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On Almost $\tau^{\star}(\sigma_1, \sigma_2)$-Continuity and Weak $\tau^{\star}(\sigma_1, \sigma_2)$-Continuity

Nongluk Viriyapong[1], Areeyuth Sama-Ae[2], Chawalit Boonpok[1,*]

[1] *Mathematics and Applied Mathematics Research Unit, Department of Mathematics, Faculty of Science, Mahasarakham University, Maha Sarakham, 44150, Thailand*
[2] *Department of Mathematics and Computer Science, Faculty of Science and Technology, Prince of Songkla University, Pattani Campus, Pattani, 94000, Thailand*

Abstract. This paper is concerned with the concepts of almost $\tau^{\star}(\sigma_1, \sigma_2)$-continuous functions and weakly $\tau^{\star}(\sigma_1, \sigma_2)$-continuous functions. Moreover, some characterizations of almost $\tau^{\star}(\sigma_1, \sigma_2)$-continuous functions and weakly $\tau^{\star}(\sigma_1, \sigma_2)$-continuous functions are investigated. Furthermore, the relationships between almost $\tau^{\star}(\sigma_1, \sigma_2)$-continuity and weak $\tau^{\star}(\sigma_1, \sigma_2)$-continuity are considered.

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1. Introduction

In 1968, Singal and Singal [1] introduced the concept of almost continuous functions as a generalization of continuity. Munshi and Bassan [2] studied the notion of almost semi-continuous functions. Noiri [3] introduced and investigated the concept of almost α-continuous functions. Nasef and Noiri [4] introduced two classes of functions, namely almost precontinuous functions and almost β-continuous functions. The class of almost precontinuity is a generalization of almost α-continuity. The class of almost β-continuity is a generalization of almost semi-continuity. Levine [5] introduced and investigated the concept of weakly continuous functions. Husain [6] introduced and studied the notion of almost continuous functions. Janković [7] introduced almost weak continuity as a generalization of both weak continuity and almost continuity. Noiri [8] investigated several characterizations of almost weakly continuous functions. Rose [9] introduced the notion of subweakly continuous functions and investigated the relationships between subweak continuity and weak continuity. Popa and Noiri [10] introduced the concept of weakly

*Corresponding author.
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Email addresses: nongluk.h@msu.ac.th (N. Viriyapong), areeyuth.s@psu.ac.th (A. Sama-Ae), chawalit.b@msu.ac.th (C. Boonpok)

(τ, m)-continuous functions as functions from a topological space into a set satisfying some minimal conditions and investigated several characterizations of weakly (τ, m)-continuous functions. Ekici et al. [11] introduced and studied the concept of weakly λ-continuous functions. In 1992, Abd El-Monsef et al. [12] introduced and studied the notions of $\mathscr{I}$-closed sets and $\mathscr{I}$-continuous functions. Semi-$\mathscr{I}$-open sets, pre-$\mathscr{I}$-open sets, α-$\mathscr{I}$-open sets, β-$\mathscr{I}$-open sets and δ-$\mathscr{I}$-open sets play an important role in the research of generalizations of continuity. Using these notions many authors introduced and studied various types of generalizations of continuity for functions and multifunctions. Hatir and Noiri [13] introduced and investigated the notions of weakly pre-$\mathscr{I}$-open sets and weakly pre-$\mathscr{I}$-continuous functions. Moreover, Hatir and Noiri [14] investigated further properties of semi-$\mathscr{I}$-open sets and semi-$\mathscr{I}$-continuous functions [15]. On the other hand, the present author introduced and studied the concepts of $\star$-continuous functions [16], θ-$\mathscr{I}$-continuous functions [17], weakly $\star$-continuous functions [18], $\theta(\star)$-continuous functions [18], almost $\star$-precontinuous functions [19], weakly $\star$-precontinuous functions [19], $p\imath$-continuous functions [20] and weakly $p\imath$-continuous functions [20]. Recently, Boonpok and Srisarakham [21] introduced and investigated the notion of (τ_1, τ_2)-continuous functions. Furthermore, some characterizations of almost (τ_1, τ_2)-continuous functions and weakly (τ_1, τ_2)-continuous functions were established in [22] and [23], respectively. In this paper, we introduce the concepts of almost $\tau^\star(\sigma_1, \sigma_2)$-continuous functions and weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous functions. We also investigate several characterizations of almost $\tau^\star(\sigma_1, \sigma_2)$-continuous functions and weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous functions.

2. Preliminaries

Throughout the present paper, spaces (X, τ_1, τ_2) and (Y, σ_1, σ_2) (or simply X and Y) always mean bitopological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a bitopological space (X, τ_1, τ_2). The closure of A and the interior of A with respect to τ_i are denoted by τ_i-Cl(A) and τ_i-Int(A), respectively, for $i = 1, 2$. A subset A of a bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$-*closed* [24] if $A = \tau_1\text{-Cl}(\tau_2\text{-Cl}(A))$. The complement of a $\tau_1\tau_2$-closed set is called $\tau_1\tau_2$-*open*. The intersection of all $\tau_1\tau_2$-closed sets of X containing A is called the $\tau_1\tau_2$-*closure* [24] of A and is denoted by $\tau_1\tau_2$-Cl(A). The union of all $\tau_1\tau_2$-open sets of X contained in A is called the $\tau_1\tau_2$-*interior* [24] of A and is denoted by $\tau_1\tau_2$-Int(A).

Lemma 1. [24] *Let A and B be subsets of a bitopological space (X, τ_1, τ_2). For the $\tau_1\tau_2$-closure, the following properties hold:*

(1) $A \subseteq \tau_1\tau_2\text{-}Cl(A)$ and $\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Cl(A)) = \tau_1\tau_2\text{-}Cl(A)$.

(2) If $A \subseteq B$, then $\tau_1\tau_2\text{-}Cl(A) \subseteq \tau_1\tau_2\text{-}Cl(B)$.

(3) $\tau_1\tau_2\text{-}Cl(A)$ is $\tau_1\tau_2$-closed.

(4) A is $\tau_1\tau_2$-closed if and only if $A = \tau_1\tau_2\text{-}Cl(A)$.

(5) $\tau_1\tau_2$-*Cl*$(X - A) = X - \tau_1\tau_2$-*Int*$(A)$.

A subset A of a bitopological space (X, τ_1, τ_2) is said to be $(\tau_1, \tau_2)r$-*open* [25] (resp. $(\tau_1, \tau_2)s$-*open* [26], $(\tau_1, \tau_2)p$-*open* [26], $(\tau_1, \tau_2)\beta$-*open* [26]) if $A = \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A))$ (resp. $A \subseteq \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(A))$, $A \subseteq \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A))$, $A \subseteq \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A)))$). The complement of a $(\tau_1, \tau_2)r$-open (resp. $(\tau_1, \tau_2)s$-open, $(\tau_1, \tau_2)p$-open, $(\tau_1, \tau_2)\beta$-open) set is said to be $(\tau_1, \tau_2)r$-*closed* (resp. $(\tau_1, \tau_2)s$-*closed*, $(\tau_1, \tau_2)p$-*closed*, $(\tau_1, \tau_2)\beta$-*closed*). A subset A of a bitopological space (X, τ_1, τ_2) is said to be $\alpha(\tau_1, \tau_2)$-*open* [27] if $A \subseteq \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(A)))$. The complement of an $\alpha(\tau_1, \tau_2)$-open set is said to be $\alpha(\tau_1, \tau_2)$-*closed*. A subset A of a bitopological space (X, τ_1, τ_2) is said to be $\tau_1\tau_2$-δ-*open* [16] if A is the union of $(\tau_1, \tau_2)r$-open sets of X. The complement of a $\tau_1\tau_2$-δ-open set is called $\tau_1\tau_2$-δ-*closed* [16]. The union of all $\tau_1\tau_2$-δ-open sets of X contained in A is called the $\tau_1\tau_2$-δ-*interior* [16] of A and is denoted by $\tau_1\tau_2$-δ-Int(A). The intersection of all $\tau_1\tau_2$-δ-closed sets of X containing A is called the $\tau_1\tau_2$-δ-*closure* [16] of A and is denoted by $\tau_1\tau_2$-δ-Cl(A). Let A be a subset of a bitopological space (X, τ_1, τ_2). A point $x \in X$ is called a $(\tau_1, \tau_2)\theta$-*cluster point* [25] of A if $\tau_1\tau_2\text{-Cl}(U) \cap A \neq \emptyset$ for every $\tau_1\tau_2$-open set U containing x. The set of all $(\tau_1, \tau_2)\theta$-cluster points of A is called the $(\tau_1, \tau_2)\theta$-*closure* [25] of A and is denoted by $(\tau_1, \tau_2)\theta$-Cl(A). A subset A of a bitopological space (X, τ_1, τ_2) is said to be $(\tau_1, \tau_2)\theta$-*closed* [25] if $(\tau_1, \tau_2)\theta\text{-Cl}(A) = A$. The complement of a $(\tau_1, \tau_2)\theta$-closed set is said to be $(\tau_1, \tau_2)\theta$-*open*. The union of all $(\tau_1, \tau_2)\theta$-open sets of X contained in A is called the $(\tau_1, \tau_2)\theta$-*interior* [25] of A and is denoted by $(\tau_1, \tau_2)\theta$-Int(A).

An ideal $\mathscr{I}$ on a topological space (X, τ) is a nonempty collection of subsets of X satisfying the following properties: (1) $A \in \mathscr{I}$ and $B \subseteq A$ imply $B \in \mathscr{I}$; (2) $A \in \mathscr{I}$ and $B \in \mathscr{I}$ imply $A \cup B \in \mathscr{I}$. A topological space (X, τ) with an ideal $\mathscr{I}$ on X is called an ideal topological space and is denoted by $(X, \tau, \mathscr{I})$. For an ideal topological space $(X, \tau, \mathscr{I})$ and a subset A of X, $A^\star(\mathscr{I})$ is defined as follows:

$$A^\star(\mathscr{I}) = \{x \in X : U \cap A \notin \mathscr{I} \text{ for every open neighbourhood } U \text{ of } x\}.$$

In case there is no chance for confusion, $A^\star(\mathscr{I})$ is simply written as $A^\star$. In [28], $A^\star$ is called the local function of A with respect to $\mathscr{I}$ and τ and $\text{Cl}^\star(A) = A^\star \cup A$ defines a Kuratowski closure operator for a topology $\tau^\star(\mathscr{I})$ finer than τ. A subset A is said to be $\star$-closed [29] if $A^\star \subseteq A$. The interior of a subset A in $(X, \tau^\star(\mathscr{I}))$ is denoted by Int$^\star(A)$. A subset A of an ideal topological space $(X, \tau, \mathscr{I})$ is said to be *semi**-$\mathscr{I}$-*open* [30] (resp. *semi*-$\mathscr{I}$-*open* [15]) if $A \subseteq \text{Cl}(\text{Int}^\star(A))$ (resp. $A \subseteq \text{Cl}^\star(\text{Int}(A))$). The complement of a semi*-$\mathscr{I}$-open (resp. semi-$\mathscr{I}$-open) set is said to be *semi**-$\mathscr{I}$-*closed* [30] (resp. semi-$\mathscr{I}$-closed [15]).

3. On almost $\tau^\star(\sigma_1, \sigma_2)$-continuous functions

In this section, we introduce the concept of almost $\tau^\star(\sigma_1, \sigma_2)$-continuous functions. Moreover, some characterizations of almost $\tau^\star(\sigma_1, \sigma_2)$-continuous functions are discussed.

Definition 1. *A function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is said to be almost $\tau^\star(\sigma_1, \sigma_2)$-continuous at a point $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y containing $f(x)$, there* *exists a $\star$-open set U of X containing x such that $f(U) \subseteq \sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V))$. A function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is said to be almost $\tau^\star(\sigma_1, \sigma_2)$-continuous if f is almost $\tau^\star(\sigma_1, \sigma_2)$-continuous at each point x of X.*

Lemma 2. [31] *Let A be a subset of a bitopological space (X, τ_1, τ_2). If A is $\tau_1\tau_2$-open in X, then (τ_1, τ_2)-sCl$(A) = \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))$.*

Theorem 1. *For a function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) f is almost $\tau^\star(\sigma_1, \sigma_2)$-continuous at $x \in X$;

(2) $x \in$ Int$^\star(f^{-1}(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V))))$ for every $\sigma_1\sigma_2$-open set V of Y containing $f(x)$;

(3) $x \in$ Int$^\star(f^{-1}((\sigma_1, \sigma_2)$-sCl$(V)))$ for every $\sigma_1\sigma_2$-open set V of Y containing $f(x)$;

(4) $x \in$ Int$^\star(f^{-1}(V))$ for every $(\sigma_1, \sigma_2)r$-open set V of Y containing $f(x)$;

(5) for each $(\sigma_1, \sigma_2)r$-open set V of Y containing $f(x)$, there exists a $\star$-open set U of X containing x such that $f(U) \subseteq V$.

Proof. (1) $\Rightarrow$ (2): Let V be any $\sigma_1\sigma_2$-open set of Y containing $f(x)$. Thus by (1), there exists a $\star$-open set U of X containing x such that $f(U) \subseteq \sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V))$. Therefore, $x \in U \subseteq f^{-1}(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V)))$ and hence $x \in$ Int$^\star(f^{-1}(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V))))$.

(2) $\Rightarrow$ (3): This follows from Lemma 2.

(3) $\Rightarrow$ (4): Let V be any $(\sigma_1, \sigma_2)r$-open set of Y containing $f(x)$. It follows from Lemma 2 that $V = \sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V)) = (\sigma_1, \sigma_2)$-sCl$(V)$.

(4) $\Rightarrow$ (5): Let V be any $(\sigma_1, \sigma_2)r$-open set of Y containing $f(x)$. Then by (4), we have $x \in$ Int$^\star(f^{-1}(V))$ and there exists a $\star$-open set U of X containing x such that $U \subseteq f^{-1}(V)$; hence $f(U) \subseteq V$.

(5) $\Rightarrow$ (1): Let V be any $\sigma_1\sigma_2$-open set of Y containing $f(x)$. Since $\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V))$ is $(\sigma_1, \sigma_2)r$-open, there exists a $\star$-open set U of X containing x such that

$$f(U) \subseteq \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)).$$

This shows that f is almost $\tau^\star(\sigma_1, \sigma_2)$-continuous at x.

Theorem 2. *For a function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) f is almost $\tau^\star(\sigma_1, \sigma_2)$-continuous;

(2) $f^{-1}(V) \subseteq$ Int$^\star(f^{-1}(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V))))$ for every $\sigma_1\sigma_2$-open set V of Y;

(3) Cl$^\star(f^{-1}(\sigma_1\sigma_2$-Cl$(\sigma_1\sigma_2$-Int$(K)))) \subseteq f^{-1}(K)$ for every $\sigma_1\sigma_2$-closed set K of Y;

(4) $Cl^\star(f^{-1}(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(B))))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-}Cl(B))$ *for every subset* B *of* Y*;*

(5) $f^{-1}(\sigma_1\sigma_2\text{-}Int(B)) \subseteq Int^\star(f^{-1}(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(B)))))$ *for every subset* B *of* Y*;*

(6) $f^{-1}(V)$ *is* $\star$*-open in* X *for every* $(\sigma_1, \sigma_2)r$*-open set* V *of* Y*;*

(7) $f^{-1}(K)$ *is* $\star$*-closed in* X *for every* $(\sigma_1, \sigma_2)r$*-closed set* K *of* Y*.*

Proof. (1) ⇒ (2): Let V be any $\sigma_1\sigma_2$-open set of Y and $x \in f^{-1}(V)$. Then, $f(x) \in V$. Thus by Theorem 1, we have $x \in \text{Int}^\star(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))))$ and hence

$$f^{-1}(V) \subseteq \text{Int}^\star(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))).$$

(2) ⇒ (3): Let K be any $\sigma_1\sigma_2$-closed set of Y. Then, $Y - K$ is $\sigma_1\sigma_2$-open in Y and by (2), $X - f^{-1}(K) = f^{-1}(Y - K) \subseteq \text{Int}^\star(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(Y - K)))) = \text{Int}^\star(X - f^{-1}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K)))) = X - \text{Cl}^\star(f^{-1}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K))))$. Thus, $\text{Cl}^\star(f^{-1}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K)))) \subseteq f^{-1}(K)$.

(3) ⇒ (4): Let B be any subset of Y. Then, $\sigma_1\sigma_2\text{-Cl}(B)$ is a $\sigma_1\sigma_2$-closed set of Y and by (3), $\text{Cl}^\star(f^{-1}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(B))))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(B))$.

(4) ⇒ (5): Let B be any subset of Y. Then by (4),

$$\begin{aligned} f^{-1}(\sigma_1\sigma_2\text{-Int}(B)) &= X - f^{-1}(\sigma_1\sigma_2\text{-Cl}(Y - B)) \\ &\subseteq X - \text{Cl}^\star(f^{-1}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(Y - B))))) \\ &= X - \text{Cl}^\star(f^{-1}(Y - \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(B))))) \\ &= \text{Int}^\star(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(B))))). \end{aligned}$$

(5) ⇒ (6): Let V be any $(\sigma_1, \sigma_2)r$-open set of Y. By (5), we have $f^{-1}(V) \subseteq \text{Int}^\star(f^{-1}(V))$ and hence $f^{-1}(V)$ is $\star$-open in X.

(6) ⇒ (7): The proof is obvious.

(7) ⇒ (1): Let $x \in X$ and V be any $(\sigma_1, \sigma_2)r$-open set of Y containing $f(x)$. Since $Y - V$ is $(\sigma_1, \sigma_2)r$-closed and by (7), $X - f^{-1}(V) = f^{-1}(Y - V)$ is $\star$-closed in X. Thus, $f^{-1}(V)$ is $\star$-open and hence $x \in \text{Int}^\star(F^+(V))$. Then, there exists a $\star$-open set U of X containing x such that $f(U) \subseteq V$. It follows from Theorem 1 that f is almost $\tau^\star(\sigma_1, \sigma_2)$-continuous.

Theorem 3. *For a function* $f : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$*, the following properties are equivalent:*

(1) f *is almost* $\tau^\star(\sigma_1, \sigma_2)$*-continuous;*

(2) $Cl^\star(f^{-1}(V)) \subseteq f^{-1}(\sigma_1\sigma_2\text{-}Cl(V))$ *for every* $(\sigma_1, \sigma_2)\beta$*-open set* V *of* Y*;*

(3) $Cl^\star(f^{-1}(V)) \subseteq f^{-1}(\sigma_1\sigma_2\text{-}Cl(V))$ *for every* $(\sigma_1, \sigma_2)s$*-open set* V *of* Y*.*

Proof. (1) ⇒ (2): Let V be any $(\sigma_1, \sigma_2)\beta$-open set of Y. Then, $\sigma_1\sigma_2$-Cl(V) is a $(\sigma_1, \sigma_2)r$-closed set of Y. Since f is almost $\tau^\star(\sigma_1, \sigma_2)$-continuous, by Theorem 2 we have $f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))$ is $\star$-closed in X. Thus, $\text{Cl}^\star(f^{-1}(V)) \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))$.

(2) ⇒ (3): The proof is obvious.

(3) ⇒ (1): Let K be any $(\sigma_1, \sigma_2)r$-closed set of Y. Then, K is $(\sigma_1, \sigma_2)s$-open in Y. Then by (3), $\text{Cl}^\star(f^{-1}(K)) \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(K)) = f^{-1}(K)$ and hence $f^{-1}(K)$ is $\star$-closed in X. By Theorem 2, f is almost $\tau^\star(\sigma_1, \sigma_2)$-continuous.

Lemma 3. [32] *For a bitopological space (X, τ_1, τ_2), the following properties hold:*

(1) $\alpha(\tau_1, \tau_2)$-Cl$(V) = \tau_1\tau_2$-Cl(V) for every $(\tau_1, \tau_2)\beta$-open set V of X;

(2) (τ_1, τ_2)-pCl$(V) = \tau_1\tau_2$-Cl(V) for every $(\tau_1, \tau_2)s$-open set V of X.

Corollary 1. *For a function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) f is almost $\tau^\star(\sigma_1, \sigma_2)$-continuous;

(2) $\text{Cl}^\star(f^{-1}(V)) \subseteq f^{-1}(\alpha(\sigma_1, \sigma_2)\text{-Cl}(V))$ for every $(\sigma_1, \sigma_2)\beta$-open set V of Y;

(3) $\text{Cl}^\star(f^{-1}(V)) \subseteq f^{-1}((\sigma_1, \sigma_2)\text{-pCl}(V))$ for every $(\sigma_1, \sigma_2)s$-open set V of Y.

Theorem 4. *For a function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) f is almost $\tau^\star(\sigma_1, \sigma_2)$-continuous;

(2) $\text{Cl}^\star(f^{-1}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))$ for every $(\sigma_1, \sigma_2)p$-open set V of Y;

(3) $\text{Cl}^\star(f^{-1}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(V)))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))$ for every $(\sigma_1, \sigma_2)p$-open set V of Y;

(4) $f^{-1}(V) \subseteq \text{Int}^\star(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))))$ for every $(\sigma_1, \sigma_2)p$-open set V of Y.

Proof. (1) ⇒ (2): Let V be any $(\sigma_1, \sigma_2)p$-open set of Y. Then, we have $\sigma_1\sigma_2$-Cl(V) is $\sigma_1\sigma_2$-closed in Y and by Theorem 2,

$$\text{Cl}^\star(f^{-1}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)).$$

(2) ⇒ (3): Let V be any $(\sigma_1, \sigma_2)p$-open set of Y. By (2),

$$\begin{aligned}\text{Cl}^\star(f^{-1}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(V)))) &\subseteq \text{Cl}^\star(f^{-1}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))))\\ &\subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)).\end{aligned}$$

(3) ⇒ (4): Let V be any $(\sigma_1, \sigma_2)p$-open set of Y. Thus by (3), we have

$$X - \text{Int}^\star(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) = \text{Cl}^\star(X - f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))))$$

$$
\begin{aligned}
&= \mathrm{Cl}^\star(f^{-1}(Y - \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))))\\
&= \mathrm{Cl}^\star(f^{-1}(\sigma_1\sigma_2\text{-Cl}(Y - \sigma_1\sigma_2\text{-Cl}(V))))\\
&= \mathrm{Cl}^\star(f^{-1}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(Y - \sigma_1\sigma_2\text{-Cl}(V)))))\\
&\subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(Y - \sigma_1\sigma_2\text{-Cl}(V)))\\
&= f^{-1}(Y - \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))\\
&\subseteq f^{-1}(Y - V)\\
&= X - f^{-1}(V)
\end{aligned}
$$

and hence $f^{-1}(V) \subseteq \mathrm{Int}^\star(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))))$.

(4) ⇒ (1): Let V be any $(\sigma_1, \sigma_2)r$-open set of Y. Then, V is $(\sigma_1, \sigma_2)p$-open in Y and by (4), $f^{-1}(V) \subseteq \mathrm{Int}^\star(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) = \mathrm{Int}^\star(f^{-1}(V))$. Thus, $f^{-1}(V)$ is $\star$-open in X. It follows from Theorem 2 that f is almost $\tau^\star(\sigma_1, \sigma_2)$-continuous.

Lemma 4. [33] *Let A be a subset of a bitopological space (X, τ_1, τ_2). Then, the following properties hold:*

(1) If A is $\tau_1\tau_2$-open in X, then $\tau_1\tau_2\text{-}Cl(A) = \tau_1\tau_2\text{-}\delta\text{-}Cl(A)$.

(2) $\tau_1\tau_2\text{-}\delta\text{-}Cl(A)$ is $\tau_1\tau_2$-closed.

Theorem 5. *For a function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) f is almost $\tau^\star(\sigma_1, \sigma_2)$-continuous;

(2) $Cl^\star(f^{-1}(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}\delta\text{-}Cl(B))))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-}\delta\text{-}Cl(B))$ for every subset B of Y;

(3) $Cl^\star(f^{-1}(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(B))))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-}\delta\text{-}Cl(B))$ for every subset B of Y.

Proof. (1) ⇒ (2): Let B be any subset of Y. By Lemma 4, $\sigma_1\sigma_2\text{-}\delta\text{-Cl}(B)$ is $\sigma_1\sigma_2$-closed in Y and by Theorem 2, $\mathrm{Cl}^\star(f^{-1}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-}\delta\text{-Cl}(B))))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-}\delta\text{-Cl}(B))$.

(2) ⇒ (3): This is obvious since $\sigma_1\sigma_2\text{-Cl}(B) \subseteq \sigma_1\sigma_2\text{-}\delta\text{-Cl}(B)$ for every subset B of Y.

(3) ⇒ (1): Let K be any $(\sigma_1, \sigma_2)r$-closed set of Y. Then by (3), we have

$$
\begin{aligned}
\mathrm{Cl}^\star(f^{-1}(K)) &= \mathrm{Cl}^\star(f^{-1}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K))))\\
&= \mathrm{Cl}^\star(f^{-1}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(K)))))\\
&\subseteq f^{-1}(\sigma_1\sigma_2\text{-}\delta\text{-Cl}(K)) = f^{-1}(K)
\end{aligned}
$$

and hence $f^{-1}(K)$ is $\star$-closed in X. By Theorem 2, f is almost $\tau^\star(\sigma_1, \sigma_2)$-continuous.

4. On weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous functions

In this section, we introduce the notion of weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous functions. Moreover, some characterizations of weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous functions are discussed.

Definition 2. *A function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is said to be weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous at a point $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y containing $f(x)$, there exists a $\star$-open set U of X containing x such that $f(U) \subseteq \sigma_1\sigma_2\text{-}Cl(V)$. A function*

$$f : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$$

is said to be weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous if f is weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous at each point x of X.

Remark 1. *For a function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following implication holds:*

almost $\tau^\star(\sigma_1, \sigma_2)$-continuity $\Rightarrow$ weakly $\tau^\star(\sigma_1, \sigma_2)$-continuity.

The converse of the implication is not true in general. We give an example for the implication as follows.

Example 1. *Let $X = \{1, 2, 3\}$ with a topology $\tau = \{\emptyset, \{2\}, \{1, 3\}, X\}$ and an ideal $\mathscr{I} = \{\emptyset\}$. Let $Y = \{a, b, c\}$ with topologies $\sigma_1 = \{\emptyset, \{a\}, \{a, b\}, Y\}$ and*

$$\sigma_2 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, Y\}.$$

A function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is defined as follows: $f(1) = a$, $f(2) = b$ and $f(3) = c$. Then, f is weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous but f is not almost $\tau^\star(\sigma_1, \sigma_2)$-continuous.

Theorem 6. *For a function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) f is weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous;

(2) $f^{-1}(V) \subseteq Int^\star(f^{-1}(\sigma_1\sigma_2\text{-}Cl(V)))$ for every $\sigma_1\sigma_2$-open set V of Y;

(3) $Cl^\star(f^{-1}(\sigma_1\sigma_2\text{-}Int(K))) \subseteq f^{-1}(K)$ for every $\sigma_1\sigma_2$-closed set K of Y;

(4) $Cl^\star(f^{-1}(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(B)))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-}Cl(B))$ for every subset B of Y;

(5) $f^{-1}(\sigma_1\sigma_2\text{-}Int(B)) \subseteq Int^\star(f^{-1}(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(B))))$ for every subset B of Y;

(6) $Cl^\star(f^{-1}(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V)))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-}Cl(V))$ for every $\sigma_1\sigma_2$-open set V of Y;

(7) $Cl^\star(f^{-1}(V)) \subseteq f^{-1}(\sigma_1\sigma_2\text{-}Cl(V))$ for every $\sigma_1\sigma_2$-open set V of Y;

(8) $Cl^\star(f^{-1}(\sigma_1\sigma_2\text{-}Int(K))) \subseteq f^{-1}(K)$ for every $(\sigma_1, \sigma_2)r$-closed set K of Y.

Proof. (1) ⇒ (2): Let V be any $\sigma_1\sigma_2$-open set of Y such that $x \in f^{-1}(V)$. Then, we have $f(x) \in V$. By (1), there exists a $\star$-open set U of X containing x such that $f(U) \subseteq \sigma_1\sigma_2\text{-Cl}(V)$. Therefore, $U \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))$. Since U is $\star$-open, we have

$$x \in \text{Int}^\star(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)))$$

and hence $f^{-1}(V) \subseteq \text{Int}^\star(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)))$.

(2) ⇒ (3): Let K be any $\sigma_1\sigma_2$-closed set of Y. Then, $Y - K$ is $\sigma_1\sigma_2$-open in Y. By (2), $X - f^{-1}(K) = f^{-1}(Y - K) \subseteq \text{Int}^\star(f^{-1}(\sigma_1\sigma_2\text{-Cl}(Y - K))) = X - \text{Cl}^\star(f^{-1}(\sigma_1\sigma_2\text{-Int}(K)))$. Thus, $\text{Cl}^\star(f^{-1}(\sigma_1\sigma_2\text{-Int}(K))) \subseteq f^{-1}(K)$.

(3) ⇒ (4): Let B be any subset of Y. Then, $\sigma_1\sigma_2\text{-Cl}(B)$ is a $\sigma_1\sigma_2$-closed set of Y and by (3), $\text{Cl}^\star(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(B)))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(B))$.

(4) ⇒ (5): Let B be any subset of Y. Thus by (4), we have

$$\begin{aligned} X - \text{Int}^\star(f^{-1}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(B)))) &= \text{Cl}^\star(X - f^{-1}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(B)))) \\ &= \text{Cl}^\star(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(Y - B)))) \\ &\subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(Y - B)) \\ &= X - f^{-1}(\sigma_1\sigma_2\text{-Int}(B)) \end{aligned}$$

and hence $f^{-1}(\sigma_1\sigma_2\text{-Int}(B)) \subseteq \text{Int}^\star(f^{-1}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(B))))$.

(5) ⇒ (1): Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y containing $f(x)$. By (5), $x \in f^{-1}(V) \subseteq \text{Int}^\star(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)))$ and there exists a $\star$-open set U of X containing x such that $U \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))$. Thus, $f(U) \subseteq \sigma_1\sigma_2\text{-Cl}(V)$ and hence f is weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous.

(4) ⇒ (6) and (6) ⇒ (7): The proofs are obvious.

(7) ⇒ (8): Let K be any $(\sigma_1, \sigma_2)r$-closed set of Y. Thus by (7),

$$\text{Cl}^\star(f^{-1}(\sigma_1\sigma_2\text{-Int}(K))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K))) = f^{-1}(K).$$

(8) ⇒ (3): Let K be any $\sigma_1\sigma_2$-closed set of Y. Then, $\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K))$ is $(\sigma_1, \sigma_2)r$-closed in Y and $\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K))) = \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(K)) = \sigma_1\sigma_2\text{-Int}(K)$. By (8),

$$\begin{aligned} \text{Cl}^\star(f^{-1}(\sigma_1\sigma_2\text{-Int}(K))) &= \text{Cl}^\star(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K))))) \\ &\subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K))) \subseteq f^{-1}(K). \end{aligned}$$

Theorem 7. *For a function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) f is weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous;

(2) $Cl^\star(f^{-1}(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V)))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-}Cl(V))$ for every $(\sigma_1, \sigma_2)\beta$-open set V of Y;

(3) $\text{Cl}^\star(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))$ *for every* $(\sigma_1, \sigma_2)s$*-open set* V *of* Y.

Proof. (1) ⇒ (2): This follows from (4) of Theorem 6.

(2) ⇒ (3): The proof is obvious since every $(\sigma_1, \sigma_2)s$-open set is $(\sigma_1, \sigma_2)\beta$-open.

(3) ⇒ (1): Since every $\sigma_1\sigma_2$-open set is $(\sigma_1, \sigma_2)s$-open, the proof is obvious by (7) of Theorem 6.

Theorem 8. *For a function* $f : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$*, the following properties are equivalent:*

(1) f *is weakly* $\tau^\star(\sigma_1, \sigma_2)$*-continuous;*

(2) $\text{Cl}^\star(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))$ *for every* $(\sigma_1, \sigma_2)p$*-open set* V *of* Y*;*

(3) $\text{Cl}^\star(f^{-1}(V)) \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))$ *for every* $(\sigma_1, \sigma_2)p$*-open set* V *of* Y*;*

(4) $f^{-1}(V) \subseteq \text{Int}^\star(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)))$ *for every* $(\sigma_1, \sigma_2)p$*-open set* V *of* Y.

Proof. (1) ⇒ (2): Let V be any $(\sigma_1, \sigma_2)p$-open set of Y. Since $\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))$ is $\sigma_1\sigma_2$-open, by Theorem 6 (7)

$$\begin{aligned}\text{Cl}^\star(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) &\subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) \\ &\subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)).\end{aligned}$$

(2) ⇒ (3): Let V be any $(\sigma_1, \sigma_2)p$-open set of Y. By (2), we have

$$\text{Cl}^\star(f^{-1}(V)) \subseteq \text{Cl}^\star(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)).$$

(3) ⇒ (4): Let V be any $(\sigma_1, \sigma_2)p$-open set of Y. Thus by (3),

$$\begin{aligned}X - \text{Int}^\star(f^{-1}(\text{Cl}^\star(V))) &= \text{Cl}^\star(X - f^{-1}(\text{Cl}^\star(V))) \\ &= \text{Cl}^\star(f^{-1}(Y - \text{Cl}^\star(V))) \\ &\subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(Y - \sigma_1\sigma_2\text{-Cl}(V))) \\ &= X - f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))) \\ &\subseteq X - f^{-1}(V)\end{aligned}$$

and so $f^{-1}(V) \subseteq \text{Int}^\star(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)))$.

(4) ⇒ (1): Let V be any $\sigma_1\sigma_2$-open set of Y. Then, V is $(\sigma_1, \sigma_2)p$-open in Y and by (4), $f^{-1}(V) \subseteq \text{Int}^\star(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)))$. By Theorem 6 (2), f is weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous.

Lemma 5. [25] *For a subset* A *of a bitopological space* (X, τ_1, τ_2)*, the following properties hold:*

(1) If A is $\tau_2\tau_2$-open in X, then $\tau_1\tau_2$-Cl$(A) = (\tau_1, \tau_2)\theta$-Cl$(A)$.

(2) $(\tau_1, \tau_2)\theta$-Cl(A) is $\tau_1\tau_2$-closed in X.

Theorem 9. *For a function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) f is weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous;

(2) Cl$^\star(f^{-1}(\sigma_1\sigma_2$-Int$((\sigma_1, \sigma_2)\theta$-Cl$(B)))) \subseteq f^{-1}((\sigma_1, \sigma_2)\theta$-Cl$(B))$ for every subset B of Y;

(3) Cl$^\star(f^{-1}(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(B)))) \subseteq f^{-1}((\sigma_1, \sigma_2)\theta$-Cl$(B))$ for every subset B of Y.

Proof. (1) ⇒ (2): Let B be any subset of Y. By Lemma 5, $(\sigma_1, \sigma_2)\theta$-Cl$(B)$ is $\sigma_1\sigma_2$-closed in Y and by Theorem 6, Cl$^\star(f^{-1}(\sigma_1\sigma_2$-Int$((\sigma_1, \sigma_2)\theta$-Cl$(B)))) \subseteq f^{-1}((\sigma_1, \sigma_2)\theta$-Cl$(B))$.

(2) ⇒ (3): The proof is obvious.

(3) ⇒ (1): Let K be any $(\sigma_1, \sigma_2)r$-closed set of Y. Then, we have

$$(\sigma_1, \sigma_2)\theta\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K)) = \sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K)) = K$$

and hence

$$\begin{aligned}\text{Cl}^\star(f^{-1}(\sigma_1\sigma_2\text{-Int}(K))) &= \text{Cl}^\star(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K))))) \\ &\subseteq f^{-1}((\sigma_1, \sigma_2)\theta\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K))) \\ &= f^{-1}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K))) \\ &= f^{-1}(K).\end{aligned}$$

Thus, by Theorem 6 f is weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous.

Theorem 10. *A function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous at $x \in X$ if and only if $x \in$ Int$^\star(f^{-1}(\sigma_1\sigma_2$-Cl$(V)))$ for every $\sigma_1\sigma_2$-open set V of Y containing $f(x)$.*

Proof. Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y containing $f(x)$. Then, there exists a $\star$-open set U of X containing x such that $f(U) \subseteq \sigma_1\sigma_2$-Cl$(V)$. Thus, $U \subseteq f^{-1}(\sigma_1\sigma_2$-Cl$(V))$ and hence $x \in$ Int$^\star(f^{-1}(\sigma_1\sigma_2$-Cl$(V)))$.

Conversely, let V be any $\sigma_1\sigma_2$-open set of Y containing $f(x)$. By the hypothesis, $x \in$ Int$^\star(f^{-1}(\sigma_1\sigma_2$-Cl$(V)))$. Then, there exists a $\star$-open set U of X such that

$$x \in U \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)).$$

Thus, $f(U) \subseteq \sigma_1\sigma_2$-Cl$(V)$ and so f is weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous at x.

Theorem 11. *A function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous if and only if $f^{-1}(V) \subseteq$ Int$^\star(f^{-1}(\sigma_1\sigma_2$-Cl$(V)))$ for every $\sigma_1\sigma_2$-open set V of Y.*

Proof. Let V be any $\sigma_1\sigma_2$-open set of Y and $x \in f^{-1}(V)$. Then, $f(x) \in V$. Since f is weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous at x, by Theorem 10 we have $x \in \text{Int}^\star(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)))$ and hence $f^{-1}(V) \subseteq \text{Int}^\star(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)))$.

Conversely, let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y containing $f(x)$. Then, we have $x \in f^{-1}(V) \subseteq \text{Int}^\star(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)))$. By Theorem 10, f is weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous at x. This shows that f is weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous.

Theorem 12. *A function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous if and only if $Cl^\star(f^{-1}(V)) \subseteq f^{-1}(\sigma_1\sigma_2\text{-}Cl(V))$ for every $\sigma_1\sigma_2$-open set V of Y.*

Proof. Let V be any $\sigma_1\sigma_2$-open set of Y. Suppose that $\text{Cl}^\star(f^{-1}(V)) \nsubseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))$. There exists $x \in \text{Cl}^\star(f^{-1}(V))$, but $x \notin f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))$. Then, $f(x) \notin \sigma_1\sigma_2\text{-Cl}(V)$ and there exists a $\star$-open set W of Y containing $f(x)$ such that $W \cap V = \emptyset$. Thus, $\sigma_1\sigma_2\text{-Cl}(W) \cap V = \emptyset$. Since f is weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous at x, there exists a $\star$-open set U of X containing x such that $f(U) \subseteq \sigma_1\sigma_2\text{-Cl}(W)$. Therefore, $f(U) \cap V = \emptyset$. Since $x \in \text{Cl}^\star(f^{-1}(V))$, $U \cap f^{-1}(V) \neq \emptyset$ and $f(U) \cap V \neq \emptyset$, which is a contradiction. This shows that $\text{Cl}^\star(f^{-1}(V)) \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))$.

Conversely, let V be any $\sigma_1\sigma_2$-open set of Y. Then, $Y - \sigma_1\sigma_2\text{-Cl}(V)$ is $\sigma_1\sigma_2$-open in Y. By the hypothesis, $\text{Cl}^\star(f^{-1}(Y - \sigma_1\sigma_2\text{-Cl}(V))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(Y - \sigma_1\sigma_2\text{-Cl}(V)))$. Thus, $X - \text{Int}^\star(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))) \subseteq X - f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))) \subseteq X - f^{-1}(V)$ and hence $f^{-1}(V) \subseteq \text{Int}^\star(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)))$. By Theorem 11, f is weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous.

Theorem 13. *For a function $(X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) f is weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous;

(2) $f^{-1}(V) \subseteq Int^\star(f^{-1}(\sigma_1\sigma_2\text{-}Cl(V)))$ for every $\sigma_1\sigma_2$-open set V of Y;

(3) $Cl^\star(f^{-1}(\sigma_1\sigma_2\text{-}Int(K))) \subseteq f^{-1}(K)$ for every $\sigma_1\sigma_2$-closed set K of Y;

(4) $Cl^\star(f^{-1}(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(B)))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-}Cl(B))$ for every subset B of Y;

(5) $f^{-1}(\sigma_1\sigma_2\text{-}Int(B)) \subseteq Int^\star(f^{-1}(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(B))))$ for every subset B of Y;

(6) $Cl^\star(f^{-1}(V)) \subseteq f^{-1}(\sigma_1\sigma_2\text{-}Cl(V))$ for every $\sigma_1\sigma_2$-open set V of Y.

Proof. (1) ⇒ (2): It follows from Theorem 11.

(2) ⇒ (3): Let K be any $\sigma_1\sigma_2$-closed set of Y. Then, $Y - K$ is $\sigma_1\sigma_2$-open in Y and by (2),

$$\begin{aligned} X - f^{-1}(K) = f^{-1}(Y - K) &\subseteq \text{Int}^\star(f^{-1}(\sigma_1\sigma_2\text{-Cl}(Y - K))) \\ &= \text{Int}^\star(f^{-1}(Y - \sigma_1\sigma_2\text{-Int}(K))) \\ &= X - \text{Cl}^\star(f^{-1}(\sigma_1\sigma_2\text{-Int}(K))). \end{aligned}$$

Thus, $\text{Cl}^\star(f^{-1}(\sigma_1\sigma_2\text{-Int}(K))) \subseteq f^{-1}(K)$.

(3) ⇒ (4): Let B be any subset of Y. Then, $\sigma_1\sigma_2$-Cl(B) is $\sigma_1\sigma_2$-closed in Y. By (3), $\text{Cl}^\star(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(B)))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(B))$.

(4) ⇒ (5): Let B be any subset of Y. Thus by (4),

$$\begin{aligned} f^{-1}(\sigma_1\sigma_2\text{-Int}(B)) &= X - f^{-1}(\sigma_1\sigma_2\text{-Cl}(Y - B)) \\ &\subseteq X - \text{Cl}^\star(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(Y - B)))) \\ &= \text{Int}^\star(f^{-1}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(B)))). \end{aligned}$$

(5) ⇒ (6): Let V be any $\sigma_1\sigma_2$-open set of Y and $x \notin f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))$. Then, there exists a $\sigma_1\sigma_2$-open set U of Y containing $f(x)$ such that $U \cap V = \emptyset$. By (5), $x \in f^{-1}(U) \subseteq \text{Int}^\star(f^{-1}(\sigma_1\sigma_2\text{-Cl}(U)))$ and there exists a $\star$-open set G of X containing x such that $f(G) \subseteq \sigma_1\sigma_2\text{-Cl}(U)$. Thus, $G \cap f^{-1}(V) = \emptyset$ and so $x \notin \text{Cl}^\star(f^{-1}(V))$. This shows that $\text{Cl}^\star(f^{-1}(V)) \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))$.

(6) ⇒ (1): Let $x \in X$ and V be any be any $\sigma_1\sigma_2$-open set of Y containing $f(x)$. Since $V = \sigma_1\sigma_2\text{-Int}(V) \subseteq \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))$, by (6) we have

$$\begin{aligned} x \in f^{-1}(V) &\subseteq f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))) \\ &= X - f^{-1}(\sigma_1\sigma_2\text{-Cl}(Y - \sigma_1\sigma_2\text{-Cl}(V))) \\ &\subseteq X - \text{Cl}^\star(f^{-1}(Y - \sigma_1\sigma_2\text{-Cl}(V))) \\ &= \text{Int}^\star(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))). \end{aligned}$$

There exists a $\star$-open set U of X containing x such that $U \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))$; hence $f(U) \subseteq \sigma_1\sigma_2\text{-Cl}(V)$. Thus, f is weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous at x. This shows that f is weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous.

Theorem 14. *For a function $(X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) f is weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous;

(2) $Cl^\star(f^{-1}(\sigma_1\sigma_2\text{-}Int(K))) \subseteq f^{-1}(K)$ for every $(\sigma_1, \sigma_2)r$-closed set K of Y;

(3) $Cl^\star(f^{-1}(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V)))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-}Cl(V))$ for every $(\sigma_1, \sigma_2)\beta$-open set V of Y;

(4) $Cl^\star(f^{-1}(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V)))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-}Cl(V))$ for every $(\sigma_1, \sigma_2)s$-open set V of Y.

Proof. (1) ⇒ (2): Let K be any $(\sigma_1, \sigma_2)r$-closed set of Y. Then, $\sigma_1\sigma_2\text{-Int}(K)$ is $\sigma_1\sigma_2$-open in Y, by Theorem 13 (6) we have

$$\text{Cl}^\star(f^{-1}(\sigma_1\sigma_2\text{-Int}(K))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K))) = f^{-1}(K).$$

(2) ⇒ (3): Let V be any $(\sigma_1, \sigma_2)\beta$-open set of Y. Then, we have

$$\sigma_1\sigma_2\text{-Cl}(V) \subseteq \sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))) \subseteq \sigma_1\sigma_2\text{-Cl}(V)$$

and hence $\sigma_1\sigma_2$-Cl(V) is $(\sigma_1, \sigma_2)r$-closed. By (2),

$$\text{Cl}^\star(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)).$$

(3) $\Rightarrow$ (4): This is obvious.

(4) $\Rightarrow$ (1): Let V be any $\sigma_1\sigma_2$-open set of Y. Then, we have V is $(\sigma_1, \sigma_2)s$-open in Y. By (4), $\text{Cl}^\star(f^{-1}(V)) \subseteq \text{Cl}^\star(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))$ and by Theorem 13 (6), f is weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous.

Theorem 15. *For a function $(X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) f is weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous;

(2) $Cl^\star(f^{-1}(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V)))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-}Cl(V))$ for every $(\sigma_1, \sigma_2)p$-open set V of Y;

(3) $Cl^\star(f^{-1}(V)) \subseteq f^{-1}(\sigma_1\sigma_2\text{-}Cl(V))$ for every $(\sigma_1, \sigma_2)p$-open set V of Y;

(4) $f^{-1}(V) \subseteq Int^\star(f^{-1}(\sigma_1\sigma_2\text{-}Cl(V)))$ for every $(\sigma_1, \sigma_2)p$-open set V of Y.

Proof. (1) $\Rightarrow$ (2): Let V be any $(\sigma_1, \sigma_2)p$-open set of Y. Then, we have

$$\sigma_1\sigma_2\text{-Cl}(V) \subseteq \sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))$$

and hence $\sigma_1\sigma_2$-Cl(V) is $(\sigma_1, \sigma_2)r$-closed in Y. Thus, by Theorem 14 (2) we have

$$\text{Cl}^\star(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)).$$

(2) $\Rightarrow$ (3): The proof is obvious.

(3) $\Rightarrow$ (4): Let V be any $(\sigma_1, \sigma_2)p$-open set of Y. By (3),

$$\begin{aligned} f^{-1}(V) &\subseteq f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))) \\ &= X - f^{-1}(\sigma_1\sigma_2\text{-Cl}(Y - \sigma_1\sigma_2\text{-Cl}(V))) \\ &\subseteq X - \text{Cl}^\star(f^{-1}(Y - \sigma_1\sigma_2\text{-Cl}(V))) \\ &= \text{Int}^\star(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))). \end{aligned}$$

(4) $\Rightarrow$ (1): Let V be any $\sigma_1\sigma_2$-open set of Y. Then, V is $(\sigma_1, \sigma_2)p$-open in Y. Thus by (4) and Theorem 13 (2), f is weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous.

Theorem 16. *For a function $(X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) f is weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous;

(2) $Cl^\star(f^{-1}(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(B)))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-}Cl(B))$ for every subset B of Y;

(3) $\mathrm{Cl}^\star(f^{-1}(\sigma_1\sigma_2\text{-}\mathrm{Int}(K))) \subseteq f^{-1}(K)$ *for every* $(\sigma_1, \sigma_2)r$*-closed set* K *of* Y*;*

(4) $\mathrm{Cl}(f^{-1}(V)) \subseteq f^{-1}(\sigma_1\sigma_2\text{-}\mathrm{Cl}(V))$ *for every* $\sigma_1\sigma_2$*-open set* V *of* Y*;*

(5) $f^{-1}(V) \subseteq \mathrm{Int}^\star(f^{-1}(\sigma_1\sigma_2\text{-}\mathrm{Cl}(V)))$ *for every* $\sigma_1\sigma_2$*-open set* V *of* Y*;*

(6) $\mathrm{Cl}^\star(f^{-1}(V)) \subseteq f^{-1}(\sigma_1\sigma_2\text{-}\mathrm{Cl}(V))$ *for every* $(\sigma_1, \sigma_2)p$*-open set* V *of* Y*;*

(7) $f^{-1}(V) \subseteq \mathrm{Int}^\star(f^{-1}(\sigma_1\sigma_2\text{-}\mathrm{Cl}(V)))$ *for every* $(\sigma_1, \sigma_2)p$*-open set* V *of* Y*.*

Proof. (1) $\Rightarrow$ (2): Let B be any subset of Y and $x \notin f^{-1}(\sigma_1\sigma_2\text{-}\mathrm{Cl}(B))$. Then, we have $f(x) \notin \sigma_1\sigma_2\text{-}\mathrm{Cl}(B)$ and there exists a $\sigma_1\sigma_2$-open set U of Y containing $f(x)$ such that $U \cap B = \emptyset$. Therefore, $\sigma_1\sigma_2\text{-}\mathrm{Cl}(U) \cap \sigma_1\sigma_2\text{-}\mathrm{Int}(\sigma_1\sigma_2\text{-}\mathrm{Cl}(B)) = \emptyset$. Since f is weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous at x, there exists a $\star$-open set W of X containing x such that $f(W) \subseteq \sigma_1\sigma_2\text{-}\mathrm{Cl}(U)$. Thus, $W \cap f^{-1}(\sigma_1\sigma_2\text{-}\mathrm{Int}(\sigma_1\sigma_2\text{-}\mathrm{Cl}(B))) = \emptyset$ and hence

$$x \notin \mathrm{Cl}^\star(f^{-1}(\sigma_1\sigma_2\text{-}\mathrm{Int}(\sigma_1\sigma_2\text{-}\mathrm{Cl}(B)))).$$

This shows that $\mathrm{Cl}^\star(f^{-1}(\sigma_1\sigma_2\text{-}\mathrm{Int}(\sigma_1\sigma_2\text{-}\mathrm{Cl}(B)))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-}\mathrm{Cl}(B))$.

(2) $\Rightarrow$ (3): Let K be any $(\sigma_1, \sigma_2)r$-closed set of Y. Then by (2), we have

$$\begin{aligned}\mathrm{Cl}^\star(f^{-1}(\sigma_1\sigma_2\text{-}\mathrm{Int}(K))) &= \mathrm{Cl}^\star(f^{-1}(\sigma_1\sigma_2\text{-}\mathrm{Int}(\sigma_1\sigma_2\text{-}\mathrm{Cl}(\sigma_1\sigma_2\text{-}\mathrm{Int}(K))))) \\ &\subseteq f^{-1}(\sigma_1\sigma_2\text{-}\mathrm{Cl}(\sigma_1\sigma_2\text{-}\mathrm{Int}(K))) \\ &= f^{-1}(K).\end{aligned}$$

(3) $\Rightarrow$ (4): Let V be any $\sigma_1\sigma_2$-open set of Y. Then, $\sigma_1\sigma_2\text{-}\mathrm{Cl}(V)$ is $(\sigma_1, \sigma_2)r$-closed in Y. By (3), $\mathrm{Cl}^\star(f^{-1}(V)) \subseteq \mathrm{Cl}^\star(f^{-1}(\sigma_1\sigma_2\text{-}\mathrm{Int}(\sigma_1\sigma_2\text{-}\mathrm{Cl}(V)))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-}\mathrm{Cl}(V))$.

(4) $\Rightarrow$ (5): Let V be any $\sigma_1\sigma_2$-open set of Y. Since $Y - \sigma_1\sigma_2\text{-}\mathrm{Cl}(V)$ is $\sigma_1\sigma_2$-open in Y, by (4) we have

$$\begin{aligned}X - \mathrm{Int}^\star(f^{-1}(\sigma_1\sigma_2\text{-}\mathrm{Cl}(V))) &= \mathrm{Cl}^\star(f^{-1}(Y - \sigma_1\sigma_2\text{-}\mathrm{Cl}(V))) \\ &\subseteq f^{-1}(\sigma_1\sigma_2\text{-}\mathrm{Cl}(Y - \sigma_1\sigma_2\text{-}\mathrm{Cl}(V))) \\ &\subseteq X - f^{-1}(V)\end{aligned}$$

and hence $f^{-1}(V) \subseteq \mathrm{Int}^\star(f^{-1}(\sigma_1\sigma_2\text{-}\mathrm{Cl}(V)))$.

(5) $\Rightarrow$ (1): Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y containing $f(x)$. By (5), $x \in f^{-1}(V) \subseteq \mathrm{Int}^\star(f^{-1}(\sigma_1\sigma_2\text{-}\mathrm{Cl}(V)))$. Put $W = \mathrm{Int}^\star(f^{-1}(\sigma_1\sigma_2\text{-}\mathrm{Cl}(V)))$. Then, W is $\star$-open set of X containing x such that $f(W) \subseteq \sigma_1\sigma_2\text{-}\mathrm{Cl}(V)$. Thus, f is weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous at x. This shows that f is weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous.

(1) $\Rightarrow$ (6): Let V be any $(\sigma_1, \sigma_2)p$-open set of Y and $x \notin f^{-1}(\sigma_1\sigma_2\text{-}\mathrm{Cl}(V))$. Then, $f(x) \notin \sigma_1\sigma_2\text{-}\mathrm{Cl}(V)$ and there exists a $\sigma_1\sigma_2$-open set G of Y containing $f(x)$ such that $G \cap V = \emptyset$. Since V is $(\sigma_1, \sigma_2)p$-open, we have

$$\begin{aligned}V \cap \sigma_1\sigma_2\text{-}\mathrm{Cl}(G) &\subseteq \sigma_1\sigma_2\text{-}\mathrm{Int}(\sigma_1\sigma_2\text{-}\mathrm{Cl}(V)) \cap \sigma_1\sigma_2\text{-}\mathrm{Cl}(G) \\ &\subseteq \sigma_1\sigma_2\text{-}\mathrm{Cl}(\sigma_1\sigma_2\text{-}\mathrm{Int}(\sigma_1\sigma_2\text{-}\mathrm{Cl}(V)) \cap G)\end{aligned}$$

$$\begin{aligned}&\subseteq \sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)\cap G))\\&\subseteq \sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V\cap G)))\\&\subseteq \sigma_1\sigma_2\text{-Cl}(V\cap G)=\emptyset.\end{aligned}$$

Since f is weakly $\tau^\star(\sigma_1,\sigma_2)$-continuous at x, there exists a $\star$-open set W of X containing x such that $f(W)\subseteq \sigma_1\sigma_2\text{-Cl}(G)$. Thus, $f(W)\cap V=\emptyset$ and hence $W\cap f^{-1}(V)=\emptyset$. Therefore, $x\notin \text{Cl}^\star(f^{-1}(V)$. This shows that $\text{Cl}^\star(f^{-1}(V))\subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))$.

(6) $\Rightarrow$ (7): Let V be any $(\sigma_1,\sigma_2)p$-open set of Y. Then, $Y-\sigma_1\sigma_2\text{-Cl}(V)$ is $\sigma_1\sigma_2$-open and hence $Y-\sigma_1\sigma_2\text{-Cl}(V)$ is $(\sigma_1,\sigma_2)p$-open in Y. Then by (6), we have

$$\begin{aligned}X-\text{Int}^\star(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)))&=\text{Cl}^\star(X-f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)))\\&=\text{Cl}^\star(f^{-1}(Y-\sigma_1\sigma_2\text{-Cl}(V)))\\&\subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(Y-\sigma_1\sigma_2\text{-Cl}(V)))\\&=f^{-1}(Y-\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))\\&=X-f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))\\&\subseteq X-f^{-1}(V)\end{aligned}$$

and hence $f^{-1}(V)\subseteq \text{Int}^\star(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)))$.

(7) $\Rightarrow$ (1): Let $x\in X$ and V be any $\sigma_1\sigma_2$-open set of Y containing $f(x)$. Since V is $(\sigma_1,\sigma_2)p$-open in Y and by (7), $x\in f^{-1}(V)\subseteq \text{Int}^\star(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)))$. Put

$$U=\text{Int}^\star(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))).$$

Then, U is a $\star$-open set of X containing x such that $f(U)\subseteq \sigma_1\sigma_2\text{-Cl}(V)$. Thus, f is weakly $\tau^\star(\sigma_1,\sigma_2)$-continuous at x. This shows that f is weakly $\tau^\star(\sigma_1,\sigma_2)$-continuous.

Theorem 17. *For a function $(X,\tau,\mathscr{I})\to(Y,\sigma_1,\sigma_2)$, the following properties are equivalent:*

(1) f is weakly $\tau^\star(\sigma_1,\sigma_2)$-continuous;

(2) $f(Cl^\star(A))\subseteq(\sigma_1,\sigma_2)\theta\text{-}Cl(f(A))$ for every subset A of X;

(3) $Cl^\star(f^{-1}(B))\subseteq f^{-1}((\sigma_1,\sigma_2)\theta\text{-}Cl(B))$ for every subset B of Y.

Proof. (1) $\Rightarrow$ (2): Let A be any subset of X. Suppose that $x\in\text{Cl}^\star(A)$ and G is any $\sigma_1\sigma_2$-open set of Y containing $f(x)$. Since f is weakly $\tau^\star(\sigma_1,\sigma_2)$-continuous, there exists a $\star$-open set U of X containing x such that $f(U)\subseteq\sigma_1\sigma_2\text{-Cl}(G)$. Since $x\in\text{Cl}^\star(A)$, we have $U\cap A\neq\emptyset$. It follows that $\emptyset\neq f(U)\cap f(A)\subseteq\sigma_1\sigma_2\text{-Cl}(G)\cap f(A)$. Thus, $f(x)\in(\sigma_1,\sigma_2)\theta\text{-Cl}(f(A))$ and hence $f(\text{Cl}^\star(A))\subseteq(\sigma_1,\sigma_2)\theta\text{-Cl}(f(A))$.

(2) $\Rightarrow$ (3): Let B be any subset of Y. Then, we have

$$f(\text{Cl}^\star(f^{-1}(B)))\subseteq(\sigma_1,\sigma_2)\theta\text{-Cl}(f(f^{-1}(B)))\subseteq(\sigma_1,\sigma_2)\theta\text{-Cl}(B)$$

and hence $\text{Cl}^{\star}(f^{-1}(B)) \subseteq f^{-1}((\sigma_1, \sigma_2)\theta\text{-Cl}(B))$.

(3) ⇒ (1): Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y containing $f(x)$. Since

$$\sigma_1\sigma_2\text{-Cl}(V) \cap (Y - \sigma_1\sigma_2\text{-Cl}(V)) = \emptyset,$$

$f(x) \notin (\sigma_1, \sigma_2)\theta\text{-Cl}(Y - \sigma_1\sigma_2\text{-Cl}(V))$ and hence $x \notin f^{-1}((\sigma_1, \sigma_2)\theta\text{-Cl}(Y - \sigma_1\sigma_2\text{-Cl}(V)))$. By (3), $x \notin \text{Cl}^{\star}(f^{-1}(Y - \sigma_1\sigma_2\text{-Cl}(V)))$ and there exists a $\star$-open set U of X containing x such that $U \cap f^{-1}(Y - \sigma_1\sigma_2\text{-Cl}(V)) = \emptyset$; hence $f(U) \cap (Y - \sigma_1\sigma_2\text{-Cl}(V)) = \emptyset$. Thus, $f(U) \subseteq \sigma_1\sigma_2\text{-Cl}(V)$ and so f is weakly $\tau^{\star}(\sigma_1, \sigma_2)$-continuous at x. This shows that f is weakly $\tau^{\star}(\sigma_1, \sigma_2)$-continuous.

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