize these inequalities within the framework of the newly introduced convexity concept. To substantiate and validate our findings, we also present several remarks and special cases demonstrating that many known results from the literature can be recovered as particular instances of our general framework. This comparative analysis underscores the broader applicability and enhanced generality of the results developed in this work.

The organization of this paper is as follows. In Section 2, we revisit several fundamental concepts and definitions that form the foundation for our subsequent analysis. Section 3.1 introduces the notion of generalized convex functions (GCF) and explores their key algebraic properties. In Section 4, we present a novel Hermite–Hadamard-type inequality based on the Atangana–Baleanu fractional integral operator (ABFIO), accompanied by several interesting corollaries and remarks. Section 5 is devoted to deriving a new integral identity, which serves as the basis for refined versions of the Hermite–Hadamard inequality. In Section 6, we investigate a new class of Pachpatte-type inequalities via ABFIO, along with related corollaries and remarks. Finally, Section 8 summarizes the main findings and suggests potential directions for future research.

2. Preliminaries

This section reviews key definitions and concepts essential for our subsequent analysis, including convexity, the Hermite–Hadamard inequality, Mittag–Leffler functions, generalized convex sets, and generalized convex functions. Additionally, we discuss Condition C, Hölder's inequality, and the power mean inequality. The section concludes with a brief overview of the Caputo–Fabrizio derivative and the Atangana–Baleanu fractional integral operator (ABFIO), which are fundamental to our study.

Definition 1 ([2]). *A real-valued function* Λ *is said to be convex, if*

$$\Lambda\left(\mathfrak{x}\mathfrak{l}_a + (1-\mathfrak{x})\,\mathfrak{l}_b\right) \leq \mathfrak{x}\Lambda\left(\mathfrak{l}_a\right) + (1-\mathfrak{x})\,\Lambda\left(\mathfrak{l}_b\right), \tag{2.1}$$

holds for all $\mathfrak{l}_a, \mathfrak{l}_b \in I$ *and* $\mathfrak{x} \in [0,1]$.

The most famous inequality involving convex functions is the Hermite-Hadamard inequality [41–43] stated as:

Theorem 1. *If* $\Lambda : [\mathfrak{l}_a, \mathfrak{l}_b] \to \mathbb{R}$ *is a convex function, then*

$$\Lambda\left(\frac{\mathfrak{l}_a + \mathfrak{l}_b}{2}\right) \leq \frac{1}{\mathfrak{l}_b - \mathfrak{l}_a}\int_{\mathfrak{l}_a}^{\mathfrak{l}_b} \Lambda(x)dx \leq \frac{\Lambda(\mathfrak{l}_a) + \Lambda(\mathfrak{l}_b)}{2}. \tag{2.2}$$

The inequality has been used to establish limits and estimates in information theory, particularly in connection with quantum calculus and quantum integral inequalities. Geometric settings of this inequality help to establish relationships between the average over an interval and the value of a function at its midpoint. Hermite-Hadamard inequality is a useful tool for studying a range of economic phenomena involving convex functions, including asset pricing and optimization, income distribution, and production.
Raina [44] introduced a generalized class of functions defined as

$$\mathscr{D}^{\kappa}_{\mu,\delta}(z) = \mathscr{D}^{\kappa(0),\,\kappa(1),\ldots}_{\mu,\delta}(z) = \sum_{k=0}^{\infty} \frac{\kappa(k)}{\Gamma(\mu k + \delta)} z^k, \tag{2.3}$$

where $\kappa = (\kappa(0), \kappa(1), \ldots, \kappa(k), \ldots)$, $\mu, \delta > 0$, and $|z| < R$. Equation (2.3) serves as a generalization of the classical Mittag–Leffler function.

If $\mu = 1$, $\delta = 0$, and $\kappa(k) = \dfrac{(\eta)_k(\theta)_k}{(\xi)_k}$ for $k = 0, 1, 2, \ldots$, where η, θ, ξ are complex parameters such that $\xi \notin \{0, -1, -2, \ldots\}$, and where the Pochhammer symbol is defined by

$$(\eta)_k = \frac{\Gamma(\eta + k)}{\Gamma(\eta)} = \eta(\eta + 1)\cdots(\eta + k - 1), \quad k = 0, 1, 2, \ldots,$$

and the domain is restricted to $|z| \leq 1$ with $z \in \mathbb{C}$, then equation (2.3) reduces to the classical hypergeometric function:

$$\mathscr{D}(\eta, \theta; \xi; z) = \sum_{k=0}^{\infty} \frac{(\eta)_k(\theta)_k}{k!(\xi)_k} z^k.$$

Moreover, if $\kappa = (1, 1, \ldots)$, $\mu = \alpha$, $\delta = 1$, with $\Re(\alpha) > 0$, then equation (2.3) reduces to the one-parameter Mittag–Leffler function:

$$\mathtt{E}_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1 + \alpha k)}. \tag{2.4}$$

Equation (2.4) is referred to as a classical Mittag–Leffler function. The Mittag–Leffler function appears usually in the study of fractional calculus and especially in the studies of fractional conjecture of the kinetic equation, super diffusive transport, random walks, Lévy flights, and in the studies of complicated structures.

Cortez presented the generalized convex set and the convex function pertaining to Raina's function in [45, 46].

Definition 2 (See [46]). *Let $\varrho = (\varrho(0), \ldots, \varrho(\mathfrak{v}), \ldots)$ and $\epsilon, \sigma > 0$. A set $X \neq \emptyset$ is said to be generalized convex, if*

$$\mathfrak{l}_a + \mathfrak{x}\, \Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b - \mathfrak{l}_a) \in X, \tag{2.5}$$

for all $\mathfrak{l}_a, \mathfrak{l}_b \in X$ and $\mathfrak{x} \in [0, 1]$.

Definition 3 (See [46]). *Let ϱ represent a bounded sequence then $\varrho = (\varrho(0), \ldots, \varrho(\mathfrak{v}), \ldots)$ and $\epsilon, \sigma > 0$. If real-valued Λ holds the following inequality*

$$\Lambda\Big(\mathfrak{l}_a + \mathfrak{x}\, \Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b - \mathfrak{l}_a)\Big) \leq \mathfrak{x}\Lambda(\mathfrak{l}_b) + (1 - \mathfrak{x})\Lambda(\mathfrak{l}_a), \tag{2.6}$$

for all $\mathfrak{l}_a, \mathfrak{l}_b \in X$, where $\mathfrak{l}_a < \mathfrak{l}_b$ and $\mathfrak{x} \in [0, 1]$, then Λ is said to be generalized convex function.

Remark 1. *If $\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b - \mathfrak{l}_a) = \mathfrak{l}_b - \mathfrak{l}_a > 0$, then achieve Definition 1.*

The following Condition-A first time explored by Ahmad et.al [47].

Condition C: Let X be generalized convex subset w.r.t. $\Delta^{\varrho}_{\epsilon,\sigma}(\cdot)$. For any $\mathfrak{l}_a, \mathfrak{l}_b \in X$ and $\mathfrak{x} \in [0,1]$,

$$\Delta^{\varrho}_{\epsilon,\sigma}\Big(\mathfrak{l}_a - (\mathfrak{l}_a + \mathfrak{x}\ \Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b - \mathfrak{l}_a))\Big) = -\mathfrak{x}\ \Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b - \mathfrak{l}_a),$$

$$\Delta^{\varrho}_{\epsilon,\sigma}\Big(\mathfrak{l}_b - \big(\mathfrak{l}_a + \mathfrak{x}\ \Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b - \mathfrak{l}_a)\big)\ \Big) = (1-\mathfrak{x})\ \Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b - \mathfrak{l}_a).$$

Note that, for every $\mathfrak{l}_a$, $\mathfrak{l}_b \in X$ and for all $\mathfrak{x}_1$, $\mathfrak{x}_2 \in [0,1]$ from Condition-A, we have

$$\Delta^{\varrho}_{\epsilon,\sigma}\Big(\mathfrak{l}_a + \mathfrak{x}_2\ \Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b - \mathfrak{l}_a) - (\mathfrak{l}_a + \mathfrak{x}_1\ \Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b - \mathfrak{l}_a))\Big) = (\mathfrak{x}_2 - \mathfrak{x}_1)\ \Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b - \mathfrak{l}_a). \tag{2.7}$$

Some well-known integral inequalities such as Hölder inequality and power-mean inequality will be used.

Theorem 2. *[48] Assume that $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Assume thats $\Lambda_1, \Lambda_2 : [x_1, x_2] \to \mathbb{R}$ are such that $|\Lambda_1|^p$ and $|\Lambda_2|^q$ are integrable on $[x_1, x_2]$. Then*

$$\int_0^1 |\Lambda_1(x)\Lambda_2(x)|dx \leq \Big(\int_0^1 |\Lambda_1(x)|^p dx\Big)^{\frac{1}{p}} \Big(\int_0^1 |\Lambda_2(x)|^q dx\Big)^{\frac{1}{q}}.$$

If we get $|\Lambda_1||\Lambda_2| = (|\Lambda_1|^{\frac{1}{p}})(|\Lambda_1|^{\frac{1}{q}}|\Lambda_2|)$ in the Hölder inequality, then we obtain the following power mean integral inequality as a simple result of the Hölder integral inequality.

Theorem 3. *[48] Assume that $\Lambda \geq 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Assume thats $\Lambda_1, \Lambda_2 : [x_1, x_2] \to \mathbb{R}$ are such that $|\Lambda_1|^p$ and $|\Lambda_2|^q$ are integrable on $[x_1, x_2]$. Then*

$$\int_0^1 |\Lambda_1(x)\Lambda_2(x)|dx \leq \Big(\int_0^1 |\Lambda_1(x)|dx\Big)^{1-\frac{1}{q}} \Big(\int_0^1 |\Lambda_1(x)|dx \int_0^1 |\Lambda_2(x)|^q dx\Big)^{\frac{1}{q}}.$$

With the advancement of fractional calculus, numerous mathematicians have introduced a variety of fractional derivative and integral operators to address complex phenomena arising in real-world applications. These operators aim to capture memory effects and hereditary properties inherent in many physical, biological, and engineering systems. Over time, several notable formulations have emerged in the literature, some of which are listed below.

In Caputo-Fabrizio (C-F) derivative operator, Atangana and Baleanu utilizing the Mittag-Leffler function and investigate the new derivative operators as follows.

Definition 4. *[49] Let $\Lambda \in H^1(\mathfrak{l}_a, \mathfrak{l}_b), \mathfrak{l}_b > \mathfrak{l}_a, \gamma \in [0,1)$ then, the definition of the Caputo-Fabrizio derivative is given by*

$${}^{ABC}_{\mathfrak{l}_a}D^{\gamma}_t[\Lambda(t)] = \frac{B(\gamma)}{1-\gamma}\int_{\mathfrak{l}_a}^t \Lambda'(x)E_\gamma\left[-\gamma\frac{(t-x)^\gamma}{(1-\gamma)}\right]dx. \tag{2.8}$$

Definition 5. *[49] Let $\Lambda \in H^1(\mathfrak{l}_a, \mathfrak{l}_b), \mathfrak{l}_b > \mathfrak{l}_a, \gamma \in [0,1)$ then, the definition of the Caputo-Fabrizio derivative is given by*

$${}^{ABR}_{\mathfrak{l}_a}D^{\gamma}_t[\Lambda(t)] = \frac{B(\gamma)}{1-\gamma}\frac{d}{dt}\int_{\mathfrak{l}_a}^t \Lambda(x)E_\gamma\left[-\gamma\frac{(t-x)^\gamma}{(1-\gamma)}\right]dx. \tag{2.9}$$

Equations (2.8) and (2.9) have a non-local kernel. Also in equation (2.8) when the function is constant we get zero.

The related fractional integral operator has been defined by Atangana-Baleanu as follows.

Definition 6. *[49] The fractional integral associated to the new fractional derivative with non-local kernel of a function $\Lambda \in H^1(\mathfrak{l}_a, \mathfrak{l}_b)$ is defined:*

$$ {}^{AB}_{\mathfrak{l}_a}I^{\gamma}\{\Lambda(t)\} = \frac{1-\gamma}{B(\gamma)}\Lambda(t) + \frac{\gamma}{B(\gamma)\Gamma(\gamma)}\int_{\mathfrak{l}_a}^{t} \Lambda(y)(t-y)^{\gamma-1}dy, $$

where $b > a, \gamma \in (0,1]$.

In [50], Abdeljawad and Baleanu introduced right hand side of the integral operator as follows: The right fractional new integral with Mittag-Leffler kernel of order $\gamma \in (0,1]$ is defined by

$$ {}^{AB}I^{\gamma}_{\mathfrak{l}_b}\{\Lambda(t)\} = \frac{1-\gamma}{B(\gamma)}\Lambda(t) + \frac{\gamma}{B(\gamma)\Gamma(\gamma)}\int_{t}^{\mathfrak{l}_b} \Lambda(y)(y-t)^{\gamma-1}dy. $$

3. The major results

In this section, we present our main results.

3.1. Generalized Convex Function and Its Properties

In this section, we utilize the Definition 3 and examine some of its algebraic properties.

Theorem 4. *If Λ_1, Λ_2 are two GCF, then $(\Lambda_1 + \Lambda_2)$ is also an GCF.*

Proof. Since given that Λ_1 and Λ_2 be two GCF, then

$$ \begin{aligned} & (\Lambda_1 + \Lambda_2)\left(\mathfrak{l}_a + \mathfrak{x}\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b - \mathfrak{l}_a)\right) \\ & = \Lambda_1(\mathfrak{l}_a + \mathfrak{x}\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b - \mathfrak{l}_a)) + \Lambda_2(\mathfrak{l}_a + \mathfrak{x}\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b - \mathfrak{l}_a)) \\ & \leq (1-\mathfrak{x})\,\Lambda_1(\mathfrak{l}_a) + \mathfrak{x}\Lambda_1(\mathfrak{l}_b) + (1-\mathfrak{x})\,\Lambda_2(\mathfrak{l}_a) + \mathfrak{x}\Lambda_2(\mathfrak{l}_b) \\ & = (1-\mathfrak{x})\left[\Lambda_1(\mathfrak{l}_a) + \Lambda_2(\mathfrak{l}_a)\right] + \mathfrak{x}\left[\Lambda_1(\mathfrak{l}_b) + \Lambda_2(\mathfrak{l}_b)\right] \\ & = (1-\mathfrak{x})\,(\Lambda_1 + \Lambda_2)(\mathfrak{l}_a) + \mathfrak{x}(\Lambda_1 + \Lambda_2)(\mathfrak{l}_b). \end{aligned} $$

This completes the proof.

Theorem 5. *If Λ is GCF, then $(c\Lambda)$ is also an GCF.*

Proof. Since Λ is GCF, and c is any constant number, then

$$ (c\Lambda)\left(\mathfrak{l}_a + \mathfrak{x}\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b - \mathfrak{l}_a)\right) \leq c\Big(\,(1-\mathfrak{x})\,\Lambda(\mathfrak{l}_a) + \mathfrak{x}\Lambda(\mathfrak{l}_b)\Big) $$

$$= (1-\mathfrak{x})\, c\Lambda\left(\mathfrak{l}_a\right) + \mathfrak{x} c\Lambda\left(\mathfrak{l}_b\right)$$
$$= \left(1-\mathfrak{x}\right)\left(c\Lambda\right)\left(\mathfrak{l}_a\right) + \mathfrak{x}(c\Lambda)\left(\mathfrak{l}_b\right).$$

This completes the proof.

Theorem 6. *Composition of two GCF is also an GCF.*

Proof.

$$\begin{aligned}\left(\Lambda_2 \circ \Lambda_1\right)\left(\mathfrak{l}_a + \mathfrak{x}\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b - \mathfrak{l}_a)\right) &= \Lambda_2(\Lambda_1(\mathfrak{l}_a + \mathfrak{x}\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b - \mathfrak{l}_a)))\\ &\le \Lambda_2\bigg(\left(1-\mathfrak{x}\right)\Lambda_1\left(\mathfrak{l}_a\right) + \mathfrak{x}\Lambda_1\left(\mathfrak{l}_b\right)\bigg)\\ &\le \left(1-\mathfrak{x}\right)\Lambda_2(\Lambda_1\left(\mathfrak{l}_a\right)) + \mathfrak{x}\Lambda_2(\Lambda_1\left(\mathfrak{l}_b\right))\\ &= \left(1-\mathfrak{x}\right)\left(\Lambda_2 \circ \Lambda_1\right)\left(\mathfrak{l}_a\right) + \mathfrak{x}(\Lambda_2 \circ \Lambda_1)\left(\mathfrak{l}_b\right).\end{aligned}$$

This completes the desired proof.

Theorem 7. *Let* $0 < \mathfrak{l}_a < \mathfrak{l}_b$, $\Lambda_j : \mathbb{X} = [\mathfrak{l}_a, \mathfrak{l}_b] \to [0, +\infty)$ *be a family of GCF and* $\Lambda(u) = \sup_j \Lambda_j(u)$. *Then,* Λ *is an GCF for* $m \in (0, 1]$, $\mathfrak{x} \in [0, 1]$, *and* $U = \{\Lambda \in [\mathfrak{l}_a, \mathfrak{l}_b] : \Lambda(\Lambda_{\mathfrak{x}}) < \infty\}$ *is an interval.*

Proof. Let $\mathfrak{l}_a, \mathfrak{l}_b \in U$, and $\mathfrak{x} \in [0, 1]$, then

$$\begin{aligned}\Lambda(\mathfrak{l}_a + \mathfrak{x}\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b - \mathfrak{l}_a)) &= \sup_j \Lambda_j(\mathfrak{l}_a + \mathfrak{x}\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b - \mathfrak{l}_a))\\ &\le \left(1-\mathfrak{x}\right)\sup_j \Lambda_j\left(\mathfrak{l}_a\right) + \mathfrak{x}\sup_j \Lambda_j\left(\mathfrak{l}_b\right)\\ &= \left(1-\mathfrak{x}\right)\Lambda\left(\mathfrak{l}_a\right) + \mathfrak{x}\Lambda\left(\mathfrak{l}_b\right) < \infty.\end{aligned}$$

This completes the proof.

4. Hermite–Hadamard Inequality Pertaining to AB Fractional Integral Operator

The main goal of this portion is to provide a new sort of the Hermite-Hadamard-type inequality for a *GCF* via ABFIO.

Theorem 8. *Let* $I \subseteq \mathbb{R}$ *be an open and non-empty convex subset and* $\mathfrak{l}_a, \mathfrak{l}_b \in I$ *with* $\mathfrak{l}_a < m\mathfrak{l}_a + \Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b - \mathfrak{l}_a)$. *If* $\Lambda : [\mathfrak{l}_a, \mathfrak{l}_a + \Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b - \mathfrak{l}_a)] \to \mathbb{R}$ *is a GCF,* $\Lambda \in L\left[\mathfrak{l}_a, \mathfrak{l}_a + \Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b - \mathfrak{l}_a)\right]$ *and* $\Delta^{\varrho}_{\epsilon,\sigma}$ *satisfies Condition C, the following inequalities for ABFIO hold*

$$\begin{aligned}&\Lambda\left(\frac{2\mathfrak{l}_a + \Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b - \mathfrak{l}_a)}{2}\right)\\ &\le \frac{B(\gamma)\Gamma(\gamma)}{2\left[\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b - \mathfrak{l}_a)\right]^{\gamma}}\left[{}^{AB}_{\mathfrak{l}_a}I^{\gamma}\left\{\Lambda\left(\mathfrak{l}_a + \Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b - \mathfrak{l}_a)\right)\right\} + {}^{AB}I^{\gamma}_{\mathfrak{l}_a + \Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b - \mathfrak{l}_a)}\left\{\Lambda\left(\mathfrak{l}_a\right)\right\}\right]\end{aligned}$$

$$
\begin{aligned}
&-\frac{(1-\gamma)\Gamma(\gamma)}{2\left[\Delta_{\epsilon,\sigma}^{\varrho}(\mathfrak{l}_b-\mathfrak{l}_a)\right]^{\gamma}}\left[\Lambda\left(\mathfrak{l}_a\right)+\Lambda\left(\mathfrak{l}_a+\Delta_{\epsilon,\sigma}^{\varrho}(\mathfrak{l}_b-\mathfrak{l}_a)\right)\right] \\
&\leq \frac{\Lambda\left(\mathfrak{l}_a\right)+\Lambda\left(\mathfrak{l}_b\right)}{2}, \qquad (4.1)
\end{aligned}
$$

where $\gamma \in (0,1]$, $B(\gamma)>0$ is a normalization function, and $\Gamma(\cdot)$ denotes the Gamma function.

Proof. Since Λ is GCF on $[\mathfrak{l}_a, \mathfrak{l}_a+\Delta_{\epsilon,\sigma}^{\varrho}(\mathfrak{l}_b-\mathfrak{l}_a)]$, we can write

$$
\begin{aligned}
&2\Lambda\left(\frac{2\mathfrak{l}_a+\Delta_{\epsilon,\sigma}^{\varrho}(\mathfrak{l}_b-\mathfrak{l}_a)}{2}\right) \\
&\leq \Lambda\left(\mathfrak{l}_a+\mathfrak{x}\Delta_{\epsilon,\sigma}^{\varrho}(\mathfrak{l}_b-\mathfrak{l}_a)\right)+\Lambda\left(\mathfrak{l}_a+(1-\mathfrak{x})\Delta_{\epsilon,\sigma}^{\varrho}(\mathfrak{l}_b-\mathfrak{l}_a)\right). \qquad (4.2)
\end{aligned}
$$

Multiplying both sides of inequality (4.2) by $\frac{\gamma}{B(\gamma)\,\Gamma(\gamma)}\,\mathfrak{x}^{\gamma-1}$ and integrating the resulting expression with respect to $\mathfrak{x}$ over the interval $[0,1]$, we obtain

$$
\begin{aligned}
&\frac{2}{B(\gamma)\Gamma(\gamma)}\Lambda\left(\frac{2\mathfrak{l}_a+\Delta_{\epsilon,\sigma}^{\varrho}(\mathfrak{l}_b-\mathfrak{l}_a)}{2}\right) \\
\leq\ &\frac{\gamma}{B(\gamma)\Gamma(\gamma)}\int_0^1 \mathfrak{x}^{\gamma-1}\Lambda\left(\mathfrak{l}_a+\mathfrak{x}\Delta_{\epsilon,\sigma}^{\varrho}(\mathfrak{l}_b-\mathfrak{l}_a)\right)d\mathfrak{x} \\
&+\frac{\gamma}{B(\gamma)\Gamma(\gamma)}\int_0^1 t^{\gamma-1}\Lambda\left(\mathfrak{l}_a+(1-\mathfrak{x})\Delta_{\epsilon,\sigma}^{\varrho}(\mathfrak{l}_b-\mathfrak{l}_a)\right)d\mathfrak{x} \\
=\ &\frac{\gamma}{B(\gamma)\Gamma(\gamma)\left[\Delta_{\epsilon,\sigma}^{\varrho}(\mathfrak{l}_b-\mathfrak{l}_a)\right]^{\gamma}}\int_{\mathfrak{l}_a}^{\mathfrak{l}_a+\Delta_{\epsilon,\sigma}^{\varrho}(\mathfrak{l}_b-m\mathfrak{l}_a)}\left(x-\mathfrak{l}_a\right)^{\gamma-1}\Lambda(x)dx \\
&+\frac{\gamma}{B(\gamma)\Gamma(\gamma)\left[\Delta_{\epsilon,\sigma}^{\varrho}(\mathfrak{l}_b-\mathfrak{l}_a)\right]^{\gamma}}\int_{\mathfrak{l}_a}^{\mathfrak{l}_a+\Delta_{\epsilon,\sigma}^{\varrho}(\mathfrak{l}_b-\mathfrak{l}_a)}\left(\mathfrak{l}_a+\Delta_{\epsilon,\sigma}^{\varrho}(\mathfrak{l}_b-\mathfrak{l}_a)-y\right)^{\gamma-1}\Lambda(y)dy.
\end{aligned}
$$

Then we can write

$$
\begin{aligned}
&\frac{2}{B(\gamma)\Gamma(\gamma)}\Lambda\left(\frac{2\mathfrak{l}_a+\Delta_{\epsilon,\sigma}^{\varrho}(\mathfrak{l}_b-\mathfrak{l}_a)}{2}\right) \\
\leq\ &\frac{1}{\left[\Delta_{\epsilon,\sigma}^{\varrho}(\mathfrak{l}_b-\mathfrak{l}_a)\right]^{\gamma}}\left[\frac{\gamma}{B(\gamma)\Gamma(\gamma)}\int_{\mathfrak{l}_a}^{\mathfrak{l}_a+\Delta_{\epsilon,\sigma}^{\varrho}(\mathfrak{l}_b-\mathfrak{l}_a)}\left(x-\mathfrak{l}_a\right)^{\gamma-1}\Lambda(x)dx+\frac{(1-\gamma)}{B(\gamma)}\Lambda\left(\mathfrak{l}_a\right)\right] \\
&-\frac{(1-\gamma)}{B(\gamma)\left[\Delta_{\epsilon,\sigma}^{\varrho}(\mathfrak{l}_b-\mathfrak{l}_a)\right]^{\gamma}}\Lambda\left(\mathfrak{l}_a\right) \\
&+\frac{1}{\left[\Delta_{\epsilon,\sigma}^{\varrho}(\mathfrak{l}_b-\mathfrak{l}_a)\right]^{\gamma}}\left[\frac{\gamma}{B(\gamma)\Gamma(\gamma)}\int_{\mathfrak{l}_a}^{\mathfrak{l}_a+\Delta_{\epsilon,\sigma}^{\varrho}(\mathfrak{l}_b-m\mathfrak{l}_a)}\left(\mathfrak{l}_a+\Delta_{\epsilon,\sigma}^{\varrho}(\mathfrak{l}_b-\mathfrak{l}_a)-y\right)^{\gamma-1}\Lambda(y)dy\right. \\
&\left.+\frac{(1-\gamma)}{B(\gamma)}\Lambda\left(\mathfrak{l}_a+\Delta_{\epsilon,\sigma}^{\varrho}(\mathfrak{l}_b-\mathfrak{l}_a)\right)\right]-\frac{(1-\gamma)}{B(\gamma)\left[\Delta_{\epsilon,\sigma}^{\varrho}(\mathfrak{l}_b-\mathfrak{l}_a)\right]^{\gamma}}\Lambda\left(\mathfrak{l}_a+\Delta_{\epsilon,\sigma}^{\varrho}(\mathfrak{l}_b-\mathfrak{l}_a)\right).
\end{aligned}
$$

So, using ABFIO, we get

$$\begin{aligned}
&\frac{2}{B(\gamma)\Gamma(\gamma)}\Lambda\left(\frac{2\mathfrak{l}_a+\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)}{2}\right)\\
\leq\ &\frac{1}{[\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)]^{\gamma}}\left[{}^{AB}_{\mathfrak{l}_a}I^{\gamma}\left\{\Lambda\left(\mathfrak{l}_a+\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)\right)\right\}+{}^{AB}I^{\gamma}_{\mathfrak{l}_a+\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)}\left\{\Lambda\left(\mathfrak{l}_a\right)\right\}\right]\\
&-\frac{(1-\gamma)}{B(\gamma)\left[\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)\right]^{\gamma}}\left[\Lambda\left(\mathfrak{l}_a\right)+\Lambda\left(\mathfrak{l}_a+\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)\right)\right]
\end{aligned}$$

and the first inequality is proved.

For the proof of the second inequality in the above inequality (4.1), we first note that if Λ is a GCF, then we can write

$$\Lambda\left(\mathfrak{l}_a+\mathfrak{x}\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)\right)\leq(1-\mathfrak{x})\Lambda\left(\mathfrak{l}_a\right)+\mathfrak{x}\Lambda\left(\mathfrak{l}_b\right)$$

and

$$\Lambda\left(\mathfrak{l}_a+(1-\mathfrak{x})\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)\right)\leq m\mathfrak{x}\Lambda\left(\mathfrak{l}_a\right)+(1-\mathfrak{x})\Lambda\left(\mathfrak{l}_b\right).$$

By summing the above inequalities term by term, we arrive at

$$\Lambda\left(\mathfrak{l}_a+\mathfrak{x}\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)\right)+\Lambda\left(\mathfrak{l}_a+(1-\mathfrak{x})\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)\right)\leq\Lambda\left(\mathfrak{l}_a\right)+\Lambda\left(\mathfrak{l}_b\right). \tag{4.3}$$

Next, we multiply both sides of inequality (4.3) by $\frac{\gamma}{B(\gamma)\Gamma(\gamma)}\mathfrak{x}^{\gamma-1}$ and integrate the resulting expression with respect to $\mathfrak{x}$ over the interval $[0,1]$. This yields

$$\begin{aligned}
&\frac{\gamma}{B(\gamma)\Gamma(\gamma)}\int_0^1\mathfrak{x}^{\gamma-1}\Lambda\left(\mathfrak{l}_a+\mathfrak{x}\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)\right)d\mathfrak{x}\\
&+\frac{\gamma}{B(\gamma)\Gamma(\gamma)}\int_0^1\mathfrak{x}^{\gamma-1}\Lambda\left(\mathfrak{l}_a+(1-\mathfrak{x})\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)\right)d\mathfrak{x}\\
\leq\ &\frac{\gamma}{B(\gamma)\Gamma(\gamma)}\left[\Lambda\left(\mathfrak{l}_a\right)+\Lambda\left(\mathfrak{l}_b\right)\right]\int_0^1\mathfrak{x}^{\gamma-1}d\mathfrak{x}.
\end{aligned}$$

Then, we can write

$$\begin{aligned}
&\frac{1}{[\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)]^{\gamma}}\left[{}^{AB}_{\mathfrak{l}_a}I^{\gamma}\left\{\Lambda\left(\mathfrak{l}_a+\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)\right)\right\}+{}^{AB}I^{\gamma}_{\mathfrak{l}_a+\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)}\left\{\Lambda\left(\mathfrak{l}_a\right)\right\}\right]\\
&-\frac{(1-\gamma)}{B(\gamma)\left[\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)\right]^{\gamma}}\left[\Lambda\left(\mathfrak{l}_a\right)+\Lambda\left(\mathfrak{l}_a+\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)\right)\right]\\
\leq\ &\frac{\Lambda\left(\mathfrak{l}_a\right)+\Lambda\left(\mathfrak{l}_b\right)}{B(\gamma)\Gamma(\gamma)}.
\end{aligned}$$

So, the proof of this theorem is completed.

Remark 2. *Choosing* $\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b - \mathfrak{l}_a) = \mathfrak{l}_b - \mathfrak{l}_a$ *in the above theorem, we have the result in [51, Proposition 2.1], inequality (13).*

Remark 3. *Considering Theorem 8, we establish the following new mathematical approach of Hermite-Hadamard inequality pertaining to the classical Mittag-Leffler function via ABFIO if we pick* $\varrho = (1, 1, ...)$ *with* $\epsilon = \alpha$ *and* $\sigma = 1$*:*

$$\begin{aligned}
&\Lambda\left(\frac{2\mathfrak{l}_a + \mathtt{E}_\alpha(\mathfrak{l}_b - \mathfrak{l}_a)}{2}\right) \\
&\leq \frac{B(\gamma)\Gamma(\gamma)}{2\left[\mathtt{E}_\alpha(\mathfrak{l}_b - \mathfrak{l}_a)\right]^\gamma}\left[{}^{AB}_{m\mathfrak{l}_a}I^\gamma\left\{\Lambda\left(\mathfrak{l}_a + \mathtt{E}_\alpha(\mathfrak{l}_b - \mathfrak{l}_a)\right)\right\} + {}^{AB}I^\gamma_{\mathfrak{l}_a + \mathtt{E}_\alpha(\mathfrak{l}_b - m\mathfrak{l}_a)}\left\{\Lambda\left(\mathfrak{l}_a\right)\right\}\right] \\
&- \frac{(1-\gamma)\Gamma(\gamma)}{2\left[\mathtt{E}_\alpha(\mathfrak{l}_b - \mathfrak{l}_a)\right]^\gamma}\left[\Lambda\left(\mathfrak{l}_a\right) + \Lambda\left(\mathfrak{l}_a + \mathtt{E}_\alpha(\mathfrak{l}_b - \mathfrak{l}_a)\right)\right] \\
&\leq \frac{\Lambda\left(\mathfrak{l}_a\right) + \Lambda\left(\mathfrak{l}_b\right)}{2}.
\end{aligned}$$

5. Refinements of Hermite-Hadamard Type Inequality via AB Fractional Integral Operator

This section's goal is to explore and offer a novel equality. We derive some novel enhancements of Hermite-Hadamard-type inequalities using an ABFIO based on this recently studied equality. To improve the content and grab readers' attention, we include a few remarks. First, we prove a lemma in the frame of ABFIO.

Throughout in this section, $B(\gamma)$ represents the normalization function and $\Gamma(.)$ represents the Gamma function.

Lemma 1. *Let* $I \subseteq \mathbb{R}$ *be an open, non-empty convex set, and let* $\mathfrak{l}_a, \mathfrak{l}_b \in I$ *with* $\mathfrak{l}_a < \mathfrak{l}_a + \Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b - \mathfrak{l}_a)$*. Suppose that* $\Lambda : I \to \mathbb{R}$ *is a differentiable function such that*

$$\Lambda' \in L\left[\mathfrak{l}_a,\, \mathfrak{l}_a + \Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b - \mathfrak{l}_a)\right].$$

Then, the following identity involving the Atangana–Baleanu fractional integral operator holds:

$$\begin{aligned}
&\frac{B(\gamma)\Gamma(\gamma)}{\left[\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b - \mathfrak{l}_a)\right]^{\gamma+1}}\left[{}^{AB}_{\mathfrak{l}_a}I^\gamma\left\{\Lambda\left(\mathfrak{l}_a + \Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b - \mathfrak{l}_a)\right)\right\} + {}^{AB}I^\gamma_{\mathfrak{l}_a + \Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b - \mathfrak{l}_a)}\left\{\Lambda\left(\mathfrak{l}_a\right)\right\}\right] \\
&- \left(\frac{\left[\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b - \mathfrak{l}_a)\right]^\gamma + (1-\gamma)\Gamma(\gamma)}{\left[\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b - \mathfrak{l}_a)\right]^{\gamma+1}}\right)\left[\Lambda\left(\mathfrak{l}_a\right) + \Lambda\left(\mathfrak{l}_a + \Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b - \mathfrak{l}_a)\right)\right] \\
= \;& \int_0^1 (1-\mathfrak{x})^\gamma \Lambda'\left(\mathfrak{l}_a + \mathfrak{x}\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b - \mathfrak{l}_a)\right)d\mathfrak{x} - \int_0^1 \mathfrak{x}^\gamma \Lambda'\left(\mathfrak{l}_a + \mathfrak{x}\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b - \mathfrak{l}_a)\right)d\mathfrak{x}
\end{aligned}$$

where $\gamma \in (0,1], \mathfrak{x} \in [0,1]$.

Proof. By using integration, we have

$$\begin{aligned}
&\int_0^1 (1-\mathfrak{x})^{\gamma}\Lambda'\left(\mathfrak{l}_a+\mathfrak{x}\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)\right)d\mathfrak{x}\\
=&\left.\frac{(1-\mathfrak{x})^{\gamma}\Lambda\left(\mathfrak{l}_a+\mathfrak{x}\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)\right)}{\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)}\right|_0^1\\
&+\frac{\gamma}{\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)}\int_0^1 \Lambda\left(\mathfrak{l}_a+\mathfrak{x}\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)\right)(1-\mathfrak{x})^{\gamma-1}d\mathfrak{x}\\
=&-\frac{\Lambda(\mathfrak{l}_a)}{\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)}+\frac{\gamma}{\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)}\int_0^1 (1-\mathfrak{x})^{\gamma-1}\Lambda\left(\mathfrak{l}_a+\mathfrak{x}\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)\right)d\mathfrak{x}\\
=&-\frac{\Lambda(\mathfrak{l}_a)}{\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)}\\
&+\frac{\gamma}{[\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)]^{\gamma+1}}\int_{\mathfrak{l}_a}^{\mathfrak{l}_a+\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)}\left(\mathfrak{l}_a+\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)-x\right)^{\gamma-1}\Lambda(x)dx. \qquad (5.1)
\end{aligned}$$

Multiplying both sides of inequality (5.1) by $\frac{1}{B(\gamma)\Gamma(\gamma)}$, we obtain

$$\begin{aligned}
&\frac{1}{B(\gamma)\Gamma(\gamma)}\int_0^1 (1-\mathfrak{x})^{\gamma}\Lambda'\left(\mathfrak{l}_a+\mathfrak{x}\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)\right)d\mathfrak{x}=-\frac{\Lambda(\mathfrak{l}_a)}{B(\gamma)\Gamma(\gamma)\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-m\mathfrak{l}_a)}\\
&+\frac{\gamma}{B(\gamma)\Gamma(\gamma)\,[\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)]^{\gamma+1}}\int_{\mathfrak{l}_a}^{\mathfrak{l}_a+\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)}\left(\mathfrak{l}_a+\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)-x\right)^{\gamma-1}\Lambda(x)dx.
\end{aligned}$$

Then, we can write

$$\begin{aligned}
&\frac{1}{B(\gamma)\Gamma(\gamma)}\int_0^1 (1-\mathfrak{x})^{\gamma}\Lambda'\left(\mathfrak{l}_a+\mathfrak{x}\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)\right)d\mathfrak{x}\\
=&-\frac{\Lambda(\mathfrak{l}_a)}{B(\gamma)\Gamma(\gamma)\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)}\\
&+\frac{1}{[\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)]^{\gamma+1}}\left[\frac{\gamma}{B(\gamma)\Gamma(\gamma)}\int_{\mathfrak{l}_a}^{\mathfrak{l}_a+\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)}\left(\mathfrak{l}_a+\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)-x\right)^{\gamma-1}\Lambda(x)dx\right.\\
&\left.+\frac{(1-\gamma)}{B(\gamma)}\Lambda\left(\mathfrak{l}_a+\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)\right)\right]-\frac{(1-\gamma)}{B(\gamma)\,[\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)]^{\gamma+1}}\Lambda\left(\mathfrak{l}_a+\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)\right).
\end{aligned}$$

Using ABFIO, we have

$$\begin{aligned}
&\frac{1}{B(\gamma)\Gamma(\gamma)}\int_0^1 (1-\mathfrak{x})^{\gamma}\Lambda'\left(\mathfrak{l}_a+\mathfrak{x}\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)\right)d\mathfrak{x}\\
=&-\frac{\Lambda(\mathfrak{l}_a)}{B(\gamma)\Gamma(\gamma)\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)}+\frac{1}{[\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)]^{\gamma+1}}\left[{}^{AB}_{\mathfrak{l}_a}I^{\gamma}\left\{\Lambda\left(\mathfrak{l}_a+\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)\right)\right\}\right]\\
&-\frac{(1-\gamma)}{B(\gamma)\,[\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)]^{\gamma+1}}\Lambda\left(m\mathfrak{l}_a+\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)\right). \qquad (5.2)
\end{aligned}$$

Similarly, using integration, we get

$$\begin{aligned}
&\int_0^1 \mathfrak{r}^{\gamma}\Lambda'\left(\mathfrak{l}_a+\mathfrak{r}\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)\right)d\mathfrak{r}\\
=&\left.\frac{\mathfrak{r}^{\gamma}\Lambda\left(\mathfrak{l}_a+\mathfrak{r}\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)\right)}{\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)}\right|_0^1-\frac{\gamma}{\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)}\int_0^1\Lambda\left(\mathfrak{l}_a+\mathfrak{r}\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)\right)\mathfrak{r}^{\gamma-1}d\mathfrak{r}\\
=&\frac{\Lambda\left(\mathfrak{l}_a+\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)\right)}{\Omega\left(\mathfrak{l}_b,\mathfrak{l}_a\right)}-\frac{\gamma}{\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)}\int_0^1\mathfrak{r}^{\gamma-1}\Lambda\left(\mathfrak{l}_a+\mathfrak{r}\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)\right)d\mathfrak{r}\\
=&\frac{\Lambda\left(\mathfrak{l}_a+\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)\right)}{\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)}-\frac{\gamma}{\left[\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)\right]^{\gamma+1}}\int_{\mathfrak{l}_a}^{\mathfrak{l}_a+\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)}(u-\mathfrak{l}_a)^{\gamma-1}\Lambda(u)du. \qquad (5.3)
\end{aligned}$$

Multiplying both sides of inequality (5.3) by $-\frac{1}{B(\gamma)\Gamma(\gamma)}$, we have

$$\begin{aligned}
&-\frac{1}{B(\gamma)\Gamma(\gamma)}\int_0^1 \mathfrak{r}^{\gamma}\Lambda'\left(\mathfrak{l}_a+\mathfrak{r}\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)\right)d\mathfrak{r}\\
=&-\frac{\Lambda\left(\mathfrak{l}_a+\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)\right)}{B(\gamma)\Gamma(\gamma)\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)}\\
&+\frac{\gamma}{B(\gamma)\Gamma(\gamma)\left[\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)\right]^{\gamma+1}}\int_{\mathfrak{l}_a}^{\mathfrak{l}_a+\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)}(u-\mathfrak{l}_a)^{\gamma-1}\Lambda(u)du.
\end{aligned}$$

Then we can write

$$\begin{aligned}
&-\frac{1}{B(\gamma)\Gamma(\gamma)}\int_0^1 \mathfrak{r}^{\gamma}\Lambda'\left(\mathfrak{l}_a+\mathfrak{r}\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)\right)d\mathfrak{r}\\
=&-\frac{\Lambda\left(\mathfrak{l}_a+\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)\right)}{B(\gamma)\Gamma(\gamma)\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)}\\
&+\frac{1}{\left[\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)\right]^{\gamma+1}}\left[\frac{\gamma}{B(\gamma)\Gamma(\gamma)}\int_{\mathfrak{l}_a}^{\mathfrak{l}_a+\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)}(u-\mathfrak{l}_a)^{\gamma-1}\Lambda(u)du\right.\\
&\left.+\frac{(1-\gamma)}{B(\gamma)}\Lambda\left(\mathfrak{l}_a\right)\right]-\frac{(1-\gamma)}{B(\gamma)\left[\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)\right]^{\gamma+1}}\Lambda\left(\mathfrak{l}_a\right).
\end{aligned}$$

Using ABFIO, we have

$$\begin{aligned}
&-\frac{1}{B(\gamma)\Gamma(\gamma)}\int_0^1 \mathfrak{r}^{\gamma}\Lambda'\left(\mathfrak{l}_a+\mathfrak{r}\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)\right)d\mathfrak{r}\\
=&-\frac{\Lambda\left(\mathfrak{l}_a+\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)\right)}{B(\gamma)\Gamma(\gamma)\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)}+\frac{1}{\left[\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)\right]^{\gamma+1}}\left[{}^{AB}I^{\gamma}_{\mathfrak{l}_a+\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)}\left\{\Lambda\left(\mathfrak{l}_a\right)\right\}\right]\\
&-\frac{(1-\gamma)}{B(\gamma)\left[\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)\right]^{\gamma+1}}\Lambda\left(\mathfrak{l}_a\right). \qquad (5.4)
\end{aligned}$$

By adding identities (5.2) and (5.4), we obtain the proof of Lemma 1.

Remark 4. *If we put* $\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)=\mathfrak{l}_b-\mathfrak{l}_a$ *in Lemma 1 then we have the result in [51, Theorem 3.1], equality (29).*

Theorem 9. *Let* $I\subseteq\mathbb{R}$ *be an open and non-empty convex set and* $\mathfrak{l}_a,\mathfrak{l}_b\in I$ *with* $\mathfrak{l}_a<\mathfrak{l}_a+\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)$. *Suppose that* $\Lambda: I\to\mathbb{R}$ *is a differentiable function and* $\Lambda'\in L\left[\mathfrak{l}_a,\mathfrak{l}_a+\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)\right]$. *If* $|\Lambda'|$ *is* $GCRF$, *then we have the following inequality for AB-FIO*

$$\begin{aligned}
&\left|\frac{B(\gamma)\Gamma(\gamma)}{[\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)]^{\gamma+1}}\left[{}^{AB}_{m\mathfrak{l}_a}I^{\gamma}\left\{\Lambda\left(\mathfrak{l}_a+\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)\right)\right\}+{}^{AB}I^{\gamma}_{\mathfrak{l}_a+\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)}\left\{\Lambda\left(\mathfrak{l}_a\right)\right\}\right]\right.\\
&\left.-\left(\frac{[\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)]^{\gamma}+(1-\gamma)\Gamma(\gamma)}{[\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)]^{\gamma+1}}\right)\left[\Lambda\left(\mathfrak{l}_a\right)+\Lambda\left(\mathfrak{l}_a+\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)\right)\right]\right|\\
\leq\ &\frac{|\Lambda'(\mathfrak{l}_a)|+|\Lambda'(\mathfrak{l}_b)|}{\gamma+1},
\end{aligned}$$

where $\gamma\in(0,1]$.

Proof. Using the identity given in Lemma 1 and the properties of the modulus, we can write

$$\begin{aligned}
&\left|\frac{B(\gamma)\Gamma(\gamma)}{[\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)]^{\gamma+1}}\left[{}^{AB}_{\mathfrak{l}_a}I^{\gamma}\left\{\Lambda\left(\mathfrak{l}_a+\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)\right)\right\}+{}^{AB}I^{\gamma}_{\mathfrak{l}_a+\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)}\left\{\Lambda\left(\mathfrak{l}_a\right)\right\}\right]\right.\\
&\left.-\left(\frac{[\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)]^{\gamma}+(1-\gamma)\Gamma(\gamma)}{[\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)]^{\gamma+1}}\right)\left[\Lambda\left(\mathfrak{l}_a\right)+\Lambda\left(\mathfrak{l}_a+\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)\right)\right]\right|\\
=\ &\left|\int_0^1(1-\mathfrak{x})^{\gamma}\Lambda'\left(\mathfrak{l}_a+\mathfrak{x}\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)\right)d\mathfrak{x}-\int_0^1\mathfrak{x}^{\gamma}\Lambda'\left(\mathfrak{l}_a+\mathfrak{x}\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)\right)d\mathfrak{x}\right|\\
\leq\ &\int_0^1(1-\mathfrak{x})^{\gamma}\left|\Lambda'\left(\mathfrak{l}_a+\mathfrak{x}\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)\right)\right|d\mathfrak{x}+\int_0^1\mathfrak{x}^{\gamma}\left|\Lambda'\left(\mathfrak{l}_a+\mathfrak{x}\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)\right)\right|d\mathfrak{x}.
\end{aligned}$$

Since $|\Lambda'|$ is $GCRF$, we obtain

$$\begin{aligned}
&\left|\frac{B(\gamma)\Gamma(\gamma)}{[\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)]^{\gamma+1}}\left[{}^{AB}_{\mathfrak{l}_a}I^{\gamma}\left\{\Lambda\left(\mathfrak{l}_a+\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)\right)\right\}+{}^{AB}I^{\gamma}_{\mathfrak{l}_a+\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)}\left\{\Lambda\left(\mathfrak{l}_a\right)\right\}\right]\right.\\
&\left.-\left(\frac{[\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)]^{\gamma}+(1-\gamma)\Gamma(\gamma)}{[\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)]^{\gamma+1}}\right)\left[\Lambda\left(\mathfrak{l}_a\right)+\Lambda\left(\mathfrak{l}_a+\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)\right)\right]\right|\\
\leq\ &\int_0^1(1-\mathfrak{x})^{\gamma}\left[(1-\mathfrak{x})\left|\Lambda'(\mathfrak{l}_a)\right|+\mathfrak{x}\left|\Lambda'(\mathfrak{l}_b)\right|\right]d\mathfrak{x}+\int_0^1\mathfrak{x}^{\gamma}\left[(1-\mathfrak{x})\left|\Lambda'(\mathfrak{l}_a)\right|+\mathfrak{x}\left|\Lambda'(\mathfrak{l}_b)\right|\right]d\mathfrak{x}\\
=\ &\frac{|\Lambda'(\mathfrak{l}_a)|+|\Lambda'(\mathfrak{l}_b)|}{\gamma+1}.
\end{aligned}$$

So, the proof is completed.

Remark 5. *Considering Theorem 9, we establish the following new mathematical approach of Hermite-Hadamard inequality pertaining to the classical Mittag-Leffler function via ABFIO if we pick* $\varrho = (1, 1, \ldots)$ *with* $\epsilon = \alpha$ *and* $\sigma = 1$*:*

$$\begin{aligned}&\left|\frac{B(\gamma)\Gamma(\gamma)}{[\mathrm{E}_\alpha(\mathfrak{l}_b - \mathfrak{l}_a)]^{\gamma+1}}\left[{}^{AB}_{\mathfrak{l}_a}I^\gamma\left\{\Lambda\left(\mathfrak{l}_a + \mathrm{E}_\alpha(\mathfrak{l}_b - \mathfrak{l}_a)\right)\right\} + {}^{AB}I^\gamma_{\mathfrak{l}_a+\mathrm{E}_\alpha(\mathfrak{l}_b-\mathfrak{l}_a)}\left\{\Lambda\left(\mathfrak{l}_a\right)\right\}\right]\right.\\ &\left.-\left(\frac{[\mathrm{E}_\alpha(\mathfrak{l}_b - \mathfrak{l}_a)]^\gamma + (1-\gamma)\Gamma(\gamma)}{[\mathrm{E}_\alpha(\mathfrak{l}_b - \mathfrak{l}_a)]^{\gamma+1}}\right)\left[\Lambda\left(\mathfrak{l}_a\right) + \Lambda\left(\mathfrak{l}_a + \mathrm{E}_\alpha(\mathfrak{l}_b - \mathfrak{l}_a)\right)\right]\right|\\ \leq\ &\frac{|\Lambda'\left(\mathfrak{l}_a\right)| + |\Lambda'\left(\mathfrak{l}_b\right)|}{\gamma+1},\end{aligned}$$

Corollary 1. *In the above Theorem 9, if we choose* $\Delta^\varrho_{\epsilon,\sigma}(\mathfrak{l}_b - \mathfrak{l}_a) = \mathfrak{l}_b - \mathfrak{l}_a$*, then we obtain*

$$\begin{aligned}&\left|\frac{B(\gamma)\Gamma(\gamma)}{(\mathfrak{l}_b - \mathfrak{l}_a)^{\gamma+1}}\left[{}^{AB}_{\mathfrak{l}_a}I^\gamma\left\{\Lambda\left(\mathfrak{l}_b\right)\right\} + {}^{AB}I^\gamma_{\mathfrak{l}_b}\left\{\Lambda\left(\mathfrak{l}_a\right)\right\}\right]\right.\\ &\left.-\left(\frac{(\mathfrak{l}_b - \mathfrak{l}_a)^\gamma + (1-\gamma)\Gamma(\gamma)}{(\mathfrak{l}_b - \mathfrak{l}_a)^{\gamma+1}}\right)\left[\Lambda\left(\mathfrak{l}_a\right) + \Lambda\left(\mathfrak{l}_b\right)\right]\right|\\ \leq\ &\frac{|\Lambda'\left(\mathfrak{l}_a\right)| + |\Lambda'\left(\mathfrak{l}_b\right)|}{\gamma+1}.\end{aligned}$$

Theorem 10. *Let* $I \subseteq \mathbb{R}$ *be an open and non-empty covex set and* $\mathfrak{l}_a, \mathfrak{l}_b \in I$ *with* $\mathfrak{l}_a < \mathfrak{l}_a + \Delta^\varrho_{\epsilon,\sigma}(\mathfrak{l}_b - m\mathfrak{l}_a)$*. Suppose that* $\Lambda : I \to \mathbb{R}$ *is a differentiable function and* $\Lambda' \in L\Big[\mathfrak{l}_a, \mathfrak{l}_a + \Delta^\varrho_{\epsilon,\sigma}(\mathfrak{l}_b - \mathfrak{l}_a)\Big]$*. If* $|\Lambda'|^q$ *is a GCF, then we have the following inequality for ABFIO*

$$\begin{aligned}&\left|\frac{B(\gamma)\Gamma(\gamma)}{[\Delta^\varrho_{\epsilon,\sigma}(\mathfrak{l}_b - \mathfrak{l}_a)]^{\gamma+1}}\left[{}^{AB}_{\mathfrak{l}_a}I^\gamma\left\{\Lambda\left(\mathfrak{l}_a + \Delta^\varrho_{\epsilon,\sigma}(\mathfrak{l}_b - \mathfrak{l}_a)\right)\right\} + {}^{AB}I^\gamma_{\mathfrak{l}_a+\Delta^\varrho_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)}\left\{\Lambda\left(m\mathfrak{l}_a\right)\right\}\right]\right.\\ &\left.-\left(\frac{[\Delta^\varrho_{\epsilon,\sigma}(\mathfrak{l}_b - \mathfrak{l}_a)]^\gamma + (1-\gamma)\Gamma(\gamma)}{[\Delta^\varrho_{\epsilon,\sigma}(\mathfrak{l}_b - \mathfrak{l}_a)]^{\gamma+1}}\right)\left[\Lambda\left(\mathfrak{l}_a\right) + \Lambda\left(\mathfrak{l}_a + \Delta^\varrho_{\epsilon,\sigma}(\mathfrak{l}_b - \mathfrak{l}_a)\right)\right]\right|\\ \leq\ & 2\left(\frac{1}{\gamma p+1}\right)^{\frac{1}{p}}\left(\frac{|\Lambda'\left(\mathfrak{l}_a\right)|^q + |\Lambda'\left(\mathfrak{l}_b\right)|^q}{2}\right)^{\frac{1}{q}},\end{aligned}$$

where $p^{-1} + q^{-1} = 1, q > 1, \gamma \in (0, 1]$.

Proof. By using Lemma 1, we get

$$\begin{aligned}&\left|\frac{B(\gamma)\Gamma(\gamma)}{[\Delta^\varrho_{\epsilon,\sigma}(\mathfrak{l}_b - \mathfrak{l}_a)]^{\gamma+1}}\left[{}^{AB}_{\mathfrak{l}_a}I^\gamma\left\{\Lambda\left(\mathfrak{l}_a + \Delta^\varrho_{\epsilon,\sigma}(\mathfrak{l}_b - \mathfrak{l}_a)\right)\right\} + {}^{AB}I^\gamma_{\mathfrak{l}_a+\Delta^\varrho_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)}\left\{\Lambda\left(\mathfrak{l}_a\right)\right\}\right]\right.\\ &\left.-\left(\frac{[\Delta^\varrho_{\epsilon,\sigma}(\mathfrak{l}_b - \mathfrak{l}_a)]^\gamma + (1-\gamma)\Gamma(\gamma)}{[\Delta^\varrho_{\epsilon,\sigma}(\mathfrak{l}_b - \mathfrak{l}_a)]^{\gamma+1}}\right)\left[\Lambda\left(\mathfrak{l}_a\right) + \Lambda\left(\mathfrak{l}_a + \Delta^\varrho_{\epsilon,\sigma}(\mathfrak{l}_b - \mathfrak{l}_a)\right)\right]\right|\\ \leq\ &\int_0^1 (1-\mathfrak{x})^\gamma\left|\Lambda'\left(\mathfrak{l}_a + \mathfrak{x}\Delta^\varrho_{\epsilon,\sigma}(\mathfrak{l}_b - \mathfrak{l}_a)\right)\right| d\mathfrak{x} + \int_0^1 \mathfrak{x}^\gamma\left|\Lambda'\left(\mathfrak{l}_a + \mathfrak{x}\Delta^\varrho_{\epsilon,\sigma}(\mathfrak{l}_b - \mathfrak{l}_a)\right)\right| d\mathfrak{x}.\end{aligned}$$

By applying Hölder inequality, we get

$$\begin{aligned}&\left|\frac{B(\gamma)\Gamma(\gamma)}{[\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)]^{\gamma+1}}\left[{}^{AB}_{\mathfrak{l}_a}I^{\gamma}\left\{\Lambda\left(\mathfrak{l}_a+\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)\right)\right\}+{}^{AB}I^{\gamma}_{\mathfrak{l}_a+\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)}\left\{\Lambda\left(\mathfrak{l}_a\right)\right\}\right]\right.\\&\left.-\left(\frac{[\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)]^{\gamma}+(1-\gamma)\Gamma(\gamma)}{[\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)]^{\gamma+1}}\right)\left[\Lambda\left(\mathfrak{l}_a\right)+\Lambda\left(\mathfrak{l}_a+\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)\right)\right]\right|\\\leq\;&\left(\int_0^1(1-\mathfrak{x})^{\gamma p}d\mathfrak{x}\right)^{\frac{1}{p}}\left(\int_0^1\left|\Lambda'\left(\mathfrak{l}_a+\mathfrak{x}\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)\right)\right|^q d\mathfrak{x}\right)^{\frac{1}{q}}\\&+\left(\int_0^1\mathfrak{x}^{\gamma p}d\mathfrak{x}\right)^{\frac{1}{p}}\left(\int_0^1\left|\Lambda'\left(\mathfrak{l}_a+\mathfrak{x}\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)\right)\right|^q d\mathfrak{x}\right)^{\frac{1}{q}}.\end{aligned}$$

By using G_mCF of $|\Lambda'|^q$, we obtain

$$\begin{aligned}&\left|\frac{B(\gamma)\Gamma(\gamma)}{[\Omega\left(\mathfrak{l}_b,\mathfrak{l}_a\right)]^{\gamma+1}}\left[{}^{AB}_{\mathfrak{l}_a}I^{\gamma}\left\{\Lambda\left(\mathfrak{l}_a+\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)\right)\right\}+{}^{AB}I^{\gamma}_{\mathfrak{l}_a+\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)}\left\{\Lambda\left(\mathfrak{l}_a\right)\right\}\right]\right.\\&\left.-\left(\frac{[\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)]^{\gamma}+(1-\gamma)\Gamma(\gamma)}{[\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)]^{\gamma+1}}\right)\left[\Lambda\left(\mathfrak{l}_a\right)+\Lambda\left(\mathfrak{l}_a+\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)\right)\right]\right|\\\leq\;&\left(\int_0^1(1-\mathfrak{x})^{\gamma p}d\mathfrak{x}\right)^{\frac{1}{p}}\left(\int_0^1\left[(1-\mathfrak{x})\left|\Lambda'\left(\mathfrak{l}_a\right)\right|^q+\mathfrak{x}\left|\Lambda'\left(\mathfrak{l}_b\right)\right|^q\right]d\mathfrak{x}\right)^{\frac{1}{q}}\\&+\left(\int_0^1\mathfrak{x}^{\gamma p}d\mathfrak{x}\right)^{\frac{1}{p}}\left(\int_0^1\left[(1-\mathfrak{x})\left|\Lambda'\left(\mathfrak{l}_a\right)\right|^q+\mathfrak{x}\left|\Lambda'\left(\mathfrak{l}_b\right)\right|^q\right]d\mathfrak{x}\right)^{\frac{1}{q}}.\end{aligned}$$

By calculating the integrals in the above inequality, we get the desired result.

Remark 6. *Considering Theorem 10, we establish the following new mathematical approach of Hermite-Hadamard inequality pertaining to the classical Mittag-Leffler function via ABFIO if we pick* $\varrho=(1,1,...)$ *with* $\epsilon=\alpha$ *and* $\sigma=1$*:*

$$\begin{aligned}&\left|\frac{B(\gamma)\Gamma(\gamma)}{[\mathrm{E}_{\alpha}(\mathfrak{l}_b-\mathfrak{l}_a)]^{\gamma+1}}\left[{}^{AB}_{\mathfrak{l}_a}I^{\gamma}\left\{\Lambda\left(\mathfrak{l}_a+\mathrm{E}_{\alpha}(\mathfrak{l}_b-\mathfrak{l}_a)\right)\right\}+{}^{AB}I^{\gamma}_{\mathfrak{l}_a+\mathrm{E}_{\alpha}(\mathfrak{l}_b-\mathfrak{l}_a)}\left\{\Lambda\left(\mathfrak{l}_a\right)\right\}\right]\right.\\&\left.-\left(\frac{[\mathrm{E}_{\alpha}(\mathfrak{l}_b-\mathfrak{l}_a)]^{\gamma}+(1-\gamma)\Gamma(\gamma)}{[\mathrm{E}_{\alpha}(\mathfrak{l}_b-\mathfrak{l}_a)]^{\gamma+1}}\right)\left[\Lambda\left(\mathfrak{l}_a\right)+\Lambda\left(\mathfrak{l}_a+\mathrm{E}_{\alpha}(\mathfrak{l}_b-\mathfrak{l}_a)\right)\right]\right|\\\leq\;&2\left(\frac{1}{\gamma p+1}\right)^{\frac{1}{p}}\left(\frac{\left|\Lambda'\left(\mathfrak{l}_a\right)\right|^q+\left|\Lambda'\left(\mathfrak{l}_b\right)\right|^q}{2}\right)^{\frac{1}{q}}.\end{aligned}$$

Corollary 2. *In the above Theorem, if we choose* $\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)=\mathfrak{l}_b-\mathfrak{l}_a$*, then we obtain*

$$\left|\frac{B(\gamma)\Gamma(\gamma)}{(\mathfrak{l}_b-\mathfrak{l}_a)^{\gamma+1}}\left[{}^{AB}_{\mathfrak{l}_a}I^{\gamma}\left\{\Lambda\left(\mathfrak{l}_b\right)\right\}+{}^{AB}I^{\gamma}_{\mathfrak{l}_b}\left\{\Lambda\left(\mathfrak{l}_a\right)\right\}\right]\right.$$

$$-\left(\frac{(\mathfrak{l}_b-\mathfrak{l}_a)^{\gamma}+(1-\gamma)\Gamma(\gamma)}{(\mathfrak{l}_b-\mathfrak{l}_a)^{\gamma+1}}\right)\left[\Lambda\left(\mathfrak{l}_a\right)+\Lambda\left(\mathfrak{l}_b\right)\right]\Bigg|$$
$$\leq\ 2\left(\frac{1}{\gamma p+1}\right)^{\frac{1}{p}}\left(\frac{|\Lambda'\left(\mathfrak{l}_a\right)|^q+|\Lambda'\left(\mathfrak{l}_b\right)|^q}{2}\right)^{\frac{1}{q}}.$$

Theorem 11. *Let $I\subseteq\mathbb{R}$ be an open and non-empty convex set and $\mathfrak{l}_a,\mathfrak{l}_b\in I$ with $\mathfrak{l}_a<\mathfrak{l}_a+\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)$. Suppose that $\Lambda:I\to\mathbb{R}$ is a differentiable function and $\Lambda'\in L\left[\mathfrak{l}_a,\mathfrak{l}_a+\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)\right]$. If $|\Lambda'|^q$ is a GCF, then we have the following inequality for AB-FIO*

$$\Bigg|\frac{B(\gamma)\Gamma(\gamma)}{\left[\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)\right]^{\gamma+1}}\left[{}^{AB}_{\mathfrak{l}_a}I^{\gamma}\left\{\Lambda\left(\mathfrak{l}_a+\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)\right)\right\}+{}^{AB}I^{\gamma}_{\mathfrak{l}_a+\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)}\left\{\Lambda\left(\mathfrak{l}_a\right)\right\}\right]$$
$$-\left(\frac{\left[\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)\right]^{\gamma}+(1-\gamma)\Gamma(\gamma)}{\left[\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)\right]^{\gamma+1}}\right)\left[\Lambda\left(\mathfrak{l}_a\right)+\Lambda\left(\mathfrak{l}_a+\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)\right)\right]\Bigg|$$
$$\leq\ \left(\frac{1}{\gamma+1}\right)^{1-\frac{1}{q}}\left[\left(\frac{|\Lambda'\left(\mathfrak{l}_a\right)|^q}{\gamma+2}+\frac{|\Lambda'\left(\mathfrak{l}_b\right)|^q}{(\gamma+1)(\gamma+2)}\right)^{\frac{1}{q}}+\left(\frac{|\Lambda'\left(\mathfrak{l}_a\right)|^q}{(\gamma+1)(\gamma+2)}+\frac{|\Lambda'\left(\mathfrak{l}_b\right)|^q}{\gamma+2}\right)^{\frac{1}{q}}\right],$$

where $\gamma\in(0,1],q\geq1$.

Proof. Employing Lemma 1 and utilizing the power mean inequality, we get

$$\Bigg|\frac{B(\gamma)\Gamma(\gamma)}{\left[\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)\right]^{\gamma+1}}\left[{}^{AB}_{\mathfrak{l}_a}I^{\gamma}\left\{\Lambda\left(\mathfrak{l}_a+\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)\right)\right\}+{}^{AB}I^{\gamma}_{\mathfrak{l}_a+\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)}\left\{\Lambda\left(\mathfrak{l}_a\right)\right\}\right]$$
$$-\left(\frac{\left[\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)\right]^{\gamma}+(1-\gamma)\Gamma(\gamma)}{\left[\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)\right]^{\gamma+1}}\right)\left[\Lambda\left(\mathfrak{l}_a\right)+\Lambda\left(\mathfrak{l}_a+\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)\right)\right]\Bigg|$$
$$\leq\ \int_0^1(1-\mathfrak{x})^{\gamma}\left|\Lambda'\left(\mathfrak{l}_a+\mathfrak{x}\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)\right)\right|d\mathfrak{x}+\int_0^1\mathfrak{x}^{\gamma}\left|\Lambda'\left(\mathfrak{l}_a+\mathfrak{x}\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)\right)\right|d\mathfrak{x}$$
$$\leq\ \left(\int_0^1(1-\mathfrak{x})^{\gamma}d\mathfrak{x}\right)^{1-\frac{1}{q}}\left(\int_0^1(1-\mathfrak{x})^{\gamma}\left|\Lambda'\left(\mathfrak{l}_a+\mathfrak{x}\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)\right)\right|^q d\mathfrak{x}\right)^{\frac{1}{q}}$$
$$+\left(\int_0^1\mathfrak{x}^{\gamma}d\mathfrak{x}\right)^{1-\frac{1}{q}}\left(\int_0^1\mathfrak{x}^{\gamma}\left|\Lambda'\left(\mathfrak{l}_a+\mathfrak{x}\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)\right)\right|^q d\mathfrak{x}\right)^{\frac{1}{q}}.$$

By using GCF of $|\Lambda'|^q$, we have

$$\Bigg|\frac{B(\gamma)\Gamma(\gamma)}{\left[\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)\right]^{\gamma+1}}\left[{}^{AB}_{\mathfrak{l}_a}I^{\gamma}\left\{\Lambda\left(\mathfrak{l}_a+\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)\right)\right\}+{}^{AB}I^{\gamma}_{\mathfrak{l}_a+\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)}\left\{\Lambda\left(\mathfrak{l}_a\right)\right\}\right]$$
$$-\left(\frac{\left[\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)\right]^{\gamma}+(1-\gamma)\Gamma(\gamma)}{\left[\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)\right]^{\gamma+1}}\right)\left[\Lambda\left(\mathfrak{l}_a\right)+\Lambda\left(\mathfrak{l}_a+\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)\right)\right]\Bigg|$$
$$\leq\ \left(\int_0^1(1-\mathfrak{x})^{\gamma}d\mathfrak{x}\right)^{1-\frac{1}{q}}\left(\int_0^1(1-\mathfrak{x})^{\gamma}\left[(1-\mathfrak{x})\left|\Lambda'\left(\mathfrak{l}_a\right)\right|^q+\mathfrak{x}\left|\Lambda'\left(\mathfrak{l}_b\right)\right|^q\right]d\mathfrak{x}\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1 \mathfrak{x}^{\gamma} d\mathfrak{x}\right)^{1-\frac{1}{q}} \left(\int_0^1 \mathfrak{x}^{\gamma}\left[(1-\mathfrak{x})\left|\Lambda'(\mathfrak{l}_a)\right|^q + \mathfrak{x}\left|\Lambda'(\mathfrak{l}_b)\right|^q\right] d\mathfrak{x}\right)^{\frac{1}{q}}$$
$$= \left(\frac{1}{\gamma+1}\right)^{1-\frac{1}{q}} \left[\left(\frac{|\Lambda'(\mathfrak{l}_a)|^q}{\gamma+2} + \frac{|\Lambda'(\mathfrak{l}_b)|^q}{(\gamma+1)(\gamma+2)}\right)^{\frac{1}{q}} + \left(\frac{|\Lambda'(\mathfrak{l}_a)|^q}{(\gamma+1)(\gamma+2)} + \frac{|\Lambda'(\mathfrak{l}_b)|^q}{\gamma+2}\right)^{\frac{1}{q}}\right].$$

Th proof is completed.

Remark 7. *Considering Theorem 11, we establish the following new mathematical approach of Hermite-Hadamard inequality pertaining to the classical Mittag-Leffler function via ABFIO if we pick $\varrho = (1, 1, \dots)$ with $\epsilon = \alpha$ and $\sigma = 1$:*

$$\Bigg|\frac{B(\gamma)\Gamma(\gamma)}{[\mathrm{E}_{\alpha}(\mathfrak{l}_b - \mathfrak{l}_a)]^{\gamma+1}} \left[{}^{AB}_{\mathfrak{l}_a}I^{\gamma}\left\{\Lambda\left(\mathfrak{l}_a + \mathrm{E}_{\alpha}(\mathfrak{l}_b - \mathfrak{l}_a)\right)\right\} + {}^{AB}I^{\gamma}_{\mathfrak{l}_a + \mathrm{E}_{\alpha}(\mathfrak{l}_b - \mathfrak{l}_a)}\left\{\Lambda\left(m\mathfrak{l}_a\right)\right\}\right]$$
$$- \left(\frac{[\mathrm{E}_{\alpha}(\mathfrak{l}_b - \mathfrak{l}_a)]^{\gamma} + (1-\gamma)\Gamma(\gamma)}{[\mathrm{E}_{\alpha}(\mathfrak{l}_b - \mathfrak{l}_a)]^{\gamma+1}}\right)\left[\Lambda\left(\mathfrak{l}_a\right) + \Lambda\left(\mathfrak{l}_a + \mathrm{E}_{\alpha}(\mathfrak{l}_b - \mathfrak{l}_a)\right)\right]\Bigg|$$
$$\leq \left(\frac{1}{\gamma+1}\right)^{1-\frac{1}{q}} \left[\left(\frac{|\Lambda'(\mathfrak{l}_a)|^q}{\gamma+2} + \frac{|\Lambda'(\mathfrak{l}_b)|^q}{(\gamma+1)(\gamma+2)}\right)^{\frac{1}{q}} + \left(\frac{|\Lambda'(\mathfrak{l}_a)|^q}{(\gamma+1)(\gamma+2)} + \frac{|\Lambda'(\mathfrak{l}_b)|^q}{\gamma+2}\right)^{\frac{1}{q}}\right],$$

Corollary 3. *In the above Theorem, if we choose $\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b - \mathfrak{l}_a) = \mathfrak{l}_b - \mathfrak{l}_a$, we obtain*

$$\Bigg|\frac{B(\gamma)\Gamma(\gamma)}{(\mathfrak{l}_b - \mathfrak{l}_a)^{\gamma+1}} \left[{}^{AB}_{\mathfrak{l}_a}I^{\gamma}\left\{\Lambda\left(\mathfrak{l}_b\right)\right\} + {}^{AB}I^{\gamma}_{\mathfrak{l}_b}\left\{\Lambda\left(\mathfrak{l}_a\right)\right\}\right]$$
$$- \left(\frac{(\mathfrak{l}_b - \mathfrak{l}_a)^{\gamma} + (1-\gamma)\Gamma(\gamma)}{(\mathfrak{l}_b - \mathfrak{l}_a)^{\gamma+1}}\right)\left[\Lambda\left(\mathfrak{l}_a\right) + \Lambda\left(\mathfrak{l}_b\right)\right]\Bigg|$$
$$\leq \left(\frac{1}{\gamma+1}\right)^{1-\frac{1}{q}} \left[\left(\frac{|\Lambda'(\mathfrak{l}_a)|^q}{\gamma+2} + \frac{|\Lambda'(\mathfrak{l}_b)|^q}{(\gamma+1)(\gamma+2)}\right)^{\frac{1}{q}} + \left(\frac{|\Lambda'(\mathfrak{l}_a)|^q}{(\gamma+1)(\gamma+2)} + \frac{|\Lambda'(\mathfrak{l}_b)|^q}{\gamma+2}\right)^{\frac{1}{q}}\right].$$

Theorem 12. *Let $I \subseteq \mathbb{R}$ be an open and non-empty convex set and $\mathfrak{l}_a, \mathfrak{l}_b \in I$ with $\mathfrak{l}_a < \mathfrak{l}_a + \Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b - \mathfrak{l}_a)$. Suppose that $\Lambda : I \to \mathbb{R}$ is a differentiable function and $\Lambda' \in L\left[\mathfrak{l}_a, \mathfrak{l}_a + \Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b - \mathfrak{l}_a)\right]$. If $|\Lambda'|^q$ is a GCF, then we have the following inequality for AB-FIO*

$$\Bigg|\frac{B(\gamma)\Gamma(\gamma)}{[\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b - \mathfrak{l}_a)]^{\gamma+1}} \left[{}^{AB}_{\mathfrak{l}_a}I^{\gamma}\left\{\Lambda\left(\mathfrak{l}_a + \Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b - \mathfrak{l}_a)\right)\right\} + {}^{AB}I^{\gamma}_{\mathfrak{l}_a + \Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b - \mathfrak{l}_a)}\left\{\Lambda\left(\mathfrak{l}_a\right)\right\}\right]$$
$$- \left(\frac{[\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b - \mathfrak{l}_a)]^{\gamma} + (1-\gamma)\Gamma(\gamma)}{[\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b - \mathfrak{l}_a)]^{\gamma+1}}\right)\left[\Lambda\left(\mathfrak{l}_a\right) + \Lambda\left(\mathfrak{l}_a + \Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b - \mathfrak{l}_a)\right)\right]\Bigg|$$
$$\leq \frac{2}{p(\gamma p+1)} + \frac{|\Lambda'(\mathfrak{l}_a)|^q + |\Lambda'(\mathfrak{l}_b)|^q}{q},$$

where $p^{-1} + q^{-1} = 1, q > 1, \gamma \in (0, 1]$.

Proof. By using the identity given in Lemma 1 and applying the Young inequality $xy \leq \frac{1}{p}x^p + \frac{1}{q}y^q$, we get

$$\begin{aligned}
&\left| \frac{B(\gamma)\Gamma(\gamma)}{[\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b - \mathfrak{l}_a)]^{\gamma+1}} \left[{}^{AB}_{\mathfrak{l}_a}I^{\gamma}\left\{\Lambda\left(\mathfrak{l}_a + \Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b - \mathfrak{l}_a)\right)\right\} + {}^{AB}I^{\gamma}_{\mathfrak{l}_a+\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)}\left\{\Lambda\left(\mathfrak{l}_a\right)\right\}\right] \right. \\
&\left. - \left(\frac{[\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b - \mathfrak{l}_a)]^{\gamma} + (1-\gamma)\Gamma(\gamma)}{[\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b - \mathfrak{l}_a)]^{\gamma+1}}\right)\left[\Lambda\left(\mathfrak{l}_a\right) + \Lambda\left(\mathfrak{l}_a + \Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b - \mathfrak{l}_a)\right)\right] \right| \\
\leq\ & \int_0^1 (1-\mathfrak{x})^{\gamma}\left|\Lambda'\left(\mathfrak{l}_a + \mathfrak{x}\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b - \mathfrak{l}_a)\right)\right| d\mathfrak{x} + \int_0^1 \mathfrak{x}^{\gamma}\left|\Lambda'\left(\mathfrak{l}_a + \mathfrak{x}\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b - \mathfrak{l}_a)\right)\right| d\mathfrak{x}. \\
\leq\ & \frac{1}{p}\int_0^1 (1-\mathfrak{x})^{\gamma p} d\mathfrak{x} + \frac{1}{q}\int_0^1 \left|\Lambda'\left(\mathfrak{l}_a + \mathfrak{x}\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b - \mathfrak{l}_a)\right)\right|^q d\mathfrak{x} \\
&+ \frac{1}{p}\int_0^1 \mathfrak{x}^{\gamma p} d\mathfrak{x} + \frac{1}{q}\int_0^1 \left|\Lambda'\left(\mathfrak{l}_a + \mathfrak{x}\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b - \mathfrak{l}_a)\right)\right|^q d\mathfrak{x}.
\end{aligned}$$

By using GCF of $|\Lambda'|^q$ and by a simple computation, we have the desired result.

Remark 8. *Considering Theorem 12, we establish the following new mathematical approach of Hermite-Hadamard inequality pertaining to the classical Mittag-Leffler function via ABFIO if we pick $\varrho = (1, 1, \ldots)$ with $\epsilon = \alpha$ and $\sigma = 1$:*

$$\begin{aligned}
&\left| \frac{B(\gamma)\Gamma(\gamma)}{[\mathrm{E}_{\alpha}(\mathfrak{l}_b - \mathfrak{l}_a)]^{\gamma+1}} \left[{}^{AB}_{\mathfrak{l}_a}I^{\gamma}\left\{\Lambda\left(\mathfrak{l}_a + \mathrm{E}_{\alpha}(\mathfrak{l}_b - \mathfrak{l}_a)\right)\right\} + {}^{AB}I^{\gamma}_{\mathfrak{l}_a+\Omega(\mathfrak{l}_b,\mathfrak{l}_a)}\left\{\Lambda\left(\mathfrak{l}_a\right)\right\}\right] \right. \\
&\left. - \left(\frac{[\mathrm{E}_{\alpha}(\mathfrak{l}_b - \mathfrak{l}_a)]^{\gamma} + (1-\gamma)\Gamma(\gamma)}{[\mathrm{E}_{\alpha}(\mathfrak{l}_b - \mathfrak{l}_a)]^{\gamma+1}}\right)\left[\Lambda\left(\mathfrak{l}_a\right) + \Lambda\left(\mathfrak{l}_a + \Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b - \mathfrak{l}_a)\right)\right] \right| \\
\leq\ & \frac{2}{p(\gamma p + 1)} + \frac{|\Lambda'(\mathfrak{l}_a)|^q + |\Lambda'(\mathfrak{l}_b)|^q}{q},
\end{aligned}$$

Corollary 4. *In the above Theorem 12, if we choose $\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b - \mathfrak{l}_a) = \mathfrak{l}_b - \mathfrak{l}_a$, we obtain*

$$\begin{aligned}
&\left| \frac{B(\gamma)\Gamma(\gamma)}{(\mathfrak{l}_b - \mathfrak{l}_a)^{\gamma+1}} \left[{}^{AB}_{\mathfrak{l}_a}I^{\gamma}\left\{\Lambda\left(\mathfrak{l}_b\right)\right\} + {}^{AB}I^{\gamma}_{\mathfrak{l}_b}\left\{\Lambda\left(\mathfrak{l}_a\right)\right\}\right] \right. \\
&\left. - \left(\frac{(\mathfrak{l}_b - \mathfrak{l}_a)^{\gamma} + (1-\gamma)\Gamma(\gamma)}{(\mathfrak{l}_b - \mathfrak{l}_a)^{\gamma+1}}\right)\left[\Lambda\left(\mathfrak{l}_a\right) + \Lambda\left(\mathfrak{l}_b\right)\right] \right| \\
\leq\ & \frac{2}{p(\gamma p + 1)} + \frac{|\Lambda'(\mathfrak{l}_a)|^q + |\Lambda'(\mathfrak{l}_b)|^q}{q}.
\end{aligned}$$

6. Pachpatte-Type Inequality via AB Fractional Integral Operator

In this section, We study and explore the Pachpatte-type inequality via ABFIO. We enhance this section's utility through the notes that are provided.

Theorem 13. *Let $I \subseteq \mathbb{R}$ be an open and non-empty convex set and $\mathfrak{l}_a, \mathfrak{l}_b \in I$ with $\mathfrak{l}_a < \mathfrak{l}_a + \Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b - \mathfrak{l}_a)$. If $\Lambda_1, \Lambda_2 : [\mathfrak{l}_a, \mathfrak{l}_a + \Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b - \mathfrak{l}_a)] \to \mathbb{R}$ are GCF, $\Lambda_1, \Lambda_2 \in L[\mathfrak{l}_a, \mathfrak{l}_a + \Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b - \mathfrak{l}_a)]$, then the following inequality for ABFIO holds*

$$\begin{aligned}
&\frac{1}{[\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b - \mathfrak{l}_a)]^{\gamma}} \left[{}^{AB}_{\ \mathfrak{l}_a} I^{\gamma} \left\{ \Lambda_1\Lambda_2 \left(\mathfrak{l}_a + \Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b - \mathfrak{l}_a) \right) \right\} + {}^{AB}I^{\gamma}_{\mathfrak{l}_a + \Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b - \mathfrak{l}_a)} \left\{ \Lambda_1\Lambda_2 \left(\mathfrak{l}_a \right) \right\} \right] \\
\leq\ & \frac{\gamma}{B(\gamma)\Gamma(\gamma)} \left[\left[\Lambda_1(\mathfrak{l}_a)\Lambda_2(\mathfrak{l}_a) + \Lambda_1(\mathfrak{l}_b)\Lambda_2(\mathfrak{l}_b) \right] \left(\frac{2}{\gamma(\gamma+1)(\gamma+2)} + \frac{1}{\gamma+2} \right) \right. \\
& \left. + 2\frac{[\Lambda_1(\mathfrak{l}_a)\Lambda_2(\mathfrak{l}_b) + \Lambda_1(\mathfrak{l}_b)\Lambda_2(\mathfrak{l}_a)]}{(\gamma+1)(\gamma+2)} \right] \\
& + \frac{(1-\gamma)}{B(\gamma)\left[\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b - \mathfrak{l}_a)\right]^{\gamma}} \Big[\Lambda_1(\mathfrak{l}_a)\Lambda_2(\mathfrak{l}_a) \\
& + \Lambda_1\left(\mathfrak{l}_a + \Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b - \mathfrak{l}_a) \right) \Lambda_2\left(\mathfrak{l}_a + \Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b - \mathfrak{l}_a) \right) \Big],
\end{aligned}$$

where $\gamma \in (0, 1]$.

Proof. Since Λ_1 and Λ_2 are *GCF* on $[\mathfrak{l}_a, \mathfrak{l}_a + \Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b - \mathfrak{l}_a)]$, we get

$$\Lambda_1\left(\mathfrak{l}_a + \mathfrak{x}\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b - \mathfrak{l}_a) \right) \leq (1-\mathfrak{x})\Lambda_1(\mathfrak{l}_a) + \mathfrak{x}\Lambda_1(\mathfrak{l}_b) \tag{6.1}$$

and

$$\Lambda_2\left(\mathfrak{l}_a + \mathfrak{x}\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b - \mathfrak{l}_a) \right) \leq (1-\mathfrak{x})\Lambda_2(\mathfrak{l}_a) + \mathfrak{x}\Lambda_2(\mathfrak{l}_b). \tag{6.2}$$

By multiplying both inequalities 6.1 and 6.2 side by side, we get

$$\begin{aligned}
&\Lambda_1\left(\mathfrak{l}_a + \mathfrak{x}\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b - \mathfrak{l}_a) \right) \Lambda_2\left(\mathfrak{l}_a + \mathfrak{x}\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b - \mathfrak{l}_a) \right) \\
&\leq (1-\mathfrak{x})^2\Lambda_1(\mathfrak{l}_a)\Lambda_2(\mathfrak{l}_a) + \mathfrak{x}^2\Lambda_1(\mathfrak{l}_b)\Lambda_2(\mathfrak{l}_b) \\
&\quad + \mathfrak{x}(1-\mathfrak{x})\left[\Lambda_1(\mathfrak{l}_a)\Lambda_2(\mathfrak{l}_b) + \Lambda_1(\mathfrak{l}_b)\Lambda_2(\mathfrak{l}_a) \right].
\end{aligned} \tag{6.3}$$

By multiplying both sides of (6.3) with $(1-\mathfrak{x})^{\gamma-1}$ and integrating the resulting inequality w.r.t. $\mathfrak{x}$ over $[0, 1]$, we obtain

$$\begin{aligned}
&\int_0^1 (1-\mathfrak{x})^{\gamma-1}\Lambda_1\left(\mathfrak{l}_a + \mathfrak{x}\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b - \mathfrak{l}_a) \right) \Lambda_2\left(\mathfrak{l}_a + \mathfrak{x}\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b - \mathfrak{l}_a) \right) d\mathfrak{x} \\
\leq\ & \int_0^1 (1-\mathfrak{x})^{\gamma-1} \left[(1-\mathfrak{x})^2\Lambda_1(\mathfrak{l}_a)\Lambda_2(\mathfrak{l}_a) + \mathfrak{x}^2\Lambda_1(\mathfrak{l}_b)\Lambda_2(\mathfrak{l}_b) \right. \\
& \left. + \mathfrak{x}(1-\mathfrak{x})\left[\Lambda_1(\mathfrak{l}_a)\Lambda_2(\mathfrak{l}_b) + \Lambda_1(\mathfrak{l}_b)\Lambda_2(\mathfrak{l}_a) \right] \right] d\mathfrak{x} \\
=\ & \frac{\Lambda_1(\mathfrak{l}_a)\Lambda_2(\mathfrak{l}_a)}{\gamma+2} + 2\frac{\Lambda_1(\mathfrak{l}_b)\Lambda_2(\mathfrak{l}_b)}{\gamma(\gamma+1)(\gamma+2)} + \frac{[\Lambda_1(\mathfrak{l}_a)\Lambda_2(\mathfrak{l}_b) + \Lambda_1(\mathfrak{l}_b)\Lambda_2(\mathfrak{l}_a)]}{(\gamma+1)(\gamma+2)}.
\end{aligned}$$

By changing the variable $\mathfrak{l}_a + \mathfrak{x}\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b - \mathfrak{l}_a) = x$, we can write the inequality in (6.4) as

$$\frac{1}{[\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b - \mathfrak{l}_a)]^{\gamma}} \int_{\mathfrak{l}_a}^{\mathfrak{l}_a + \Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b - \mathfrak{l}_a)} \left(\mathfrak{l}_a + \Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b - \mathfrak{l}_a) - x \right)^{\gamma-1} \Lambda_1(x)\Lambda_2(x)dx$$

$$\leq \frac{\Lambda_1(\mathfrak{l}_a)\Lambda_2(\mathfrak{l}_a)}{\gamma+2} + 2\frac{\Lambda_1(\mathfrak{l}_b)\Lambda_2(\mathfrak{l}_b)}{\gamma(\gamma+1)(\gamma+2)} + \frac{[\Lambda_1(\mathfrak{l}_a)\Lambda_2(\mathfrak{l}_b)+\Lambda_1(\mathfrak{l}_b)\Lambda_2(\mathfrak{l}_a)]}{(\gamma+1)(\gamma+2)}. \tag{6.4}$$

By multiplying the both sides of (6.4) by $\frac{\gamma}{B(\gamma)\Gamma(\gamma)}$ and then adding the term $\frac{(1-\gamma)}{B(\gamma)\left[\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)\right]^{\gamma}}\Lambda_1\left(\mathfrak{l}_a+\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)\right)\Lambda_2\left(\mathfrak{l}_a+\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)\right)$ to both sides of (6.4) and finally using ABFIO, we get

$$\begin{aligned}
&\frac{1}{\left[\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)\right]^{\gamma}}\left[{}^{AB}_{\mathfrak{l}_a}I^{\gamma}\left\{\Lambda_1\Lambda_2\left(\mathfrak{l}_a+\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)\right)\right\}\right]\\
\leq\ & \frac{\gamma}{B(\gamma)\Gamma(\gamma)}\left[\frac{\Lambda_1(\mathfrak{l}_a)\Lambda_2(\mathfrak{l}_a)}{\gamma+2} + 2\frac{\Lambda_1(\mathfrak{l}_b)\Lambda_2(\mathfrak{l}_b)}{\gamma(\gamma+1)(\gamma+2)} + \frac{[\Lambda_1(\mathfrak{l}_a)\Lambda_2(\mathfrak{l}_b)+\Lambda_1(\mathfrak{l}_b)\Lambda_2(\mathfrak{l}_a)]}{(\gamma+1)(\gamma+2)}\right]\\
&+\frac{(1-\gamma)}{B(\gamma)\left[\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)\right]^{\gamma}}\Lambda_1\left(\mathfrak{l}_a+\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)\right)\Lambda_2\left(\mathfrak{l}_a+\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)\right).
\end{aligned}\tag{6.5}$$

Similarly, by multiplying both sides of (6.3) with $\mathfrak{r}^{\gamma-1}$ and integrating the resulting inequality w.r.t. $\mathfrak{r}$ over $[0,1]$, we obtain

$$\begin{aligned}
&\int_0^1 \mathfrak{r}^{\gamma-1}\Lambda_1\left(\mathfrak{l}_a+\mathfrak{r}\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)\right)\Lambda_2\left(\mathfrak{l}_a+\mathfrak{r}\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)\right)d\mathfrak{r}\\
\leq\ & \int_0^1 \mathfrak{r}^{\gamma-1}\left[(1-\mathfrak{r})^2\Lambda_1(\mathfrak{l}_a)\Lambda_2(\mathfrak{l}_a)+\mathfrak{r}^2\Lambda_1(\mathfrak{l}_b)\Lambda_2(\mathfrak{l}_b)\right.\\
&\left.+\mathfrak{r}(1-\mathfrak{r})\left[\Lambda_1(\mathfrak{l}_a)\Lambda_2(\mathfrak{l}_b)+\Lambda_1(\mathfrak{l}_b)\Lambda_2(\mathfrak{l}_a)\right]\right]d\mathfrak{r}\\
=\ & 2\frac{\Lambda_1(\mathfrak{l}_a)\Lambda_2(\mathfrak{l}_a)}{\gamma(\gamma+1)(\gamma+2)} + \frac{\Lambda_1(\mathfrak{l}_b)\Lambda_2(\mathfrak{l}_b)}{\gamma+2} + \frac{[\Lambda_1(\mathfrak{l}_a)\Lambda_2(\mathfrak{l}_b)+\Lambda_1(\mathfrak{l}_b)\Lambda_2(\mathfrak{l}_a)]}{(\gamma+1)(\gamma+2)}.
\end{aligned}$$

By making calculations similar to those in the proof of (6.5), we obtain

$$\begin{aligned}
&\frac{1}{\left[\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)\right]^{\gamma}}\left[{}^{AB}I^{\gamma}_{\mathfrak{l}_a+\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)}\left\{\Lambda_1\Lambda_2(\mathfrak{l}_a)\right\}\right]\\
\leq\ & \frac{\gamma}{B(\gamma)\Gamma(\gamma)}\left[2\frac{\Lambda_1(\mathfrak{l}_a)\Lambda_2(\mathfrak{l}_a)}{\gamma(\gamma+1)(\gamma+2)} + 2\frac{\Lambda_1(\mathfrak{l}_b)\Lambda_2(\mathfrak{l}_b)}{\gamma+2} + \frac{[\Lambda_1(\mathfrak{l}_a)\Lambda_2(\mathfrak{l}_b)+\Lambda_1(\mathfrak{l}_b)\Lambda_2(\mathfrak{l}_a)]}{(\gamma+1)(\gamma+2)}\right]\\
&+\frac{(1-\gamma)}{B(\gamma)\left[\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)\right]^{\gamma}}\Lambda_1(\mathfrak{l}_a)\Lambda_2(\mathfrak{l}_a).
\end{aligned}\tag{6.6}$$

Adding (6.5) and (6.6) side by side, we get

$$\begin{aligned}
&\frac{1}{\left[\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)\right]^{\gamma}}\left[{}^{AB}_{\mathfrak{l}_a}I^{\gamma}\left\{\Lambda_1\Lambda_2\left(\mathfrak{l}_a+\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)\right)\right\}\right.\\
&\left.+{}^{AB}I^{\gamma}_{\mathfrak{l}_a+\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)}\left\{\Lambda_1\Lambda_2(\mathfrak{l}_a)\right\}\right]\\
\leq\ & \frac{\gamma}{B(\gamma)\Gamma(\gamma)}\left[\left[\Lambda_1(\mathfrak{l}_a)\Lambda_2(\mathfrak{l}_a)+\Lambda_1(\mathfrak{l}_b)\Lambda_2(\mathfrak{l}_b)\right]\left(\frac{2}{\gamma(\gamma+1)(\gamma+2)}+\frac{1}{\gamma+2}\right)\right.\\
&\left.+2\frac{[\Lambda_1(\mathfrak{l}_a)\Lambda_2(\mathfrak{l}_b)+\Lambda_1(\mathfrak{l}_b)\Lambda_2(\mathfrak{l}_a)]}{(\gamma+1)(\gamma+2)}\right]
\end{aligned}$$

$$+\frac{(1-\gamma)}{B(\gamma)\left[\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)\right]^{\gamma}}\Big[\Lambda_1\left(\mathfrak{l}_a\right)\Lambda_2\left(\mathfrak{l}_a\right)$$
$$+\Lambda_1\left(\mathfrak{l}_a+\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)\right)\Lambda_2\left(\mathfrak{l}_a+\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)\right)\Big].$$

The proof is completed.

Remark 9. *Considering Theorem 13, we establish the following new mathematical approach of Pachpatte-type inequality pertaining to the classical Mittag-Leffler function via ABFIO if we pick $\varrho=(1,1,...)$ with $\epsilon=\alpha$ and $\sigma=1$:*

$$\frac{1}{\left[\Omega\left(\mathfrak{l}_b,\mathfrak{l}_a,\right)\right]^{\gamma}}\left[{}^{AB}_{\ \mathfrak{l}_a}I^{\gamma}\left\{\Lambda_1\Lambda_2\left(\mathfrak{l}_a+\mathrm{E}_{\alpha}(\mathfrak{l}_b-\mathfrak{l}_a)\right)\right\}+{}^{AB}I^{\gamma}_{\mathfrak{l}_a+\mathrm{E}_{\alpha}(\mathfrak{l}_b-\mathfrak{l}_a)}\left\{\Lambda_1\Lambda_2\left(\mathfrak{l}_a\right)\right\}\right]$$
$$\leq \frac{\gamma}{B(\gamma)\Gamma(\gamma)}\Bigg[\left[\Lambda_1\left(\mathfrak{l}_a\right)\Lambda_2\left(\mathfrak{l}_a\right)+\Lambda_1\left(\mathfrak{l}_b\right)\Lambda_2\left(\mathfrak{l}_b\right)\right]\left(\frac{2}{\gamma(\gamma+1)(\gamma+2)}+\frac{1}{\gamma+2}\right)$$
$$+2\frac{\left[\Lambda_1\left(\mathfrak{l}_a\right)\Lambda_2\left(\mathfrak{l}_b\right)+\Lambda_1\left(\mathfrak{l}_b\right)\Lambda_2\left(\mathfrak{l}_a\right)\right]}{(\gamma+1)(\gamma+2)}\Bigg]$$
$$+\frac{(1-\gamma)}{B(\gamma)\left[\mathrm{E}_{\alpha}(\mathfrak{l}_b-\mathfrak{l}_a)\right]^{\gamma}}\Big[\Lambda_1\left(\mathfrak{l}_a\right)\Lambda_2\left(\mathfrak{l}_a\right)$$
$$+\Lambda_1\left(\mathfrak{l}_a+\mathrm{E}_{\alpha}(\mathfrak{l}_b-\mathfrak{l}_a)\right)\Lambda_2\left(\mathfrak{l}_a+\mathrm{E}_{\alpha}(\mathfrak{l}_b-\mathfrak{l}_a)\right)\Big].$$

Also, in the above Theorem 13, if we put $\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_b-\mathfrak{l}_a)=\mathfrak{l}_b-\mathfrak{l}_a$, then we get the new variant of Pachpatte-type integral inequality involving convexity via ABFIO.

Theorem 14. *If $\Lambda_1,\Lambda_2:[\mathfrak{l}_a,\mathfrak{l}_b]\to\mathbb{R}$ are convex functions, $\Lambda_1,\Lambda_2\in L\left[\mathfrak{l}_a,\mathfrak{l}_b\right]$, then the following inequality for ABFIO holds*

$$\frac{1}{\left(\mathfrak{l}_b-\mathfrak{l}_a\right)^{\gamma}}\left[{}^{AB}_{\mathfrak{l}_a}I^{\gamma}\left\{\Lambda_1\Lambda_2\left(\mathfrak{l}_b\right)\right\}+{}^{AB}I^{\gamma}_{\mathfrak{l}_b}\left\{\Lambda_1\Lambda_2\left(\mathfrak{l}_a\right)\right\}\right]$$
$$\leq \frac{\gamma}{B(\gamma)\Gamma(\gamma)}\Bigg[\left[\Lambda_1\left(\mathfrak{l}_a\right)\Lambda_2\left(\mathfrak{l}_a\right)+\Lambda_1\left(\mathfrak{l}_b\right)\Lambda_2\left(\mathfrak{l}_b\right)\right]\left(\frac{2}{\gamma(\gamma+1)(\gamma+2)}+\frac{1}{\gamma+2}\right)$$
$$+2\frac{\left[\Lambda_1\left(\mathfrak{l}_a\right)\Lambda_2\left(\mathfrak{l}_b\right)+\Lambda_1\left(\mathfrak{l}_b\right)\Lambda_2\left(\mathfrak{l}_a\right)\right]}{(\gamma+1)(\gamma+2)}\Bigg]$$
$$+\frac{(1-\gamma)}{B(\gamma)\left(\mathfrak{l}_b-\mathfrak{l}_a\right)^{\gamma}}\left[\Lambda_1\left(\mathfrak{l}_a\right)\Lambda_2\left(\mathfrak{l}_a\right)+\Lambda_1\left(\mathfrak{l}_b\right)\Lambda_2\left(\mathfrak{l}_b\right)\right].$$

7. Applications to entropy

Entropy is important because it provides a fundamental measure of uncertainty, disorder, or information content across various scientific disciplines. It also has a strong relation with integral inequalities, particularly in the study of convex functions and functional analysis. For example, many entropy-based inequalities, such as the Gibbs inequality or logarithmic Sobolev inequalities, are special cases or extensions of classical integral inequalities. These relationships allow researchers to analyze the behavior of complex systems by connecting probabilistic concepts with analytical tools. This interplay between entropy and integral inequalities is crucial in various fields, including mathematical physics, optimization, statistical mechanics, and information geometry, where it helps to derive bounds, characterize stability, and study the distribution of measures. Consider the exponential random variable $\mathcal{E}_\lambda$ with parameter $\lambda > 0$, whose probability density function (PDF) is

$$\mathtt{f}(\mathtt{x}) = \lambda e^{-\lambda \mathtt{x}}, \quad \mathtt{x} \geq 0.$$

The Shannon entropy of $\mathcal{E}_\lambda$ is defined by

$$H(\mathcal{E}_\lambda) = -\int_0^\infty \mathtt{f}(\mathtt{x}) \log \mathtt{f}(\mathtt{x})\, d\mathtt{x} = -\int_0^\infty \lambda e^{-\lambda \mathtt{x}} \log\left(\lambda e^{-\lambda \mathtt{x}}\right) d\mathtt{x}.$$

Compute the Shannon entropy explicitly using standard integration,

$$\begin{aligned} H(\mathcal{E}_\lambda) &= -\int_0^\infty \lambda e^{-\lambda \mathtt{x}} \big(\log \lambda - \lambda \mathtt{x} \big)\, d\mathtt{x} \\ &= -\log \lambda \int_0^\infty \lambda e^{-\lambda \mathtt{x}}\, d\mathtt{x} + \lambda \int_0^\infty \lambda \mathtt{x} e^{-\lambda \mathtt{x}}\, d\mathtt{x} \\ &= -\log \lambda \cdot 1 + \lambda \cdot \frac{1}{\lambda} = 1 - \log \lambda. \end{aligned}$$

Define the function $\mathtt{\Lambda}$ related to the integrand and consider

$$\mathtt{\Lambda}(\mathtt{x}) := -\log \mathtt{f}(\mathtt{x}) = -\log\left(\lambda e^{-\lambda \mathtt{x}}\right) = \lambda \mathtt{x} - \log \lambda,$$

which is linear (hence convex) on $[0, \infty)$.
Choose the interval and Mittag-Leffler deformation and restrict the domain to the compact interval

$$[\mathfrak{l}_{\mathtt{a}}, \mathfrak{l}_{\mathtt{b}}] := \left[0, \frac{1}{\lambda}\right].$$

Define the Mittag-Leffler type deformation

$$\Delta^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_{\mathtt{b}}) := \sum_{k=0}^{\infty} \frac{\varrho(k)}{\Gamma(\epsilon k + \sigma)} \left(\frac{1}{\lambda}\right)^k,$$

with parameters

$$\varrho(k) = 1, \quad \epsilon = \alpha \in (0,1), \quad \sigma = 1,$$

so that

$$\Delta^{(1)}_{\alpha,1}\left(\frac{1}{\lambda}\right) = \sum_{k=0}^{\infty} \frac{\left(\frac{1}{\lambda}\right)^k}{\Gamma(1 + \alpha k)} =: \mathtt{E}_\alpha\left(\frac{1}{\lambda}\right),$$

the classical Mittag-Leffler function.

Applying `Theorem 8` to $\mathtt{\Lambda}$. From the integral inequality theorem for generalized-convex functions and AB-fractional integrals, we have the double inequality:

$$\begin{aligned}\mathtt{\Lambda}\left(\frac{2\mathfrak{l}_{\mathtt{a}} + \mathtt{\Delta}^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_{\mathtt{b}})}{2}\right) &\leq \frac{B(\gamma)\Gamma(\gamma)}{2\left[\mathtt{\Delta}^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_{\mathtt{b}})\right]^{\gamma}}\left[0\mathtt{\Delta}^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_{\mathtt{b}})\mathtt{\Lambda}\left(\mathtt{\Delta}^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_{\mathtt{b}})\right) + \mathtt{\Delta}^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_{\mathtt{b}})0\mathtt{\Lambda}(0)\right] \\ &\quad - \frac{(1-\gamma)\Gamma(\gamma)}{2\left[\mathtt{\Delta}^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_{\mathtt{b}})\right]^{\gamma}}\left[\mathtt{\Lambda}(0) + \mathtt{\Lambda}\left(\mathtt{\Delta}^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_{\mathtt{b}})\right)\right] \leq \frac{\mathtt{\Lambda}(0) + \mathtt{\Lambda}(\mathfrak{l}_{\mathtt{b}})}{2}.\end{aligned} \tag{7.1}$$

Substitute the values:

$$\mathfrak{l}_{\mathtt{a}} = 0, \quad \mathfrak{l}_{\mathtt{b}} = \frac{1}{\lambda}, \quad \mathtt{\Delta}^{\varrho}_{\epsilon,\sigma}(\mathfrak{l}_{\mathtt{b}}) = \mathtt{E}_{\alpha}\left(\frac{1}{\lambda}\right),$$

and recall

$$\mathtt{\Lambda}(\mathtt{x}) = \lambda\mathtt{x} - \log\lambda.$$

Thus,

$$\mathtt{\Lambda}\left(\frac{\mathtt{E}_{\alpha}\left(\frac{1}{\lambda}\right)}{2}\right) = \lambda \cdot \frac{\mathtt{E}_{\alpha}\left(\frac{1}{\lambda}\right)}{2} - \log\lambda,$$

and

$$\frac{\mathtt{\Lambda}(0) + \mathtt{\Lambda}\left(\frac{1}{\lambda}\right)}{2} = \frac{(-\log\lambda) + (1 - \log\lambda)}{2} = \frac{1 - 2\log\lambda}{2}.$$

Therefore, inequality (7.1) becomes

$$\boxed{\lambda \cdot \frac{\mathtt{E}_{\alpha}\left(\frac{1}{\lambda}\right)}{2} - \log\lambda \leq (\text{fractional AB integral expression}) \leq \frac{1 - 2\log\lambda}{2}.}$$

This inequality provides fractional integral bounds on the entropy-related function $\mathtt{\Lambda}$, linking fractional calculus, Mittag-Leffler functions, and information-theoretic measures.

8. Conclusions

Fractional calculus has attracted considerable attention from researchers and scholars across a wide range of disciplines. At the same time, convexity theory has emerged as a powerful analytical framework for constructing novel numerical models that address complex problems in both pure and applied sciences. The growing interest in convex analysis and its related inequalities is driven by ongoing theoretical advancements, generalizations, and diverse applications. In this work, we first present a novel approach to the Hermite–Hadamard inequality via generalized convex functions (GCF) over the Atangana–Baleanu fractional integral operator (ABFIO), accompanied by several remarks and corollaries. Secondly, we establish a new identity involving the Raina function and derive refined versions of the Hermite–Hadamard inequality using classical tools such as Hölder's inequality, the power mean inequality, and Young's inequality. Thirdly, we propose a new modification of a Pachpatte-type inequality based on a recently introduced concept within the

ABFIO framework. Moreover, the established inequalities have potential implications in the contexts of interval analysis and quantum calculus. Integral inequalities, in particular, represent a rapidly evolving area of research. The integration of interval-valued analysis and quantum calculus into this field opens up intriguing avenues for further development, likely to captivate future investigations.

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