EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 3, Article Number 6565
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Upper and Lower Continuous Multifunctions Defined Between an Ideal Topological Space and a Bitopological Space

Jeeranunt Khampakdee[1], Areeyuth Sama-Ae[2], Chawalit Boonpok[1,*]

[1] *Mathematics and Applied Mathematics Research Unit, Department of Mathematics, Faculty of Science, Mahasarakham University, Maha Sarakham, 44150, Thailand*
[2] *Department of Mathematics and Computer Science, Faculty of Science and Technology, Prince of Songkla University, Pattani Campus, Pattani, 94000, Thailand*

Abstract. This paper presents new concepts of continuous multifunctions defined from an ideal topological space into a bitopological space, called upper $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions and lower $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions. Furthermore, several characterizations and some properties concerning upper $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions and lower $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions are investigated.

2020 Mathematics Subject Classifications: 54C08, 54C60

Key Words and Phrases: Upper $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunction, lower $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunction

1. Introduction

The field of the mathematical science which goes under the name of topology is concerned with all questions directly or indirectly related to continuity. The concept of ideals in topological spaces has been introduced and studied by Kuratowski [1] and Vaidyanathaswamy [2] which is one of the important areas of research in the branch of mathematics. Stronger and weaker forms of open sets in ideal topological spaces such as semi-$\mathscr{I}$-open sets, pre-$\mathscr{I}$-open sets, α-$\mathscr{I}$-open sets, β-$\mathscr{I}$-open sets and δ-$\mathscr{I}$-open sets play an important role in the research of generalizations of continuity. Using these notions many authors introduced and studied various types of generalizations of continuity for functions and multifunctions. Hatir and Noiri [3] introduced and investigated the notions of weakly pre-$\mathscr{I}$-open sets and weakly pre-$\mathscr{I}$-continuous functions. Moreover, Hatir and Noiri [4] investigated further properties of semi-$\mathscr{I}$-open sets and semi-$\mathscr{I}$-continuous

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v18i3.6565

Email addresses: jeeranunt.k@msu.ac.th (J. Khampakdee), areeyuth.s@psu.ac.th (A. Sama-Ae), chawalit.b@msu.ac.th (C. Boonpok)

functions. On the other hand, the present author [5] introduced new classes of multifunctions between ideal topological spaces, namely upper $\star$-continuous multifunctions and lower $\star$-continuous multifunctions. Furthermore, several characterizations of upper $\star$-continuous multifunctions, lower $\star$-continuous multifunctions, upper almost $\star$-continuous multifunctions, lower almost $\star$-continuous multifunctions, upper weakly $\star$-continuous multifunctions and lower weakly $\star$-continuous multifunctions were considered in [5]. Quite recently, the present author [6] introduced and investigated the notions of $p\imath$-continuous multifunctions and weakly $p\imath$-continuous multifunctions. Pue-on et al. [7] introduced and studied the concepts of upper (τ_1, τ_2)-continuous multifunctions and lower (τ_1, τ_2)-continuous multifunctions. Klanarong et al. [8] investigated several characterizations of upper (τ_1, τ_2)-continuous multifunctions and lower (τ_1, τ_2)-continuous multifunctions by utilizing the notions of $(\tau_1, \tau_2)\theta$-closed sets and $(\tau_1, \tau_2)\theta$-open sets. Thongmoon et al. [9] studied some characterizations of upper (τ_1, τ_2)-continuous multifunctions and lower (τ_1, τ_2)-continuous multifunctions by using $\tau_1\tau_2$-δ-open sets and $\tau_1\tau_2$-δ-closed sets. In this paper, we introduce the concepts of upper $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions and lower $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions. We also investigate several characterizations of upper $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions and lower $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions.

2. Preliminaries

Throughout the present paper, spaces (X, τ_1, τ_2) and (Y, σ_1, σ_2) (or simply X and Y) always mean bitopological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a bitopological space (X, τ_1, τ_2). The closure of A and the interior of A with respect to τ_i are denoted by τ_i-Cl(A) and τ_i-Int(A), respectively, for $i = 1, 2$. A subset A of a bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$-*closed* [10] if $A = \tau_1$-Cl$(\tau_2$-Cl$(A))$. The complement of a $\tau_1\tau_2$-closed set is called $\tau_1\tau_2$-*open*. The intersection of all $\tau_1\tau_2$-closed sets of X containing A is called the $\tau_1\tau_2$-*closure* [10] of A and is denoted by $\tau_1\tau_2$-Cl(A). The union of all $\tau_1\tau_2$-open sets of X contained in A is called the $\tau_1\tau_2$-*interior* [10] of A and is denoted by $\tau_1\tau_2$-Int(A).

Lemma 1. [10] *Let A and B be subsets of a bitopological space (X, τ_1, τ_2). For the $\tau_1\tau_2$-closure, the following properties hold:*

(1) $A \subseteq \tau_1\tau_2$-Cl(A) and $\tau_1\tau_2$-Cl$(\tau_1\tau_2$-Cl$(A)) = \tau_1\tau_2$-Cl(A).

(2) If $A \subseteq B$, then $\tau_1\tau_2$-Cl$(A) \subseteq \tau_1\tau_2$-Cl$(B)$.

(3) $\tau_1\tau_2$-Cl(A) is $\tau_1\tau_2$-closed.

(4) A is $\tau_1\tau_2$-closed if and only if $A = \tau_1\tau_2$-Cl(A).

(5) $\tau_1\tau_2$-Cl$(X - A) = X - \tau_1\tau_2$-Int$(A)$.

A subset A of a bitopological space (X, τ_1, τ_2) is said to be $(\tau_1, \tau_2)r$-*open* [11] (resp. $(\tau_1, \tau_2)s$-*open* [12], $(\tau_1, \tau_2)p$-*open* [12], $(\tau_1, \tau_2)\beta$-*open* [12]) if $A = \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))$ (resp. $A \subseteq \tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(A))$, $A \subseteq \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))$, $A \subseteq \tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))))$.

The complement of a $(\tau_1, \tau_2)r$-open (resp. $(\tau_1, \tau_2)s$-open, $(\tau_1, \tau_2)p$-open, $(\tau_1, \tau_2)\beta$-open) set is said to be $(\tau_1, \tau_2)r$*-closed* (resp. $(\tau_1, \tau_2)s$*-closed*, $(\tau_1, \tau_2)p$*-closed*, $(\tau_1, \tau_2)\beta$*-closed*). A subset A of a bitopological space (X, τ_1, τ_2) is said to be $\alpha(\tau_1, \tau_2)$*-open* [13] if $A \subseteq \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(A)))$. The complement of an $\alpha(\tau_1, \tau_2)$-open set is said to be $\alpha(\tau_1, \tau_2)$*-closed*. A subset A of a bitopological space (X, τ_1, τ_2) is said to be $\tau_1\tau_2$*-*δ*-open* if A is the union of $(\tau_1, \tau_2)r$-open sets of X. The complement of a $\tau_1\tau_2$-δ-open set is called $\tau_1\tau_2$*-*δ*-closed*. The union of all $\tau_1\tau_2$-δ-open sets of X contained in A is called the $\tau_1\tau_2$*-*δ*-interior* of A and is denoted by $\tau_1\tau_2\text{-}\delta\text{-Int}(A)$. The intersection of all $\tau_1\tau_2$-δ-closed sets of X containing A is called the $\tau_1\tau_2$*-*δ*-closure* of A and is denoted by $\tau_1\tau_2\text{-}\delta\text{-Cl}(A)$ [14]. For a subset A of a bitopological space (X, τ_1, τ_2), a point $x \in X$ is called a $(\tau_1, \tau_2)\theta$*-cluster point* of A if $\tau_1\tau_2\text{-Cl}(U) \cap A \neq \emptyset$ for every $\tau_1\tau_2$-open set U containing x. The set of all $(\tau_1, \tau_2)\theta$-cluster points of A is called the $(\tau_1, \tau_2)\theta$*-closure* of A and is denoted by $(\tau_1, \tau_2)\theta\text{-Cl}(A)$. A subset A of a bitopological space (X, τ_1, τ_2) is said to be $(\tau_1, \tau_2)\theta$*-closed* if $(\tau_1, \tau_2)\theta\text{-Cl}(A) = A$. The complement of a $(\tau_1, \tau_2)\theta$-closed set is said to be $(\tau_1, \tau_2)\theta$*-open*. The union of all $(\tau_1, \tau_2)\theta$-open sets of X contained in A is called the $(\tau_1, \tau_2)\theta$*-interior* of A and is denoted by $(\tau_1, \tau_2)\theta\text{-Int}(A)$ [11].

An ideal $\mathscr{I}$ on a topological space (X, τ) is a nonempty collection of subsets of X satisfying the following properties: (1) $A \in \mathscr{I}$ and $B \subseteq A$ imply $B \in \mathscr{I}$; (2) $A \in \mathscr{I}$ and $B \in \mathscr{I}$ imply $A \cup B \in \mathscr{I}$. A topological space (X, τ) with an ideal $\mathscr{I}$ on X is called an ideal topological space and is denoted by $(X, \tau, \mathscr{I})$. For an ideal topological space $(X, \tau, \mathscr{I})$ and a subset A of X, $A^\star(\mathscr{I})$ is defined as follows:

$$A^\star(\mathscr{I}) = \{x \in X : U \cap A \notin \mathscr{I} \text{ for every open neighbourhood } U \text{ of } x\}.$$

In case there is no chance for confusion, $A^\star(\mathscr{I})$ is simply written as $A^\star$. In [1], $A^\star$ is called the local function of A with respect to $\mathscr{I}$ and τ and $\text{Cl}^\star(A) = A^\star \cup A$ defines a Kuratowski closure operator for a topology $\tau^\star(\mathscr{I})$ finer than τ. A subset A is said to be $\star$-closed [15] if $A^\star \subseteq A$. The interior of a subset A in $(X, \tau^\star(\mathscr{I}))$ is denoted by $\text{Int}^\star(A)$. A subset A of an ideal topological space $(X, \tau, \mathscr{I})$ is said to be *semi***-*$\mathscr{I}$*-open* [16] (resp. *semi-*$\mathscr{I}$*-open* [4]) if $A \subseteq \text{Cl}(\text{Int}^\star(A))$ (resp. $A \subseteq \text{Cl}^\star(\text{Int}(A))$). The complement of a semi*-$\mathscr{I}$-open (resp. semi-$\mathscr{I}$-open) set is said to be *semi***-*$\mathscr{I}$*-closed* [16] (resp. semi-$\mathscr{I}$-closed [4]).

By a multifunction $F : X \to Y$, we mean a point-to-set correspondence from X into Y, and we always assume that $F(x) \neq \emptyset$ for all $x \in X$. For a multifunction $F : X \to Y$, we shall denote the upper and lower inverse of a set B of Y by $F^+(B)$ and $F^-(B)$, respectively, that is, $F^+(B) = \{x \in X \mid F(x) \subseteq B\}$ and $F^-(B) = \{x \in X \mid F(x) \cap B \neq \emptyset\}$. In particular, $F^-(y) = \{x \in X \mid y \in F(x)\}$ for each point $y \in Y$. For each $A \subseteq X$, $F(A) = \cup_{x \in A} F(x)$.

3. Upper and lower $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions

In this section, we introduce the concepts of upper $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions and lower $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions. Furthermore, several characterizations of upper $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions and lower $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions are discussed.

Definition 1. *A multifunction* $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ *is called upper* $\tau^\star(\sigma_1, \sigma_2)$*-continuous at a point* $x \in X$ *if for each* $\sigma_1\sigma_2$*-open set* V *of* Y *such that* $F(x) \subseteq V$*, there exists a* $\star$*-open set* U *of* X *containing* x *such that* $F(U) \subseteq V$*. A multifunction* $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ *is called upper* $\tau^\star(\sigma_1, \sigma_2)$*-continuous if* F *is upper* $\tau^\star(\sigma_1, \sigma_2)$*-continuous at each point* x *of* X*.*

Theorem 1. *For a multifunction* $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$*, the following properties are equivalent:*

(1) F *is upper* $\tau^\star(\sigma_1, \sigma_2)$*-continuous;*

(2) $F^+(V)$ *is* $\star$*-open in* X *for every* $\sigma_1\sigma_2$*-open set* V *of* Y*;*

(3) $F^-(K)$ *is* $\star$*-closed in* X *for every* $\sigma_1\sigma_2$*-closed set* K *of* Y*;*

(4) $Cl^\star(F^-(B)) \subseteq F^-(\sigma_1\sigma_2\text{-}Cl(B))$ *for every subset* B *of* Y*;*

(5) $F^+(\sigma_1\sigma_2\text{-}Int(B)) \subseteq Int^\star(F^+(B))$ *for every subset* B *of* Y*.*

Proof. (1) ⇒ (2): Let V be any $\sigma_1\sigma_2$-open set of Y and $x \in F^+(V)$. Then, $F(x) \subseteq V$ and by (1), there exists a $\star$-open set U of X containing x such that $F(U) \subseteq V$. Thus, $x \in U \subseteq F^+(V)$ and hence $x \in \text{Int}^\star(F^+(V))$. Therefore, $F^+(V) \subseteq \text{Int}^\star(F^+(V))$. This shows that $F^+(V)$ is $\star$-open in X.

(2) ⇒ (3): This follows from the fact that $F^+(Y - B) = X - F^-(B)$ for every subset B of Y.

(3) ⇒ (4): Let B be any subset of Y. Then, $\sigma_1\sigma_2\text{-Cl}(B)$ is $\sigma_1\sigma_2$-closed in Y and by (3), $\text{Cl}^\star(F^-(B)) \subseteq \text{Cl}^\star(F^-(\sigma_1\sigma_2\text{-Cl}(B))) = F^-(\sigma_1\sigma_2\text{-Cl}(B))$.

(4) ⇒ (5): Let B be any subset of Y. Thus by (4), we have $X - \text{Int}^\star(F^+(B)) = \text{Cl}^\star(X - F^+(B)) = \text{Cl}^\star(F^-(Y - B)) \subseteq F^-(\sigma_1\sigma_2\text{-Cl}(Y - B)) = F^-(Y - \sigma_1\sigma_2\text{-Int}(B)) = X - F^+(\sigma_1\sigma_2\text{-Int}(B))$ and hence $F^+(\sigma_1\sigma_2\text{-Int}(B)) \subseteq \text{Int}^\star(F^+(B))$.

(5) ⇒ (1): Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y such that $F(x) \subseteq V$. Then, $x \in F^+(V) = \text{Int}^\star(F^+(V))$. There exists a $\star$-open set U of X containing x such that $U \subseteq F^+(V)$; hence $F(U) \subseteq V$. This shows that F is upper $\tau^\star(\sigma_1, \sigma_2)$-continuous.

Definition 2. *A multifunction* $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ *is called lower* $\tau^\star(\sigma_1, \sigma_2)$*-continuous at a point* $x \in X$ *if for each* $\sigma_1\sigma_2$*-open set* V *of* Y *such that* $F(x) \cap V \neq \emptyset$*, there exists a* $\star$*-open set* U *of* X *containing* x *such that* $F(z) \cap V \neq \emptyset$ *for every* $z \in U$*. A multifunction* $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ *is called lower* $\tau^\star(\sigma_1, \sigma_2)$*-continuous if* F *is lower* $\tau^\star(\sigma_1, \sigma_2)$*-continuous at each point* x *of* X*.*

Theorem 2. *For a multifunction* $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$*, the following properties are equivalent:*

(1) F *is lower* $\tau^\star(\sigma_1, \sigma_2)$*-continuous;*

(2) $F^-(V)$ *is* $\star$*-open in* X *for every* $\sigma_1\sigma_2$*-open set* V *of* Y*;*

(3) $F^+(K)$ *is* $\star$*-closed in* X *for every* $\sigma_1\sigma_2$*-closed set* K *of* Y*;*

(4) $Cl^\star(F^+(B)) \subseteq F^+(\sigma_1\sigma_2\text{-}Cl(B))$ *for every subset* B *of* Y*;*

(5) $F(Cl^\star(A)) \subseteq \sigma_1\sigma_2\text{-}Cl(F(A))$ *for every subset* A *of* X*;*

(6) $F^-(\sigma_1\sigma_2\text{-}Int(B)) \subseteq Int^\star(F^-(B))$ *for every subset* B *of* Y*.*

Proof. We prove only the implications (4) ⇒ (5) and (5) ⇒ (6) being the proofs of the other similar to those of Theorem 1.

(4) ⇒ (5): Let A be any subset of X. Then by (4), we have $\text{Cl}^\star(A) \subseteq \text{Cl}^\star(F^+(F(A))) \subseteq F^+(\text{Cl}^\star(F(A)))$ and so $F(\text{Cl}^\star(A)) \subseteq \sigma_1\sigma_2\text{-Cl}(F(A))$.

(5) ⇒ (6): Let B be any subset of Y. By (5),

$$F(\text{Cl}^\star(F^+(Y-B))) \subseteq \sigma_1\sigma_2\text{-Cl}(F(F^+(Y-B))) \subseteq \sigma_1\sigma_2\text{-Cl}(Y-B) = Y - \sigma_1\sigma_2\text{-Int}(B).$$

Since $F(\text{Cl}^\star(F^+(Y-B))) = F(\text{Cl}^\star(X - F^-(B))) = F(X - \text{Int}^\star(F^-(B)))$, we have

$$X - \text{Int}^\star(F^-(B)) \subseteq F^+(Y - \sigma_1\sigma_2\text{-Int}(B)) = X - F^-(\sigma_1\sigma_2\text{-Int}(B))$$

and hence $F^-(\sigma_1\sigma_2\text{-Int}(B)) \subseteq \text{Int}^\star(F^-(B))$.

Definition 3. *A function* $f : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ *is said to be* $\tau^\star(\sigma_1, \sigma_2)$*-continuous at a point* $x \in X$ *if for each* $\sigma_1\sigma_2$*-open set* V *of* Y *containing* $f(x)$*, there exists a* $\star$*-open set* U *of* X *containing* x *such that* $f(U) \subseteq V$*. A function* $f : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ *is said to be* $\tau^\star(\sigma_1, \sigma_2)$*-continuous if* f *is* $\tau^\star(\sigma_1, \sigma_2)$*-continuous at each point* x *of* X*.*

Corollary 1. *For a function* $f : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$*, the following properties are equivalent:*

(1) f *is* $\tau^\star(\sigma_1, \sigma_2)$*-continuous;*

(2) $f^{-1}(V)$ *is* $\star$*-open in* X *for every* $\sigma_1\sigma_2$*-open set* V *of* Y*;*

(3) $f^{-1}(K)$ *is* $\star$*-closed in* X *for every* $\sigma_1\sigma_2$*-closed set* K *of* Y*;*

(4) $Cl^\star(f^{-1}(B)) \subseteq f^{-1}(\sigma_1\sigma_2\text{-}Cl(B))$ *for every subset* B *of* Y*;*

(5) $f(Cl^\star(A)) \subseteq \sigma_1\sigma_2\text{-}Cl(f(A))$ *for every subset* A *of* X*;*

(6) $f^{-1}(\sigma_1\sigma_2\text{-}Int(B)) \subseteq Int^\star(f^{-1}(B))$ *for every subset* B *of* Y*.*

Definition 4. [9] *A bitopological space* (X, τ_1, τ_2) *is said to be* $(\tau_1, \tau_2)s$*-regular if for each* $(\tau_1, \tau_2)s$*-closed set* F *and each* $x \notin F$*, there exist disjoint* $(\tau_1, \tau_2)s$*-open sets* U *and* V *such that* $x \in U$ *and* $F \subseteq V$*.*

Lemma 2. [9] *A bitopological space* (X, τ_1, τ_2) *is* $(\tau_1, \tau_2)s$*-regular if and only if for each* $x \in X$ *and each* $(\tau_1, \tau_2)s$*-open set* U *containing* x*, there exists a* $(\tau_1, \tau_2)s$*-open set* V *such that* $x \in V \subseteq (\tau_1, \tau_2)\text{-}sCl(V) \subseteq U$*.*

Lemma 3. [9] *Let (X, τ_1, τ_2) be a (τ_1, τ_2)s-regular space. Then, the following properties hold:*

(1) $\tau_1\tau_2$-Cl$(A) = \tau_1\tau_2$-δ-Cl(A) for every subset A of X.

(2) Every $\tau_1\tau_2$-open set is $\tau_1\tau_2$-δ-open.

Theorem 3. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, where (Y, σ_1, σ_2) is (σ_1, σ_2)s-regular, the following properties are equivalent:*

(1) F is upper $\tau^\star(\sigma_1, \sigma_2)$-continuous;

(2) $F^-(\sigma_1\sigma_2$-δ-Cl$(B))$ is $\star$-closed in X for every subset B of Y;

(3) $F^-(K)$ is $\star$-closed in X for every $\sigma_1\sigma_2$-δ-closed set K of Y;

(4) $F^+(V)$ is $\star$-open in X for every $\sigma_1\sigma_2$-δ-open set V of Y.

Proof. (1) $\Rightarrow$ (2): Let B be any subset of Y. By Lemma 3, $\sigma_1\sigma_2$-δ-Cl(B) is $\sigma_1\sigma_2$-closed in Y. Since F is upper $\tau^\star(\sigma_1, \sigma_2)$-continuous, by Theorem 1 $F^-(\sigma_1\sigma_2$-δ-Cl$(B))$ is $\star$-closed in X.

(2) $\Rightarrow$ (3): Let K be any $\sigma_1\sigma_2$-δ-closed set of Y. Then, $\sigma_1\sigma_2$-δ-Cl$(K) = K$ and by (2), we have $F^-(K)$ is $\star$-closed in X.

(3) $\Rightarrow$ (4): This follows from the fact that $F^+(Y - B) = X - F^-(B)$ for any subset B of Y.

(4) $\Rightarrow$ (1): Let V be any $\sigma_1\sigma_2$-open set of Y. Since (Y, σ_1, σ_2) is (σ_1, σ_2)s-regular, we have V is $\sigma_1\sigma_2$-δ-open in Y and by (4), $F^+(V)$ is $\star$-open in X. Thus, F is upper $\tau^\star(\sigma_1, \sigma_2)$-continuous by Theorem 1.

Theorem 4. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, where (Y, σ_1, σ_2) is (σ_1, σ_2)s-regular, the following properties are equivalent:*

(1) F is lower $\tau^\star(\sigma_1, \sigma_2)$-continuous;

(2) $F^+(\sigma_1\sigma_2$-δ-Cl$(B))$ is $\star$-closed in X for every subset B of Y;

(3) $F^+(K)$ is $\star$-closed in X for every $\sigma_1\sigma_2$-δ-closed set K of Y;

(4) $F^-(V)$ is $\star$-open in X for every $\sigma_1\sigma_2$-δ-open set V of Y.

Proof. The proof is similar to that of Theorem 3.

Corollary 2. *For a function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, where (Y, σ_1, σ_2) is (σ_1, σ_2)s-regular, the following properties are equivalent:*

(1) f is $\tau^\star(\sigma_1, \sigma_2)$-continuous;

(2) $f^{-1}(\sigma_1\sigma_2$-δ-Cl$(B))$ is $\star$-closed in X for every subset B of Y;

(3) $f^{-1}(K)$ *is* $\star$*-closed in* X *for every* $\sigma_1\sigma_2$*-*δ*-closed set* K *of* Y*;*

(4) $f^{-1}(V)$ *is* $\star$*-open in* X *for every* $\sigma_1\sigma_2$*-*δ*-open set* V *of* Y*.*

Definition 5. [17] *A bitopological space* (X, τ_1, τ_2) *is said to be* (τ_1, τ_2)*-regular if for each* $\tau_1\tau_2$*-closed set* F *and each* $x \notin F$*, there exist disjoint* $\tau_1\tau_2$*-open sets* U *and* V *such that* $x \in U$ *and* $F \subseteq V$*.*

Lemma 4. [17] *A bitopological space* (X, τ_1, τ_2) *is* (τ_1, τ_2)*-regular if and only if for each* $x \in X$ *and each* $\tau_1\tau_2$*-open set* U *containing* x*, there exists a* $\tau_1\tau_2$*-open set* V *such that* $x \in V \subseteq \tau_1\tau_2\text{-}Cl(V) \subseteq U$*.*

Lemma 5. [8] *Let* (X, τ_1, τ_2) *be a* (τ_1, τ_2)*-regular space. Then, the following properties hold:*

(1) $\tau_1\tau_2\text{-}Cl(A) = (\tau_1, \tau_2)\theta\text{-}Cl(A)$ *for every subset* A *of* X*.*

(2) Every $\tau_1\tau_2$*-open set is* $(\tau_1, \tau_2)\theta$*-open.*

Theorem 5. *For a multifunction* $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$*, where* (Y, σ_1, σ_2) *is* (σ_1, σ_2)*-regular, the following properties are equivalent:*

(1) F *is upper* $\tau^\star(\sigma_1, \sigma_2)$*-continuous;*

(2) $F^-((\sigma_1, \sigma_2)\theta\text{-}Cl(B))$ *is* $\star$*-closed in* X *for every subset* B *of* Y*;*

(3) $F^-(K)$ *is* $\star$*-closed in* X *for every* $(\sigma_1, \sigma_2)\theta$*-closed set* K *of* Y*;*

(4) $F^+(V)$ *is* $\star$*-open in* X *for every* $(\sigma_1, \sigma_2)\theta$*-open set* V *of* Y*.*

Proof. (1) $\Rightarrow$ (2): Let B be any subset of Y. By Lemma 5, $(\sigma_1, \sigma_2)\theta\text{-Cl}(B)$ is $\sigma_1\sigma_2$-closed in Y. Since F is upper $\tau^\star(\sigma_1, \sigma_2)$-continuous, by Theorem 1 $F^-((\sigma_1, \sigma_2)\theta\text{-Cl}(B))$ is $\star$-closed in X.

(2) $\Rightarrow$ (3): Let K be any $(\sigma_1, \sigma_2)\theta$-closed set of Y. Then, $(\sigma_1, \sigma_2)\theta\text{-Cl}(K) = K$ and by (2), we have $F^-(K)$ is $\star$-closed in X.

(3) $\Rightarrow$ (4): This follows from the fact that $F^+(Y - B) = X - F^-(B)$ for every subset B of Y.

(4) $\Rightarrow$ (1): Let V be any $\sigma_1\sigma_2$-open set of Y. Since (Y, σ_1, σ_2) is (σ_1, σ_2)-regular, we have V is $(\sigma_1, \sigma_2)\theta$-open in Y and by (4), $F^+(V)$ is $\star$-open in X. Thus, F is upper $\tau^\star(\sigma_1, \sigma_2)$-continuous by Theorem 1.

Theorem 6. *For a multifunction* $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$*, where* (Y, σ_1, σ_2) *is* (σ_1, σ_2)*-regular, the following properties are equivalent:*

(1) F *is lower* $\tau^\star(\sigma_1, \sigma_2)$*-continuous;*

(2) $F^+((\sigma_1, \sigma_2)\theta\text{-}Cl(B))$ *is* $\star$*-closed in* X *for every subset* B *of* Y*;*

(3) $F^+(K)$ *is* $\star$*-closed in* X *for every* $(\sigma_1, \sigma_2)\theta$*-closed set* K *of* Y*;*

(4) $F^-(V)$ *is* $\star$*-open in* X *for every* $(\sigma_1, \sigma_2)\theta$*-open set* V *of* Y*.*

Proof. The proof is similar to that of Theorem 5.

Corollary 3. *For a function* $f : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$*, where* (Y, σ_1, σ_2) *is* (σ_1, σ_2)*-regular, the following properties are equivalent:*

(1) f *is* $\tau^\star(\sigma_1, \sigma_2)$*-continuous;*

(2) $f^{-1}((\sigma_1, \sigma_2)\theta\text{-}Cl(B))$ *is* $\star$*-closed in* X *for every subset* B *of* Y*;*

(3) $f^{-1}(K)$ *is* $\star$*-closed in* X *for every* $(\sigma_1, \sigma_2)\theta$*-closed set* K *of* Y*;*

(4) $f^{-1}(V)$ *is* $\star$*-open in* X *for every* $(\sigma_1, \sigma_2)\theta$*-open set* V *of* Y*.*

Acknowledgements

This research project was financially supported by Mahasarakham University.

References

[1] K. Kuratowski. *Topology, Vol. I.* Academic Press, New York, 1966.

[2] R. Vaidyanathaswamy. The localisation theory in set-topology. *Proceedings of the Indian Academy of Sciences-Section A*, 20:51–61, 1944.

[3] E. Hatir and T. Noiri. Weakly pre-I-open sets and decomposition of continuity. *Acta Mathematica Hungarica*, 106(3):227–238, 2005.

[4] E. Hatir and T. Noiri. On decompositions of continuity via idealization. *Acta Mathematica Hungarica*, 96:341–349, 2002.

[5] C. Boonpok. On continuous multifunctions in ideal topological spaces. *Lobachevskii Journal of Mathematics*, 40(1):24–35, 2019.

[6] C. Boonpok. $p\imath$-continuity and weak $p\imath$-continuity. *Carpathian Mathematical Publications*, 17(1):171–186, 2025.

[7] P. Pue-on, S. Sompong, and C. Boonpok. Upper and lower (τ_1, τ_2)-continuous multifunctions. *International Journal of Mathematics and Computer Science*, 19(4):1305–1310, 2024.

[8] C. Klanarong, S. Sompong, and C. Boonpok. (τ_1, τ_2)-continuity and $(\tau_1, \tau_2)\theta$-closed sets. *International Journal of Mathematics and Computer Science*, 19(4):1299–1304, 2024.

[9] M. Thongmoon, S. Sompong, and C. Boonpok. (τ_1, τ_2)-continuous multifunctions and $\tau_1\tau_2$-δ-open sets. *International Journal of Mathematics and Computer Science*, 19(4):1369–1375, 2024.

[10] C. Boonpok, C. Viriyapong, and M. Thongmoon. On upper and lower (τ_1, τ_2)-precontinuous multifunctions. *Journal of Mathematics and Computer Science*, 18:282–293, 2018.

[11] C. Viriyapong and C. Boonpok. $(\tau_1, \tau_2)\alpha$-continuity for multifunctions. *Journal of Mathematics*, 2020:6285763, 2020.

[12] C. Boonpok. $(\tau_1, \tau_2)\delta$-semicontinuous multifunctions. *Heliyon*, 6:e05367, 2020.

[13] N. Viriyapong, S. Sompong, and C. Boonpok. (τ_1, τ_2)-extremal disconnectedness in bitopological spaces. *International Journal of Mathematics and Computer Science*, 19(3):855–860, 2024.

[14] C. Boonpok and P. Pue-on. Characterizations of almost (τ_1, τ_2)-continuous multifunctions. *International Journal of Analysis and Applications*, 22:33, 2024.

[15] D. Janković and T. R. Hamlett. New topologies from old via ideals. *The American Mathematical Monthly*, 97:295–310, 1990.

[16] E. Ekici and T. Noiri. $\star$-extremally disconnected ideal topological spaces. *Acta Mathematica Hungarica*, 122:81–90, 2009.

[17] M. Chiangpradit, S. Sompong, and C. Boonpok. On characterizations of (τ_1, τ_2)-regular spaces. *International Journal of Mathematics and Computer Science*, 19(4):1329–1334, 2024.