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Upper and Lower Quasi $\theta\tau^{\star}(\sigma_1, \sigma_2)$-continuity

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Abstract. This paper presents new concepts of continuous multifunctions, called upper quasi $\theta\tau^{\star}(\sigma_1, \sigma_2)$-continuous multifunctions and lower quasi $\theta\tau^{\star}(\sigma_1, \sigma_2)$-continuous multifunctions. Moreover, several characterizations and some properties concerning upper quasi $\theta\tau^{\star}(\sigma_1, \sigma_2)$-continuous multifunctions and lower quasi $\theta\tau^{\star}(\sigma_1, \sigma_2)$-continuous multifunctions are considered.

2020 Mathematics Subject Classifications: 54C08, 54C60

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1. Introduction

In 1963, Levine [1] introduced and studied the notion of semi-continuous functions. Arya and Bhamini [2] introduced the concept of θ-semi-continuity as a generalization of semi-continuity. Noiri [3] and Jafari and Noiri [4] have further investigated some characterizations of θ-semi-continuous functions. Marcus [5] introduced and investigated the notion of quasi continuous functions. Popa [6] introduced and studied the notion of almost quasi continuous functions. Neubrunnovaá [7] showed that quasi continuity is equivalent to semi-continuity due to Levine [1]. Popa and Stan [8] introduced and investigated the notion of weakly quasi continuous functions. Weak quasi continuity is implied by quasi continuity and weak continuity [9] which are independent of each other. Popa [10] extended the concept of quasicontinuous functions to the setting of multifunctions. Popa and Noiri [11] introduced the concept of almost quasi continuous multifunctions and investigated some characterizations of such multifunctions. Noiri and Popa [12] introduced and studied the notion of weakly quasi continuous multifunctions. Popa and Noiri [13] introduced the notion of θ-quasicontinuous multifunctions and investigated several further properties of such multifunctions. Moreover, some characterizations of upper and

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lower θ-quasicontinuous multifunctions were presented in [14]. Semi-$\mathscr{I}$-open sets, pre-$\mathscr{I}$-open sets, α-$\mathscr{I}$-open sets, β-$\mathscr{I}$-open sets and δ-$\mathscr{I}$-open sets play an important role in the research of generalizations of continuity in ideal topological spaces. Hatir and Noiri [15] introduced and investigated the notions of weakly pre-$\mathscr{I}$-open sets and weakly pre-$\mathscr{I}$-continuous functions. Furthermore, Hatir and Noiri [16] investigated further properties of semi-$\mathscr{I}$-open sets and semi-$\mathscr{I}$-continuous functions. In 2019, the present author [17] introduced new classes of multifunctions between ideal topological spaces, namely upper $\star$-continuous multifunctions and lower $\star$-continuous multifunctions. In particular, several characterizations of upper $\star$-continuous multifunctions, lower $\star$-continuous multifunctions, upper almost $\star$-continuous multifunctions, lower $\star$-continuous multifunctions, upper weakly $\star$-continuous multifunctions and lower weakly $\star$-continuous multifunctions were considered in [17]. On the other hand, the present author introduced and investigated the notions of $p\imath$-continuous multifunctions [18] and weakly $p\imath$-continuous multifunctions [18]. Pue-on et al. [19] introduced and studied the concepts of upper (τ_1, τ_2)-continuous multifunctions and lower (τ_1, τ_2)-continuous multifunctions. Klanarong et al. [20] introduced and investigated the notions of upper almost (τ_1, τ_2)-continuous multifunctions and lower almost (τ_1, τ_2)-continuous multifunctions. Thongmoon et al. [21] introduced and studied the concepts of upper weakly (τ_1, τ_2)-continuous multifunctions and lower weakly (τ_1, τ_2)-continuous multifunctions. In this paper, we introduce the notions of upper quasi $\theta\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions and lower quasi $\theta\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions. We also investigate several characterizations of upper quasi $\theta\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions and lower quasi $\theta\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions.

2. Preliminaries

Throughout the present paper, spaces (X, τ_1, τ_2) and (Y, σ_1, σ_2) (or simply X and Y) always mean bitopological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a bitopological space (X, τ_1, τ_2). The closure of A and the interior of A with respect to τ_i are denoted by τ_i-Cl(A) and τ_i-Int(A), respectively, for $i = 1, 2$. A subset A of a bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$-*closed* [22] if $A = \tau_1$-Cl$(\tau_2$-Cl$(A))$. The complement of a $\tau_1\tau_2$-closed set is called $\tau_1\tau_2$-*open*. Let A be a subset of a bitopological space (X, τ_1, τ_2). The intersection of all $\tau_1\tau_2$-closed sets of X containing A is called the $\tau_1\tau_2$-*closure* [22] of A and is denoted by $\tau_1\tau_2$-Cl(A). The union of all $\tau_1\tau_2$-open sets of X contained in A is called the $\tau_1\tau_2$-*interior* [22] of A and is denoted by $\tau_1\tau_2$-Int(A). A subset A of a bitopological space (X, τ_1, τ_2) is said to be $\tau_1\tau_2$-*clopen* [22] if A is both $\tau_1\tau_2$-open and $\tau_1\tau_2$-closed. A subset A of a bitopological space (X, τ_1, τ_2) is said to be $(\tau_1, \tau_2)r$-*open* [23] (resp. $(\tau_1, \tau_2)s$-*open* [24], $(\tau_1, \tau_2)p$-*open* [24], $(\tau_1, \tau_2)\beta$-*open* [24]) if $A = \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))$ (resp. $A \subseteq \tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(A))$, $A \subseteq \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))$, $A \subseteq \tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))))$. The complement of a $(\tau_1, \tau_2)r$-open (resp. $(\tau_1, \tau_2)s$-open, $(\tau_1, \tau_2)p$-open, $(\tau_1, \tau_2)\beta$-open) set is called $(\tau_1, \tau_2)r$-*closed* (resp. $(\tau_1, \tau_2)s$-*closed*, $(\tau_1, \tau_2)p$-*closed*, $(\tau_1, \tau_2)\beta$-*closed*). A subset A of a bitopological space (X, τ_1, τ_2) is said to be $\alpha(\tau_1, \tau_2)$-*open* [25] if $A \subseteq \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(A)))$. The complement of an $\alpha(\tau_1, \tau_2)$-open set is said to be $\alpha(\tau_1, \tau_2)$-*closed*.

Let A be a subset of a bitopological space (X, τ_1, τ_2). A point $x \in X$ is called a $(\tau_1, \tau_2)\theta$*-cluster point* [23] of A if $\tau_1\tau_2$-Cl$(U) \cap A \neq \emptyset$ for every $\tau_1\tau_2$-open set U containing x. The set of all $(\tau_1, \tau_2)\theta$-cluster points of A is called the $(\tau_1, \tau_2)\theta$*-closure* [23] of A and is denoted by $(\tau_1, \tau_2)\theta$-Cl(A). A subset A of a bitopological space (X, τ_1, τ_2) is said to be $(\tau_1, \tau_2)\theta$*-closed* [23] if $(\tau_1, \tau_2)\theta$-Cl$(A) = A$. The complement of a $(\tau_1, \tau_2)\theta$-closed set is said to be $(\tau_1, \tau_2)\theta$*-open*. The union of all $(\tau_1, \tau_2)\theta$-open sets of X contained in A is called the $(\tau_1, \tau_2)\theta$*-interior* [23] of A and is denoted by $(\tau_1, \tau_2)\theta$-Int(A).

Lemma 1. [23] *For a subset A of a bitopological space (X, τ_1, τ_2), the following properties hold:*

(1) If A is $\tau_1\tau_2$-open in X, then $\tau_1\tau_2$-Cl$(A) = (\tau_1, \tau_2)\theta$-Cl$(A)$.

(2) $(\tau_1, \tau_2)\theta$-Cl(A) is $\tau_1\tau_2$-closed in X.

An ideal $\mathscr{I}$ on a topological space (X, τ) is a nonempty collection of subsets of X satisfying the following properties: (1) $A \in \mathscr{I}$ and $B \subseteq A$ imply $B \in \mathscr{I}$; (2) $A \in \mathscr{I}$ and $B \in \mathscr{I}$ imply $A \cup B \in \mathscr{I}$. A topological space (X, τ) with an ideal $\mathscr{I}$ on X is called an ideal topological space and is denoted by $(X, \tau, \mathscr{I})$. For an ideal topological space $(X, \tau, \mathscr{I})$ and a subset A of X, $A^\star(\mathscr{I})$ is defined as follows:

$$A^\star(\mathscr{I}) = \{x \in X : U \cap A \notin \mathscr{I} \text{ for every open neighbourhood } U \text{ of } x\}.$$

In case there is no chance for confusion, $A^\star(\mathscr{I})$ is simply written as $A^\star$. In [26], $A^\star$ is called the local function of A with respect to $\mathscr{I}$ and τ and $\mathrm{Cl}^\star(A) = A^\star \cup A$ defines a Kuratowski closure operator for a topology $\tau^\star(\mathscr{I})$ finer than τ. A subset A is said to be $\star$-closed [27] if $A^\star \subseteq A$. The interior of a subset A in $(X, \tau^\star(\mathscr{I}))$ is denoted by $\mathrm{Int}^\star(A)$. A subset A of an ideal topological space $(X, \tau, \mathscr{I})$ is said to be *semi***-$\mathscr{I}$-open* [28] (resp. *semi-$\mathscr{I}$-open* [16]) if $A \subseteq \mathrm{Cl}(\mathrm{Int}^\star(A))$ (resp. $A \subseteq \mathrm{Cl}^\star(\mathrm{Int}(A))$). The complement of a semi*-$\mathscr{I}$-open (resp. semi-$\mathscr{I}$-open) set is said to be *semi***-$\mathscr{I}$-closed* [28] (resp. semi-$\mathscr{I}$-closed [16]). A subset A of an ideal topological space $(X, \tau, \mathscr{I})$ is called *semi-$\mathscr{I}^\star$-open* [29] if $A \subseteq \mathrm{Cl}^\star(\mathrm{Int}^\star(A))$. The complement of a semi-$\mathscr{I}^\star$-open set is called *semi-$\mathscr{I}^\star$-closed*. For a subset A of an ideal topological space $(X, \tau, \mathscr{I})$, the intersection of all semi-$\mathscr{I}^\star$-closed sets containing A is called the *semi-$\mathscr{I}^\star$-closure* [29] of A and is denoted by $\mathrm{sCl}^\star(A)$ ($\mathrm{sCl}_{\mathscr{I}^\star}(A)$ [29]). The union of all semi-$\mathscr{I}^\star$-open sets contained in A is called the *semi-$\mathscr{I}^\star$-interior* [29] of A and is denoted by $\mathrm{sInt}^\star(A)$ ($s\mathrm{Int}_{\mathscr{I}^\star}(A)$ [29]). The family of all semi-$\mathscr{I}^\star$-open sets of an ideal topological space $(X, \tau, \mathscr{I})$ is denoted by $S_{\mathscr{I}^\star}O(X)$. Let A be a subset of an ideal topological space $(X, \tau, \mathscr{I})$. The *semi-$\theta(\star)$-closure* [30] of A, $\star_\theta s\mathrm{Cl}(A)$ and the *semi-$\theta(\star)$-interior* [30] of A, $\star_\theta s\mathrm{Int}(A)$ are defined as follows:

$$\star_\theta\, s\mathrm{Cl}(A) = \{x \in X \mid A \cap s\mathrm{Cl}^\star(U) \neq \emptyset \text{ for every } U \in S_{\mathscr{I}^\star}O(X, x)\},$$
$$\star_\theta\, s\mathrm{Int}(A) = \{x \in X \mid s\mathrm{Cl}^\star(U) \subseteq A \text{ for some } U \in S_{\mathscr{I}^\star}O(X, x)\},$$

where $S_{\mathscr{I}^\star}O(X, x) = \{U \mid x \in U \text{ and } U \in S_{\mathscr{I}^\star}O(X)\}$.

By a multifunction $F : X \to Y$, we mean a point-to-set correspondence from X into Y, and always assume that $F(x) \neq \emptyset$ for all $x \in X$. For a multifunction $F : X \to Y$, we shall denote the upper and lower inverse of a set B of Y by $F^+(B)$ and $F^-(B)$, respectively, that is, $F^+(B) = \{x \in X \mid F(x) \subseteq B\}$ and $F^-(B) = \{x \in X \mid F(x) \cap B \neq \emptyset\}$.

3. Upper and lower quasi $\theta\tau^{\star}(\sigma_1, \sigma_2)$-continuous multifunctions

In this section, we introduce the notions of upper quasi $\theta\tau^{\star}(\sigma_1, \sigma_2)$-continuous multifunctions and lower quasi $\theta\tau^{\star}(\sigma_1, \sigma_2)$-continuous multifunctions. Moreover, several characterizations of upper quasi $\theta\tau^{\star}(\sigma_1, \sigma_2)$-continuous multifunctions and lower quasi $\theta\tau^{\star}(\sigma_1, \sigma_2)$-continuous multifunctions are discussed.

Definition 1. *A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is said to be upper quasi $\theta\tau^{\star}(\sigma_1, \sigma_2)$-continuous if for each $x \in X$ and each $\sigma_1\sigma_2$-open set V of Y containing $F(x)$, there exists a semi-$\mathscr{I}^{\star}$-open set U of X containing x such that $F(sCl^{\star}(U)) \subseteq \sigma_1\sigma_2\text{-}Cl(V)$.*

Theorem 1. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is upper quasi $\theta\tau^{\star}(\sigma_1, \sigma_2)$-continuous;

(2) $\star_\theta sCl(F^-(\sigma_1\sigma_2\text{-}Int((\sigma_1, \sigma_2)\theta\text{-}Cl(B)))) \subseteq F^-((\sigma_1, \sigma_2)\theta\text{-}Cl(B))$ for every subset B of Y;

(3) $\star_\theta sCl(F^-(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V)))) \subseteq F^-(\sigma_1\sigma_2\text{-}Cl(V))$ for every $\sigma_1\sigma_2$-open set V of Y;

(4) $\star_\theta sCl(F^-(\sigma_1\sigma_2\text{-}Int(K))) \subseteq F^-(K)$ for every $(\sigma_1, \sigma_2)r$-closed set K of Y;

(5) $F^+(V) \subseteq \star_\theta sInt(F^+(\sigma_1\sigma_2\text{-}Cl(V)))$ for every $\sigma_1\sigma_2$-open set V of Y;

(6) $\star_\theta sCl(F^-(\sigma_1\sigma_2\text{-}Int(K))) \subseteq F^-(K)$ for every $\sigma_1\sigma_2$-closed set K of Y;

(7) $\star_\theta sCl(F^-(V)) \subseteq F^-(\sigma_1\sigma_2\text{-}Cl(V))$ for every $\sigma_1\sigma_2$-open set V of Y.

Proof. (1) ⇒ (2): Let B be any subset of Y. Suppose that $x \notin F^-((\sigma_1, \sigma_2)\theta\text{-Cl}(B))$. Then, $x \in X - F^-((\sigma_1, \sigma_2)\theta\text{-Cl}(B))$ and $F(x) \subseteq Y - (\sigma_1, \sigma_2)\theta\text{-Cl}(B)$. Since $(\sigma_1, \sigma_2)\theta\text{-Cl}(B)$ is $\sigma_1\sigma_2$-closed in Y, by (1) there exists a semi-$\mathscr{I}^{\star}$-open set U of X containing x such that $F(\text{sCl}^{\star}(U)) \subseteq \sigma_1\sigma_2\text{-Cl}(Y - (\sigma_1, \sigma_2)\theta\text{-Cl}(B)) = Y - \sigma_1\sigma_2\text{-Int}((\sigma_1, \sigma_2)\theta\text{-Cl}(B))$. Thus, we have $F(\text{sCl}^{\star}(U)) \cap \sigma_1\sigma_2\text{-Int}((\sigma_1, \sigma_2)\theta\text{-Cl}(B)) = \emptyset$ and

$$\text{sCl}^{\star}(U) \cap F^-(\sigma_1\sigma_2\text{-Int}((\sigma_1, \sigma_2)\theta\text{-Cl}(B))) = \emptyset.$$

This shows that $x \notin \star_\theta\text{sCl}(F^-(\sigma_1\sigma_2\text{-Int}((\sigma_1, \sigma_2)\theta\text{-Cl}(B))))$. Thus,

$$\star_\theta\text{sCl}(F^-(\sigma_1\sigma_2\text{-Int}((\sigma_1, \sigma_2)\theta\text{-Cl}(B)))) \subseteq F^-((\sigma_1, \sigma_2)\theta\text{-Cl}(B)).$$

(2) ⇒ (3): This is obvious since $\sigma_1\sigma_2\text{-Cl}(V) = (\sigma_1, \sigma_2)\theta\text{-Cl}(V)$ for every $\sigma_1\sigma_2$-open set V of Y.

(3) ⇒ (4): Let K be any $(\sigma_1, \sigma_2)r$-closed set of Y. Thus by (3), we have

$$\begin{aligned}\star_\theta\text{sCl}(F^-(\sigma_1\sigma_2\text{-Int}(K))) &= \star_\theta\text{sCl}(F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K))))) \\ &\subseteq F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K))) = F^-(K).\end{aligned}$$

(4) ⇒ (5): Let V be any $\sigma_1\sigma_2$-open set of Y. Then, we have

$$\begin{aligned} X - \star_\theta\text{sInt}(F^+(\sigma_1\sigma_2\text{-Cl}(V))) &= \star_\theta\text{sCl}(X - F^+(\sigma_1\sigma_2\text{-Cl}(V))) \\ &= \star_\theta\text{sCl}(F^-(Y - \sigma_1\sigma_2\text{-Cl}(V))), \end{aligned}$$

$Y - \sigma_1\sigma_2\text{-Cl}(V) = \sigma_1\sigma_2\text{-Int}(Y - \sigma_1\sigma_2\text{-Cl}(V)) \subseteq \sigma_1\sigma_2\text{-Int}(Y - \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))$ and $Y - \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))$ is $(\sigma_1, \sigma_2)r$-closed in Y. Thus by (4),

$$\begin{aligned} \star_\theta\text{sCl}(F^-(\sigma_1\sigma_2\text{-Int}(Y - \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))))) &\subseteq F^-(Y - \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))) \\ &= X - F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))) \\ &\subseteq X - F^+(V) \end{aligned}$$

and hence $F^+(V) \subseteq \star_\theta\text{sInt}(F^+(\sigma_1\sigma_2\text{-Cl}(V)))$.

(5) ⇒ (6): Let K be any $\sigma_1\sigma_2$-closed set of Y. Then by (5), we have

$$\begin{aligned} X - F^-(K) = F^+(Y - K) &\subseteq \star_\theta\text{sInt}(F^+(\sigma_1\sigma_2\text{-Cl}(Y - K))) \\ &= \star_\theta\text{sInt}(F^+(Y - \sigma_1\sigma_2\text{-Int}(K))) \\ &= \star_\theta\text{sInt}(X - F^-(\sigma_1\sigma_2\text{-Int}(K))) \\ &= X - \star_\theta\text{sCl}(F^-(\sigma_1\sigma_2\text{-Int}(K))). \end{aligned}$$

Thus, $\star_\theta\text{sCl}(F^-(\sigma_1\sigma_2\text{-Int}(K))) \subseteq F^-(K)$.

(6) ⇒ (7): Let V be any $\sigma_1\sigma_2$-closed set of Y. Then, we have $\sigma_1\sigma_2\text{-Cl}(V)$ is $\sigma_1\sigma_2$-closed in Y and by (6), $\star_\theta\text{sCl}(F^-(V)) \subseteq \star_\theta\text{sCl}(F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) \subseteq F^-(\sigma_1\sigma_2\text{-Cl}(V))$.

(7) ⇒ (1): Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y containing $F(x)$. Then, $\sigma_1\sigma_2\text{-Cl}(Y - \sigma_1\sigma_2\text{-Cl}(V)) \cap F(x) = \emptyset$ and $x \notin F^-(\sigma_1\sigma_2\text{-Cl}(Y - \sigma_1\sigma_2\text{-Cl}(V)))$. It follows from (7) that $x \notin \star_\theta\text{sCl}(F^-(Y - \sigma_1\sigma_2\text{-Cl}(V)))$. Then, there exists a semi-$\mathscr{I}^\star$-open set U of X containing x such that $\text{sCl}^\star(U) \cap F^-(Y - \sigma_1\sigma_2\text{-Cl}(V)) = \emptyset$; hence $F(\text{sCl}^\star(U)) \subseteq \sigma_1\sigma_2\text{-Cl}(V)$. This shows that F is upper quasi $\theta\tau^\star(\sigma_1, \sigma_2)$-continuous.

Definition 2. *A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is said to be lower quasi $\theta\tau^\star(\sigma_1, \sigma_2)$-continuous if for each $x \in X$ and each $\sigma_1\sigma_2$-open set V of Y such that*

$$V \cap F(x) \neq \emptyset,$$

there exists a semi-$\mathscr{I}^\star$-open set U of X containing x such that $\sigma_1\sigma_2$-Cl$(V) \cap F(z) \neq \emptyset$ for every $z \in$ sCl$^\star(U)$.

Lemma 2. *If $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is lower quasi $\theta\tau^\star(\sigma_1, \sigma_2)$-continuous, then for each $x \in X$ and each subset B of Y with $(\sigma_1, \sigma_2)\theta$-Int$(B) \cap F(x) \neq \emptyset$ there exists a semi-$\mathscr{I}^\star$-open set U of X containing x such that sCl$^\star(U) \subseteq F^-(B)$.*

Proof. Since $(\sigma_1, \sigma_2)\theta\text{-Int}(B) \cap F(x) \neq \emptyset$, there exists a $\sigma_1\sigma_2$-open set V of Y such that $V \subseteq \sigma_1\sigma_2\text{-Cl}(V) \subseteq B$ and $V \cap F(x) \neq \emptyset$. Since F is lower quasi $\theta\tau^\star(\sigma_1, \sigma_2)$-continuous, there exists a semi-$\mathscr{I}^\star$-open set U of X containing x such that $\sigma_1\sigma_2\text{-Cl}(V) \cap F(z) \neq \emptyset$ for every $z \in \text{sCl}^\star(U)$ and hence $\text{sCl}^\star(U) \subseteq F^-(B)$.

Theorem 2. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is lower quasi $\theta\tau^\star(\sigma_1, \sigma_2)$-continuous;

(2) $\star_\theta s Cl(F^+(B)) \subseteq F^+((\sigma_1, \sigma_2)\theta\text{-}Cl(B))$ for every subset B of Y;

(3) $\star_\theta s Cl(F^+(V)) \subseteq F^+(\sigma_1\sigma_2\text{-}Cl(V))$ for every $\sigma_1\sigma_2$-open set V of Y;

(4) $F^-(V) \subseteq \star_\theta s Int(F^-(\sigma_1\sigma_2\text{-}Cl(V)))$ for every $\sigma_1\sigma_2$-open set V of Y;

(5) $F(\star_\theta s Cl(A)) \subseteq (\sigma_1, \sigma_2)\theta\text{-}Cl(F(A))$ for every subset A of X;

(6) $\star_\theta s Cl(F^+(\sigma_1\sigma_2\text{-}Int((\sigma_1, \sigma_2)\theta\text{-}Cl(B)))) \subseteq F^+((\sigma_1, \sigma_2)\theta\text{-}Cl(B))$ for every subset B of Y;

(7) $\star_\theta s Cl(F^+(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V)))) \subseteq F^+(\sigma_1\sigma_2\text{-}Cl(V))$ for every $\sigma_1\sigma_2$-open set V of Y;

(8) $\star_\theta s Cl(F^+(\sigma_1\sigma_2\text{-}Int(K))) \subseteq F^+(K)$ for every $(\sigma_1, \sigma_2)r$-closed set K of Y;

(9) $\star_\theta s Cl(F^+(\sigma_1\sigma_2\text{-}Int(K))) \subseteq F^+(K)$ for every $\sigma_1\sigma_2$-closed set K of Y.

Proof. (1) ⇒ (2): Let B be any subset of Y. Suppose that $x \notin F^+((\sigma_1, \sigma_2)\theta\text{-Cl}(B))$. Then, $x \in F^-(Y - (\sigma_1, \sigma_2)\theta\text{-Cl}(B)) = F^-((\sigma_1, \sigma_2)\theta\text{-Int}(Y - B))$. Since F is lower quasi $\theta\tau^\star(\sigma_1, \sigma_2)$-continuous, by Lemma 2 there exists a semi-$\mathscr{I}^\star$-open set U of X containing x such that $\text{sCl}^\star(U) \subseteq F^-(Y - B) = X - F^+(B)$. Thus, $\text{sCl}^\star(U) \cap F^+(B) = \emptyset$ and hence $x \notin \star_\theta\text{sCl}(F^+(B))$.

(2) ⇒ (3): This is obvious since $\sigma_1\sigma_2\text{-Cl}(V) = (\sigma_1, \sigma_2)\theta\text{-Cl}(V)$ for every $\sigma_1\sigma_2$-open set V of Y.

(3) ⇒ (4): Let V be any $\sigma_1\sigma_2$-open set of Y. Then by (3), we have

$$\begin{aligned} X - \star_\theta\text{sInt}(F^-(\sigma_1\sigma_2\text{-Cl}(V))) &= \star_\theta\text{sCl}(X - F^-(\sigma_1\sigma_2\text{-Cl}(V))) \\ &= \star_\theta\text{sCl}(F^+(Y - \sigma_1\sigma_2\text{-Cl}(V))) \\ &\subseteq F^+(\sigma_1\sigma_2\text{-Cl}(Y - \sigma_1\sigma_2\text{-Cl}(V))) \\ &\subseteq F^+(\sigma_1\sigma_2\text{-Cl}(Y - V)) \\ &= F^+(Y - V) = X - F^-(V) \end{aligned}$$

and hence $F^-(V) \subseteq \star_\theta\text{sInt}(F^-(\sigma_1\sigma_2\text{-Cl}(V)))$.

(4) ⇒ (1): Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y such that $F(x) \cap V \neq \emptyset$. By (4), $x \in F^-(V) \subseteq \star_\theta\text{sInt}(F^-(\sigma_1\sigma_2\text{-Cl}(V)))$. Then, there exists a semi-$\mathscr{I}^\star$-open set U of X containing x such that $\text{sCl}^\star(U) \subseteq F^-(\sigma_1\sigma_2\text{-Cl}(V))$; hence $\sigma_1\sigma_2\text{-Cl}(V) \cap F(z) \neq \emptyset$ for every $z \in \text{sCl}^\star(U)$. This shows that F is lower quasi $\theta\tau^\star(\sigma_1, \sigma_2)$-continuous.

(2) ⇒ (5): Let A be any subset of X. By replacing B in (2) by $F(A)$, we have $\star_\theta\text{sCl}(A) \subseteq \star_\theta\text{sCl}(F^+(F(A))) \subseteq F^+((\sigma_1, \sigma_2)\theta\text{-Cl}(F(A)))$. Thus,

$$F(\star_\theta\text{sCl}(A)) \subseteq (\sigma_1, \sigma_2)\theta\text{-Cl}(F(A)).$$

(5) ⇒ (2): Let B be any subset of Y. Replacing A in (5) by $F^+(B)$, we have $F(\star_\theta\text{sCl}(F^+(B))) \subseteq (\sigma_1, \sigma_2)\theta\text{-Cl}(F(F^+(B))) \subseteq (\sigma_1, \sigma_2)\theta\text{-Cl}(B)$ and hence

$$\star_\theta\text{sCl}(F^+(B)) \subseteq F^+((\sigma_1, \sigma_2)\theta\text{-Cl}(B)).$$

(3) ⇒ (6): Let B be any subset of Y. Put $V = \sigma_1\sigma_2\text{-Int}((\sigma_1, \sigma_2)\theta\text{-Cl}(B))$ in (3). Then, since $(\sigma_1, \sigma_2)\theta\text{-Cl}(B)$ is $\sigma_1\sigma_2$-closed in Y, we have

$$\begin{aligned}\star_\theta\text{sCl}(F^+(\sigma_1\sigma_2\text{-Int}((\sigma_1, \sigma_2)\theta\text{-Cl}(B)))) &\subseteq F^+(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}((\sigma_1, \sigma_2)\theta\text{-Cl}(B)))) \\ &\subseteq F^+((\sigma_1, \sigma_2)\theta\text{-Cl}(B)).\end{aligned}$$

(6) ⇒ (7): This is obvious since $\sigma_1\sigma_2\text{-Cl}(V) = (\sigma_1, \sigma_2)\theta\text{-Cl}(V)$ for every $\sigma_1\sigma_2$-open set V of Y.

(7) ⇒ (8): Let K be any $(\sigma_1, \sigma_2)r$-closed set of Y. Then by (7), we have

$$\begin{aligned}\star_\theta\text{sCl}(F^+(\sigma_1\sigma_2\text{-Int}(K))) &= \star_\theta\text{sCl}(F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K))))) \\ &\subseteq F^+(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K))) = F^+(K).\end{aligned}$$

(8) ⇒ (9): Let K be any $\sigma_1\sigma_2$-closed set of Y. Then, $\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K))$ is $(\sigma_1, \sigma_2)r$-closed in Y and by (8),

$$\begin{aligned}\star_\theta\text{sCl}(F^+(\sigma_1\sigma_2\text{-Int}(K))) &= \star_\theta\text{sCl}(F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K))))) \\ &\subseteq F^+(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K))) \subseteq F^+(K).\end{aligned}$$

(9) ⇒ (4): Let V be any $\sigma_1\sigma_2$-open set of Y. Then, $Y - V$ is $\sigma_1\sigma_2$-closed in Y and by (9), $\star_\theta\text{sCl}(F^+(\sigma_1\sigma_2\text{-Int}(Y - V))) \subseteq F^+(Y - V) = X - F^-(V)$. Moreover, we have

$$\begin{aligned}\star_\theta\text{sCl}(F^+(\sigma_1\sigma_2\text{-Int}(Y - V))) &= \star_\theta\text{sCl}(F^+(Y - \sigma_1\sigma_2\text{-Cl}(V))) \\ &= \star_\theta\text{sCl}(X - F^-(\sigma_1\sigma_2\text{-Cl}(V))) \\ &= X - \star_\theta\text{sInt}(F^-(\sigma_1\sigma_2\text{-Cl}(V))).\end{aligned}$$

Thus, $F^-(V) \subseteq \star_\theta\text{sInt}(F^-(\sigma_1\sigma_2\text{-Cl}(V)))$.

Theorem 3. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is upper quasi $\theta\tau^\star(\sigma_1, \sigma_2)$-continuous;

(2) $\star_\theta sCl(F^-(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V)))) \subseteq F^-(\sigma_1\sigma_2\text{-}Cl(V))$ for every $(\sigma_1, \sigma_2)\beta$-open set V of Y;

(3) $\star_\theta sCl(F^-(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V)))) \subseteq F^-(\sigma_1\sigma_2\text{-}Cl(V))$ for every $(\sigma_1, \sigma_2)s$-open set V of Y.

Proof. (1) ⇒ (2): Let V be any $(\sigma_1, \sigma_2)\beta$-open set of Y. Then,

$$V \subseteq \sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))$$

and hence $\sigma_1\sigma_2\text{-Cl}(V) = \sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))$. Since $\sigma_1\sigma_2\text{-Cl}(V)$ is $(\sigma_1, \sigma_2)r$-closed in Y, by Theorem 1 we have

$$\star_\theta\text{sCl}(F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) \subseteq F^-(\sigma_1\sigma_2\text{-Cl}(V)).$$

(2) ⇒ (3): The proof is obvious.

(3) ⇒ (1): Let V be any $\sigma_1\sigma_2$-open set of Y. Then, V is $(\sigma_1, \sigma_2)s$-open in Y and by (3), $\star_\theta\text{sCl}(F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) \subseteq F^-(\sigma_1\sigma_2\text{-Cl}(V))$. Thus by Theorem 1, F is upper quasi $\theta\tau^\star(\sigma_1, \sigma_2)$-continuous.

Theorem 4. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is lower quasi $\theta\tau^\star(\sigma_1, \sigma_2)$-continuous;

(2) $\star_\theta sCl(F^+(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V)))) \subseteq F^+(\sigma_1\sigma_2\text{-}Cl(V))$ for every $(\sigma_1, \sigma_2)\beta$-open set V of Y;

(3) $\star_\theta sCl(F^+(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V)))) \subseteq F^+(\sigma_1\sigma_2\text{-}Cl(V))$ for every $(\sigma_1, \sigma_2)s$-open set V of Y.

Proof. The proof is similar to that of Theorem 3.

Theorem 5. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is upper quasi $\theta\tau^\star(\sigma_1, \sigma_2)$-continuous;

(2) $\star_\theta sCl(F^-(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V)))) \subseteq F^-(\sigma_1\sigma_2\text{-}Cl(V))$ for every $(\sigma_1, \sigma_2)p$-open set V of Y;

(3) $\star_\theta sCl(F^-(V)) \subseteq F^-(\sigma_1\sigma_2\text{-}Cl(V))$ for every $(\sigma_1, \sigma_2)p$-open set V of Y;

(4) $F^+(V) \subseteq \star_\theta sInt(F^+(\sigma_1\sigma_2\text{-}Cl(V)))$ for every $(\sigma_1, \sigma_2)p$-open set V of Y.

Proof. (1) ⇒ (2): Let V be any $(\sigma_1, \sigma_2)p$-open set of Y. Since $\sigma_1\sigma_2\text{-Cl}(V)$ is a $\sigma_1\sigma_2$-open set of Y, by Theorem 3 we have

$$\begin{aligned}\star_\theta\text{sCl}(F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) &\subseteq F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))))\\ &= F^-(\sigma_1\sigma_2\text{-Cl}(V)).\end{aligned}$$

(2) ⇒ (3): Let V be any $(\sigma_1, \sigma_2)p$-open set of Y. Then, $V \subseteq \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))$ and by (2),

$$\star_\theta\text{sCl}(F^-(V)) \subseteq \star_\theta\text{sCl}(F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))))$$

$$\subseteq F^{-}(\sigma_1\sigma_2\text{-Cl}(V)).$$

(3) ⇒ (4): Let V be any $(\sigma_1, \sigma_2)p$-open set of Y. Then by (3), we have

$$\begin{aligned} X - \star_\theta\text{sInt}(F^{+}(\sigma_1\sigma_2\text{-Cl}(V))) &= \star_\theta\text{sCl}(X - F^{+}(\sigma_1\sigma_2\text{-Cl}(V))) \\ &= \star_\theta\text{sCl}(F^{-}(Y - \sigma_1\sigma_2\text{-Cl}(V))) \\ &\subseteq F^{-}(\sigma_1\sigma_2\text{-Cl}(Y - \sigma_1\sigma_2\text{-Cl}(V))) \\ &= X - F^{+}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))) \\ &\subseteq X - F^{+}(V) \end{aligned}$$

and hence $F^{+}(V) \subseteq \star_\theta\text{sInt}(F^{+}(\sigma_1\sigma_2\text{-Cl}(V)))$.

(4) ⇒ (1): Let V be any $\sigma_1\sigma_2$-open set of Y. Then, V is $(\sigma_1, \sigma_2)p$-open in Y and by (4), we have $F^{+}(V) \subseteq \star_\theta\text{sInt}(F^{+}(\sigma_1\sigma_2\text{-Cl}(V)))$. By Theorem 1, F is upper quasi $\theta\tau^\star(\sigma_1, \sigma_2)$-continuous.

Theorem 6. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is lower quasi $\theta\tau^\star(\sigma_1, \sigma_2)$-continuous;

(2) $\star_\theta sCl(F^{+}(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V)))) \subseteq F^{+}(\sigma_1\sigma_2\text{-}Cl(V))$ for every $(\sigma_1, \sigma_2)p$-open set V of Y;

(3) $\star_\theta sCl(F^{+}(V)) \subseteq F^{+}(\sigma_1\sigma_2\text{-}Cl(V))$ for every $(\sigma_1, \sigma_2)p$-open set V of Y;

(4) $F^{-}(V) \subseteq \star_\theta sInt(F^{-}(\sigma_1\sigma_2\text{-}Cl(V)))$ for every $(\sigma_1, \sigma_2)p$-open set V of Y.

Proof. The proof is similar to that of Theorem 5.

Recall that a bitopological space (X, τ_1, τ_2) is said to be $\tau_1\tau_2$-*compact* [22] if for every cover of X by $\tau_1\tau_2$-open sets of X has a finite subcover. A bitopological space (X, τ_1, τ_2) is said to be *quasi* (τ_1, τ_2)-$\mathscr{H}$-*closed* [21] if for every $\tau_1\tau_2$-open cover $\{U_\gamma \mid \gamma \in \Gamma\}$, there exists a finite subset Γ_0 of Γ such that $X = \cup\{\tau_1\tau_2\text{-Cl}(U_\gamma) \mid \gamma \in \Gamma_0\}$. An ideal topological space $(X, \tau, \mathscr{I})$ is called $s^\star$-*closed* [29] if for every semi-$\mathscr{I}^\star$-open cover $\{V_\alpha \mid \alpha \in \nabla\}$ of X, there exists a finite subset ∇_0 of ∇ such that $X = \cup\{\text{sCl}^\star(V_\alpha) \mid \alpha \in \nabla_0\}$.

Theorem 7. *Let $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ be an upper quasi $\theta\tau^\star(\sigma_1, \sigma_2)$-continuous surjective multifunction such that $F(x)$ is $\sigma_1\sigma_2$-compact for each $x \in X$. If $(X, \tau, \mathscr{I})$ is $s^\star$-closed, then (Y, σ_1, σ_2) is quasi (σ_1, σ_2)-$\mathscr{H}$-closed.*

Proof. Let $\{V_\gamma \mid \gamma \in \Gamma\}$ be any $\sigma_1\sigma_2$-open cover of Y. For each $x \in X$, $F(x)$ is $\sigma_1\sigma_2$-compact and there exists a finite subset $\Gamma(x)$ of Γ such that $F(x) \subseteq \cup\{V_\gamma \mid \gamma \in \Gamma(x)\}$. Put $V(x) = \cup\{V_\gamma \mid \gamma \in \Gamma(x)\}$. Then, $F(x) \subseteq V(x)$ and $V(x)$ is $\sigma_1\sigma_2$-open in Y. Since F is upper quasi $\theta\tau^\star(\sigma_1, \sigma_2)$-continuous, there exists a semi-$\mathscr{I}^\star$-open set $U(x)$ of X containing

x such that $F(\text{sCl}^\star(U(x))) \subseteq \sigma_1\sigma_2\text{-Cl}(V(x))$. The family $\{U(x) \mid x \in X\}$ is a semi-$\mathscr{I}^\star$-open cover of X. Since $(X, \tau, \mathscr{I})$ is $s^\star$-closed, there exists a finite number of points, say, $x_1, x_2, ..., x_n$ in X such that $X = \cup\{\text{sCl}^\star(U(x_i)) \mid i = 1, 2, ..., n\}$. Since F is surjective,

$$\begin{aligned} Y = F(X) = F(\bigcup_{i=1}^{n} \text{sCl}^\star(U(x_i))) = \bigcup_{i=1}^{n} F(\text{sCl}^\star(U(x_i))) &\subseteq \bigcup_{i=1}^{n} \sigma_1\sigma_2\text{-Cl}(V(x_i)) \\ &= \bigcup_{i=1}^{n} \cup_{\gamma\in\Gamma(x_i)} \sigma_1\sigma_2\text{-Cl}(V_\gamma). \end{aligned}$$

This shows that (Y, σ_1, σ_2) is quasi (σ_1, σ_2)-$\mathscr{H}$-closed.

For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, a multifunction

$$\text{sCl}F_\circledast : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$$

is defined as follows: $\text{sCl}F_\circledast(x) = (\sigma_1, \sigma_2)\text{-sCl}(F(x))$ for each $x \in X$.

Lemma 3. *Let $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ be a multifunction. Then, $sClF^-_\circledast(V) = F^-(V)$ for ever $(\sigma_1, \sigma_2)s$-open set V of Y.*

Proof. Let V be any $(\sigma_1, \sigma_2)s$-open set of Y. Let $x \in \text{sCl}F^-_\circledast(V)$. Then,

$$(\sigma_1, \sigma_2)\text{-sCl}(F(x)) \cap V = \text{sCl}F_\circledast(x) \cap V \neq \emptyset.$$

Since V is $(\sigma_1, \sigma_2)s$-open in Y, we have $V \cap F(x) \neq \emptyset$ and hence $x \in F^-(V)$. This shows that $\text{sCl}F^-_\circledast(V) \subseteq F^-(V)$. On the other hand, let $x \in F^-(V)$. Then,

$$\emptyset \neq F(x) \cap V \subseteq (\sigma_1, \sigma_2)\text{-sCl}(F(x)) \cap V.$$

Thus, $x \in \text{sCl}F^-_\circledast(V)$ and so $\text{sCl}F^-_\circledast(V) = F^-(V)$.

Theorem 8. *A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is lower quasi $\theta\tau^\star(\sigma_1, \sigma_2)$-continuous if and only if $sClF_\circledast : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is lower quasi $\theta\tau^\star(\sigma_1, \sigma_2)$-continuous.*

Proof. Suppose that F is lower quasi $\theta\tau^\star(\sigma_1, \sigma_2)$-continuous. Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y such that $\text{sCl}F_\circledast(x) \cap V \neq \emptyset$. By Lemma 3, we have $F(x) \cap V \neq \emptyset$. Since F is lower quasi $\theta\tau^\star(\sigma_1, \sigma_2)$-continuous, there exists a semi-$\mathscr{I}^\star$-open set U of X containing x such that $\sigma_1\sigma_2\text{-Cl}(V) \cap F(z) \neq \emptyset$ for every $z \in \text{sCl}^\star(U)$. Since $\sigma_1\sigma_2\text{-Cl}(V)$ is $(\sigma_1, \sigma_2)s$-open in Y, by Lemma 3 we have $\text{sCl}^\star(U) \subseteq F^-(\sigma_1\sigma_2\text{-Cl}(V)) = \text{sCl}F^-_\circledast(\sigma_1\sigma_2\text{-Cl}(V))$ and hence $\text{sCl}F_\circledast(z) \cap \sigma_1\sigma_2\text{-Cl}(V) \neq \emptyset$ for every $z \in \text{sCl}^\star(U)$. This shows that $\text{sCl}F_\circledast$ is lower quasi $\theta\tau^\star(\sigma_1, \sigma_2)$-continuous.

Conversely, suppose that $\text{sCl}F_\circledast$ is lower quasi $\theta\tau^\star(\sigma_1, \sigma_2)$-continuous. Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y such that $F(x) \cap V \neq \emptyset$. Then, $(\sigma_1, \sigma_2)\text{-sCl}(F(x)) \cap V \neq \emptyset$. Since $\text{sCl}F_\circledast$ is lower quasi $\theta\tau^\star(\sigma_1, \sigma_2)$-continuous, there exists a semi-$\mathscr{I}^\star$-open set of X containing x such that $\text{sCl}F_\circledast(z) \cap \sigma_1\sigma_2\text{-Cl}(V) \neq \emptyset$ for every $z \in \text{sCl}^\star(U)$. Since $\sigma_1\sigma_2\text{-Cl}(V)$ is $(\sigma_1, \sigma_2)s$-open in Y and by Lemma 3, $\text{sCl}^\star(U) \subseteq \text{sCl}F^-_\circledast(\sigma_1\sigma_2\text{-Cl}(V)) = F^-(\sigma_1\sigma_2\text{-Cl}(V))$ and hence $\sigma_1\sigma_2\text{-Cl}(V) \cap F(z) \neq \emptyset$ for every $z \in \text{sCl}^\star(U)$. Thus, F is lower quasi $\theta\tau^\star(\sigma_1, \sigma_2)$-continuous.

Definition 3. [22] *A subset A of a bitopological space (X, τ_1, τ_2) is said to be:*

(1) $\tau_1\tau_2$-paracompact if every cover of A by $\tau_1\tau_2$-open sets of X is refined by a cover of A which consists of $\tau_1\tau_2$-open sets of X and is $\tau_1\tau_2$-locally finite in X;

(2) $\tau_1\tau_2$-regular if for each $x \in A$ and each $\tau_1\tau_2$-open set U of X containing x, there exists a $\tau_1\tau_2$-open set V of X such that $x \in V \subseteq \tau_1\tau_2\text{-}Cl(V) \subseteq U$.

Lemma 4. [22] *If A is a $\tau_1\tau_2$-regular $\tau_1\tau_2$-paracompact set of a bitopological space (X, τ_1, τ_2) and U is a $\tau_1\tau_2$-open neighborhood of A, then there exists a $\tau_1\tau_2$-open set V of X such that $A \subseteq V \subseteq \tau_1\tau_2\text{-}Cl(V) \subseteq U$.*

Lemma 5. *If $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is a multifunction such that $F(x)$ is $\sigma_1\sigma_2$-regular and $\sigma_1\sigma_2$-paracompact for each $x \in X$, then $sClF^+_{\circledast}(V) = F^+(V)$ for each $\sigma_1\sigma_2$-open set V of Y.*

Proof. Let V be any $\sigma_1\sigma_2$-open set of Y and $x \in \text{sCl}F^+_{\circledast}(V)$. Then, $\text{sCl}F^+_{\circledast}(x) \subseteq V$ and $F(x) \subseteq (\sigma_1, \sigma_2)\text{-sCl}(F(x)) = \text{sCl}F^+_{\circledast}(x) \subseteq V$. Thus, $x \in F^+(V)$ and so $\text{sCl}F^+_{\circledast}(V) \subseteq F^+(V)$. On the other hand, let $x \in F^+(V)$. Then, $F(x) \subseteq V$ and by Lemma 4, there exists a $\sigma_1\sigma_2$-open set W of Y such that $F(x) \subseteq W \subseteq \sigma_1\sigma_2\text{-Cl}(W) \subseteq V$; hence

$$\text{sCl}F^+_{\circledast}(x) = (\sigma_1, \sigma_2)\text{-sCl}(F(x)) \subseteq \sigma_1\sigma_2\text{-Cl}(W) \subseteq V.$$

Thus, $x \in \text{sCl}F^+_{\circledast}(V)$ and hence $F^+(V) \subseteq \text{sCl}F^+_{\circledast}(V)$. Therefore, $F^+(V) = \text{sCl}F^+_{\circledast}(V)$.

Theorem 9. *Let $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ be a multifunction such that $F(x)$ is $\sigma_1\sigma_2$-paracompact and $\sigma_1\sigma_2$-regular for each $x \in X$. Then, F is upper quasi $\theta\tau^\star(\sigma_1, \sigma_2)$-continuous if and only if $sClF_{\circledast} : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is upper quasi $\theta\tau^\star(\sigma_1, \sigma_2)$-continuous.*

Proof. Suppose that F is upper quasi $\theta\tau^\star(\sigma_1, \sigma_2)$-continuous. It follows from Theorem 1 and Lemma 5 that for every $\sigma_1\sigma_2$-open set V of Y,

$$\text{sCl}F^+_{\circledast}(V) = F^+(V) \subseteq \star_\theta\text{sInt}(F^+(\sigma_1\sigma_2\text{-Cl}(V))) = \star_\theta\text{sInt}(\text{sCl}F^+_{\circledast}(\sigma_1\sigma_2\text{-Cl}(V))).$$

Thus by Theorem 1, $\text{sCl}F_{\circledast}$ is upper quasi $\theta\tau^\star(\sigma_1, \sigma_2)$-continuous.

Conversely, suppose that $\text{sCl}F_{\circledast}$ is upper quasi $\theta\tau^\star(\sigma_1, \sigma_2)$-continuous. It follows from Theorem 1 and Lemma 5 that for every $\sigma_1\sigma_2$-open set V of Y,

$$F^+(V) = \text{sCl}F^+_{\circledast}(V) \subseteq \star_\theta\text{sInt}(\text{sCl}F^+_{\circledast}(\sigma_1\sigma_2\text{-Cl}(V))) = \star_\theta\text{sInt}(F^+(\sigma_1\sigma_2\text{-Cl}(V))).$$

By Theorem 1, F is upper quasi $\theta\tau^\star(\sigma_1, \sigma_2)$-continuous.

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