EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 3, Article Number 6567
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

On Weak Forms of Upper and Lower Continuous Multifunctions between an Ideal Topological Space and a Bitopological Space

Prapart Pue-on[1], Areeyuth Sama-Ae[2], Chawalit Boonpok[1,*]

[1] *Mathematics and Applied Mathematics Research Unit, Department of Mathematics, Faculty of Science, Mahasarakham University, Maha Sarakham, 44150, Thailand*
[2] *Department of Mathematics and Computer Science, Faculty of Science and Technology, Prince of Songkla University, Pattani Campus, Pattani, 94000, Thailand*

Abstract. This paper presents new concepts of continuous multifunctions defined from an ideal topological space into a bitopological space, called upper weakly $\tau^\star(\sigma_1,\sigma_2)$-continuous multifunctions and lower weakly $\tau^\star(\sigma_1,\sigma_2)$-continuous multifunctions. Furthermore, several characterizations and some properties concerning upper weakly $\tau^\star(\sigma_1,\sigma_2)$-continuous multifunctions and lower weakly $\tau^\star(\sigma_1,\sigma_2)$-continuous multifunctions are investigated. Moreover, the relationships between almost $\tau^\star(\sigma_1,\sigma_2)$-continuity and weak $\tau^\star(\sigma_1,\sigma_2)$-continuity are established.

2020 Mathematics Subject Classifications: 54C08, 54C60

Key Words and Phrases: Upper weakly $\tau^\star(\sigma_1,\sigma_2)$-continuous multifunction, lower weakly $\tau^\star(\sigma_1,\sigma_2)$-continuous multifunction

---

1. Introduction

The concept of weakly continuous functions was introduced by Levine [1]. Husain [2] introduced and studied the notion of almost continuous functions. Janković [3] introduced almost weak continuity as a generalization of both weak continuity and almost continuity. Noiri [4] investigated several characterizations of almost weakly continuous functions. Rose [5] introduced the notion of subweakly continuous functions and investigated the relationships between subweak continuity and weak continuity. Popa and Noiri [6] introduced the concept of weakly (τ, m)-continuous functions as functions from a topological space into a set satisfying some minimal conditions and investigated several characterizations of weakly (τ, m)-continuous functions. Ekici et al. [7] introduced and studied the concept of weakly λ-continuous functions. Popa [8] and Smithson [9] independently introduced the notion of

---

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v18i3.6567

Email addresses: prapart.p@msu.ac.th (P. Pue-on), areeyuth.s@psu.ac.th (A. Sama-Ae), chawalit.b@msu.ac.th (C. Boonpok)

weakly continuous multifunctions. Popa and Noiri [10] introduced a class of multifunctions called weakly α-continuous multifunctions. Furthermore, Popa and Noiri [11] investigated some characterizations of upper and lower weakly β-continuous multifunctions. Noiri and Popa [12] introduced and investigated the notion of weakly m-continuous multifunctions as a multifunction from a set satisfying certain minimal condition into a topological space. Pue-on et al. [13] introduced and studied the concepts of upper (τ_1, τ_2)-continuous multifunctions and lower (τ_1, τ_2)-continuous multifunctions. Klanarong et al. [14] introduced and investigated the notions of upper almost (τ_1, τ_2)-continuous multifunctions and lower almost (τ_1, τ_2)-continuous multifunctions. Thongmoon et al. [15] introduced and studied the concepts of upper weakly (τ_1, τ_2)-continuous multifunctions and lower weakly (τ_1, τ_2)-continuous multifunctions. The concept of ideal topological spaces was introduced and studied by Kuratowski [16] and Vaidyanathaswamy [17]. Weaker and stronger forms of open sets in ideal topological spaces such as semi-$\mathscr{I}$-open sets, pre-$\mathscr{I}$-open sets, α-$\mathscr{I}$-open sets, β-$\mathscr{I}$-open sets and δ-$\mathscr{I}$-open sets play an important role in the research of generalizations of continuity. Using these notions many authors introduced and studied various types of generalizations of continuity for functions and multifunctions. Hatir and Noiri [18] introduced and investigated the notions of weakly pre-$\mathscr{I}$-open sets and weakly pre-$\mathscr{I}$-continuous functions. Moreover, Hatir and Noiri [19] investigated further properties of semi-$\mathscr{I}$-open sets and semi-$\mathscr{I}$-continuous functions. On the other hand, the present author [20] introduced the notions of upper $\star$-continuous multifunctions and lower $\star$-continuous multifunctions. Furthermore, several characterizations of upper $\star$-continuous multifunctions, lower $\star$-continuous multifunctions, upper almost $\star$-continuous multifunctions, lower almost $\star$-continuous multifunctions, upper weakly $\star$-continuous multifunctions and lower weakly $\star$-continuous multifunctions were considered in [20]. Quite recently, the present author [21] introduced and studied the notions of $p\iota$-continuous multifunctions and weakly $p\iota$-continuous multifunctions. In this paper, we introduce new classes of multifunctions between an ideal topological space and a bitopological space, namely upper weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions and lower weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions. We also investigate several characterizations of upper weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions and lower weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions.

2. Preliminaries

Throughout the present paper, spaces (X, τ_1, τ_2) and (Y, σ_1, σ_2) (or simply X and Y) always mean bitopological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a bitopological space (X, τ_1, τ_2). The closure of A and the interior of A with respect to τ_i are denoted by τ_i-Cl(A) and τ_i-Int(A), respectively, for $i = 1, 2$. A subset A of a bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$-*closed* [22] if $A = \tau_1\text{-Cl}(\tau_2\text{-Cl}(A))$. The complement of a $\tau_1\tau_2$-closed set is called $\tau_1\tau_2$-*open*. The intersection of all $\tau_1\tau_2$-closed sets of X containing A is called the $\tau_1\tau_2$-*closure* [22] of A and is denoted by $\tau_1\tau_2$-Cl(A). The union of all $\tau_1\tau_2$-open sets of X contained in A is called the $\tau_1\tau_2$-*interior* [22] of A and is denoted by $\tau_1\tau_2$-Int(A).

Lemma 1. [22] *Let A and B be subsets of a bitopological space (X, τ_1, τ_2). For the $\tau_1\tau_2$-*

closure, the following properties hold:

(1) $A \subseteq \tau_1\tau_2$-*Cl*(A) *and* $\tau_1\tau_2$-*Cl*$(\tau_1\tau_2$-*Cl*$(A)) = \tau_1\tau_2$-*Cl*(A).

(2) If $A \subseteq B$, *then* $\tau_1\tau_2$-*Cl*$(A) \subseteq \tau_1\tau_2$-*Cl*$(B)$.

(3) $\tau_1\tau_2$-*Cl*(A) *is* $\tau_1\tau_2$-*closed.*

(4) A *is* $\tau_1\tau_2$-*closed if and only if* $A = \tau_1\tau_2$-*Cl*(A).

(5) $\tau_1\tau_2$-*Cl*$(X - A) = X - \tau_1\tau_2$-*Int*$(A)$.

A subset A of a bitopological space (X, τ_1, τ_2) is said to be $(\tau_1, \tau_2)r$-*open* [23] (resp. $(\tau_1, \tau_2)s$-*open* [24], $(\tau_1, \tau_2)p$-*open* [24], $(\tau_1, \tau_2)\beta$-*open* [24]) if $A = \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A))$ (resp. $A \subseteq \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(A))$, $A \subseteq \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A))$, $A \subseteq \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A)))$). The complement of a $(\tau_1, \tau_2)r$-open (resp. $(\tau_1, \tau_2)s$-open, $(\tau_1, \tau_2)p$-open, $(\tau_1, \tau_2)\beta$-open) set is said to be $(\tau_1, \tau_2)r$-*closed* (resp. $(\tau_1, \tau_2)s$-*closed*, $(\tau_1, \tau_2)p$-*closed*, $(\tau_1, \tau_2)\beta$-*closed*). A subset A of a bitopological space (X, τ_1, τ_2) is said to be $\alpha(\tau_1, \tau_2)$-*open* [25] if $A \subseteq \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(A)))$. The complement of an $\alpha(\tau_1, \tau_2)$-open set is said to be $\alpha(\tau_1, \tau_2)$-*closed*. Let A be a subset of a bitopological space (X, τ_1, τ_2). The intersection of all $(\tau_1, \tau_2)p$-closed (resp. $(\tau_1, \tau_2)s$-closed, $\alpha(\tau_1, \tau_2)$-closed) sets of X containing A is called the $(\tau_1, \tau_2)p$-*closure* [26] (resp. $(\tau_1, \tau_2)s$-*closure* [24], $\alpha(\tau_1, \tau_2)$-*closure* [27]) of A and is denoted by $(\tau_1, \tau_2)\text{-pCl}(A)$ (resp. $(\tau_1, \tau_2)\text{-sCl}(A)$, $\alpha(\tau_1, \tau_2)\text{-Cl}(A)$). The union of all $(\tau_1, \tau_2)p$-open (resp. $(\tau_1, \tau_2)s$-open, $\alpha(\tau_1, \tau_2)$-open) sets of X contained in A is called the $(\tau_1, \tau_2)p$-*interior* [26] (resp. $(\tau_1, \tau_2)s$-*interior* [24], $\alpha(\tau_1, \tau_2)$-*interior* [27]) of A and is denoted by $(\tau_1, \tau_2)\text{-pInt}(A)$ (resp. $(\tau_1, \tau_2)\text{-sInt}(A)$, $\alpha(\tau_1, \tau_2)\text{-Int}(A)$). For a subset A of a bitopological space (X, τ_1, τ_2), a point $x \in X$ is called a $(\tau_1, \tau_2)\theta$-*cluster point* of A if $\tau_1\tau_2\text{-Cl}(U) \cap A \neq \emptyset$ for every $\tau_1\tau_2$-open set U containing x. The set of all $(\tau_1, \tau_2)\theta$-cluster points of A is called the $(\tau_1, \tau_2)\theta$-*closure* of A and is denoted by $(\tau_1, \tau_2)\theta\text{-Cl}(A)$. A subset A of a bitopological space (X, τ_1, τ_2) is said to be $(\tau_1, \tau_2)\theta$-*closed* if $(\tau_1, \tau_2)\theta\text{-Cl}(A) = A$. The complement of a $(\tau_1, \tau_2)\theta$-closed set is said to be $(\tau_1, \tau_2)\theta$-*open*. The union of all $(\tau_1, \tau_2)\theta$-open sets of X contained in A is called the $(\tau_1, \tau_2)\theta$-*interior* of A and is denoted by $(\tau_1, \tau_2)\theta\text{-Int}(A)$ [23].

An ideal $\mathscr{I}$ on a topological space (X, τ) is a nonempty collection of subsets of X satisfying the following properties: (1) $A \in \mathscr{I}$ and $B \subseteq A$ imply $B \in \mathscr{I}$; (2) $A \in \mathscr{I}$ and $B \in \mathscr{I}$ imply $A \cup B \in \mathscr{I}$. A topological space (X, τ) with an ideal $\mathscr{I}$ on X is called an ideal topological space and is denoted by $(X, \tau, \mathscr{I})$. For an ideal topological space $(X, \tau, \mathscr{I})$ and a subset A of X, $A^\star(\mathscr{I})$ is defined as follows:

$$A^\star(\mathscr{I}) = \{x \in X : U \cap A \notin \mathscr{I} \text{ for every open neighbourhood } U \text{ of } x\}.$$

In case there is no chance for confusion, $A^\star(\mathscr{I})$ is simply written as $A^\star$. In [16], $A^\star$ is called the local function of A with respect to $\mathscr{I}$ and τ and $\text{Cl}^\star(A) = A^\star \cup A$ defines a Kuratowski closure operator for a topology $\tau^\star(\mathscr{I})$ finer than τ. A subset A is said to be $\star$-closed [3] if $A^\star \subseteq A$. The interior of a subset A in $(X, \tau^\star(\mathscr{I}))$ is denoted by $\text{Int}^\star(A)$. A subset A of an ideal topological space $(X, \tau, \mathscr{I})$ is said to be *semi**-$\mathscr{I}$-*open* [28] (resp. *semi*-$\mathscr{I}$-*open*

[19]) if $A \subseteq \text{Cl}(\text{Int}^\star(A))$ (resp. $A \subseteq \text{Cl}^\star(\text{Int}(A))$). The complement of a semi*-$\mathscr{I}$-open (resp. semi-$\mathscr{I}$-open) set is said to be *semi*-$\mathscr{I}$-closed* [28] (resp. semi-$\mathscr{I}$-closed [19]).

By a multifunction $F : X \to Y$, we mean a point-to-set correspondence from X into Y, and we always assume that $F(x) \neq \emptyset$ for all $x \in X$. For a multifunction $F : X \to Y$, we shall denote the upper and lower inverse of a set B of Y by $F^+(B)$ and $F^-(B)$, respectively, that is, $F^+(B) = \{x \in X \mid F(x) \subseteq B\}$ and $F^-(B) = \{x \in X \mid F(x) \cap B \neq \emptyset\}$. In particular, $F^-(y) = \{x \in X \mid y \in F(x)\}$ for each point $y \in Y$. For each $A \subseteq X$, $F(A) = \cup_{x\in A} F(x)$.

3. Upper and lower weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions

In this section, we introduce the concepts of upper weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions and lower weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions. Furthermore, several characterizations of upper weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions and lower weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions are discussed.

Definition 1. *A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is said to be upper weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous at a point $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y such that $F(x) \subseteq V$, there exists a $\star$-open set U of X containing x such that $F(U) \subseteq \sigma_1\sigma_2$-Cl$(V)$. A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is said to be upper weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous if F is upper weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous at each point x of X.*

Definition 2. [29] *A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is said to be:*

(i) *upper almost $\tau^\star(\sigma_1, \sigma_2)$-continuous if for each point $x \in X$ and each $\sigma_1\sigma_2$-open set V of Y such that $F(x) \subseteq V$, there exists a $\star$-open set U of X containing x such that $F(U) \subseteq \sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V))$;*

(ii) *lower almost $\tau^\star(\sigma_1, \sigma_2)$-continuous if for each point $x \in X$ and each $\sigma_1\sigma_2$-open set V of Y such that $V \cap F(x) \neq \emptyset$, there exists a $\star$-open set U of X containing x such that $\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V)) \cap F(z) \neq \emptyset$ for every $z \in U$.*

Remark 1. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following implication holds:*

upper almost $\tau^\star(\sigma_1, \sigma_2)$-continuity $\Rightarrow$ upper weakly $\tau^\star(\sigma_1, \sigma_2)$-continuity.

The converse of the implication is not true in general. We give an example for the implication as follows.

Example 1. *Let $X = \{1, 2, 3\}$ with a topology $\tau = \{\emptyset, \{2\}, \{1, 3\}, X\}$ and an ideal $\mathscr{I} = \{\emptyset, \{2\}\}$. Let $Y = \{a, b, c\}$ with topologies $\sigma_1 = \{\emptyset, \{a\}, \{a, b\}, Y\}$ and*

$$\sigma_2 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, Y\}.$$

A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is defined as follows: $F(1) = \{a\}$, $F(2) = \{b\}$ and $F(3) = \{a, c\}$. Then, F is upper weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous but F is not upper almost $\tau^\star(\sigma_1, \sigma_2)$-continuous.

Theorem 1. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is upper weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous;

(2) $F^+(V) \subseteq Int^\star(F^+(\sigma_1\sigma_2\text{-}Cl(V)))$ for every $\sigma_1\sigma_2$-open set V of Y;

(3) $Cl^\star(F^-(\sigma_1\sigma_2\text{-}Int(K))) \subseteq F^-(K)$ for every $\sigma_1\sigma_2$-closed set K of Y;

(4) $Cl^\star(F^-(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(B)))) \subseteq F^-(\sigma_1\sigma_2\text{-}Cl(B))$ for every subset B of Y;

(5) $F^+(\sigma_1\sigma_2\text{-}Int(B)) \subseteq Int^\star(F^+(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(B))))$ for every subset B of Y;

(6) $Cl^\star(F^-(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V)))) \subseteq F^-(\sigma_1\sigma_2\text{-}Cl(V))$ for every $\sigma_1\sigma_2$-open set V of Y;

(7) $Cl^\star(F^-(V)) \subseteq F^-(\sigma_1\sigma_2\text{-}Cl(V))$ for every $\sigma_1\sigma_2$-open set V of Y;

(8) $Cl^\star(F^-(\sigma_1\sigma_2\text{-}Int(K))) \subseteq F^-(K)$ for every $(\sigma_1, \sigma_2)r$-closed set K of Y.

Proof. (1) $\Rightarrow$ (2): Let V be any $\sigma_1\sigma_2$-open set of Y such that $x \in F^+(V)$. Then, we have $F(x) \subseteq V$ and by (1), there exists a $\star$-open set U of X containing x such that $F(U) \subseteq \sigma_1\sigma_2\text{-Cl}(V)$. Thus, $U \subseteq F^+(\sigma_1\sigma_2\text{-Cl}(V))$. Since U is $\star$-open, we have $x \in \text{Int}^\star(F^+(\sigma_1\sigma_2\text{-Cl}(V)))$ and so $F^+(V) \subseteq \text{Int}^\star(F^+(\sigma_1\sigma_2\text{-Cl}(V)))$.

(2) $\Rightarrow$ (3): Let K be any $\sigma_1\sigma_2$-closed set of Y. Then, $Y - K$ is $\sigma_1\sigma_2$-open in Y. By (2), $X - F^-(K) = F^+(Y - K) \subseteq \text{Int}^\star(F^+(\sigma_1\sigma_2\text{-Cl}(Y - K))) = X - \text{Cl}^\star(F^-(\sigma_1\sigma_2\text{-Int}(K)))$. Thus, $\text{Cl}^\star(F^-(\sigma_1\sigma_2\text{-Int}(K))) \subseteq F^-(K)$.

(3) $\Rightarrow$ (4): Let B be any subset of Y. Then, $\sigma_1\sigma_2\text{-Cl}(B)$ is a $\sigma_1\sigma_2$-closed set of Y and by (3), $\text{Cl}^\star(F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(B)))) \subseteq F^-(\sigma_1\sigma_2\text{-Cl}(B))$.

(4) $\Rightarrow$ (5): Let B be any subset of Y. Thus by (4), we have

$$\begin{aligned}
X - \text{Int}^\star(F^+(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(B)))) &= \text{Cl}^\star(X - F^+(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(B)))) \\
&= \text{Cl}^\star(F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(Y - B)))) \\
&\subseteq F^-(\sigma_1\sigma_2\text{-Cl}(Y - B)) \\
&= X - F^+(\sigma_1\sigma_2\text{-Int}(B))
\end{aligned}$$

and hence $F^+(\sigma_1\sigma_2\text{-Int}(B)) \subseteq \text{Int}^\star(F^+(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(B))))$.

(5) $\Rightarrow$ (1): Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y such that $F(x) \subseteq V$. By (5), $x \in F^+(V) \subseteq \text{Int}^\star(F^+(\sigma_1\sigma_2\text{-Cl}(V)))$ and there exists a $\star$-open set U of X containing x such that $U \subseteq F^+(\sigma_1\sigma_2\text{-Cl}(V))$. Thus, $F(U) \subseteq \sigma_1\sigma_2\text{-Cl}(V)$ and hence F is upper weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous.

(4) $\Rightarrow$ (6) and (6) $\Rightarrow$ (7): The proofs are obvious.

(7) $\Rightarrow$ (8): Let K be any $(\sigma_1, \sigma_2)r$-closed set of Y. Thus by (7),

$$\text{Cl}^\star(F^-(\sigma_1\sigma_2\text{-Int}(K))) \subseteq F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K))) = F^-(K).$$

(8) $\Rightarrow$ (3): Let K be any $\sigma_1\sigma_2$-closed set of Y. Then, $\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K))$ is $(\sigma_1, \sigma_2)r$-closed in Y and $\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K))) = \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(K)) = \sigma_1\sigma_2\text{-Int}(K)$. By (8),

$$\begin{aligned}\text{Cl}^\star(F^-(\sigma_1\sigma_2\text{-Int}(K))) &= \text{Cl}^\star(F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K))))) \\ &\subseteq F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K))) \\ &\subseteq F^-(K).\end{aligned}$$

Definition 3. *A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is said to be lower weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous at a point $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y such that*

$$V \cap F(x) \neq \emptyset,$$

there exists a $\star$-open set U of X containing x such that $\sigma_1\sigma_2$-Cl$(V) \cap F(z) \neq \emptyset$ for every $z \in U$. A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is said to be lower weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous if F is lower weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous at each point x of X.

Theorem 2. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is lower weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous;

(2) $F^-(V) \subseteq \text{Int}^\star(F^-(\sigma_1\sigma_2\text{-Cl}(V)))$ for every $\sigma_1\sigma_2$-open set V of Y;

(3) $\text{Cl}^\star(F^+(\sigma_1\sigma_2\text{-Int}(K))) \subseteq F^+(K)$ for every $\sigma_1\sigma_2$-closed set K of Y;

(4) $\text{Cl}^\star(F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(B)))) \subseteq F^+(\sigma_1\sigma_2\text{-Cl}(B))$ for every subset B of Y;

(5) $F^-(\sigma_1\sigma_2\text{-Int}(B)) \subseteq \text{Int}^\star(F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(B))))$ for every subset B of Y;

(6) $\text{Cl}^\star(F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) \subseteq F^+(\sigma_1\sigma_2\text{-Cl}(V))$ for every $\sigma_1\sigma_2$-open set V of Y;

(7) $\text{Cl}^\star(F^+(V)) \subseteq F^+(\sigma_1\sigma_2\text{-Cl}(V))$ for every $\sigma_1\sigma_2$-open set V of Y;

(8) $\text{Cl}^\star(F^+(\sigma_1\sigma_2\text{-Int}(K))) \subseteq F^+(K)$ for every $(\sigma_1, \sigma_2)r$-closed set K of Y.

Proof. The proof is similar to that of Theorem 1.

Theorem 3. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is upper weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous;

(2) $\text{Cl}^\star(F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) \subseteq F^-(\sigma_1\sigma_2\text{-Cl}(V))$ for every $(\sigma_1, \sigma_2)\beta$-open set V of Y;

(3) $\text{Cl}^\star(F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) \subseteq F^-(\sigma_1\sigma_2\text{-Cl}(V))$ for every $(\sigma_1, \sigma_2)s$-open set V of Y.

Proof. (1) $\Rightarrow$ (2): This follows from (4) of Theorem 1.

(2) $\Rightarrow$ (3): The proof is obvious since every $(\sigma_1, \sigma_2)s$-open set is $(\sigma_1, \sigma_2)\beta$-open.

(3) $\Rightarrow$ (1): Since every $\sigma_1\sigma_2$-open set is $(\sigma_1, \sigma_2)s$-open, the proof is obvious by (7) of Theorem 1.

Theorem 4. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is lower weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous;

(2) $\mathrm{Cl}^\star(F^+(\sigma_1\sigma_2\text{-}\mathrm{Int}(\sigma_1\sigma_2\text{-}\mathrm{Cl}(V)))) \subseteq F^+(\sigma_1\sigma_2\text{-}\mathrm{Cl}(V))$ for every $(\sigma_1, \sigma_2)\beta$-open set V of Y;

(3) $\mathrm{Cl}^\star(F^+(\sigma_1\sigma_2\text{-}\mathrm{Int}(\sigma_1\sigma_2\text{-}\mathrm{Cl}(V)))) \subseteq F^+(\sigma_1\sigma_2\text{-}\mathrm{Cl}(V))$ for every $(\sigma_1, \sigma_2)s$-open set V of Y.

Proof. The proof is similar to that of Theorem 3.

Theorem 5. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is upper weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous;

(2) $\mathrm{Cl}^\star(F^-(\sigma_1\sigma_2\text{-}\mathrm{Int}(\sigma_1\sigma_2\text{-}\mathrm{Cl}(V)))) \subseteq F^-(\sigma_1\sigma_2\text{-}\mathrm{Cl}(V))$ for every $(\sigma_1, \sigma_2)p$-open set V of Y;

(3) $\mathrm{Cl}^\star(F^-(V)) \subseteq F^-(\sigma_1\sigma_2\text{-}\mathrm{Cl}(V))$ for every $(\sigma_1, \sigma_2)p$-open set V of Y;

(4) $F^+(V) \subseteq \mathrm{Int}^\star(F^+(\sigma_1\sigma_2\text{-}\mathrm{Cl}(V)))$ for every $(\sigma_1, \sigma_2)p$-open set V of Y.

Proof. (1) $\Rightarrow$ (2): Let V be any $(\sigma_1, \sigma_2)p$-open set of Y. Since $\sigma_1\sigma_2\text{-}\mathrm{Int}(\sigma_1\sigma_2\text{-}\mathrm{Cl}(V))$ is $\sigma_1\sigma_2$-open, by Theorem 1(7)

$$\begin{aligned}\mathrm{Cl}^\star(F^-(\sigma_1\sigma_2\text{-}\mathrm{Int}(\sigma_1\sigma_2\text{-}\mathrm{Cl}(V)))) &\subseteq F^-(\sigma_1\sigma_2\text{-}\mathrm{Cl}(\sigma_1\sigma_2\text{-}\mathrm{Int}(\sigma_1\sigma_2\text{-}\mathrm{Cl}(V)))) \\ &\subseteq F^-(\sigma_1\sigma_2\text{-}\mathrm{Cl}(V)).\end{aligned}$$

(2) $\Rightarrow$ (3): Let V be any $(\sigma_1, \sigma_2)p$-open set of Y. By (2), we have

$$\mathrm{Cl}^\star(F^-(V)) \subseteq \mathrm{Cl}^\star(F^-(\sigma_1\sigma_2\text{-}\mathrm{Int}(\sigma_1\sigma_2\text{-}\mathrm{Cl}(V)))) \subseteq F^-(\sigma_1\sigma_2\text{-}\mathrm{Cl}(V)).$$

(3) $\Rightarrow$ (4): Let V be any $(\sigma_1, \sigma_2)p$-open set of Y. Thus by (3),

$$\begin{aligned}X - \mathrm{Int}^\star(F^+(\mathrm{Cl}^\star(V))) &= \mathrm{Cl}^\star(X - F^+(\mathrm{Cl}^\star(V))) \\ &= \mathrm{Cl}^\star(F^-(Y - \mathrm{Cl}^\star(V))) \\ &\subseteq F^-(\sigma_1\sigma_2\text{-}\mathrm{Cl}(Y - \sigma_1\sigma_2\text{-}\mathrm{Cl}(V))) \\ &= X - F^+(\sigma_1\sigma_2\text{-}\mathrm{Int}(\sigma_1\sigma_2\text{-}\mathrm{Cl}(V)))\end{aligned}$$

$$\subseteq X - F^{+}(V)$$

and hence $F^{+}(V) \subseteq \text{Int}^{\star}(F^{+}(\sigma_1\sigma_2\text{-Cl}(V)))$.

(4) $\Rightarrow$ (1): Let V be any $\sigma_1\sigma_2$-open set of Y. Then, V is $(\sigma_1, \sigma_2)p$-open in Y and by (4), $F^{+}(V) \subseteq \text{Int}^{\star}(F^{+}(\sigma_1\sigma_2\text{-Cl}(V)))$. By Theorem 1(2), F is upper weakly $\tau^{\star}(\sigma_1, \sigma_2)$-continuous.

Theorem 6. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is lower weakly $\tau^{\star}(\sigma_1, \sigma_2)$-continuous;

(2) $\mathit{Cl}^{\star}(F^{+}(\sigma_1\sigma_2\text{-}\mathit{Int}(\sigma_1\sigma_2\text{-}\mathit{Cl}(V)))) \subseteq F^{+}(\sigma_1\sigma_2\text{-}\mathit{Cl}(V))$ for every $(\sigma_1, \sigma_2)p$-open set V of Y;

(3) $\mathit{Cl}^{\star}(F^{+}(V)) \subseteq F^{+}(\sigma_1\sigma_2\text{-}\mathit{Cl}(V))$ for every $(\sigma_1, \sigma_2)p$-open set V of Y;

(4) $F^{-}(V) \subseteq \mathit{Int}^{\star}(F^{-}(\sigma_1\sigma_2\text{-}\mathit{Cl}(V)))$ for every $(\sigma_1, \sigma_2)p$-open set V of Y.

Proof. The proof is similar to that of Theorem 5.

Lemma 2. *If $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is lower weakly $\tau^{\star}(\sigma_1, \sigma_2)$-continuous, then for each $x \in X$ and each subset B of Y with $(\sigma_1, \sigma_2)\theta\text{-}\mathit{Int}(B) \cap F(x) \neq \emptyset$, there exists a $\star$-open set U of X containing x such that $U \subseteq F^{-}(B)$.*

Proof. Since $(\sigma_1, \sigma_2)\theta\text{-Int}(B) \cap F(x) \neq \emptyset$, there exists a nonempty $\sigma_1\sigma_2$-open set V of Y such that $\sigma_1\sigma_2\text{-Cl}(V) \subseteq B$ and $V \cap F(x) \neq \emptyset$. Since F is lower weakly $\tau^{\star}(\sigma_1, \sigma_2)$-continuous, there exists a $\star$-open set U of X containing x such that $\sigma_1\sigma_2\text{-Cl}(V) \cap F(z) \neq \emptyset$ for each $z \in U$ and hence $U \subseteq F^{-}(B)$.

Theorem 7. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is lower weakly $\tau^{\star}(\sigma_1, \sigma_2)$-continuous;

(2) $\mathit{Cl}^{\star}(F^{+}(B)) \subseteq F^{+}((\sigma_1, \sigma_2)\theta\text{-}\mathit{Cl}(B))$ for every subset B of Y;

(3) $F(\mathit{Cl}^{\star}(A)) \subseteq (\sigma_1, \sigma_2)\theta\text{-}\mathit{Cl}(F(A))$ for every subset A of X.

Proof. (1) $\Rightarrow$ (2): Let B be any subset of Y. Suppose that $x \notin F^{+}((\sigma_1, \sigma_2)\theta\text{-Cl}(B))$. Then, we have $x \in F^{-}(Y - (\sigma_1, \sigma_2)\theta\text{-Cl}(B)) = F^{-}((\sigma_1, \sigma_2)\theta\text{-Int}(Y - B))$. By Lemma 3, there exists a $\star$-open set U of X containing x such that $U \subseteq F^{-}(Y - B) = X - F^{+}(B)$. Thus, $U \cap F^{+}(B) = \emptyset$ and hence $x \notin \text{Cl}^{\star}(F^{+}(B))$. This shows that

$$\text{Cl}^{\star}(F^{+}(B)) \subseteq F^{+}((\sigma_1, \sigma_2)\theta\text{-Cl}(B)).$$

(2) ⇒ (3): Let A be any subset of X. By (2), we have

$$\mathrm{Cl}^\star(A) \subseteq \mathrm{Cl}^\star(F^+(F(A))) \subseteq F^+((\sigma_1,\sigma_2)\theta\text{-}\mathrm{Cl}(F(A))).$$

Thus, $F(\mathrm{Cl}^\star(A)) \subseteq (\sigma_1,\sigma_2)\theta\text{-}\mathrm{Cl}(F(A))$.

(3) ⇒ (1): Let V be any $\sigma_1\sigma_2$-open set of Y. Then, $\sigma_1\sigma_2\text{-}\mathrm{Cl}(V) = (\sigma_1,\sigma_2)\theta\text{-}\mathrm{Cl}(V)$ and by (3), $F(\mathrm{Cl}^\star(F^+(V))) \subseteq (\sigma_1,\sigma_2)\theta\text{-}\mathrm{Cl}(F(F^+(V))) \subseteq (\sigma_1,\sigma_2)\theta\text{-}\mathrm{Cl}(V) = \sigma_1\sigma_2\text{-}\mathrm{Cl}(V)$. Thus, $\mathrm{Cl}^\star(F^+(V)) \subseteq F^+(\sigma_1\sigma_2\text{-}\mathrm{Cl}(V))$ and by Theorem 1, F is lower weakly $\tau^\star(\sigma_1,\sigma_2)$-continuous.

Theorem 8. *For a multifunction $F : (X,\tau,\mathscr{I}) \to (Y,\sigma_1,\sigma_2)$, the following properties are equivalent:*

(1) F is upper weakly $\tau^\star(\sigma_1,\sigma_2)$-continuous;

(2) $Cl^\star(F^-(\sigma_1\sigma_2\text{-}Int((\sigma_1,\sigma_2)\theta\text{-}Cl(B)))) \subseteq F^-((\sigma_1,\sigma_2)\theta\text{-}Cl(B))$ for every subset B of Y;

(3) $Cl^\star(F^-(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(B)))) \subseteq F^-((\sigma_1,\sigma_2)\theta\text{-}Cl(B))$ for every subset B of Y.

Proof. (1) ⇒ (2): Let B be any subset of Y. Then, $(\sigma_1,\sigma_2)\theta\text{-}\mathrm{Cl}(B)$ is $\sigma_1\sigma_2$-closed in Y and by Theorem 2, $\mathrm{Cl}^\star(F^-(\sigma_1\sigma_2\text{-}\mathrm{Int}((\sigma_1,\sigma_2)\theta\text{-}\mathrm{Cl}(B)))) \subseteq F^-((\sigma_1,\sigma_2)\theta\text{-}\mathrm{Cl}(B))$.

(2) ⇒ (3): The proof is obvious.

(3) ⇒ (1): Let K be any $(\sigma_1,\sigma_2)r$-closed set of Y. Then, we have

$$(\sigma_1,\sigma_2)\theta\text{-}\mathrm{Cl}(\sigma_1\sigma_2\text{-}\mathrm{Int}(K)) = \sigma_1\sigma_2\text{-}\mathrm{Cl}(\sigma_1\sigma_2\text{-}\mathrm{Int}(K)) = K$$

and hence

$$\begin{aligned}\mathrm{Cl}^\star(F^-(\sigma_1\sigma_2\text{-}\mathrm{Int}(K))) &= \mathrm{Cl}^\star(\sigma_1\sigma_2\text{-}\mathrm{Int}(\sigma_1\sigma_2\text{-}\mathrm{Cl}(\sigma_1\sigma_2\text{-}\mathrm{Int}(K))))\\ &\subseteq F^-((\sigma_1,\sigma_2)\theta\text{-}\mathrm{Cl}(\sigma_1\sigma_2\text{-}\mathrm{Int}(K)))\\ &= F^-(\sigma_1\sigma_2\text{-}\mathrm{Cl}(\sigma_1\sigma_2\text{-}\mathrm{Int}(K)))\\ &= F^-(K).\end{aligned}$$

Thus by Theorem 2, F is upper weakly $\tau^\star(\sigma_1,\sigma_2)$-continuous.

Theorem 9. *For a multifunction $F : (X,\tau,\mathscr{I}) \to (Y,\sigma_1,\sigma_2)$, the following properties are equivalent:*

(1) F is lower weakly $\tau^\star(\sigma_1,\sigma_2)$-continuous;

(2) $Cl^\star(F^+(\sigma_1\sigma_2\text{-}Int((\sigma_1,\sigma_2)\theta\text{-}Cl(B)))) \subseteq F^+((\sigma_1,\sigma_2)\theta\text{-}Cl(B))$ for every subset B of Y;

(3) $Cl^\star(F^+(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(B)))) \subseteq F^+((\sigma_1,\sigma_2)\theta\text{-}Cl(B))$ for every subset B of Y.

Proof. The proof is similar to that of Theorem 10.

Recall that a subset A of a bitopological space (X, τ_1, τ_2) is said to be $\tau_1\tau_2$-*paracompact* [22] if every cover of A by $\tau_1\tau_2$-open sets of X is refined by a cover of A which consists of $\tau_1\tau_2$-open sets of X and is $\tau_1\tau_2$-locally finite in X. A subset A of a bitopological space (X, τ_1, τ_2) is said to be $\tau_1\tau_2$-*regular* [22] if for each $x \in A$ and each $\tau_1\tau_2$-open set U of X containing x, there exists a $\tau_1\tau_2$-open set V of X such that $x \in V \subseteq \tau_1\tau_2\text{-Cl}(V) \subseteq U$.

Lemma 3. [22] *If A is a $\tau_1\tau_2$-regular $\tau_1\tau_2$-paracompact set of a bitopological space (X, τ_1, τ_2) and U is a $\tau_1\tau_2$-open neighbourhood of A, then there exists a $\tau_1\tau_2$-open set V of X such that $A \subseteq V \subseteq \tau_1\tau_2\text{-}Cl(V) \subseteq U$.*

Theorem 10. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ such that $F(x)$ is a $\sigma_1\sigma_2$-regular $\sigma_1\sigma_2$-paracompact set of Y for each point $x \in X$, the following properties are equivalent:*

(1) F is upper $\tau^\star(\sigma_1, \sigma_2)$-continuous;

(2) F is upper almost $\tau^\star(\sigma_1, \sigma_2)$-continuous;

(3) F is upper weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous.

Proof. We show only the implication (3) $\Rightarrow$ (1) since the others are obvious. Suppose that F is upper weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous. Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y such that $F(x) \subseteq V$. Since $F(x)$ is $\sigma_1\sigma_2$-regular $\sigma_1\sigma_2$-paracompact, by Lemma 3 there exists a $\sigma_1\sigma_2$-open set W of Y such that $F(x) \subseteq W \subseteq \sigma_1\sigma_2\text{-Cl}(W) \subseteq V$. Since F is upper weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous, there exists a $\star$-open set U of X containing x such that $F(U) \subseteq \sigma_1\sigma_2\text{-Cl}(W)$; hence $F(U) \subseteq V$. This shows that F is upper $\tau^\star(\sigma_1, \sigma_2)$-continuous.

Recall that a bitopological space (X, τ_1, τ_2) is said to be $\tau_1\tau_2$-*compact* [22] if every cover of X by $\tau_1\tau_2$-open sets of X has a finite subcover.

Definition 4. [30] *A bitopological space (X, τ_1, τ_2) is said to be (τ_1, τ_2)-regular if for each $\tau_1\tau_2$-closed set F and each $x \in X - F$, there exist disjoint $\tau_1\tau_2$-open sets U and V such that $x \in U$ and $F \subseteq V$.*

Corollary 1. *Let $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ be a multifunction such that $F(x)$ is $\sigma_1\sigma_2$-compact for each point $x \in X$ and (Y, σ_1, σ_2) is (σ_1, σ_2)-regular. Then, the following properties are equivalent:*

(1) F is upper $\tau^\star(\sigma_1, \sigma_2)$-continuous;

(2) F is upper almost $\tau^\star(\sigma_1, \sigma_2)$-continuous;

(3) F is upper weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous.

Lemma 4. [31] *If A is a $\tau_1\tau_2$-regular set of a bitopological space (X, τ_1, τ_2), then for each $\tau_1\tau_2$-open set G which intersect A, there exists a $\tau_1\tau_2$-open set W such that $A \cap W \neq \emptyset$ and $\tau_1\tau_2\text{-}Cl(W) \subseteq G$.*

Theorem 11. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ such that $F(x)$ is a $\sigma_1\sigma_2$-regular set of Y for each $x \in X$, the following properties are equivalent:*

(1) F is lower $\tau^\star(\sigma_1, \sigma_2)$-continuous;

(2) F is lower almost $\tau^\star(\sigma_1, \sigma_2)$-continuous;

(3) F is lower weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous.

Proof. We show only the implication (3) $\Rightarrow$ (1) since the others are obvious. Suppose that F is lower weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous. Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y such that $V \cap F(x) \neq \emptyset$. Since $F(x)$ is $\sigma_1\sigma_2$-regular, by Lemma 4 there exists a $\sigma_1\sigma_2$-open set W of Y such that $F(x) \cap W \neq \emptyset$ and $\sigma_1\sigma_2$-Cl$(W) \subseteq V$. Since F is lower weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous, there exists a $\star$-open set U of X containing x such that $\sigma_1\sigma_2$-Cl$(W) \cap F(z) \neq \emptyset$; hence $F(z) \cap V \neq \emptyset$ for each $z \in U$. This shows that F is lower $\tau^\star(\sigma_1, \sigma_2)$-continuous.

Definition 5. [32] *A bitopological space (X, τ_1, τ_2) is said to be (τ_1, τ_2)-normal if for each pair of disjoint $\tau_1\tau_2$-closed sets F and F', there exist disjoint $\tau_1\tau_2$-open sets U and V such that $F \subseteq U$ and $F' \subseteq V$.*

Theorem 12. *If $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is a multifunction such that $F(x)$ is $\sigma_1\sigma_2$-closed in Y for each $x \in X$ and (Y, σ_1, σ_2) is a (σ_1, σ_2)-normal space, then the following properties are equivalent:*

(1) F is upper $\tau^\star(\sigma_1, \sigma_2)$-continuous;

(2) F is upper almost $\tau^\star(\sigma_1, \sigma_2)$-continuous;

(3) F is upper weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous.

Proof. As in Theorem 10, we prove only the implication (3) $\Rightarrow$ (1). Suppose that F is upper weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous. Let $x \in X$ and G be any $\sigma_1\sigma_2$-open set of Y containing $F(x)$. Since $F(x)$ is $\sigma_1\sigma_2$-closed in Y, by the (σ_1, σ_2)-normality of Y there exists a $\sigma_1\sigma_2$-open set V of Y such that $F(x) \subseteq V \subseteq \sigma_1\sigma_2$-Cl$(V) \subseteq G$. Since F is upper weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous, there exists a $\star$-open set U of X containing x such that $F(U) \subseteq \sigma_1\sigma_2$-Cl$(V) \subseteq G$. This shows that F is upper $\tau^\star(\sigma_1, \sigma_2)$-continuous.

Acknowledgements

This research project was financially supported by Mahasarakham University.

References

[1] N. Levine. A decomposition of continuity in topological spaces. *The American Mathematical Monthly*, 68:44–46, 1961.

[2] T. Husain. Almost continuous mappings. *Prace Matematyczne*, 10:1–7, 1966.

[3] D. S. Janković. θ-regular spaces. *International Journal of Mathematics and Mathematical Sciences*, 8:615–619, 1985.

[4] T. Noiri. Properties of some weak forms of continuity. *International Journal of Mathematics and Mathematical Sciences*, 10(1):97–111, 1987.

[5] D. A. Rose. Weak continuity and almost continuity. *International Journal of Mathematics and Mathematical Sciences*, 7:311–318, 1984.

[6] V. Popa and T. Noiri. On weakly (τ, m)-continuous functions. *Rendiconti del Circolo Matematico di Palermo Series 2*, 51:295–316, 2002.

[7] E. Ekici, S. Jafari, M. Caldas, and T. Noiri. Weakly λ-continuous functions. *Novi Sad Journal of Mathematics*, 38:47–56, 2008.

[8] V. Popa. Weakly continuous multifunctions. *Bollettino dell' Unione Matematica Italiana (5)*, 15(A):379–388, 1978.

[9] R. E. Smithson. Almost and weak continuity for multifunctions. *Bulletin of the Calcutta Mathematical Society*, 70:383–390, 1978.

[10] V. Popa and T. Noiri. On upper and lower weakly α-continuous multifunctions. *Novi Sad Journal of Mathematics*, 32(1):7–24, 2002.

[11] V. Popa and T. Noiri. On upper and lower weakly β-continuous multifunctions. *Annales Universitatis Scientiarum Budapestinensis de Rolando Eötvös Nominatae Sectio Mathematica*, 43:25–48, 2000.

[12] T. Noiri and V. Popa. A unified theory of weak continuity for multifunctions. *Studii şi Cercetări Ştiinţific. Seria: Matematică. Universitatea din Bacău*, 16:167–200, 2006.

[13] P. Pue-on, S. Sompong, and C. Boonpok. Upper and lower (τ_1, τ_2)-continuous multifunctions. *International Journal of Mathematics and Computer Science*, 19(4):1305–1310, 2024.

[14] C. Klanarong, S. Sompong, and C. Boonpok. Upper and lower almost (τ_1, τ_2)-continuous multifunctions. *European Journal of Pure and Applied Mathematics*, 17(2):1244–1253, 2024.

[15] M. Thongmoon, S. Sompong, and C. Boonpok. Upper and lower weak (τ_1, τ_2)-continuity. *European Journal of Pure and Applied Mathematics*, 17(3):1705–1716, 2024.

[16] K. Kuratowski. *Topology, Vol. I.* Academic Press, New York, 1966.

[17] R. Vaidyanathaswamy. The localisation theory in set-topology. *Proceedings of the Indian Academy of Sciences-Section A*, 20:51–61, 1944.

[18] E. Hatir and T. Noiri. Weakly pre-I-open sets and decomposition of continuity. *Acta Mathematica Hungarica*, 106(3):227–238, 2005.

[19] E. Hatir and T. Noiri. On decompositions of continuity via idealization. *Acta Mathematica Hungarica*, 96:341–349, 2002.

[20] C. Boonpok. On continuous multifunctions in ideal topological spaces. *Lobachevskii*

Journal of Mathematics, 40(1):24–35, 2019.

[21] C. Boonpok. $p\imath$-continuity and weak $p\imath$-continuity. *Carpathian Mathematical Publications*, 17(1):171–186, 2025.

[22] C. Boonpok, C. Viriyapong, and M. Thongmoon. On upper and lower (τ_1, τ_2)-precontinuous multifunctions. *Journal of Mathematics and Computer Science*, 18:282–293, 2018.

[23] C. Viriyapong and C. Boonpok. $(\tau_1, \tau_2)\alpha$-continuity for multifunctions. *Journal of Mathematics*, 2020:6285763, 2020.

[24] C. Boonpok. $(\tau_1, \tau_2)\delta$-semicontinuous multifunctions. *Heliyon*, 6:e05367, 2020.

[25] N. Viriyapong, S. Sompong, and C. Boonpok. (τ_1, τ_2)-extremal disconnectedness in bitopological spaces. *International Journal of Mathematics and Computer Science*, 19(3):855–860, 2024.

[26] N. Viriyapong, S. Sompong, and C. Boonpok. Upper and lower s-$(\tau_1, \tau_2)p$-continuous multifunctions. *European Journal of Pure and Applied Mathematics*, 17(3):2210–2220, 2024.

[27] C. Viriyapong, S. Sompong, and C. Boonpok. Upper and lower slight $\alpha(\tau_1, \tau_2)$-continuity. *European Journal of Pure and Applied Mathematics*, 17(3):2142–2154, 2024.

[28] E. Ekici, , and T. Noiri. $\star$-extremally disconnected ideal topological spaces. *Acta Mathematica Hungarica*, 122:81–90, 2009.

[29] C. Viriyapong, A. Sama-Ae, and C. Boonpok. Almost continuity for multifunctions defined from an ideal topological space into a bitopological space. (submitted).

[30] M. Chiangpradit, S. Sompong, and C. Boonpok. On characterizations of (τ_1, τ_2)-regular spaces. *International Journal of Mathematics and Computer Science*, 19(4):1329–1334, 2024.

[31] B. Kong-ied, S. Sompong, and C. Boonpok. (τ_1, τ_2)-continuity and weak (τ_1, τ_2)-continuity. *International Journal of Mathematics and Computer Science*, 19(4):1321–1327, 2024.

[32] M. Chiangpradit, S. Sompong, and C. Boonpok. On characterizations of (τ_1, τ_2)-normal spaces. *International Journal of Mathematics and Computer Science*, 19(4):1315–1320, 2024.