EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 4, Article Number 6557
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

A New Definition of (α, β)-Fractional Derivatives for Complex-Valued Functions and Their Analytic Properties

Waseem Ghazi Alshanti[1,*], Ma′mon Abu Hammad[1], Roshdi Khalil[2]

[1] *Department of Mathematics, Al Zaytoonah University of Jordan, Amman, Jordan*
[2] *Department of Mathematics, The University of Jordan, Amman, Jordan*

Abstract. In this paper, we present a new definition of fractional derivatives of complex-valued functions, defined via two parameters (α, β). We introduce the concepts of both (α, β)-Cauchy-Riemann equations and (α, β)-fractional analytic complex valued function.
2020 Mathematics Subject Classifications: 32Q02, 30K99, 30J15
Key Words and Phrases: (α, β)-derivative, (α, β)-Cauchy-Riemann equations, (α, β)-analytic complex valued function

1. Introduction

In mathematics, the positive integer ordered derivative $\frac{dy}{dx} = f'(x)$ of a function $y = f(x)$ is a concept by which we can find the instantaneous rate of change of a vertical variable y with respect to a horizontal variable x, specifically, we write

$$f'(x) = \lim_{\triangle x \to 0} \frac{\triangle y}{\triangle x} = \lim_{\triangle x \to 0} \frac{f(x + \triangle x) - f(x)}{\triangle x},$$

where $\triangle x$ and $\triangle y$ are, respectively, the corresponding increments of the variable x and the variable y [1]. Dynamical systems whose state evolves over time, such as population growth and the flowing of a fluid through a pipe, can be described and studied by differential equations (ordinary or partial) that involve derivative of a function. However, some of these dynamical systems that represent phenomena like electromagnetism, economy and finance, and signal processing cannot be handled accurately using standard differential equations that include positive integer ordered derivatives. In such cases, differential equations in which the included derivatives possess fractional orders come into play. It is known that, the most widely used definitions for fractional derivatives are the old ones Riemann–Liouville and Caputo definitions and the latest one [1–4], the so called conformable fractional derivative [5]. These definitions can be stated as follows:

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v18i4.6557

Email addresses: w.alshanti@zuj.edu.jo (W. G. Alshanti), m.abuhammad@zuj.edu.jo (M. A. Hammad), roshdi@ju.edu.jo (R. Khalil)

(i) Riemann–Liouville definition [1]. For $\alpha \in [n-1, n)$, the α derivative for f is

$$D_a^{\alpha}(f)(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(x)}{(t-x)^{\alpha-n+1}} dx.$$

(ii) Caputo definition [2]. For $\alpha \in [n-1, n)$, the α derivative of f is

$$D_a^{\alpha}(f)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(x)}{(t-x)^{\alpha-n+1}} dx.$$

(iii) Khalil definition [5]. For $\alpha \in (0, 1)$, the α-conformable derivative of f is

$$D^{\alpha}u(x) = \lim_{\epsilon \to 0} \frac{u(x + \epsilon x^{1-\alpha}) - u(x)}{\epsilon}.$$

There is still no general consensus on the concept of complex-order derivatives. The past century has seen few contributions related to this concept. For example in [6], the author assumed, together with certain assumption, that the derivative of complex order w of a complex function $f(z)$, to be defined by the generalized Cauchy integral

$$D^{(w)}f(z) = \frac{\Gamma(1+w)}{2\pi i} \int_{\partial D} f(\eta)(\eta - z)^{-w-1} d\eta,$$

where D is a closed region within the complex plane $\mathbb{C}$. Moreover, as an application in solving hypergeometric integral equations, the concept of derivatives of purely imaginary orders were also studied [7]. Recently, the concept of α-fractional analytic function was carried out with some nice results [8].

In this paper, we present the definition of (α, β)-fractional derivative of a complex function $f(z)$ defined on a region $G \subseteq \mathbb{C}$.

2. The Definition / Proof of the basic result

Throughout this paper, let $E = \{(x, y) : x, y > 0\}$ and $f : G \subseteq E \to \mathbb{C}$ be a complex valued function defined on a region $G \subseteq \mathbb{C}$. Let $z_\circ = (x_\circ, y_\circ) \in G^\circ$, the interior of G, and $x_\circ, y_\circ > 0$.

Definition 1. *A function $f : G \subseteq \mathbb{C} \to \mathbb{C}$ is said to be (α, β)-differentiable at $z_\circ = x_\circ + iy_\circ \in G^\circ$ and denoted by $f^{(\alpha,\beta)}(z_\circ)$, where $|\alpha|, |\beta| < 1$, if*

$$\begin{aligned} f^{(\alpha,\beta)}(z_\circ) &= D^{(\alpha,\beta)}f(z_\circ) \\ &= \lim_{(\epsilon_1,\epsilon_2)\to(0,0)} \frac{f(x_\circ + \epsilon_1 x_\circ^{1-\alpha}, y_\circ + \epsilon_2 y_\circ^{1-\beta}) - f(x_\circ, y_\circ)}{\epsilon_1 + i\epsilon_2}, \end{aligned} \tag{1}$$

exists .

Accordingly, let us assume $f(z) = u(x,y) + iv(x,y)$ such that it is (α,β)-differentiable at $z_\circ = (x_\circ, y_\circ)$. Thus, the above limit (1) exists along any bath toward $z_\circ$ and all are equal. Hence, if we move along $(x, y_\circ)$ towards $z_\circ = (x_\circ, y_\circ)$, we get

$$\begin{aligned} D^{(\alpha,\beta)} f(z_\circ) &= \lim_{\epsilon_1 \to 0} \frac{\left[u(x_\circ + \epsilon_1 x_\circ^{1-\alpha}, y_\circ) + iv(x_\circ + \epsilon_1 x_\circ^{1-\alpha}, y_\circ)\right] - \left[u(x_\circ, y_\circ) + iv(x_\circ, y_\circ)\right]}{\epsilon_1} \\ &= u_x^\alpha(x_\circ, y_\circ) + iv_x^\alpha(x_\circ, y_\circ), \end{aligned} \tag{2}$$

where $u_x^\alpha(x_\circ, y_\circ)$ is the α-fractional partial derivative of u with respect to x at $(x_\circ, y_\circ)$ and $v_x^\alpha(x_\circ, y_\circ)$ is the α-fractional partial derivative of v with respect to x at $(x_\circ, y_\circ)$.

Similarly, if we move along $(x_\circ, y)$ towards $z_\circ = (x_\circ, y_\circ)$, we get

$$\begin{aligned} D^{(\alpha,\beta)} f(z_\circ) &= \lim_{\epsilon_2 \to 0} \frac{\left[u(x_\circ, y_\circ + \epsilon_2 y_\circ^{1-\beta}) + iv(x_\circ, y_\circ + \epsilon_2 y_\circ^{1-\beta})\right] - \left[u(x_\circ, y_\circ) + iv(x_\circ, y_\circ)\right]}{i\epsilon_2} \\ &= v_y^\beta(x_\circ, y_\circ) - iu_y^\beta(x_\circ, y_\circ), \end{aligned} \tag{3}$$

where $v_y^\beta(x_\circ, y_\circ)$ is the β-fractional partial derivative of v with respect to y at $(x_\circ, y_\circ)$ and $u_y^\beta(x_\circ, y_\circ)$ is the β-fractional partial derivative of v with respect to y at $(x_\circ, y_\circ)$.

Therefore, for (α,β)-differentiability of f at $z_\circ = (x_\circ, y_\circ)$, we must have

$$\begin{aligned} u_x^\alpha(z_\circ) &= v_y^\beta(z_\circ), \\ u_y^\beta(z_\circ) &= -v_x^\alpha(z_\circ). \end{aligned} \tag{4}$$

Equations (4) will be called the (α,β)-Cauchy-Riemann equtions. Noting that if $\alpha = \beta$, then the two equations (4) represent the α-Cauchy-Riemann equations that are reported in [8].

Example 1. Consider $f(z) = f(x,y) = \frac{\beta^2 x^{2\alpha} - \alpha^2 y^{2\beta}}{\alpha^2 \beta^2} + i2\frac{x^\alpha y^\beta}{\alpha\beta}$, where $|\alpha|, |\beta| < 1$. Then we have $u_x^\alpha = v_y^\beta = \frac{2}{\alpha} x^\alpha$, and $u_y^\beta = -v_x^\alpha = -\frac{2}{\beta} y^\beta$ [5]. Hence, u_x^α, u_y^β, v_x^α, and v_y^β are all exist and continuous at $z_\circ = x_\circ + iy_\circ \in G^\circ$. Moreover, they satisfy the (α,β)-Cauchy-Riemann equations (4).

Example 2. For $f(z) = f(x,y) = e^{\left(\frac{x^\alpha}{\alpha} + i\frac{y^\beta}{\beta}\right)} = e^{\frac{x^\alpha}{\alpha}} \cos \frac{y^\beta}{\beta} + ie^{\frac{x^\alpha}{\alpha}} \sin \frac{y^\beta}{\beta}$, where $|\alpha|, |\beta| < 1$. We have $u_x^\alpha = v_y^\beta = e^{\frac{x^\alpha}{\alpha}} \cos \frac{y^\beta}{\beta}$, and $u_y^\beta = -v_x^\alpha = -e^{\frac{x^\alpha}{\alpha}} \sin \frac{y^\beta}{\beta}$ [5]. Hence, u_x^α, u_y^β, v_x^α, and v_y^β are all exist and continuous at $z_\circ = x_\circ + iy_\circ \in G^\circ$. Moreover, they satisfy the (α,β)-Cauchy-Riemann equations (4).

Clearly, as in the classical case, (α,β)-Cauchy-Riemann equations are necessary condition for (α,β)-differentiability of $f(z)$ at $z_\circ$ but not a sufficient one. This leads to the following result.

Theorem 1. *(Sufficient conditions for (α,β)-differentiability) Let $f : G \subseteq \mathbb{C} \to \mathbb{C}$ be a complex valued function defined on a region $G \subseteq \mathbb{C}$. Let $z_\circ = (x_\circ, y_\circ) \in G^\circ$ the interior of G. If*

i) u_x^α, u_y^β, v_x^α, and v_y^β all exist and are continuous at $z_\circ$,
ii) (α,β)-Cauchy-Riemann equations (4) are satisfied at $z_\circ$,

then, f is (α,β)-differentiable at $z_\circ$.

Proof. Suppose that the two conditions of the Theorem 1 hold. Now,

$$\begin{aligned}
&\frac{f(x_\circ + \frac{x-x_\circ}{x_\circ^{1-\alpha}}x_\circ^{1-\alpha} + i\left[y_\circ + \frac{y-y_\circ}{y_\circ^{1-\beta}}y_\circ^{1-\beta}\right]) - f(x_\circ, y_\circ)}{\frac{x-x_\circ}{x_\circ^{1-\alpha}} + i\frac{y-y_\circ}{y_\circ^{1-\beta}}} \\
=\ &\frac{u(x_\circ + \frac{x-x_\circ}{x_\circ^{1-\alpha}}x_\circ^{1-\alpha}, y_\circ + \frac{y-y_\circ}{y_\circ^{1-\beta}}y_\circ^{1-\beta}) - u(x_\circ, y_\circ)}{\frac{x-x_\circ}{x_\circ^{1-\alpha}} + i\frac{y-y_\circ}{y_\circ^{1-\beta}}} \\
&+ i\frac{v(x_\circ + \frac{x-x_\circ}{x_\circ^{1-\alpha}}x_\circ^{1-\alpha}, y_\circ + \frac{y-y_\circ}{y_\circ^{1-\beta}}y_\circ^{1-\beta}) - v(x_\circ, y_\circ)}{\frac{x-x_\circ}{x_\circ^{1-\alpha}} + i\frac{y-y_\circ}{y_\circ^{1-\beta}}}.
\end{aligned}$$

But,

$$\begin{aligned}
&u(x_\circ + \frac{x-x_\circ}{x_\circ^{1-\alpha}}x_\circ^{1-\alpha}, y_\circ + \frac{y-y_\circ}{y_\circ^{1-\beta}}y_\circ^{1-\beta}) - u(x_\circ, y_\circ) \\
=\ &u(x,y) - u(x_\circ, y_\circ) \\
=\ &\frac{x-x_\circ}{x_\circ^{1-\alpha}}u_x^\alpha(x_\circ, y_\circ) + \frac{y-y_\circ}{y_\circ^{1-\beta}}u_y^\beta(x_\circ, y_\circ) + \sqrt{(x-x_\circ)^2 + (y-y_\circ)^2}\,\delta_1(z),
\end{aligned}$$

and

$$\begin{aligned}
&v(x_\circ + \frac{x-x_\circ}{x_\circ^{1-\alpha}}x_\circ^{1-\alpha}, y_\circ + \frac{y-y_\circ}{y_\circ^{1-\beta}}y_\circ^{1-\beta}) - v(x_\circ, y_\circ) \\
=\ &v(x,y) - v(x_\circ, y_\circ) \\
=\ &\frac{x-x_\circ}{x_\circ^{1-\alpha}}v_x^\alpha(x_\circ, y_\circ) + \frac{y-y_\circ}{y_\circ^{1-\beta}}v_y^\beta(x_\circ, y_\circ) + \sqrt{(x-x_\circ)^2 + (y-y_\circ)^2}\,\delta_2(z),
\end{aligned}$$

where $z = (x,y)$.

By the first condition (i) of Theorem 1, that is u_x^α, u_y^β, v_x^α, and v_y^β exist and continuous at $z_\circ$, we deduce that both $\lim_{z\to z_\circ} \delta_1(z) = \lim_{z\to z_\circ} \delta_2(z) = 0$. In other words, as $\left(\frac{x-x_\circ}{x_\circ^{1-\alpha}} + i\frac{y-y_\circ}{y_\circ^{1-\beta}}\right) \to 0$, we have $\delta_1(z) \to 0$ and $\delta_2(z) \to 0$. Moreover, by the second condition (ii) of Theorem 1, we have

$$u(x_\circ + \frac{x-x_\circ}{x_\circ^{1-\alpha}}x_\circ^{1-\alpha}, y_\circ + \frac{y-y_\circ}{y_\circ^{1-\beta}}y_\circ^{1-\beta}) - u(x_\circ, y_\circ)$$

$$\begin{aligned}
&+i\left[v(x_\circ+\frac{x-x_\circ}{x_\circ^{1-\alpha}}x_\circ^{1-\alpha},y_\circ+\frac{y-y_\circ}{y_\circ^{1-\beta}}y_\circ^{1-\beta})-v(x_\circ,y_\circ)\right]\\
=&\ \frac{x-x_\circ}{x_\circ^{1-\alpha}}\left(u_x^\alpha(x_\circ,y_\circ)+iv_x^\alpha(x_\circ,y_\circ)\right)+\frac{y-y_\circ}{y_\circ^{1-\beta}}\left(u_y^\beta(x_\circ,y_\circ)+iv_y^\beta(x_\circ,y_\circ)\right)\\
&+\sqrt{(x-x_\circ)^2+(y-y_\circ)^2}\left(\delta_1(z)+i\delta_2(z)\right)\\
=&\ \frac{x-x_\circ}{x_\circ^{1-\alpha}}\left(u_x^\alpha(x_\circ,y_\circ)+iv_x^\alpha(x_\circ,y_\circ)\right)+\frac{y-y_\circ}{y_\circ^{1-\beta}}\left(-v_x^\alpha(x_\circ,y_\circ)+iu_x^\alpha(x_\circ,y_\circ)\right)\\
&+\sqrt{(x-x_\circ)^2+(y-y_\circ)^2}\left(\delta_1(z)+i\delta_2(z)\right)\\
=&\ \left(\frac{x-x_\circ}{x_\circ^{1-\alpha}}+i\frac{y-y_\circ}{y_\circ^{1-\beta}}\right)\left(u_x^\alpha(x_\circ,y_\circ)+iv_x^\alpha(x_\circ,y_\circ)\right)\\
&+\sqrt{(x-x_\circ)^2+(y-y_\circ)^2}\left(\delta_1(z)+i\delta_2(z)\right).
\end{aligned}$$

Therefore,

$$u_x^\alpha(x_\circ,y_\circ)+iv_x^\alpha(x_\circ,y_\circ)+\frac{\sqrt{(x-x_\circ)^2+(y-y_\circ)^2}\left(\delta_1(z)+i\delta_2(z)\right)}{\frac{x-x_\circ}{x_\circ^{1-\alpha}}+i\frac{y-y_\circ}{y_\circ^{1-\beta}}}.$$

Which, as $\left(\frac{x-x_\circ}{x_\circ^{1-\alpha}}+i\frac{y-y_\circ}{y_\circ^{1-\beta}}\right)\to 0$, implies

$$\begin{aligned}
D^{(\alpha,\beta)}f(z_\circ) &= u_x^\alpha(x_\circ,y_\circ)+iv_x^\alpha(x_\circ,y_\circ)\\
&= v_y^\beta(x_\circ,y_\circ)-iu_y^\beta(x_\circ,y_\circ).
\end{aligned}$$

Example 3. Let $f(z)=f(x,y)=\frac{\beta^2x^{2\alpha}-\alpha^2y^{2\beta}}{\alpha^2\beta^2}$, where $|\alpha|,|\beta|<1$. Then (α,β)-Cauchy-Riemann equations are not satisfied for all $z=x+iy$ such that $x,\ y\neq 0$ since $u_x^\alpha=\frac{2}{\alpha}x^\alpha\neq 0=v_y^\beta$, and $u_y^\beta=-\frac{2}{\beta}y^\beta\neq 0=-v_x^\alpha$. Consequently, by Theorem 1, f is not (α,β)-differentiable at any non-zero point. However, at $z=0$ the two conditions of Theorem 1 are satisfied. So, $f^{(\alpha,\beta)}(0)$ exists such that $f^{(\alpha,\beta)}(0)=u_x^\alpha(0,0)+iv_x^\alpha(0,0)=v_y^\beta(0,0)-iu_y^\beta(0,0)=0$.

Definition 2. *A function $f:G\subseteq E\to\mathbb{C}$ is said to be (α,β)-differentiable at on a domain G if f is (α,β)-differentiable at every $z\in G$.*

Clearly, that both functions in examples 1, 2 are (α,β)-differentiable over $\mathbb{C}$.

Now, to restate the sufficient conditions theorem for (α,β)-differentiability in polar coordinates, we proceed as follows:

Let $x=r\cos\theta$ and $y=r\sin\theta$. Then $z=x+iy=r(\cos\theta+i\sin\theta)=re^{i\theta}$ and

$$f(z)=f(re^{i\theta})=u(r,\theta)+iv(r,\theta). \tag{5}$$

The (α,β)-derivative of (5), with respect to r and with respect to θ are, respectively,

$$f^{(\alpha,\beta)}(re^{i\theta})r^{1-\alpha}e^{i\theta} = u_r^\alpha(r,\theta)+iv_r^\alpha(r,\theta),$$

$$f^{(\alpha,\beta)}(re^{i\theta})ir\theta^{1-\beta}e^{i\theta} = u_{\theta}^{\beta}(r,\theta) + iv_{\theta}^{\beta}(r,\theta). \tag{6}$$

Hence, by equating both equations in (6), we get the polar version of the (α, β)-Cauchy-Riemann equations, namely,

$$\begin{aligned} u_r^{\alpha} &= \frac{\theta^{\beta-1}}{r^{\alpha}} v_{\theta}^{\beta}, \\ u_{\theta}^{\beta} &= -\frac{r^{\alpha}}{\theta^{\beta-1}} v_r^{\alpha}. \end{aligned} \tag{7}$$

Definition 3. *A function $f : G \subseteq E \to \mathbb{C}$ is said to be (α, β)-analytic at $z_\circ = x_\circ + iy_\circ \in G^\circ$ where $|\alpha|, |\beta| < 1$, if*

i) f is (α, β)-differentiable at $z_\circ$,
ii) there exists $\epsilon > 0$ such that f is (α, β)-differentiable for all $z \in D(\epsilon, z_\circ)$.

Clearly, both functions in examples 1 and 2 are (α, β)-analytic.

Definition 4. *A function $f : G \subseteq \mathbb{C} \to \mathbb{C}$ is said to be (α, β)-analytic the domain G where $|\alpha|, |\beta| < 1$, if it is (α, β)-analytic at every $z \in G$.*

3. Applications, Discussion, and Implications

The concept of the (α, β)-fractional derivative offers promising potential in a variety of scientific and engineering domains. In the area of signal and image processing, classical derivatives often fail to capture the memory and hereditary properties inherent in certain signals and textures. By introducing two parameters that independently influence the phase and magnitude in two dimensions, the (α, β)-fractional derivative provides finer control over 2D signal representations. This approach can be leveraged for more precise techniques in edge detection, texture segmentation, and in adaptations of the fractional Fourier transform.

In fractional control theory, modern controllers such as fractional PID systems already benefit from generalized derivatives to describe systems with memory (See [9], [10]). Extending these ideas to the complex domain, the (α, β)-fractional derivative could be incorporated into the design of complex transfer functions and impulse responses for systems with asymmetrical time scales—where, for example, position and velocity evolve under different fractional dynamics.

In the study of complex dynamics and fractals, the generalized derivative opens a pathway to modeling non-local interactions and generating fractional Julia or Mandelbrot sets. The flexibility of the parameters α and β may lead to the creation of entirely new families of fractals whose dimensions and symmetries can be tuned, with possible applications in data compression, encryption, and chaotic modeling.

Physical modeling in areas such as fluid flow or electromagnetic field theory may also benefit from the (α, β)-Cauchy-Riemann framework. In inhomogeneous or anisotropic media, conventional calculus often cannot account for asymmetric diffusion or non-local interactions. The polar-coordinate form derived in this paper suggests that the new derivative could capture these effects with greater accuracy.

Finally, from a purely mathematical perspective, the new definition unifies conformable derivatives with the classical Cauchy-Riemann theory, thereby bridging real and complex fractional calculus. It invites further investigation into its relationships with other fractional analytic function classes, such as Hadamard-type or Caputo-type, potentially contributing to a more comprehensive theory of fractional complex analysis.

4. Conclusions

This work has introduced and developed the concept of the (α, β)-derivative for complex-valued functions, along with its associated (α, β)-Cauchy-Riemann equations and the definition of (α, β)-analytic functions. By generalizing the idea of fractional differentiation to allow independent fractional orders in two orthogonal directions, we have extended both the theoretical framework and the potential applicability of fractional complex analysis.

The proposed derivative not only preserves many of the desirable properties of classical complex derivatives but also adds new degrees of freedom for modeling phenomena where asymmetry, non-locality, and memory effects are present. Its polar-coordinate formulation further enriches the theory by connecting it naturally to problems with radial and angular components.

Looking ahead, there are several promising directions for future work. Analytically, it would be valuable to explore the connections between (α, β)-analyticity and other generalized analytic classes, as well as to investigate the spectral and mapping properties of operators defined via this derivative. Numerically, developing efficient algorithms for computing (α, β)-derivatives could enable applications in image analysis, control system simulation, and computational physics. From an applied standpoint, the flexibility of the new framework could be harnessed to model asymmetric processes in engineering, physics, and finance, where traditional integer-order calculus fails to capture the observed behavior.

In essence, the (α, β)-fractional derivative provides a robust and adaptable mathematical tool that has the potential to deepen our understanding of complex-valued functions and to inspire innovative solutions across multiple disciplines.

Acknowledgements

The authors would like to thank the editor of EJPAM as well as the anonymous reviewers for their valuable comments and suggestions.

References

[1] K. S. Miller. *An Introduction to Fractional Calculus and Fractional Differential Equations*. J. Wiley and Sons, New York, USA, 1993.

[2] K. Oldham and J. Spanier. *The Fractional Calculus: Theory and Applications of Differentiation and Integration of Arbitrary Order*. Elsevier, Massachusetts, USA, 1974.

[3] A. Kilbas, H. Srivastava, and J. Trujillo. *Theory and Applications of Fractional Differential Equations*. Elsevier, Massachusetts, USA, 2006.

[4] I. Podlubny. *Fractional Differential Equations*. Elsevier, Philadelphia, USA, 1999.

[5] R. Khalil, M. Al Horani, A. Yousef, and M. Sababheh. A new definition of fractional derivative. *Journal of Computational and Applied Mathematics*, 264:65–70, 2014.

[6] L. Campos. On a concept of derivative of complex order with applications to special functions. *IMA Journal of Applied Mathematics*, 33(2):109–133, 1984.

[7] E. Love. Fractional derivatives of imaginary order. *Journal of the London Mathematical Society*, 2(2):241–259, 1972.

[8] M. Adm and R. Khalil. New definition of fractional analytic functions. *Missouri Journal of Mathematical Sciences*, 35(2):194–209, 2023.

[9] I. Batiha, S. Njadat, R. Batyha, A. Zraiqat, A. Dababneh, and S. Momani. Design fractional-order pid controllers for single-joint robot arm model. *International Journal of Advances in Soft Computing and its Applications*, 14(2):241–259, 2022.

[10] I. Batiha, J. Oudetallah, A. Ouannas, A. Al-Nana, and I. Jebril. Tuning the fractional-order pid-controller for blood glucose level of diabetic patients. *International Journal of Advances in Soft Computing and its Applications*, 13(2):1–10, 2021.