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Second-Order Abstract Cauchy Problem in Two Variables

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Abstract. In this paper, we consider the following second-order abstract Cauchy problem in two variables

$$D^2_{ss}u(s,t) + D^2_{tt}u(s,t) + 2D^2_{st}u(s,t) + A\left[D_s u(s,t) + D_t u(s,t)\right] = Bu(s,t),$$

where A, B are closed linear operators on a Banach space X such that

$$\begin{aligned} A &: Dom(A) \subseteq X \to X, \\ B &: Dom(B) \subseteq X \to X, \end{aligned}$$

$u(s,t) : [0,1] \times [0,1] \to X$ is an unknown twice-continuously partially differentiable function on $[0,1] \times [0,1] \subseteq \mathbb{R}^2$ where $Rang(u) \subseteq Dom(A) \cap Dom(B)$. Utilizing properties of atomic operators, an atomic solution is obtained.

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1. Introduction

Ordinary and partial differential equations in which the unknown function and its derivatives take values in some abstract space such as Hilbert space or Banach space are called abstract differential equations [1, 2]. Among these classical vector valued abstract differential equations, the abstract Cauchy problem (ACP) is considered as the most famous one. The classical form of the ACP can be presented as follows:

$$\begin{aligned} &\frac{du}{dt} = Lu(t), \quad t \geq 0, \\ &u(0) = t_\circ, \end{aligned} \tag{1}$$

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where $L : Dom(L) \subseteq X \to X$ is a linear operator of an appropriate type such that $Rang(u) \subseteq Dom(L)$, $t_\circ \in X$ is given and $u : [0, \infty) \to X$ is the unknown function. For both linear and nonlinear ACPs, there are many applications in engineering and applied sciences [3, 4].

In spite of the fact that, the ACPs have a standing investigate history that has profound roots in time, a common hypothesis that controls the solutions of such problems has not,however, been confirmed yet. Early studies, principally, were conducted by means of both Laplace transform techniques [5] and the concept of semigroup of linear operators [6], [7].

Lately, in 2010, modern strategy has been suggested [8], [9], [10], [11] to manipulate specific types of ordinary and fractional ACPs. The new approach is carried out by utilizing the theory of atoms operators in Banach spaces and the obtained solutions in such cases are called atomic solutions.

Let X is a Banach space and A, B are closed linear operators on X such that $A : Dom(A) \subseteq X \to X$, $B : Dom(B) \subseteq X \to X$. Moreover, assume $u(s,t) : [0,1] \times [0,1] \to X$ is an unknown twice-continuously partially differentiable function on $[0,1] \times [0,1] \subseteq \mathbb{R}^2$ where $Rang(u) \subseteq Dom(A) \cap Dom(B)$. Our main result discusses the atomic solution of the following second-order ACP in two variables,

$$D^2_{ss}u(s,t) + D^2_{tt}u(s,t) + 2D^2_{st}u(s,t) + A\left[D_s u(s,t) + D_t u(s,t)\right] = Bu(s,t).$$

Before we present the main result, some definitions and results from atoms operators concept are presented.

2. Preliminaries

Definition 1. [12] *Let X and Y be any two Banach spaces and X^* is the dual space of X. For $\zeta \in X$ and $\eta \in Y$, the operator $\zeta \otimes \eta : X^* \to Y$, defined by $\zeta \otimes \eta(z^*) = z^*(\zeta)\eta = \langle \zeta, z^* \rangle \eta$, is a bounded one rank linear operator. Such operators are called atoms.*

Now, if we assume two equal atoms, namely, $\zeta_1 \otimes \eta_1 = \zeta_2 \otimes \eta_2$, then for any $z^* \in X^*$ we have

$$\langle \zeta_1, z^* \rangle \eta_1 = \langle \zeta_2, z^* \rangle \eta_2.$$

Thus, $\eta_1 = \eta_2$. Similarly, one can prove $\zeta_1 = \zeta_2$. This leads us to the following interesting Lemma.

Lemma 1. [13] *Let $\zeta_1 \otimes \eta_1$ and $\zeta_2 \otimes \eta_2$ be two nonzero atoms in $X \otimes Y$ such that $\zeta_1 \otimes \eta_1 + \zeta_2 \otimes \eta_2 = \zeta_3 \otimes \eta_3$. Then either $\zeta_1 = \zeta_2 = \zeta_3$ or $\eta_1 = \eta_2 = \eta_3$.*

Finally, one of the most interesting theorem that lies in the heart of functional analysis as well as approximation theory and guarantees that any continuous function of several variables can be written as a sum of products of continuous separated functions.

Theorem 1. [12] *Let I, J be two compact intervals, and $C(I)$, $C(J)$, and $C(I \times J)$ be, respectively, the spaces of continuous functions on I, J, and $I \times J$. Then every $f \in$*

$C(I \times J)$ can be written in the form $f(x,y) = \sum_{i=1}^{\infty} u_i(x) v_i(y)$, where $u_i(x) \in C(I)$ and $v_i(y) \in C(J)$.

Hahn-Banach theorem is known as one of the most important results in functional analysis. This theorem enables us to extend bounded linear functionals that is defined on a subspace of a vector space to the whole space. Consequently, the dual space X^* is a non-trivial space associated with X.

Theorem 2. [14] *(Hahn-Banach Theorem) Let M be a linear subspace of a normed linear space N and let $f \in M^*$. Then f can be extended to a bounded linear functional $F \in N^*$ (defined on the whole linear space N) such that $F|_M = f$ and $\|f\|_{M^*} = \|F\|_{N^*}$.*

A consequence of Hahn-Banach theorem, is the following corollary.

Corollary 1. [14] *Let N be a normed linear space. Then for all $x \in N$ there exists $x^* \in N^*$ such that $\|x^*\|_{N^*} = 1$ and $x^*(x) = \|x\|_N$.*

3. Main Results

Theorem 3. *Consider the following ACP in two variables*

$$D_{ss}^2 u(s,t) + D_{tt}^2 u(s,t) + 2D_{st}^2 u(s,t) + A\left[D_s u(s,t) + D_t u(s,t)\right] = Bu(s,t), \tag{2}$$

where A, B are closed linear operators on a Banach space X such that

$$\begin{aligned} A &: Dom(A) \subseteq X \to X, \\ B &: Dom(B) \subseteq X \to X, \end{aligned}$$

$u(s,t) : [0,1] \times [0,1] \to X$ is an unknown twice-continuously partially differentiable function on $[0,1] \times [0,1] \subseteq \mathbb{R}^2$ where $Rang(u) \subseteq Dom(A) \cap Dom(B)$.

Then, for all $\omega \neq 0 \in X$, (2) has non-trivial atomic solution of the form

$$u(s,t) = \frac{e^{s-3t}}{2}\left[\left(1-\sqrt{5}\right)e^{\frac{3-\sqrt{5}}{2}t} - \left(1+\sqrt{5}\right)e^{\frac{3+\sqrt{5}}{2}t}\right] \otimes \omega.$$

Proof. Let $P \in C^2[0,1]$ and $Q \in C^2[0,1]$ such that

$$\begin{aligned} P &= P(s) : [0,1] \to X, \\ Q &= Q(t) : [0,1] \to X. \end{aligned}$$

Suppose $u(s,t)$, the solution to (2), be in the following atom form $u = P \otimes Q \otimes \omega$, where $\omega \neq 0 \in X$. On substituting this atom operator form into (2) we obtain the following tensor product equation

$$\begin{gathered} \left[P''(s) \otimes Q(t) + P(s) \otimes Q''(t) + 2P'(s) \otimes Q'(t)\right] \otimes \omega \\ + \left[P'(s) \otimes Q(t) + P(s) \otimes Q'(t)\right] \otimes A\omega \\ = P(s) \otimes Q(t) \otimes B\omega. \end{gathered} \tag{3}$$

Now, let us assume that $\{\omega, A\omega, B\omega\}$ are independent in X. Hence, by Corollary 1, one can assume that there exists $x^* \in X$ such that $\langle x^*, \omega\rangle = \langle x^*, A\omega\rangle = \langle x^*, B\omega\rangle = 1$, where $\langle x^*, \cdot\rangle := x^*(\cdot)$. Therefore, on applying such x^* to both sides of (3), we get

$$\begin{aligned} &[P''(s) \otimes Q(t) + P(s) \otimes Q''(t) + 2P'(s) \otimes Q'(t)] \\ &\quad + [P'(s) \otimes Q(t) + P(s) \otimes Q'(t)] \\ &\quad = P(s) \otimes Q(t), \end{aligned} \tag{4}$$

which can be simplified to

$$P''(s) \otimes Q(t) + P'(s) \otimes \left[2Q'(t) + Q(t)\right] = P(s) \otimes \left[Q(t) - Q'(t) - Q''(t)\right]. \tag{5}$$

But, equation (5) reveals that the sum of two atoms is an atom. Hence, by Lemma 1, we get the following two cases

$$\begin{array}{lll} \text{either} & (i) & P''(s) = P'(s) = P(s), \\ \text{or} & (ii) & Q(t) = 2Q'(t) + Q(t) = Q(t) - Q'(t) - Q''(t). \end{array} \tag{6}$$

Before we start handling the two cases, let us consider, without loss of generality, the following initial conditions,

$$\begin{aligned} P(0) &= P'(0) = 1, \\ Q(0) &= Q'(0) = 1. \end{aligned} \tag{7}$$

Case (i): For this case, there are three sub-cases which are $P'' = P'$, $P'' = P$, and $P' = P$ together with appropriate conditions from (7) are, simply, the following classical initial value problems

$$\begin{array}{ll} P''(s) - P'(s) = 0, & P'(0) = P(0) = 1, \\ P''(s) - P(s) = 0, & P'(0) = P(0) = 1, \\ P'(s) - P(s) = 0, & P(0) = 1. \end{array} \tag{8}$$

All intial value problems listed in (8) have the same solution which is

$$P(s) = e^s. \tag{9}$$

This implies that an atomic solution can be obtained for this particular case. This atomic solution can be found by substituting the solution $P(s) = e^s$ into (5) we have

$$Q''(t) + 3Q'(t) + Q(t) = 0. \tag{10}$$

Equation (10) is a second order ordinary differential equation with constant coefficients. Together with appropriate initial conditions form (7) namely, $Q(0) = Q'(0) = 1$, can be solved to give

$$Q(t) = \frac{e^{-3t}}{2}\left[\left(1 - \sqrt{5}\right) e^{\frac{3-\sqrt{5}}{2}t} - \left(1 + \sqrt{5}\right) e^{\frac{3+\sqrt{5}}{2}t}\right]. \tag{11}$$

Therefore, for all $\omega \neq 0 \in X$, an atomic solution, of the form $u = P \otimes Q \otimes \omega$ to (2), can be obtained, by considering (9) and (11), as

$$u(s,t) = \frac{e^{s-3t}}{2}\left[\left(1-\sqrt{5}\right)e^{\frac{3-\sqrt{5}}{2}t} - \left(1+\sqrt{5}\right)e^{\frac{3+\sqrt{5}}{2}t}\right] \otimes \omega. \tag{12}$$

Case (ii): This case reveals three sub-cases, namely, $Q(t) = 2Q'(t) + Q(t)$, $Q(t) = Q(t) - Q'(t) - Q''(t)$, and $2Q'(t) + Q(t) = Q(t) - Q'(t) - Q''(t)$. Correspondingly, we have $Q'(t) = 0$, $Q''(t) + Q'(t) = 0$, and $Q''(t) + 3Q'(t) = 0$. Hence, together with appropriate conditions from (7), we get the following classical initial value problems

$$\begin{array}{cc} Q'(t) = 0, & Q(0) = 1, \\ Q''(t) + Q'(t) = 0, & Q'(0) = Q(0) = 1, \\ Q''(t) + 3Q'(t) = 0, & Q'(0) = Q(0) = 1. \end{array} \tag{13}$$

The corresponding solution to each initial value problem, respectively, is $Q(t) = 1$, $Q(t) = 2 - e^{-t}$, and $Q(t) = \frac{4}{3} - \frac{e^{-3t}}{3}$. Different form of $Q(t)$ for each particular sub-case means that case(ii) does not admit an atomic solution. So, the only atomic solution of the form $u = PQ \otimes \omega$, where $\omega \neq 0 \in X$ to (2) is the one obtained by case(i) which is described in (12). Therefore, the proof is complete.

4. Application

In this section, we present an example to illustrate the analytical method described in the previous section.

Let X is a Banach space and A, B are closed linear operators on X such that $A : Dom(A) \subseteq X \to X$, $B : Dom(B) \subseteq X \to X$. Moreover, assume $u(t) : [0,1] \to X$ is an unknown twice-continuously differentiable function on $[0,1] \subset \mathbb{R}$ where $Rang(u) \subseteq Dom(A) \cap Dom(B)$. Now, consider the following first-order ACP in one variable,

$$u''(t) + Au'(t) + Bu(t) = 0, \tag{14}$$

Let $v \in C^2[0,1]$ such that $v := v(t) : [0,1] \to X$. Suppose $u(t)$, the solution to (14), be in the following atom form $u = v \otimes \omega$, where $\omega \neq 0 \in X$. On substituting this atom operator form into (14) we obtain the following tensor product equation

$$v''(t) \otimes \omega + v'(t) \otimes A\omega + v(t) \otimes B\omega = 0. \tag{15}$$

Assume that $\{\omega, A\omega, B\omega\}$ are independent in X. Hence, by Corollary 1, one can assume that there exists $x^* \in X$ such that $\langle x^*, \omega \rangle = \langle x^*, A\omega \rangle = \langle x^*, B\omega \rangle = 1$, where $\langle x^*, \cdot \rangle := x^*(\cdot)$. Therefore, on applying such x^* to both sides of (15), we get

$$v''(t) + v'(t) + v(t) = 0. \tag{16}$$

By assuming $v(0) = 1$, we have $v(t) = 2e^{\frac{-t}{2}} \sin\left(\frac{\sqrt{3}}{2}t + \frac{\pi}{6}\right)$ and therefore, (14) has the following atomic solution

$$u = v \otimes \omega$$

$$= \quad 2e^{\frac{-t}{2}} \sin\left(\frac{\sqrt{3}}{2}t + \frac{\pi}{6}\right) \otimes \omega, \quad \text{where } \omega \neq 0 \in X.$$

Apart from the example discussed above, the ideas developed here may be applied in several directions. In control theory, many systems described by partial differential equations (PDEs) can be formulated as ACPs in Banach or Hilbert spaces. This is often the case for distributed parameter systems such as vibrating beams, heat exchangers, or models involving fluid structure interaction [15], [16]. By expressing solutions in the atomic form used here, the governing equations can sometimes be reduced to simpler, separable problems. Such reductions may help in studying controllability, stability, and in designing feedback controls for complex systems.

Problems of a similar nature appear in mathematical physics, where the evolution of a system depends on both time and space variables. Examples include wave motion in elastic media, anisotropic heat conduction, or certain coupled Schrödinger equations [17], [18]. The atomic representation of solutions makes it possible to obtain exact formulas in special cases, and in more complicated settings it can lead to accurate approximations while retaining the essential structure dictated by the physical model.

There are also possible uses in abstract modeling outside traditional physics or engineering. In economics, population dynamics, or network theory, models may depend on several independent variables, and their governing equations may be expressed in operator form [19]. Decomposing these systems into simpler components, as in the present approach, can assist in analysing parameter effects, identifying model sensitivities, and constructing reduced models that are easier to work with but still capture the main behaviour of the original system.

5. Conclusions

In this paper, we utilize some properties of atoms operators to solve ACP in two variables that is

$$D^2_{ss}u(s,t) + D^2_{tt}u(s,t) + 2D^2_{st}u(s,t) + A\left[D_s u(s,t) + D_t u(s,t)\right] = Bu(s,t).$$

The use of an atomic solution of the form $u = P \otimes Q \otimes \omega$ enables us to reduce the ACP into separeted classical ordinary differential equations in tems of the functions $P(s)$ and $Q(t)$. These equations can be solved easily and the atomic solution can be then obtained. An application to the procedure is also provided.

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