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Almost Continuity for Multifunctions Defined from an Ideal Topological Space into a Bitopological Space

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Abstract. This paper presents new concepts of continuous multifunctions defined between an ideal topological space and a bitopological space, namely upper almost $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions and lower almost $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions. Moreover, several characterizations and some properties concerning upper almost $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions and lower almost $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions are considered. Furthermore, the relationships between $\tau^\star(\sigma_1, \sigma_2)$-continuity and almost $\tau^\star(\sigma_1, \sigma_2)$-continuity are discussed.

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1. Introduction

It is well-known that the branch of mathematics called topology is related to all questions directly or indirectly concerned with continuity. Singal and Singal [1] introduced the concept of almost continuous functions as a generalization of continuity. Munshi and Bassan [2] studied the notion of almost semi-continuous functions. Noiri [3] introduced and investigated the concept of almost α-continuous functions. Nasef and Noiri [4] introduced two classes of functions, namely almost precontinuous functions and almost β-continuous functions. The class of almost precontinuity is a generalization of almost α-continuity. The class of almost β-continuity is a generalization of almost semi-continuity. Popa [5] introduced and studied the concepts of upper almost continuous multifunctions and lower almost continuous multifunctions. Furthermore, Popa and Noiri [6] introduced and investigated the notions of upper almost quasi-continuous multifunctions and lower almost quasi-continuous multifunctions. Popa et al. [7] introduced the concepts of upper almost

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precontinuous multifunctions and lower almost precontinuous multifunctions. Noiri and Popa [8] introduced the concepts of upper almost β-continuous multifunctions and lower almost β-continuous multifunctions. Moreover, some characterizations of upper almost β-continuous multifunctions and lower almost β-continuous multifunctions were presented in [9]. Popa and Noiri [10] introduced and investigated the notions of upper almost α-continuous multifunctions and lower almost α-continuous multifunctions. Pue-on et al. [11] introduced and studied the concepts of upper (τ_1, τ_2)-continuous multifunctions and lower (τ_1, τ_2)-continuous multifunctions. Klanarong et al. [12] introduced and investigated the notions of upper almost (τ_1, τ_2)-continuous multifunctions and lower almost (τ_1, τ_2)-continuous multifunctions. The notion of ideal topological spaces was introduced and studied by Kuratowski [13] and Vaidyanathaswamy [14]. Stronger and weaker forms of open sets in ideal topological spaces such as semi-$\mathscr{I}$-open sets, pre-$\mathscr{I}$-open sets, α-$\mathscr{I}$-open sets, β-$\mathscr{I}$-open sets and δ-$\mathscr{I}$-open sets play an important role in the research of generalizations of continuity. Using these notions many authors introduced and studied various types of generalizations of continuity for functions and multifunctions. Hatir and Noiri [15] introduced and investigated the notions of weakly pre-$\mathscr{I}$-open sets and weakly pre-$\mathscr{I}$-continuous functions. Furthermore, Hatir and Noiri [16] investigated further properties of semi-$\mathscr{I}$-open sets and semi-$\mathscr{I}$-continuous functions. On the other hand, the present author [17] introduced the concepts of upper $\star$-continuous multifunctions and lower $\star$-continuous multifunctions. Moreover, several characterizations of upper $\star$-continuous multifunctions, lower $\star$-continuous multifunctions, upper almost $\star$-continuous multifunctions and lower almost $\star$-continuous multifunctions were established in [17]. Quite recently, the present author [18] introduced and studied the notions of $\mathscr{p}\imath$-continuous multifunctions and weakly $\mathscr{p}\imath$-continuous multifunctions. In this paper, we introduce new classes of multifunctions between an ideal topological space and a bitopological space, namely upper almost $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions and lower almost $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions. We also investigate several characterizations of upper almost $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions and lower almost $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions.

2. Preliminaries

Throughout the present paper, spaces (X, τ_1, τ_2) and (Y, σ_1, σ_2) (or simply X and Y) always mean bitopological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a bitopological space (X, τ_1, τ_2). The closure of A and the interior of A with respect to τ_i are denoted by τ_i-Cl(A) and τ_i-Int(A), respectively, for $i = 1, 2$. A subset A of a bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$-*closed* [19] if $A = \tau_1$-Cl$(\tau_2$-Cl$(A))$. The complement of a $\tau_1\tau_2$-closed set is called $\tau_1\tau_2$-*open*. The intersection of all $\tau_1\tau_2$-closed sets of X containing A is called the $\tau_1\tau_2$-*closure* [19] of A and is denoted by $\tau_1\tau_2$-Cl(A). The union of all $\tau_1\tau_2$-open sets of X contained in A is called the $\tau_1\tau_2$-*interior* [19] of A and is denoted by $\tau_1\tau_2$-Int(A).

Lemma 1. [19] *Let A and B be subsets of a bitopological space (X, τ_1, τ_2). For the $\tau_1\tau_2$-closure, the following properties hold:*

(1) $A \subseteq \tau_1\tau_2\text{-}Cl(A)$ *and* $\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Cl(A)) = \tau_1\tau_2\text{-}Cl(A)$.

(2) If $A \subseteq B$, *then* $\tau_1\tau_2\text{-}Cl(A) \subseteq \tau_1\tau_2\text{-}Cl(B)$.

(3) $\tau_1\tau_2\text{-}Cl(A)$ *is* $\tau_1\tau_2$*-closed.*

(4) A *is* $\tau_1\tau_2$*-closed if and only if* $A = \tau_1\tau_2\text{-}Cl(A)$.

(5) $\tau_1\tau_2\text{-}Cl(X - A) = X - \tau_1\tau_2\text{-}Int(A)$.

A subset A of a bitopological space (X, τ_1, τ_2) is said to be $(\tau_1, \tau_2)r$*-open* [20] (resp. $(\tau_1, \tau_2)s$*-open* [21], $(\tau_1, \tau_2)p$*-open* [21], $(\tau_1, \tau_2)\beta$*-open* [21]) if $A = \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A))$ (resp. $A \subseteq \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(A))$, $A \subseteq \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A))$, $A \subseteq \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A)))$). The complement of a $(\tau_1, \tau_2)r$-open (resp. $(\tau_1, \tau_2)s$-open, $(\tau_1, \tau_2)p$-open, $(\tau_1, \tau_2)\beta$-open) set is said to be $(\tau_1, \tau_2)r$*-closed* (resp. $(\tau_1, \tau_2)s$*-closed*, $(\tau_1, \tau_2)p$*-closed*, $(\tau_1, \tau_2)\beta$*-closed*). A subset A of a bitopological space (X, τ_1, τ_2) is said to be $\alpha(\tau_1, \tau_2)$*-open* [22] if $A \subseteq \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(A)))$. The complement of an $\alpha(\tau_1, \tau_2)$-open set is said to be $\alpha(\tau_1, \tau_2)$*-closed*. Let A be a subset of a bitopological space (X, τ_1, τ_2). The intersection of all $(\tau_1, \tau_2)p$-closed (resp. $(\tau_1, \tau_2)s$-closed, $\alpha(\tau_1, \tau_2)$-closed) sets of X containing A is called the $(\tau_1, \tau_2)p$*-closure* [23] (resp. $(\tau_1, \tau_2)s$*-closure* [21], $\alpha(\tau_1, \tau_2)$*-closure* [24]) of A and is denoted by $(\tau_1, \tau_2)\text{-pCl}(A)$ (resp. $(\tau_1, \tau_2)\text{-sCl}(A)$, $\alpha(\tau_1, \tau_2)\text{-Cl}(A)$). The union of all $(\tau_1, \tau_2)p$-open (resp. $(\tau_1, \tau_2)s$-open, $\alpha(\tau_1, \tau_2)$-open) sets of X contained in A is called the $(\tau_1, \tau_2)p$*-interior* [23] (resp. $(\tau_1, \tau_2)s$*-interior* [21], $\alpha(\tau_1, \tau_2)$*-interior* [24]) of A and is denoted by $(\tau_1, \tau_2)\text{-pInt}(A)$ (resp. $(\tau_1, \tau_2)\text{-sInt}(A)$, $\alpha(\tau_1, \tau_2)\text{-Int}(A)$).

A subset A of a bitopological space (X, τ_1, τ_2) is said to be $\tau_1\tau_2$*-*δ*-open* if A is the union of $(\tau_1, \tau_2)r$-open sets of X. The complement of a $\tau_1\tau_2$-δ-open set is called $\tau_1\tau_2$*-*δ*-closed*. The union of all $\tau_1\tau_2$-δ-open sets of X contained in A is called the $\tau_1\tau_2$*-*δ*-interior* of A and is denoted by $\tau_1\tau_2\text{-}\delta\text{-Int}(A)$. The intersection of all $\tau_1\tau_2$-δ-closed sets of X containing A is called the $\tau_1\tau_2$*-*δ*-closure* of A and is denoted by $\tau_1\tau_2\text{-}\delta\text{-Cl}(A)$ [25]. For a subset A of a bitopological space (X, τ_1, τ_2), a point $x \in X$ is called a $(\tau_1, \tau_2)\theta$*-cluster point* of A if $\tau_1\tau_2\text{-Cl}(U) \cap A \neq \emptyset$ for every $\tau_1\tau_2$-open set U containing x. The set of all $(\tau_1, \tau_2)\theta$-cluster points of A is called the $(\tau_1, \tau_2)\theta$*-closure* of A and is denoted by $(\tau_1, \tau_2)\theta\text{-Cl}(A)$. A subset A of a bitopological space (X, τ_1, τ_2) is said to be $(\tau_1, \tau_2)\theta$*-closed* if $(\tau_1, \tau_2)\theta\text{-Cl}(A) = A$. The complement of a $(\tau_1, \tau_2)\theta$-closed set is said to be $(\tau_1, \tau_2)\theta$*-open*. The union of all $(\tau_1, \tau_2)\theta$-open sets of X contained in A is called the $(\tau_1, \tau_2)\theta$*-interior* of A and is denoted by $(\tau_1, \tau_2)\theta\text{-Int}(A)$ [20].

An ideal $\mathscr{I}$ on a topological space (X, τ) is a nonempty collection of subsets of X satisfying the following properties: (1) $A \in \mathscr{I}$ and $B \subseteq A$ imply $B \in \mathscr{I}$; (2) $A \in \mathscr{I}$ and $B \in \mathscr{I}$ imply $A \cup B \in \mathscr{I}$. A topological space (X, τ) with an ideal $\mathscr{I}$ on X is called an ideal topological space and is denoted by $(X, \tau, \mathscr{I})$. For an ideal topological space $(X, \tau, \mathscr{I})$ and a subset A of X, $A^\star(\mathscr{I})$ is defined as follows:

$$A^\star(\mathscr{I}) = \{x \in X : U \cap A \notin \mathscr{I} \text{ for every open neighbourhood } U \text{ of } x\}.$$

In case there is no chance for confusion, $A^\star(\mathscr{I})$ is simply written as $A^\star$. In [13], $A^\star$ is called the local function of A with respect to $\mathscr{I}$ and τ and $\text{Cl}^\star(A) = A^\star \cup A$ defines a Kuratowski closure operator for a topology $\tau^\star(\mathscr{I})$ finer than τ. A subset A is said to be $\star$-closed [26] if $A^\star \subseteq A$. The interior of a subset A in $(X, \tau^\star(\mathscr{I}))$ is denoted by $\text{Int}^\star(A)$. A subset A of an ideal topological space $(X, \tau, \mathscr{I})$ is said to be *semi***-*$\mathscr{I}$*-open* [27] (resp. *semi-*$\mathscr{I}$*-open* [16]) if $A \subseteq \text{Cl}(\text{Int}^\star(A))$ (resp. $A \subseteq \text{Cl}^\star(\text{Int}(A))$). The complement of a semi*-$\mathscr{I}$-open (resp. semi-$\mathscr{I}$-open) set is said to be *semi***-*$\mathscr{I}$*-closed* [27] (resp. semi-$\mathscr{I}$-closed [16]).

By a multifunction $F : X \to Y$, we mean a point-to-set correspondence from X into Y, and we always assume that $F(x) \neq \emptyset$ for all $x \in X$. For a multifunction $F : X \to Y$, we shall denote the upper and lower inverse of a set B of Y by $F^+(B)$ and $F^-(B)$, respectively, that is, $F^+(B) = \{x \in X \mid F(x) \subseteq B\}$ and $F^-(B) = \{x \in X \mid F(x) \cap B \neq \emptyset\}$. In particular, $F^-(y) = \{x \in X \mid y \in F(x)\}$ for each point $y \in Y$. For each $A \subseteq X$, $F(A) = \cup_{x \in A} F(x)$.

3. Upper and lower almost $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions

In this section, we introduce the concepts of upper almost $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions and lower almost $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions. Furthermore, several characterizations of upper almost $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions and lower almost $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions are discussed.

Definition 1. *A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is said to be upper almost $\tau^\star(\sigma_1, \sigma_2)$-continuous at a point $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y such that $F(x) \subseteq V$, there exists a $\star$-open set U of X containing x such that $F(U) \subseteq \sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V))$. A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is said to be upper almost $\tau^\star(\sigma_1, \sigma_2)$-continuous if F is upper almost $\tau^\star(\sigma_1, \sigma_2)$-continuous at each point x of X.*

Lemma 2. [12] *Let A be a subset of a bitopological space (X, τ_1, τ_2). If A is $\tau_1\tau_2$-open in X, then (τ_1, τ_2)-sCl$(A) = \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))$.*

Theorem 1. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is upper almost $\tau^\star(\sigma_1, \sigma_2)$-continuous at $x \in X$;

(2) $x \in$ Int$^\star(F^+(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V))))$ for every $\sigma_1\sigma_2$-open set V of Y containing $F(x)$;

(3) $x \in$ Int$^\star(F^+((\sigma_1, \sigma_2)$-sCl$(V)))$ for every $\sigma_1\sigma_2$-open set V of Y containing $F(x)$;

(4) $x \in$ Int$^\star(F^+(V))$ for every $(\sigma_1, \sigma_2)r$-open set V of Y containing $F(x)$;

(5) for each $(\sigma_1, \sigma_2)r$-open set V of Y containing $F(x)$, there exists a $\star$-open set U of X containing x such that $F(U) \subseteq V$.

Proof. (1) $\Rightarrow$ (2): Let V be any $\sigma_1\sigma_2$-open set of Y containing $F(x)$. Thus by (1), there exists a $\star$-open set U of X containing x such that $F(U) \subseteq \sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V))$. Therefore, $x \in U \subseteq F^+(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V)))$ and so $x \in$ Int$^\star(F^+(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V))))$.

(2) ⇒ (3): This follows from Lemma 2.

(3) ⇒ (4): Let V be any $\sigma_1\sigma_2$-open set of Y containing $F(x)$. It follows from Lemma 2 that $V = \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)) = (\sigma_1, \sigma_2)\text{-sCl}(V)$.

(4) ⇒ (5): Let V be any $(\sigma_1, \sigma_2)r$-open set of Y containing $F(x)$. Then by (4), we have $x \in \text{Int}^\star(F^+(V))$ and there exists a $\star$-open set U of X containing x such that $x \in U \subseteq F^+(V)$; hence $F(U) \subseteq V$.

(5) ⇒ (1): Let V be any $\sigma_1\sigma_2$-open set of Y containing $F(x)$. Since $\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))$ is $(\sigma_1, \sigma_2)r$-open in Y and by (5), there exists a $\star$-open set U of X containing x such that $F(U) \subseteq \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))$. This shows that F is upper almost $\tau^\star(\sigma_1, \sigma_2)$-continuous at $x \in X$.

Definition 2. *A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is called lower almost $\tau^\star(\sigma_1, \sigma_2)$-continuous at a point $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y such that $V \cap F(x) \neq \emptyset$, there exists a $\star$-open set U of X containing x such that $\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V)) \cap F(z) \neq \emptyset$ for every $z \in U$. A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is called lower almost $\tau^\star(\sigma_1, \sigma_2)$-continuous if F is lower almost $\tau^\star(\sigma_1, \sigma_2)$-continuous at each point x of X.*

Theorem 2. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is lower almost $\tau^\star(\sigma_1, \sigma_2)$-continuous at $x \in X$;

(2) $x \in$ Int$^\star(F^-(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V))))$ for every $\sigma_1\sigma_2$-open set V of Y such that

$$V \cap F(x) \neq \emptyset;$$

(3) $x \in$ Int$^\star(F^-((\sigma_1, \sigma_2)$-sCl$(V)))$ for every $\sigma_1\sigma_2$-open set V of Y such that

$$V \cap F(x) \neq \emptyset;$$

(4) $x \in$ Int$^\star(F^-(V))$ for every $(\sigma_1, \sigma_2)r$-open set V of Y such that $V \cap F(x) \neq \emptyset$;

(5) for each $(\sigma_1, \sigma_2)r$-open set V of Y such that $V \cap F(x) \neq \emptyset$, there exists a $\star$-open set U of X containing x such that $U \subseteq F^-(V)$.

Proof. The proof is similar to that of Theorem 1.

Theorem 3. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is upper almost $\tau^\star(\sigma_1, \sigma_2)$-continuous;

(2) $F^+(V) \subseteq$ Int$^\star(F^+(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V))))$ for every $\sigma_1\sigma_2$-open set V of Y;

(3) Cl$^\star(F^-(\sigma_1\sigma_2$-Cl$(\sigma_1\sigma_2$-Int$(K)))) \subseteq F^-(K)$ for every $\sigma_1\sigma_2$-closed set K of Y;

(4) $\mathrm{Cl}^{\star}(F^{-}(\sigma_1\sigma_2\text{-}\mathrm{Cl}(\sigma_1\sigma_2\text{-}\mathrm{Int}(\sigma_1\sigma_2\text{-}\mathrm{Cl}(B))))) \subseteq F^{-}(\sigma_1\sigma_2\text{-}\mathrm{Cl}(B))$ *for every subset* B *of* Y*;*

(5) $F^{+}(\sigma_1\sigma_2\text{-}\mathrm{Int}(B)) \subseteq \mathrm{Int}^{\star}(F^{+}(\sigma_1\sigma_2\text{-}\mathrm{Int}(\sigma_1\sigma_2\text{-}\mathrm{Cl}(\sigma_1\sigma_2\text{-}\mathrm{Int}(B)))))$ *for every subset* B *of* Y*;*

(6) $F^{+}(V)$ *is* $\star$*-open in* X *for every* $(\sigma_1, \sigma_2)r$*-open set* V *of* Y*;*

(7) $F^{-}(K)$ *is* $\star$*-closed in* X *for every* $(\sigma_1, \sigma_2)r$*-closed set* K *of* Y*.*

Proof. (1) $\Rightarrow$ (2): Let V be any $\sigma_1\sigma_2$-open set of Y and $x \in F^{+}(V)$. Then, $F(x) \subseteq V$. Thus by Theorem 1, we have $x \in \mathrm{Int}^{\star}(F^{+}(\sigma_1\sigma_2\text{-}\mathrm{Int}(\sigma_1\sigma_2\text{-}\mathrm{Cl}(V))))$ and hence $F^{+}(V) \subseteq \mathrm{Int}^{\star}(F^{+}(\sigma_1\sigma_2\text{-}\mathrm{Int}(\sigma_1\sigma_2\text{-}\mathrm{Cl}(V))))$.

(2) $\Rightarrow$ (3): Let K be any $\sigma_1\sigma_2$-closed set of Y. Then, $Y - K$ is $\sigma_1\sigma_2$-open in Y and by (2),

$$\begin{aligned} X - F^{-}(K) = F^{+}(Y - K) &\subseteq \mathrm{Int}^{\star}(F^{+}(\sigma_1\sigma_2\text{-}\mathrm{Int}(\sigma_1\sigma_2\text{-}\mathrm{Cl}(Y - K)))) \\ &= \mathrm{Int}^{\star}(X - F^{-}(\sigma_1\sigma_2\text{-}\mathrm{Cl}(\sigma_1\sigma_2\text{-}\mathrm{Int}(K)))) \\ &= X - \mathrm{Cl}^{\star}(F^{-}(\sigma_1\sigma_2\text{-}\mathrm{Cl}(\sigma_1\sigma_2\text{-}\mathrm{Int}(K)))). \end{aligned}$$

Thus, $\mathrm{Cl}^{\star}(F^{-}(\sigma_1\sigma_2\text{-}\mathrm{Cl}(\sigma_1\sigma_2\text{-}\mathrm{Int}(K)))) \subseteq F^{-}(K)$.

(3) $\Rightarrow$ (4): Let B be any subset of Y. Then, $\sigma_1\sigma_2\text{-}\mathrm{Cl}(B)$ is a $\sigma_1\sigma_2$-closed set of Y and by (3), $\mathrm{Cl}^{\star}(F^{-}(\sigma_1\sigma_2\text{-}\mathrm{Cl}(\sigma_1\sigma_2\text{-}\mathrm{Int}(\sigma_1\sigma_2\text{-}\mathrm{Cl}(B))))) \subseteq F^{-}(\sigma_1\sigma_2\text{-}\mathrm{Cl}(B))$.

(4) $\Rightarrow$ (5): Let B be any subset of Y. Thus by (4), we have

$$\begin{aligned} F^{+}(\sigma_1\sigma_2\text{-}\mathrm{Int}(B)) &= X - F^{-}(\sigma_1\sigma_2\text{-}\mathrm{Cl}(Y - B)) \\ &\subseteq X - \mathrm{Cl}^{\star}(F^{-}(\sigma_1\sigma_2\text{-}\mathrm{Cl}(\sigma_1\sigma_2\text{-}\mathrm{Int}(\sigma_1\sigma_2\text{-}\mathrm{Cl}(Y - B))))) \\ &= X - \mathrm{Cl}^{\star}(F^{-}(Y - \sigma_1\sigma_2\text{-}\mathrm{Int}(\sigma_1\sigma_2\text{-}\mathrm{Cl}(\sigma_1\sigma_2\text{-}\mathrm{Int}(B))))) \\ &= \mathrm{Int}^{\star}(F^{+}(\sigma_1\sigma_2\text{-}\mathrm{Int}(\sigma_1\sigma_2\text{-}\mathrm{Cl}(\sigma_1\sigma_2\text{-}\mathrm{Int}(B))))). \end{aligned}$$

(5) $\Rightarrow$ (6): Let V be any $(\sigma_1, \sigma_2)r$-open set of Y. By (5), we have $F^{+}(V) \subseteq \mathrm{Int}^{\star}(F^{+}(V))$ and hence $F^{+}(V)$ is $\star$-open in X.

(6) $\Rightarrow$ (7): The proof is obvious.

(7) $\Rightarrow$ (1): Let $x \in X$ and V be any $(\sigma_1, \sigma_2)r$-open set of Y containing $F(x)$. Since $Y - V$ is $(\sigma_1, \sigma_2)r$-closed and by (7), $X - F^{+}(V) = F^{-}(Y - V)$ is $\star$-closed in X. Thus, $F^{+}(V)$ is $\star$-open and hence $x \in \mathrm{Int}^{\star}(F^{+}(V))$. Then, there exists a $\star$-open set U of X containing x such that $F(U) \subseteq V$. It follows from Theorem 1 that F is upper almost $\tau^{\star}(\sigma_1, \sigma_2)$-continuous.

Theorem 4. *For a multifunction* $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$*, the following properties are equivalent:*

(1) F *is lower almost* $\tau^{\star}(\sigma_1, \sigma_2)$*-continuous;*

(2) $F^{-}(V) \subseteq \mathrm{Int}^{\star}(F^{-}(\sigma_1\sigma_2\text{-}\mathrm{Int}(\sigma_1\sigma_2\text{-}\mathrm{Cl}(V))))$ *for every* $\sigma_1\sigma_2$*-open set* V *of* Y*;*

(3) $Cl^\star(F^+(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(K)))) \subseteq F^+(K)$ *for every* $\sigma_1\sigma_2$*-closed set* K *of* Y*;*

(4) $Cl^\star(F^+(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(B))))) \subseteq F^+(\sigma_1\sigma_2\text{-}Cl(B))$ *for every subset* B *of* Y*;*

(5) $F^-(\sigma_1\sigma_2\text{-}Int(B)) \subseteq Int^\star(F^-(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(B)))))$ *for every subset* B *of* Y*;*

(6) $F^-(V)$ *is* $\star$*-open in* X *for every* $(\sigma_1, \sigma_2)r$*-open set* V *of* Y*;*

(7) $F^+(K)$ *is* $\star$*-closed in* X *for every* $(\sigma_1, \sigma_2)r$*-closed set* K *of* Y*.*

Proof. The proof is similar to that of Theorem 3.

Theorem 5. *For a multifunction* $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$*, the following properties are equivalent:*

(1) F *is upper almost* $\tau^\star(\sigma_1, \sigma_2)$*-continuous;*

(2) $Cl^\star(F^-(V)) \subseteq F^-(\sigma_1\sigma_2\text{-}Cl(V))$ *for every* $(\sigma_1, \sigma_2)\beta$*-open set* V *of* Y*;*

(3) $Cl^\star(F^-(V)) \subseteq F^-(\sigma_1\sigma_2\text{-}Cl(V))$ *for every* $(\sigma_1, \sigma_2)s$*-open set* V *of* Y*.*

Proof. (1) ⇒ (2): Let V be any $(\sigma_1, \sigma_2)\beta$-open set of Y. Then, $\sigma_1\sigma_2$-Cl(V) is a $(\sigma_1, \sigma_2)r$-closed set of Y. Since F is upper almost $\tau^\star(\sigma_1, \sigma_2)$-continuous and by Theorem 3, $F^-(\sigma_1\sigma_2\text{-Cl}(V))$ is $\star$-closed in X. Thus, $\text{Cl}^\star(F^-(V)) \subseteq F^-(\sigma_1\sigma_2\text{-Cl}(V))$.

(2) ⇒ (3): The proof is obvious.

(3) ⇒ (1): Let K be any $(\sigma_1, \sigma_2)r$-closed set of Y. Then, K is $(\sigma_1, \sigma_2)s$-open in Y. Then by (3), $\text{Cl}^\star(F^-(K)) \subseteq F^-(\sigma_1\sigma_2\text{-Cl}(K)) = F^-(K)$ and hence $F^-(K)$ is $\star$-closed in X. By Theorem 3, F is upper almost $\tau^\star(\sigma_1, \sigma_2)$-continuous.

Theorem 6. *For a multifunction* $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$*, the following properties are equivalent:*

(1) F *is lower almost* $\tau^\star(\sigma_1, \sigma_2)$*-continuous;*

(2) $Cl^\star(F^+(V)) \subseteq F^+(\sigma_1\sigma_2\text{-}Cl(V))$ *for every* $(\sigma_1, \sigma_2)\beta$*-open set* V *of* Y*;*

(3) $Cl^\star(F^+(V)) \subseteq F^+(\sigma_1\sigma_2\text{-}Cl(V))$ *for every* $(\sigma_1, \sigma_2)s$*-open set* V *of* Y*.*

Proof. The proof is similar to that of Theorem 5.

Lemma 3. [28] *For a bitopological space* (X, τ_1, τ_2)*, the following properties hold:*

(1) $\alpha(\tau_1, \tau_2)\text{-}Cl(V) = \tau_1\tau_2\text{-}Cl(V)$ *for every* $(\tau_1, \tau_2)\beta$*-open set* V *of* X*;*

(2) $(\tau_1, \tau_2)\text{-}pCl(V) = \tau_1\tau_2\text{-}Cl(V)$ *for every* $(\tau_1, \tau_2)s$*-open set* V *of* X*.*

Corollary 1. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is upper almost $\tau^\star(\sigma_1, \sigma_2)$-continuous;

(2) $Cl^\star(F^-(V)) \subseteq F^-(\alpha(\sigma_1, \sigma_2)\text{-}Cl(V))$ for every $(\sigma_1, \sigma_2)\beta$-open set V of Y;

(3) $Cl^\star(F^-(V)) \subseteq F^-((\sigma_1, \sigma_2)\text{-}pCl(V))$ for every $(\sigma_1, \sigma_2)s$-open set V of Y.

Corollary 2. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is lower almost $\tau^\star(\sigma_1, \sigma_2)$-continuous;

(2) $Cl^\star(F^+(V)) \subseteq F^+(\alpha(\sigma_1, \sigma_2)\text{-}Cl(V))$ for every $(\sigma_1, \sigma_2)\beta$-open set V of Y;

(3) $Cl^\star(F^+(V)) \subseteq F^+((\sigma_1, \sigma_2)\text{-}pCl(V))$ for every $(\sigma_1, \sigma_2)s$-open set V of Y.

Theorem 7. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is upper almost $\tau^\star(\sigma_1, \sigma_2)$-continuous;

(2) $Cl^\star(F^-(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V))))) \subseteq F^-(\sigma_1\sigma_2\text{-}Cl(V))$ for every $(\sigma_1, \sigma_2)p$-open set V of Y;

(3) $Cl^\star(F^-(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(V)))) \subseteq F^-(\sigma_1\sigma_2\text{-}Cl(V))$ for every $(\sigma_1, \sigma_2)p$-open set V of Y;

(4) $F^+(V) \subseteq Int^\star(F^+(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V))))$ for every $(\sigma_1, \sigma_2)p$-open set V of Y.

Proof. (1) $\Rightarrow$ (2): Let V be any $(\sigma_1, \sigma_2)p$-open set of Y. Then, $\sigma_1\sigma_2\text{-Cl}(V)$ is $\sigma_1\sigma_2$-closed in Y and by Theorem 3, we have

$$\text{Cl}^\star(F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))))) \subseteq F^-(\sigma_1\sigma_2\text{-Cl}(V)).$$

(2) $\Rightarrow$ (3): Let V be any $(\sigma_1, \sigma_2)p$-open set of Y. By (2),

$$\begin{aligned} \text{Cl}^\star(F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(V)))) &\subseteq \text{Cl}^\star(F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))))) \\ &\subseteq F^-(\sigma_1\sigma_2\text{-Cl}(V)). \end{aligned}$$

(3) $\Rightarrow$ (4): Let V be any $(\sigma_1, \sigma_2)p$-open set of Y. Thus by (3), we have

$$\begin{aligned} X - \text{Int}^\star(F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) &= \text{Cl}^\star(X - F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) \\ &= \text{Cl}^\star(F^-(Y - \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) \\ &= \text{Cl}^\star(F^-(\sigma_1\sigma_2\text{-Cl}(Y - \sigma_1\sigma_2\text{-Cl}(V)))) \\ &= \text{Cl}^\star(F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(Y - \sigma_1\sigma_2\text{-Cl}(V))))) \\ &\subseteq F^-(\sigma_1\sigma_2\text{-Cl}(Y - \sigma_1\sigma_2\text{-Cl}(V))) \end{aligned}$$

$$= F^-(Y - \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))$$
$$\subseteq F^-(Y - V)$$
$$= X - F^+(V)$$

and hence $F^+(V) \subseteq \text{Int}^\star(F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))))$.

(4) ⇒ (1): Let V be any $(\sigma_1, \sigma_2)r$-open set of Y. Then, V is $(\sigma_1, \sigma_2)p$-open in Y and by (4), $F^+(V) \subseteq \text{Int}^\star(F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) = \text{Int}^\star(F^+(V))$. Thus, $F^+(V)$ is $\star$-open in X. It follows from Theorem 3 that F is upper almost $\tau^\star(\sigma_1, \sigma_2)$-continuous.

Theorem 8. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is lower almost $\tau^\star(\sigma_1, \sigma_2)$-continuous;

(2) $\text{Cl}^\star(F^+(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))))) \subseteq F^+(\sigma_1\sigma_2\text{-Cl}(V))$ for every $(\sigma_1, \sigma_2)p$-open set V of Y;

(3) $\text{Cl}^\star(F^+(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(V)))) \subseteq F^+(\sigma_1\sigma_2\text{-Cl}(V))$ for every $(\sigma_1, \sigma_2)p$-open set V of Y;

(4) $F^-(V) \subseteq \text{Int}^\star(F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))))$ for every $(\sigma_1, \sigma_2)p$-open set V of Y.

Proof. The proof is similar to that of Theorem 7.

Lemma 4. [29] *Let A be a subset of a bitopological space (X, τ_1, τ_2). Then, the following properties hold:*

(1) If A is $\tau_1\tau_2$-open in X, then $\tau_1\tau_2\text{-Cl}(A) = \tau_1\tau_2\text{-}\delta\text{-Cl}(A)$.

(2) $\tau_1\tau_2\text{-}\delta\text{-Cl}(A)$ is $\tau_1\tau_2$-closed.

Theorem 9. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is upper almost $\tau^\star(\sigma_1, \sigma_2)$-continuous;

(2) $\text{Cl}^\star(F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-}\delta\text{-Cl}(B))))) \subseteq F^-(\sigma_1\sigma_2\text{-}\delta\text{-Cl}(B))$ for every subset B of Y;

(3) $\text{Cl}^\star(F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(B))))) \subseteq F^-(\sigma_1\sigma_2\text{-}\delta\text{-Cl}(B))$ for every subset B of Y.

Proof. (1) ⇒ (2): Let B be any subset of Y. By Lemma 4, $\sigma_1\sigma_2\text{-}\delta\text{-Cl}(B)$ is $\sigma_1\sigma_2$-closed in Y and by Theorem 3, $\text{Cl}^\star(F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-}\delta\text{-Cl}(B))))) \subseteq F^-(\sigma_1\sigma_2\text{-}\delta\text{-Cl}(B))$.

(2) ⇒ (3): This is obvious since $\sigma_1\sigma_2\text{-Cl}(B) \subseteq \sigma_1\sigma_2\text{-}\delta\text{-Cl}(B)$ for every subset B of Y.

(3) ⇒ (1): Let K be any $(\sigma_1, \sigma_2)r$-closed set of Y. Then by (3), we have

$$\text{Cl}^\star(F^-(K)) = \text{Cl}^\star(F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K))))$$

$$
\begin{aligned}
&= \mathrm{Cl}^{\star}(F^{-}(\sigma_1\sigma_2\text{-}\mathrm{Cl}(\sigma_1\sigma_2\text{-}\mathrm{Int}(\sigma_1\sigma_2\text{-}\mathrm{Cl}(K)))))\\
&\subseteq F^{-}(\sigma_1\sigma_2\text{-}\delta\text{-}\mathrm{Cl}(K))\\
&= F^{-}(K)
\end{aligned}
$$

and hence $F^{-}(K)$ is $\star$-closed in X. By Theorem 3, F is upper almost $\tau^{\star}(\sigma_1, \sigma_2)$-continuous.

Theorem 10. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is lower almost $\tau^{\star}(\sigma_1, \sigma_2)$-continuous;

(2) $Cl^{\star}(F^{+}(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}\delta\text{-}Cl(B))))) \subseteq F^{+}(\sigma_1\sigma_2\text{-}\delta\text{-}Cl(B))$ for every subset B of Y;

(3) $Cl^{\star}(F^{+}(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(B))))) \subseteq F^{+}(\sigma_1\sigma_2\text{-}\delta\text{-}Cl(B))$ for every subset B of Y.

Proof. The proof is similar to that of Theorem 9.

Lemma 5. *If $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is lower almost $\tau^{\star}(\sigma_1, \sigma_2)$-continuous, then for each $x \in X$ and each subset B of Y with $\sigma_1\sigma_2\text{-}\delta\text{-}Int(B) \cap F(x) \neq \emptyset$, there exists a $\star$-open set U of X containing x such that $U \subseteq F^{-}(B)$.*

Proof. Let $x \in X$ and B be a subset of Y with $\sigma_1\sigma_2\text{-}\delta\text{-}\mathrm{Int}(B) \cap F(x) \neq \emptyset$. Since $\sigma_1\sigma_2\text{-}\delta\text{-}\mathrm{Int}(B) \cap F(x) \neq \emptyset$, there exists a nonempty $(\sigma_1, \sigma_2)r$-open set V of Y such that $V \subseteq B$ and $V \cap F(x) \neq \emptyset$. Since F is lower almost $\tau^{\star}(\sigma_1, \sigma_2)$-continuous, there exists a $\star$-open set U of X containing x such that $V \cap F(z) \neq \emptyset$ for each $z \in U$; hence $U \subseteq F^{-}(B)$.

Theorem 11. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is lower almost $\tau^{\star}(\sigma_1, \sigma_2)$-continuous;

(2) $Cl^{\star}(F^{+}(B)) \subseteq F^{+}(\sigma_1\sigma_2\text{-}\delta\text{-}Cl(B))$ for every subset B of Y;

(3) $F(Cl^{\star}(A)) \subseteq \sigma_1\sigma_2\text{-}\delta\text{-}Cl(F(A))$ for every subset A of X;

(4) $F^{+}(K)$ is $\star$-closed in X for every $\sigma_1\sigma_2$-δ-closed set K of Y;

(5) $F^{-}(V)$ is $\star$-open in X for every $\sigma_1\sigma_2$-δ-open set V of Y;

(6) $F^{-}(\sigma_1\sigma_2\text{-}\delta\text{-}Int(B)) \subseteq Int^{\star}(F^{-}(B))$ for every subset B of Y.

Proof. (1) $\Rightarrow$ (2): Let B be any subset of Y. Suppose that $x \notin F^{+}(\sigma_1\sigma_2\text{-}\delta\text{-}\mathrm{Cl}(B))$. Then, we have $x \in F^{-}(Y - \sigma_1\sigma_2\text{-}\delta\text{-}\mathrm{Cl}(B)) = F^{-}(\sigma_1\sigma_2\text{-}\delta\text{-}\mathrm{Int}(Y - B))$. There exists a $\star$-open set U of X containing x such that $U \subseteq F^{-}(Y - B) = X - F^{+}(B)$. Thus, $U \cap F^{+}(B) = \emptyset$ and hence $x \in X - \mathrm{Cl}^{\star}(F^{+}(B))$. This shows that $\mathrm{Cl}^{\star}(F^{+}(B)) \subseteq F^{+}(\sigma_1\sigma_2\text{-}\delta\text{-}\mathrm{Cl}(B))$.

(2) ⇒ (3): Let A be any subset of X. By (2), we have

$$\mathrm{Cl}^\star(A) \subseteq \tau_1\tau_2\text{-}\mathrm{Cl}(F^+(F(A))) \subseteq F^+(\sigma_1\sigma_2\text{-}\delta\text{-}\mathrm{Cl}(F(A)))$$

and hence $F(\mathrm{Cl}^\star(A)) \subseteq \sigma_1\sigma_2\text{-}\delta\text{-}\mathrm{Cl}(F(A))$.

(3) ⇒ (1): Let B be any subset of Y. Then, by the hypothesis and Lemma 4,

$$\begin{aligned}
&F(\mathrm{Cl}^\star(F^+(\sigma_1\sigma_2\text{-}\mathrm{Cl}(\sigma_1\sigma_2\text{-}\mathrm{Int}(\sigma_1\sigma_2\text{-}\mathrm{Cl}(B))))))\\
&\subseteq \tau_1\tau_2\text{-}\delta\text{-}\mathrm{Cl}(F(F^+(\sigma_1\sigma_2\text{-}\mathrm{Cl}(\sigma_1\sigma_2\text{-}\mathrm{Int}(\sigma_1\sigma_2\text{-}\mathrm{Cl}(B))))))\\
&\subseteq \sigma_1\sigma_2\text{-}\mathrm{Cl}(\sigma_1\sigma_2\text{-}\mathrm{Int}(\sigma_1\sigma_2\text{-}\mathrm{Cl}(B)))\\
&\subseteq \sigma_1\sigma_2\text{-}\mathrm{Cl}(B)
\end{aligned}$$

and hence $\mathrm{Cl}^\star(F^+(\sigma_1\sigma_2\text{-}\mathrm{Cl}(\sigma_1\sigma_2\text{-}\mathrm{Int}(\sigma_1\sigma_2\text{-}\mathrm{Cl}(B))))) \subseteq F^+(\sigma_1\sigma_2\text{-}\mathrm{Cl}(B))$. By Theorem 4, F is lower almost $\tau^\star(\sigma_1,\sigma_2)$-continuous.

(2) ⇒ (4): Let K be any $\sigma_1\sigma_2$-δ-closed set of Y. Then, $\sigma_1\sigma_2\text{-}\delta\text{-}\mathrm{Cl}(K) = K$. By (2), we have $\mathrm{Cl}^\star(F^+(K)) \subseteq F^+(\sigma_1\sigma_2\text{-}\delta\text{-}\mathrm{Cl}(K)) = F^+(K)$ and hence $F^+(K)$ is $\star$-closed in X.

(4) ⇒ (5): The proof is obvious.

(5) ⇒ (6): Let B be any subset of Y. Then by (5), we have

$$F^-(\sigma_1\sigma_2\text{-}\delta\text{-}\mathrm{Int}(B)) = \mathrm{Int}^\star(F^-(\sigma_1\sigma_2\text{-}\delta\text{-}\mathrm{Int}(B))) \subseteq \mathrm{Int}^\star(F^-(B)).$$

(6) ⇒ (1): Let V be any $(\sigma_1,\sigma_2)r$-open set of Y. Then, we have V is $\sigma_1\sigma_2$-δ-open and $\sigma_1\sigma_2\text{-}\delta\text{-}\mathrm{Int}(V) = V$. Thus by (6), $F^-(V) \subseteq \mathrm{Int}^\star(F^-(V))$ and hence $F^-(V)$ is $\star$-open in X. By Theorem 4, F is lower almost $\tau^\star(\sigma_1,\sigma_2)$-continuous.

Definition 3. [30] *A multifunction $F : (X,\tau,\mathscr{I}) \to (Y,\sigma_1,\sigma_2)$ is said to be upper $\tau^\star(\sigma_1,\sigma_2)$-continuous at a point $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y such that $F(x) \subseteq V$, there exists a $\star$-open set U of X containing x such that $F(U) \subseteq V$. A multifunction $F : (X,\tau,\mathscr{I}) \to (Y,\sigma_1,\sigma_2)$ is said to be upper $\tau^\star(\sigma_1,\sigma_2)$-continuous if F is upper $\tau^\star(\sigma_1,\sigma_2)$-continuous at each point x of X.*

Definition 4. [30] *A multifunction $F : (X,\tau,\mathscr{I}) \to (Y,\sigma_1,\sigma_2)$ is called lower $\tau^\star(\sigma_1,\sigma_2)$-continuous at a point $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists a $\star$-open set U of X containing x such that $F(z) \cap V \neq \emptyset$ for every $z \in U$. A multifunction $F : (X,\tau,\mathscr{I}) \to (Y,\sigma_1,\sigma_2)$ is called lower $\tau^\star(\sigma_1,\sigma_2)$-continuous if F is lower $\tau^\star(\sigma_1,\sigma_2)$-continuous at each point x of X.*

Remark 1. *For a multifunction $F : (X,\tau,\mathscr{I}) \to (Y,\sigma_1,\sigma_2)$, the following implication holds:*

upper $\tau^\star(\sigma_1,\sigma_2)$-continuity ⇒ upper almost $\tau^\star(\sigma_1,\sigma_2)$-continuity.

The converse of the implication is not true in general. We give an example for the implication as follows.

Example 1. *Let* $X = \{1, 2, 3\}$ *with a topology* $\tau = \{\emptyset, \{1\}, \{2\}, \{1, 2\}, X\}$ *and an ideal* $\mathscr{I} = \{\emptyset, \{1\}\}$*. Let* $Y = \{a, b, c\}$ *with topologies* $\sigma_1 = \{\emptyset, \{a\}, \{a, b\}, Y\}$ *and*

$$\sigma_2 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, Y\}.$$

A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ *is defined as follows:* $F(1) = \{c\}$ *and* $F(2) = F(3) = \{a, b\}$*. Then,* F *is upper almost* $\tau^{\star}(\sigma_1, \sigma_2)$*-continuous but* F *is not upper* $\tau^{\star}(\sigma_1, \sigma_2)$*-continuous.*

Definition 5. [31] *A bitopological space* (X, τ_1, τ_2) *is said to be* (τ_1, τ_2)*s-regular if for each* (τ_1, τ_2)*s-closed set* F *and each* $x \notin F$*, there exist disjoint* (τ_1, τ_2)*s-open sets* U *and* V *such that* $x \in U$ *and* $F \subseteq V$*.*

Lemma 6. [31] *Let* (X, τ_1, τ_2) *be a* (τ_1, τ_2)*s-regular space. Then, the following properties hold:*

(1) $\tau_1\tau_2\text{-}Cl(A) = \tau_1\tau_2\text{-}\delta\text{-}Cl(A)$ *for every subset* A *of* X*.*

(2) Every $\tau_1\tau_2$*-open set is* $\tau_1\tau_2$*-*δ*-open.*

Lemma 7. [31] *For a multifunction* $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$*, where* (Y, σ_1, σ_2) *is a* (σ_1, σ_2)*s-regular space, the following properties are equivalent:*

(1) F *is lower* $\tau^{\star}(\sigma_1, \sigma_2)$*-continuous;*

(2) $F^{+}(\sigma_1\sigma_2\text{-}\delta\text{-}Cl(B))$ *is* $\star$*-closed in* X *for every subset* B *of* Y*;*

(3) $F^{+}(K)$ *is* $\star$*-closed in* X *for every* $\sigma_1\sigma_2$*-*δ*-closed set* K *of* Y*;*

(4) $F^{-}(V)$ *is* $\star$*-open in* X *for every* $\sigma_1\sigma_2$*-*δ*-open set* V *of* Y*.*

Theorem 12. *For a multifunction* $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$*, where* (Y, σ_1, σ_2) *is a* (σ_1, σ_2)*s-regular space, the following properties are equivalent:*

(1) F *is lower* $\tau^{\star}(\sigma_1, \sigma_2)$*-continuous;*

(2) $F^{+}(\sigma_1\sigma_2\text{-}\delta\text{-}Cl(B))$ *is* $\star$*-closed in* X *for every subset* B *of* Y*;*

(3) $F^{+}(K)$ *is* $\star$*-closed in* X *for every* $\sigma_1\sigma_2$*-*δ*-closed set* K *of* Y*;*

(4) $F^{-}(V)$ *is* $\star$*-open in* X *for every* $\sigma_1\sigma_2$*-*δ*-open set* V *of* Y*;*

(5) F *is lower almost* $\tau^{\star}(\sigma_1, \sigma_2)$*-continuous.*

Proof. The proofs of the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are similar as in Lemma 7.

(4) $\Rightarrow$ (5): Let V be any $(\sigma_1, \sigma_2)r$-open set of Y. Then, V is $\sigma_1\sigma_2$-open in Y and by Lemma 6, V is $\sigma_1\sigma_2$-δ-open in Y. By (4), we have $F^{-}(V)$ is $\star$-open in X. Thus by Theorem 4, F is lower almost $\tau^{\star}(\sigma_1, \sigma_2)$-continuous.

(5) $\Rightarrow$ (1): Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y such that $V \cap F(x) \neq \emptyset$. Since (Y, σ_1, σ_2) is $(\sigma_1, \sigma_2)s$-regular, there exists a $(\sigma_1, \sigma_2)r$-open set W such that $W \cap F(x) \neq \emptyset$ and $W \subseteq V$. Since F is lower almost $\tau^\star(\sigma_1, \sigma_2)$-continuous, there exists a $\star$-open set U of X containing x such that $W \cap F(z) \neq \emptyset$ for every $z \in U$. Thus, $F(z) \cap V \neq \emptyset$ for every $z \in U$. This shows that F is lower $\tau^\star(\sigma_1, \sigma_2)$-continuous.

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