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Almost Weak Continuity for Multifunctions Defined between an Ideal Topological Space and a Bitopological Space

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Abstract. This paper presents new concepts of continuous multifunctions, called upper almost weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions and lower almost weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions. Moreover, several characterizations and some properties concerning upper almost weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions and lower almost weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions are considered.

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1. Introduction

Topology as a field of mathematics is concerned with all questions directly or indirectly related to continuity. Singal and Singal [1] introduced the concept of almost continuous functions as a generalization of continuity. Munshi and Bassan [2] studied the notion of almost semi-continuous functions. Noiri [3] introduced and investigated the concept of almost α-continuous functions. Nasef and Noiri [4] introduced two classes of functions, namely almost precontinuous functions and almost β-continuous functions. The class of almost precontinuity is a generalization of almost α-continuity. The class of almost β-continuity is a generalization of almost semi-continuity. Levine [5] introduced and investigated the concept of weakly continuous functions. Husain [6] introduced and studied the notion of almost continuous functions. Janković [7] introduced almost weak continuity as a generalization of both weak continuity and almost continuity. Noiri [8] investigated

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several characterizations of almost weakly continuous functions. Rose [9] introduced the notion of subweakly continuous functions and investigated the relationships between subweak continuity and weak continuity. In 1993, Noiri and Popa [10] extended the concept of almost weakly continuous functions to multifunctions and defined upper almost weakly continuous multifunctions and lower almost weakly continuous multifunctions. Popa and Noiri [11] investigated some characterizations and several properties concerning upper almost weakly continuous multifunctions and lower almost weakly continuous multifunctions. Abd El-Monsef et al. [12] introduced and studied the notions of $\mathscr{I}$-closed sets and $\mathscr{I}$-continuous functions. Semi-$\mathscr{I}$-open sets, pre-$\mathscr{I}$-open sets, α-$\mathscr{I}$-open sets, β-$\mathscr{I}$-open sets and δ-$\mathscr{I}$-open sets play an important role in the research of generalizations of continuity in ideal topological spaces. In 2005, Hatir and Noiri [13] introduced and investigated the notions of weakly pre-$\mathscr{I}$-open sets and weakly pre-$\mathscr{I}$-continuous functions. Furthermore, Hatir and Noiri [14] investigated further properties of semi-$\mathscr{I}$-open sets and semi-$\mathscr{I}$-continuous functions. On the other hand, the present author introduced and studied new classes of multifunctions between ideal topological spaces, namely upper $\star$-continuous multifunctions [15], lower $\star$-continuous multifunctions [15], upper almost $\star$-continuous multifunctions [15], lower almost $\star$-continuous multifunctions [15], upper weakly $\star$-continuous multifunctions [15], lower weakly $\star$-continuous multifunctions [15], $p\imath$-continuous multifunctions [16] and weakly $p\imath$-continuous multifunctions [16]. Recently, Pue-on et al. [17] extended the idea of continuous multifunctions to bitopological spaces. Klanarong et al. [18] introduced and investigated the concepts of upper almost (τ_1, τ_2)-continuous multifunctions and lower almost (τ_1, τ_2)-continuous multifunctions. Thongmoon et al. [19] introduced and studied the notions of upper weakly (τ_1, τ_2)-continuous multifunctions and lower weakly (τ_1, τ_2)-continuous multifunctions. On the other hand, the present authors introduced and investigated the concepts of upper almost weakly (τ_1, τ_2)-continuous multifunctions and lower almost weakly (τ_1, τ_2)-continuous multifunctions [20]. In this paper, we introduce the concepts of upper almost weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions and lower almost weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions. We also investigate several characterizations of upper almost weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions and lower almost weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions.

2. Preliminaries

Throughout the present paper, spaces (X, τ_1, τ_2) and (Y, σ_1, σ_2) (or simply X and Y) always mean bitopological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a bitopological space (X, τ_1, τ_2). The closure of A and the interior of A with respect to τ_i are denoted by τ_i-Cl(A) and τ_i-Int(A), respectively, for $i = 1, 2$. A subset A of a bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$-*closed* [21] if $A = \tau_1$-Cl$(\tau_2$-Cl$(A))$. The complement of a $\tau_1\tau_2$-closed set is called $\tau_1\tau_2$-*open*. Let A be a subset of a bitopological space (X, τ_1, τ_2). The intersection of all $\tau_1\tau_2$-closed sets of X containing A is called the $\tau_1\tau_2$-*closure* [21] of A and is denoted by $\tau_1\tau_2$-Cl(A). The union of all $\tau_1\tau_2$-open sets of X contained in A is called the $\tau_1\tau_2$-*interior* [21] of A and is denoted by $\tau_1\tau_2$-Int(A). A subset A of a bitopological space (X, τ_1, τ_2) is said to be $\tau_1\tau_2$-*clopen* [21] if

A is both $\tau_1\tau_2$-open and $\tau_1\tau_2$-closed. A subset A of a bitopological space (X, τ_1, τ_2) is said to be $(\tau_1, \tau_2)r$*-open* [22] (resp. $(\tau_1, \tau_2)s$*-open* [23], $(\tau_1, \tau_2)p$*-open* [23], $(\tau_1, \tau_2)\beta$*-open* [23]) if $A = \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A))$ (resp. $A \subseteq \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(A))$, $A \subseteq \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A))$, $A \subseteq \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A)))$). The complement of a $(\tau_1, \tau_2)r$-open (resp. $(\tau_1, \tau_2)s$-open, $(\tau_1, \tau_2)p$-open, $(\tau_1, \tau_2)\beta$-open) set is called $(\tau_1, \tau_2)r$*-closed* (resp. $(\tau_1, \tau_2)s$*-closed*, $(\tau_1, \tau_2)p$*-closed*, $(\tau_1, \tau_2)\beta$*-closed*). A subset A of a bitopological space (X, τ_1, τ_2) is said to be $\alpha(\tau_1, \tau_2)$*-open* [24] if $A \subseteq \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(A)))$. The complement of an $\alpha(\tau_1, \tau_2)$-open set is said to be $\alpha(\tau_1, \tau_2)$*-closed*.

For a subset A of a bitopological space (X, τ_1, τ_2), a point $x \in X$ is called a $(\tau_1, \tau_2)\theta$*-cluster point* of A if $\tau_1\tau_2\text{-Cl}(U) \cap A \neq \emptyset$ for every $\tau_1\tau_2$-open set U containing x. The set of all $(\tau_1, \tau_2)\theta$-cluster points of A is called the $(\tau_1, \tau_2)\theta$*-closure* of A and is denoted by $(\tau_1, \tau_2)\theta\text{-Cl}(A)$. A subset A of a bitopological space (X, τ_1, τ_2) is said to be $(\tau_1, \tau_2)\theta$*-closed* if $(\tau_1, \tau_2)\theta\text{-Cl}(A) = A$. The complement of a $(\tau_1, \tau_2)\theta$-closed set is said to be $(\tau_1, \tau_2)\theta$*-open*. The union of all $(\tau_1, \tau_2)\theta$-open sets of X contained in A is called the $(\tau_1, \tau_2)\theta$*-interior* of A and is denoted by $(\tau_1, \tau_2)\theta\text{-Int}(A)$ [22].

Lemma 1. [22] *For a subset A of a bitopological space (X, τ_1, τ_2), the following properties hold:*

(1) If A is $\tau_1\tau_2$-open in X, then $\tau_1\tau_2\text{-Cl}(A) = (\tau_1, \tau_2)\theta\text{-Cl}(A)$.

(2) $(\tau_1, \tau_2)\theta\text{-Cl}(A)$ is $\tau_1\tau_2$-closed in X.

An ideal $\mathscr{I}$ on a topological space (X, τ) is a nonempty collection of subsets of X satisfying the following properties: (1) $A \in \mathscr{I}$ and $B \subseteq A$ imply $B \in \mathscr{I}$; (2) $A \in \mathscr{I}$ and $B \in \mathscr{I}$ imply $A \cup B \in \mathscr{I}$. A topological space (X, τ) with an ideal $\mathscr{I}$ on X is called an ideal topological space and is denoted by $(X, \tau, \mathscr{I})$. For an ideal topological space $(X, \tau, \mathscr{I})$ and a subset A of X, $A^\star(\mathscr{I})$ is defined as follows:

$$A^\star(\mathscr{I}) = \{x \in X : U \cap A \notin \mathscr{I} \text{ for every open neighbourhood } U \text{ of } x\}.$$

In case there is no chance for confusion, $A^\star(\mathscr{I})$ is simply written as $A^\star$. In [25], $A^\star$ is called the local function of A with respect to $\mathscr{I}$ and τ and $\text{Cl}^\star(A) = A^\star \cup A$ defines a Kuratowski closure operator for a topology $\tau^\star(\mathscr{I})$ finer than τ. A subset A is said to be $\star$-closed [26] if $A^\star \subseteq A$. The interior of a subset A in $(X, \tau^\star(\mathscr{I}))$ is denoted by $\text{Int}^\star(A)$. A subset A of an ideal topological space $(X, \tau, \mathscr{I})$ is said to be *semi***-$\mathscr{I}$-open* [27] (resp. *semi-$\mathscr{I}$-open* [14]) if $A \subseteq \text{Cl}(\text{Int}^\star(A))$ (resp. $A \subseteq \text{Cl}^\star(\text{Int}(A))$). The complement of a semi*-$\mathscr{I}$-open (resp. semi-$\mathscr{I}$-open) set is said to be *semi***-$\mathscr{I}$-closed* [27] (resp. semi-$\mathscr{I}$-closed [14]). A subset A of an ideal topological space $(X, \tau, \mathscr{I})$ is called *$\mathscr{I}^\star$-preopen* [15] if $A \subseteq \text{Int}^\star(\text{Cl}^\star(A))$. The complement of a $\mathscr{I}^\star$-preopen set is called *$\mathscr{I}^\star$-preclosed*. For a subset A of an ideal topological space $(X, \tau, \mathscr{I})$, the intersection of all $\mathscr{I}^\star$-preclosed sets containing A is called the *$\star$-preclosure* of A and is denoted by $\text{pCl}^\star(A)$. The union of all $\mathscr{I}^\star$-preopen sets contained in A is called the *$\star$-preinterior* of A and is denoted by $\text{pInt}^\star(A)$.

Lemma 2. *For a subset A of an ideal topological space $(X, \tau, \mathscr{I})$, the following properties hold:*

(1) $pCl^\star(A) = A \cup Cl^\star(Int^\star(A))$.

(2) $pInt^\star(A) = A \cap Int^\star(Cl^\star(A))$.

By a multifunction $F : X \to Y$, we mean a point-to-set correspondence from X into Y, and always assume that $F(x) \neq \emptyset$ for all $x \in X$. For a multifunction $F : X \to Y$, we shall denote the upper and lower inverse of a set B of Y by $F^+(B)$ and $F^-(B)$, respectively, that is, $F^+(B) = \{x \in X \mid F(x) \subseteq B\}$ and $F^-(B) = \{x \in X \mid F(x) \cap B \neq \emptyset\}$.

3. Upper and lower almost weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions

In this section, we introduce the notions of upper almost weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions and lower almost weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions. Moreover, several characterizations of upper almost weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions and lower almost weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions are discussed.

Definition 1. *A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is said to be upper almost weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous if for each $x \in X$ and each $\sigma_1\sigma_2$-open set V of Y such that $F(x) \subseteq V$, $x \in Int^\star(Cl^\star(F^+(\sigma_1\sigma_2\text{-}Cl(V))))$.*

Theorem 1. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is upper almost weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous;

(2) $F^+(V) \subseteq Int^\star(Cl^\star(F^+(\sigma_1\sigma_2\text{-}Cl(V))))$ for every $\sigma_1\sigma_2$-open set V of Y;

(3) $Cl^\star(Int^\star(F^-(V))) \subseteq F^-(\sigma_1\sigma_2\text{-}Cl(V))$ for every $\sigma_1\sigma_2$-open set V of Y;

(4) $pCl^\star(F^-(V)) \subseteq F^-(\sigma_1\sigma_2\text{-}Cl(V))$ for every $\sigma_1\sigma_2$-open set V of Y;

(5) $F^+(V) \subseteq pInt^\star(F^+(\sigma_1\sigma_2\text{-}Cl(V)))$ for every $\sigma_1\sigma_2$-open set V of Y;

(6) for each $x \in X$ and each $\sigma_1\sigma_2$-open set V of Y containing $F(x)$, there exists an $\mathscr{I}^\star$-preopen set U of X containing x such that $F(U) \subseteq \sigma_1\sigma_2\text{-}Cl(V)$.

Proof. (1) $\Rightarrow$ (2): Let V be any $\sigma_1\sigma_2$-open set of Y and $x \in F^+(V)$. Then, $F(x) \subseteq V$ and by (1), we have $x \in \mathrm{Int}^\star(\mathrm{Cl}^\star(F^+(\sigma_1\sigma_2\text{-}\mathrm{Cl}(V))))$. Therefore,

$$F^+(V) \subseteq \mathrm{Int}^\star(\mathrm{Cl}^\star(F^+(\sigma_1\sigma_2\text{-}\mathrm{Cl}(V)))).$$

(2) $\Rightarrow$ (3): Let V be any $\sigma_1\sigma_2$-open set of Y. Since $Y - \sigma_1\sigma_2\text{-}\mathrm{Cl}(V)$ is $\sigma_1\sigma_2$-open and by (2),

$$X - F^-(\sigma_1\sigma_2\text{-}\mathrm{Cl}(V)) = F^+(Y - \sigma_1\sigma_2\text{-}\mathrm{Cl}(V))$$

$$\begin{aligned}&\subseteq \text{Int}^\star(\text{Cl}^\star(F^+(\sigma_1\sigma_2\text{-Cl}(Y-\sigma_1\sigma_2\text{-Cl}(V)))))\\&\subseteq \text{Int}^\star(\text{Cl}^\star(F^+(Y-V)))\\&= \text{Int}^\star(\text{Cl}^\star(X-F^-(V)))\\&= X-\text{Cl}^\star(\text{Int}^\star(F^-(V))).\end{aligned}$$

Thus, $\text{Cl}^\star(\text{Int}^\star(F^-(V)))\subseteq F^-(\sigma_1\sigma_2\text{-Cl}(V))$.

(3) ⇒ (4): Let V be any $\sigma_1\sigma_2$-open set of Y. By (3) and Lemma 2,

$$\text{pCl}^\star(F^-(V))=\text{Cl}^\star(\text{Int}^\star(F^-(V)))\cup F^-(V)\subseteq F^-(\sigma_1\sigma_2\text{-Cl}(V)).$$

(4) ⇒ (5): Let V be any $\sigma_1\sigma_2$-open set of Y. Then, $Y-\sigma_1\sigma_2\text{-Cl}(V)$ is $\sigma_1\sigma_2$-open in Y. Thus by (4),

$$\begin{aligned}X-\text{pInt}^\star(F^+(\sigma_1\sigma_2\text{-Cl}(V)))&=\text{pCl}^\star(X-F^+(\sigma_1\sigma_2\text{-Cl}(V)))\\&=\text{pCl}^\star(F^-(Y-\sigma_1\sigma_2\text{-Cl}(V)))\\&\subseteq F^-(\sigma_1\sigma_2\text{-Cl}(Y-\sigma_1\sigma_2\text{-Cl}(V)))\\&\subseteq F^-(Y-V)=X-F^+(V)\end{aligned}$$

and hence $F^+(V)\subseteq \text{pInt}^\star(F^+(\sigma_1\sigma_2\text{-Cl}(V)))$.

(5) ⇒ (6): Let $x\in X$ and V be any $\sigma_1\sigma_2$-open set of Y containing $F(x)$. By (5), $x\in F^+(V)\subseteq \text{pInt}^\star(F^+(\sigma_1\sigma_2\text{-Cl}(V)))$ and there exists a $\mathscr{I}^\star$-preopen set U of X containing x such that $F(U)\subseteq \sigma_1\sigma_2\text{-Cl}(V)$.

(6) ⇒ (1): Let $x\in X$ and V be any $\sigma_1\sigma_2$-open set of Y containing $F(x)$. By (6), there exists an $\mathscr{I}^\star$-preopen set U of X containing x such that $F(U)\subseteq\sigma_1\sigma_2\text{-Cl}(V)$; hence $U\subseteq F^+(\sigma_1\sigma_2\text{-Cl}(V))$. Thus, $x\in U\subseteq \text{Int}^\star(\text{Cl}^\star(U))\subseteq \text{Int}^\star(\text{Cl}^\star(F^+(\sigma_1\sigma_2\text{-Cl}(V))))$. This shows that F is upper almost weakly $\tau^\star(\sigma_1,\sigma_2)$-continuous.

Definition 2. *A multifunction $F:(X,\tau,\mathscr{I})\to(Y,\sigma_1,\sigma_2)$ is said to be lower almost weakly $\tau^\star(\sigma_1,\sigma_2)$-continuous if for each $x\in X$ and each $\sigma_1\sigma_2$-open set V of Y such that $F(x)\cap V\neq\emptyset$, $x\in Int^\star(Cl^\star(F^-(\sigma_1\sigma_2\text{-}Cl(V))))$.*

Theorem 2. *For a multifunction $F:(X,\tau,\mathscr{I})\to(Y,\sigma_1,\sigma_2)$, the following properties are equivalent:*

(1) F is lower almost weakly $\tau^\star(\sigma_1,\sigma_2)$-continuous;

(2) $F^-(V)\subseteq Int^\star(Cl^\star(F^-(\sigma_1\sigma_2\text{-}Cl(V))))$ for every $\sigma_1\sigma_2$-open set V of Y;

(3) $Cl^\star(Int^\star(F^+(V)))\subseteq F^+(\sigma_1\sigma_2\text{-}Cl(V))$ for every $\sigma_1\sigma_2$-open set V of Y;

(4) $pCl^\star(F^+(V))\subseteq F^+(\sigma_1\sigma_2\text{-}Cl(V))$ for every $\sigma_1\sigma_2$-open set V of Y;

(5) $F^-(V)\subseteq pInt^\star(F^-(\sigma_1\sigma_2\text{-}Cl(V)))$ for every $\sigma_1\sigma_2$-open set V of Y;

(6) for each $x\in X$ and each $\sigma_1\sigma_2$-open set V of Y such that $F(x)\cap V\neq\emptyset$, there exists an $\mathscr{I}^\star$-preopen set U of X containing x such that $F(z)\cap\sigma_1\sigma_2\text{-}Cl(V)\neq\emptyset$ for each $z\in U$.

Proof. The proof is similar to that of Theorem 1.

Theorem 3. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is upper almost weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous;

(2) $Cl^\star(Int^\star(F^-(\sigma_1\sigma_2\text{-}Int(K)))) \subseteq F^-(K)$ for every $\sigma_1\sigma_2$-closed set K of Y;

(3) $pCl^\star(F^-(\sigma_1\sigma_2\text{-}Int(K))) \subseteq F^-(K)$ for every $\sigma_1\sigma_2$-closed set K of Y;

(4) $pCl^\star(F^-(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2 Cl(B)))) \subseteq F^-(\sigma_1\sigma_2\text{-}Cl(B))$ for every subset B of Y;

(5) $F^+(\sigma_1\sigma_2\text{-}Int(B)) \subseteq pInt^\star(F^+(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(B))))$ for every subset B of Y.

Proof. (1) ⇒ (2): Let K be any $\sigma_1\sigma_2$-closed set of Y. Then, $Y - K$ is $\sigma_1\sigma_2$-open in Y, by Theorem 1, we have

$$\begin{aligned} X - F^-(K) = F^+(Y - K) &\subseteq \mathrm{Int}^\star(\mathrm{Cl}^\star(F^+(\sigma_1\sigma_2\text{-}\mathrm{Cl}(Y - K)))) \\ &= \mathrm{Int}^\star(\mathrm{Cl}^\star(F^+(Y - \sigma_1\sigma_2\text{-}\mathrm{Int}(K)))) \\ &= \mathrm{Int}^\star(\mathrm{Cl}^\star(X - F^-(\sigma_1\sigma_2\text{-}\mathrm{Int}(K)))) \\ &= X - \mathrm{Cl}^\star(\mathrm{Int}^\star(F^-(\sigma_1\sigma_2\text{-}\mathrm{Int}(K)))) \end{aligned}$$

and so $\mathrm{Cl}^\star(\mathrm{Int}^\star(F^-(\sigma_1\sigma_2\text{-}\mathrm{Int}(K)))) \subseteq F^-(K)$.

(2) ⇒ (3): Let K be any $\sigma_1\sigma_2$-closed set of Y. By Lemma 2, we have

$$\mathrm{pCl}^\star(F^-(\sigma_1\sigma_2\text{-}\mathrm{Int}(K))) = F^-(\sigma_1\sigma_2\text{-}\mathrm{Int}(K)) \cup \mathrm{Cl}^\star(\mathrm{Int}^\star(F^-(\sigma_1\sigma_2\text{-}\mathrm{Int}(K)))) \subseteq F^-(K).$$

(3) ⇒ (4): The proof is obvious.

(4) ⇒ (5): Let B be any subset of Y. By (4), we have

$$\begin{aligned} X - \mathrm{pInt}^\star(F^+(\sigma_1\sigma_2\text{-}\mathrm{Cl}(\sigma_1\sigma_2\text{-}\mathrm{Int}(B)))) &= \mathrm{pCl}^\star(X - F^+(\sigma_1\sigma_2\text{-}\mathrm{Cl}(\sigma_1\sigma_2\text{-}\mathrm{Int}(B)))) \\ &= \mathrm{pCl}^\star(F^-(Y - \sigma_1\sigma_2\text{-}\mathrm{Cl}(\sigma_1\sigma_2\text{-}\mathrm{Int}(B)))) \\ &= \mathrm{pCl}^\star(F^-(\sigma_1\sigma_2\text{-}\mathrm{Int}(\sigma_1\sigma_2\text{-}\mathrm{Cl}(Y - B)))) \\ &\subseteq F^-(\sigma_1\sigma_2\text{-}\mathrm{Cl}(Y - B)) \\ &= X - F^+(\sigma_1\sigma_2\text{-}\mathrm{Int}(B)). \end{aligned}$$

Thus, $F^+(\sigma_1\sigma_2\text{-}\mathrm{Int}(B)) \subseteq \mathrm{pInt}^\star(F^+(\sigma_1\sigma_2\text{-}\mathrm{Cl}(\sigma_1\sigma_2\text{-}\mathrm{Int}(B))))$.

(5) ⇒ (1): Let V be any $\sigma_1\sigma_2$-open set of Y. Then by (5), we have

$$F^+(V) \subseteq \mathrm{pInt}^\star(F^+(\sigma_1\sigma_2\text{-}\mathrm{Cl}(V)))$$

and hence F is upper almost weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous by Theorem 1.

Theorem 4. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is lower almost weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous;

(2) $Cl^\star(Int^\star(F^+(\sigma_1\sigma_2\text{-}Int(K)))) \subseteq F^+(K)$ for every $\sigma_1\sigma_2$-closed set K of Y;

(3) $pCl^\star(F^+(\sigma_1\sigma_2\text{-}Int(K))) \subseteq F^+(K)$ for every $\sigma_1\sigma_2$-closed set K of Y;

(4) $pCl^\star(F^+(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(B)))) \subseteq F^+(\sigma_1\sigma_2\text{-}Cl(B))$ for every subset B of Y;

(5) $F^-(\sigma_1\sigma_2\text{-}Int(B)) \subseteq pInt^\star(F^-(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(B))))$ for every subset B of Y.

Proof. The proof is similar to that of Theorem 3.

Theorem 5. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is upper almost weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous;

(2) $pCl^\star(F^-(\sigma_1\sigma_2\text{-}Int((\sigma_1, \sigma_2)\text{-}\theta Cl(B)))) \subseteq F^-((\sigma_1, \sigma_2)\theta\text{-}Cl(B))$ for every subset B of Y;

(3) $pCl^\star(F^-(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V)))) \subseteq F^-(\sigma_1\sigma_2\text{-}Cl(V))$ for every $\sigma_1\sigma_2$-open set V of Y;

(4) $pCl^\star(F^-(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V)))) \subseteq F^-(\sigma_1\sigma_2\text{-}Cl(V))$ for every $(\sigma_1, \sigma_2)p$-open set V of Y;

(5) $pCl^\star(F^-(\sigma_1\sigma_2\text{-}Int(K))) \subseteq F^-(K)$ for every $(\sigma_1, \sigma_2)r$-closed set K of Y.

Proof. (1) $\Rightarrow$ (2): Let B be any subset of Y. Let $x \in X - F^-((\sigma_1, \sigma_2)\theta\text{-Cl}(B))$. Then, $x \in F^+(Y - (\sigma_1, \sigma_2)\theta\text{-Cl}(B))$ and $(\sigma_1, \sigma_2)\theta\text{-Cl}(B)$ is $\sigma_1\sigma_2$-closed in Y. By Theorem 1, there exists an $\mathscr{I}^\star$-preopen set U of X containing x such that

$$\begin{aligned} U \subseteq F^+(\sigma_1\sigma_2\text{-Cl}(Y - (\sigma_1, \sigma_2)\theta\text{-Cl}(B))) &= F^+(Y - \sigma_1\sigma_2\text{-Int}((\sigma_1, \sigma_2)\theta\text{-Cl}(B))) \\ &= X - F^-(\sigma_1\sigma_2\text{-Int}((\sigma_1, \sigma_2)\theta\text{-Cl}(B))). \end{aligned}$$

Thus, $U \cap F^-(\sigma_1\sigma_2\text{-Int}((\sigma_1, \sigma_2)\theta\text{-Cl}(B))) = \emptyset$ and hence

$$x \in X - \text{pCl}^\star(F^-(\sigma_1\sigma_2\text{-Int}((\sigma_1, \sigma_2)\theta\text{-Cl}(B)))).$$

Therefore, $\text{pCl}^\star(F^-(\sigma_1\sigma_2\text{-Int}((\sigma_1, \sigma_2)\theta\text{-Cl}(B)))) \subseteq F^-((\sigma_1, \sigma_2)\theta\text{-Cl}(B))$.

(2) $\Rightarrow$ (3): The proof is obvious since $(\sigma_1, \sigma_2)\theta\text{-Cl}(V) = \sigma_1\sigma_2\text{-Cl}(V)$ for every $\sigma_1\sigma_2$-open set V of Y.

(3) $\Rightarrow$ (4): Let V be any $(\sigma_1, \sigma_2)p$-open set of Y. Then, $V \subseteq \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))$ and by (3), we have

$$\begin{aligned} \text{pCl}^\star(F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) &= \text{pCl}^\star(F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))))) \\ &\subseteq F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Int}(V)))) \\ &= F^-(\sigma_1\sigma_2\text{-Cl}(V)). \end{aligned}$$

(4) ⇒ (5): Let K be any $(\sigma_1, \sigma_2)r$-closed set of Y. Then, $\sigma_1\sigma_2$-Int(K) is $(\sigma_1, \sigma_2)p$-open in Y and by (4),

$$\begin{aligned} \text{pCl}^\star(F^-(\sigma_1\sigma_2\text{-Int}(K))) &= \text{pCl}^\star(F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K))))) \\ &\subseteq F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K))) = F^-(K). \end{aligned}$$

(5) ⇒ (1): Let V be any $\sigma_1\sigma_2$-open set of Y. Then, $\sigma_1\sigma_2$-Cl(V) is $(\sigma_1, \sigma_2)r$-closed in Y and by (5), $\text{pCl}^\star(F^-(V)) \subseteq \text{pCl}^\star(F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) \subseteq F^-(\sigma_1\sigma_2\text{-Cl}(V))$. It follows from Theorem 1 that F is upper almost weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous.

Theorem 6. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is lower almost weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous;

(2) $pCl^\star(F^+(\sigma_1\sigma_2\text{-}Int((\sigma_1, \sigma_2)\theta\text{-}Cl(B)))) \subseteq F^+((\sigma_1, \sigma_2)\theta\text{-}Cl(B))$ for every subset B of Y;

(3) $pCl^\star(F^+(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V)))) \subseteq F^+(\sigma_1\sigma_2\text{-}Cl(V))$ for every $\sigma_1\sigma_2$-open set V of Y;

(4) $pCl^\star(F^+(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V)))) \subseteq F^+(\sigma_1\sigma_2\text{-}Cl(V))$ for every $(\sigma_1, \sigma_2)p$-open set V of Y;

(5) $pCl^\star(F^+(\sigma_1\sigma_2\text{-}Int(K))) \subseteq F^+(K)$ for every $(\sigma_1, \sigma_2)r$-closed set K of Y.

Proof. The proof is similar to that of Theorem 5.

For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, by $\text{Cl}F_\imath : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ (resp. $\text{pCl}F_\imath : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$) we denote a multifunction defined as follows: $\text{Cl}F_\imath(x) = \sigma_1\sigma_2\text{-Cl}(F(x))$ (resp. $\text{pCl}F_\imath(x) = (\sigma_1, \sigma_2)\text{-pCl}(F(x))$) for each $x \in X$.

Definition 3. [21] *A subset A of a bitopological space (X, τ_1, τ_2) is said to be:*

(1) $\tau_1\tau_2$-paracompact if every cover of A by $\tau_1\tau_2$-open sets of X is refined by a cover of A which consists of $\tau_1\tau_2$-open sets of X and is $\tau_1\tau_2$-locally finite in X;

(2) $\tau_1\tau_2$-regular if for each $x \in A$ and each $\tau_1\tau_2$-open set U of X containing x, there exists a $\tau_1\tau_2$-open set V of X such that $x \in V \subseteq \tau_1\tau_2\text{-}Cl(V) \subseteq U$.

Lemma 3. [21] *If A is a $\tau_1\tau_2$-regular $\tau_1\tau_2$-paracompact set of a bitopological space (X, τ_1, τ_2) and U is a $\tau_1\tau_2$-open neighbourhood of A, then there exists a $\tau_1\tau_2$-open set V of X such that $A \subseteq V \subseteq \tau_1\tau_2\text{-}Cl(V) \subseteq U$.*

Lemma 4. *If $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is a multifunction such that $F(x)$ is $\sigma_1\sigma_2$-regular and $\sigma_1\sigma_2$-paracompact for each $x \in X$, then $ClF_\imath^+(V) = pClF_\imath^+(V) = F^+(V)$ for each $\sigma_1\sigma_2$-open set V of Y.*

Proof. It follows from Lemma 5 of [28].

Theorem 7. *Let $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ be a multifunction such that $F(x)$ is $\sigma_1\sigma_2$-paracompact and $\sigma_1\sigma_2$-regular for each $x \in X$. Then, the following properties are equivalent:*

(1) F is upper almost weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous;

(2) $p\mathrm{Cl}F_\imath$ is upper almost weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous;

(3) $\mathrm{Cl}F_\imath$ is upper almost weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous.

Proof. We put $G = \mathrm{Cl}F_\imath$ or $\mathrm{pCl}F_\imath$ in the sequel. Suppose that F is upper almost weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous. Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y containing $G(x)$. By Lemma 4, we have $x \in G^+(V) = F^+(V)$ and hence there exists an $\mathscr{I}^\star$-preopen set U containing x such that $F(U) \subseteq \sigma_1\sigma_2\text{-Cl}(V)$. Since $F(z)$ is $\sigma_1\sigma_2$-paracompact and $\sigma_1\sigma_2$-regular for each $z \in U$, by Lemma 3 there exists a $\sigma_1\sigma_2$-open set W such that $F(z) \subseteq W \subseteq \sigma_1\sigma_2\text{-Cl}(W) \subseteq V$; hence $G(z) \subseteq \sigma_1\sigma_2\text{-Cl}(W) \subseteq \sigma_1\sigma_2\text{-Cl}(V)$ for each $z \in U$. Thus, $G(U) \subseteq \sigma_1\sigma_2\text{-Cl}(V)$. This shows that G is upper almost weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous.

Conversely, suppose that G is upper almost weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous. Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y containing $G(x)$. By Lemma 4, we have $x \in F^+(V) = G^+(V)$ and hence $G(x) \subseteq V$. Then, there exists an $\mathscr{I}^\star$-preopen set U containing x such that $F(U) \subseteq \sigma_1\sigma_2\text{-Cl}(V)$. Thus, $U \subseteq G^+(V) = F^+(V)$ and hence $F(U) \subseteq \sigma_1\sigma_2\text{-Cl}(V)$. This shows that F is upper almost weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous.

Lemma 5. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, $\mathrm{Cl}F_\imath^-(V) = p\mathrm{Cl}F_\imath^-(V) = F^-(V)$ for each $\sigma_1\sigma_2$-open set V of Y.*

Proof. It follows from Lemma 3 of [28].

Theorem 8. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is lower almost weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous;

(2) $p\mathrm{Cl}F_\imath$ is lower almost weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous;

(3) $\mathrm{Cl}F_\imath$ is lower almost weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous.

Proof. By using Lemma 5 this can be shown similarly to that of Theorem 7.

The $\star$-*prefrontier* of a subset A of an ideal topological space $(X, \tau, \mathscr{I})$, denoted by $\mathrm{pfr}^\star(A)$, is defined by $\mathrm{pfr}^\star(A) = \mathrm{pCl}^\star(A) \cap \mathrm{pCl}^\star(X - A) = \mathrm{pCl}^\star(A) - \mathrm{pInt}^\star(A)$.

Theorem 9. *The set of all points x of X at which a multifunction*

$$F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$$

is not upper almost weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous is identical with the union of the $\star$-prefrontier of the upper inverse images of the $\sigma_1\sigma_2$-closure of $\sigma_1\sigma_2$-open sets containing $F(x)$.

Proof. Let $x \in X$ at which F is not upper almost weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous. There exists a $\sigma_1\sigma_2$-open set V of Y containing $F(x)$ such that $U \cap (X - F^+(V)) \neq \emptyset$ for every $\mathscr{I}^\star$-preopen set U of X containing x. Therefore, we have

$$x \in \mathrm{pCl}^\star(X - F^+(\sigma_1\sigma_2\text{-}\mathrm{Cl}(V))) = X - \mathrm{pInt}^\star(F^+(\sigma_1\sigma_2\text{-}\mathrm{Cl}(V))).$$

Since $x \in F^+(V)$, we have $x \in \mathrm{pCl}^\star(F^+(\sigma_1\sigma_2\text{-}\mathrm{Cl}(V)))$ and so $x \in \mathrm{pfr}^\star(F^+(\sigma_1\sigma_2\text{-}\mathrm{Cl}(V)))$.

Conversely, if F is upper almost weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous, then for any $\sigma_1\sigma_2$-open set V of Y containing $F(x)$ there exists an $\mathscr{I}^\star$-preopen set U of X containing x such that $F(U) \subseteq \sigma_1\sigma_2\text{-}\mathrm{Cl}(V)$; hence $U \subseteq F^+(\sigma_1\sigma_2\text{-}\mathrm{Cl}(V))$. Therefore, $x \in \mathrm{pInt}^\star(F^+(\sigma_1\sigma_2\text{-}\mathrm{Cl}(V)))$. This contradicts with the fact that $x \in \mathrm{pfr}^\star(F^+(\sigma_1\sigma_2\text{-}\mathrm{Cl}(V)))$. Thus, F is not upper almost weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous at x.

Theorem 10. *The set of all points x of X at which a multifunction*

$$F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$$

is not lower almost weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous is identical with the union of the $\star$-prefrontier of the lower inverse images of $\sigma_1\sigma_2$-closure of $\sigma_1\sigma_2$-open sets meeting $F(x)$.

Proof. The proof is similar to that of Theorem 9.

Definition 4. *A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is said to be upper $\tau^\star(\sigma_1, \sigma_2)$-precontinuous at a point $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y such that $F(x) \subseteq V$, there exists an $\mathscr{I}^\star$-preopen set U of X containing x such that $F(U) \subseteq V$. A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is said to be upper $\tau^\star(\sigma_1, \sigma_2)$-precontinuous if F is upper $\tau^\star(\sigma_1, \sigma_2)$-precontinuous at each point x of X.*

Theorem 11. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is upper $\tau^\star(\sigma_1, \sigma_2)$-precontinuous;

(2) $F^+(V)$ is $\mathscr{I}^\star$-preopen in X for every $\sigma_1\sigma_2$-open set V of Y;

(3) $F^-(K)$ is $\mathscr{I}^\star$-preclosed in X for every $\sigma_1\sigma_2$-closed set K of Y;

(4) $pCl^\star(F^-(B)) \subseteq F^-(\sigma_1\sigma_2\text{-}Cl(B))$ for every subset B of Y;

(5) $F^+(\sigma_1\sigma_2\text{-}Int(B)) \subseteq pInt^\star(F^+(B))$ *for every subset* B *of* Y*.*

Proof. (1) ⇒ (2): Let V be any $\sigma_1\sigma_2$-open set of Y and $x \in F^+(V)$. Then, $F(x) \subseteq V$ and by (1), there exists an $\mathscr{I}^\star$-preopen set U of X containing x such that $F(U) \subseteq V$. Thus, $x \in U \subseteq F^+(V)$ and hence $x \in \text{pInt}^\star(F^+(V))$. Therefore, $F^+(V) \subseteq \text{pInt}^\star(F^+(V))$. This shows that $F^+(V)$ is $\mathscr{I}^\star$-preopen in X.

(2) ⇒ (3): This follows from the fact that $F^+(Y - B) = X - F^-(B)$ for every subset B of Y.

(3) ⇒ (4): Let B be any subset of Y. Then, $\sigma_1\sigma_2\text{-Cl}(B)$ is $\sigma_1\sigma_2$-closed in Y and by (3), $\text{pCl}^\star(F^-(B)) \subseteq \text{pCl}^\star(F^-(\sigma_1\sigma_2\text{-Cl}(B))) = F^-(\sigma_1\sigma_2\text{-Cl}(B))$.

(4) ⇒ (5): Let B be any subset of Y. By (4), $X - \text{pInt}^\star(F^+(B)) = \text{pCl}^\star(X - F^+(B)) = \text{pCl}^\star(F^-(Y - B)) \subseteq F^-(\sigma_1\sigma_2\text{-Cl}(Y - B)) = F^-(Y - \sigma_1\sigma_2\text{-Int}(B)) = X - F^+(\sigma_1\sigma_2\text{-Int}(B))$ and hence $F^+(\sigma_1\sigma_2\text{-Int}(B)) \subseteq \text{pInt}^\star(F^+(B))$.

(5) ⇒ (1): Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y such that $F(x) \subseteq V$. Then, $x \in F^+(V) = \text{pInt}^\star(F^+(V))$. There exists an $\mathscr{I}^\star$-preopen set U of X containing x such that $U \subseteq F^+(V)$; hence $F(U) \subseteq V$. This shows that F is upper $\tau^\star(\sigma_1, \sigma_2)$-precontinuous.

Definition 5. *A multifunction* $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ *is said to be lower* $\tau^\star(\sigma_1, \sigma_2)$*-precontinuous at a point* $x \in X$ *if for each* $\sigma_1\sigma_2$*-open set* V *of* Y *such that* $F(x) \cap V \neq \emptyset$*, there exists an* $\mathscr{I}^\star$*-preopen set* U *of* X *containing* x *such that* $F(z) \cap V \neq \emptyset$ *for every* $z \in U$*. A multifunction* $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ *is called lower* $\tau^\star(\sigma_1, \sigma_2)$*-precontinuous if* F *is lower* $\tau^\star(\sigma_1, \sigma_2)$*-precontinuous at each point* x *of* X*.*

Theorem 12. *For a multifunction* $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$*, the following properties are equivalent:*

(1) F *is lower* $\tau^\star(\sigma_1, \sigma_2)$*-precontinuous;*

(2) $F^-(V)$ *is* $\mathscr{I}^\star$*-preopen in* X *for every* $\sigma_1\sigma_2$*-open set* V *of* Y*;*

(3) $F^+(K)$ *is* $\mathscr{I}^\star$*-preclosed in* X *for every* $\sigma_1\sigma_2$*-closed set* K *of* Y*;*

(4) $pCl^\star(F^+(B)) \subseteq F^+(\sigma_1\sigma_2\text{-}Cl(B))$ *for every subset* B *of* Y*;*

(5) $F(pCl^\star(A)) \subseteq \sigma_1\sigma_2\text{-}Cl(F(A))$ *for every subset* A *of* X*;*

(6) $F^-(\sigma_1\sigma_2\text{-}Int(B)) \subseteq pInt^\star(F^-(B))$ *for every subset* B *of* Y*.*

Proof. We prove only the implications (4) ⇒ (5) and (5) ⇒ (6) being the proofs of the other similar to those of Theorem 11.

(4) ⇒ (5): Let A be any subset of X. Thus by (4), $\text{pCl}^\star(A) \subseteq \text{pCl}^\star(F^+(F(A))) \subseteq F^+(\sigma_1\sigma_2\text{-Cl}(F(A)))$ and so $F(\text{pCl}^\star(A)) \subseteq \sigma_1\sigma_2\text{-Cl}(F(A))$.

(5) ⇒ (6): Let B be any subset of Y. By (5),

$$F(\text{pCl}^\star(F^+(Y - B))) \subseteq \sigma_1\sigma_2\text{-Cl}(F(F^+(Y - B))) \subseteq \sigma_1\sigma_2\text{-Cl}(Y - B) = Y - \sigma_1\sigma_2\text{-Int}(B).$$

Since $F(\text{pCl}^\star(F^+(Y-B))) = F(\text{pCl}^\star(X-F^-(B))) = F(X-\text{pInt}^\star(F^-(B)))$, we have

$$X-\text{pInt}^\star(F^-(B)) \subseteq F^+(Y-\sigma_1\sigma_2\text{-Int}(B)) = X-F^-(\sigma_1\sigma_2\text{-Int}(B))$$

and hence $F^-(\sigma_1\sigma_2\text{-Int}(B)) \subseteq \text{pInt}^\star(F^-(B))$.

Recall that a bitopological space (X,τ_1,τ_2) is said to be (τ_1,τ_2)-*regular* [29] if for each $\tau_1\tau_2$-closed set F and each $x \notin F$, there exist disjoint $\tau_1\tau_2$-open sets U and V such that $x \in U$ and $F \subseteq V$.

Lemma 6. [30] *Let (X,τ_1,τ_2) be a (τ_1,τ_2)-regular space. Then, the following properties hold:*

(1) $\tau_1\tau_2$-$Cl(A) = (\tau_1,\tau_2)\theta$-$Cl(A)$ for every subset A of X.

(2) Every $\tau_1\tau_2$-open set is $(\tau_1,\tau_2)\theta$-open.

Theorem 13. *For a multifunction $F : (X,\tau,\mathscr{I}) \to (Y,\sigma_1,\sigma_2)$, where (Y,σ_1,σ_2) is (σ_1,σ_2)-regular, the following properties are equivalent:*

(1) F is upper $\tau^\star(\sigma_1,\sigma_2)$-precontinuous;

(2) $F^-((\sigma_1,\sigma_2)\theta$-$Cl(B))$ is $\mathscr{I}^\star$-preclosed in X for every subset B of Y;

(3) $F^-(K)$ is $\mathscr{I}^\star$-preclosed in X for every $(\sigma_1,\sigma_2)\theta$-closed set K of Y;

(4) $F^+(V)$ is $\mathscr{I}^\star$-preopen in X for every $(\sigma_1,\sigma_2)\theta$-open set V of Y.

Proof. (1) $\Rightarrow$ (2): Let B be any subset of Y. Then, $(\sigma_1,\sigma_2)\theta$-Cl$(B)$ is $\sigma_1\sigma_2$-closed in Y and by Theorem 11, $F^-((\sigma_1,\sigma_2)\theta\text{-Cl}(B))$ is $\mathscr{I}^\star$-preclosed in X.

(2) $\Rightarrow$ (3): The proof is obvious.

(3) $\Rightarrow$ (4): Let V be any $(\sigma_1,\sigma_2)\theta$-open set of Y. By (3), $F^-(Y-V)$ is $\mathscr{I}^\star$-preclosed in X and $F^-(Y-V) = X-F^+(V)$. Thus, $F^+(V)$ is $\mathscr{I}^\star$-preopen in X.

(4) $\Rightarrow$ (1): Let V be any $\sigma_1\sigma_2$-open set of Y. Since (Y,σ_1,σ_2) is (σ_1,σ_2)-regular, by Lemma 6 we have V is $(\sigma_1,\sigma_2)\theta$-open in Y and by (4), $F^+(V)$ is $\mathscr{I}^\star$-preopen in X. Thus by Theorem 11, F is upper $\tau^\star(\sigma_1,\sigma_2)$-precontinuous.

Theorem 14. *For a multifunction $F : (X,\tau,\mathscr{I}) \to (Y,\sigma_1,\sigma_2)$, where (Y,σ_1,σ_2) is (σ_1,σ_2)-regular, the following properties are equivalent:*

(1) F is lower $\tau^\star(\sigma_1,\sigma_2)$-precontinuous;

(2) $F^+((\sigma_1,\sigma_2)\theta$-$Cl(B))$ is $\mathscr{I}^\star$-preclosed in X for every subset B of Y;

(3) $F^+(K)$ is $\mathscr{I}^\star$-preclosed in X for every $(\sigma_1,\sigma_2)\theta$-closed set K of Y;

(4) $F^-(V)$ is $\mathscr{I}^\star$-preopen in X for every $(\sigma_1,\sigma_2)\theta$-open set V of Y;

(5) F is lower almost weakly $\tau^\star(\sigma_1,\sigma_2)$-continuous.

Proof. We prove only the implication (5) $\Rightarrow$ (1), the proof of the other being similar to that of Theprem 13. The proof of the implication (4) $\Rightarrow$ (5) is obvious.

(5) $\Rightarrow$ (1): Let V be any $\sigma_1\sigma_2$-open set of Y and $x \in F^-(V)$. Then, $F(x) \cap V \neq \emptyset$. Since (Y, σ_1, σ_2) is (σ_1, σ_2)-regular, there exists a $\sigma_1\sigma_2$-open set W of Y such that $F(x) \cap W \neq \emptyset$ and $\sigma_1\sigma_2$-Cl$(W) \subseteq V$. Since F is lower almost weakly $\tau^\star(\sigma_1, \sigma_2)$-continuous, by Theorem 2 there exists an $\mathscr{I}^\star$-open set U of X containing x such that

$$U \subseteq F^-(\sigma_1\sigma_2\text{-Cl}(W)) \subseteq F^-(V).$$

Thus, $x \in U \subseteq \text{Int}^\star(\text{Cl}^\star(U)) \subseteq \text{Int}^\star(\text{Cl}^\star(F^-(V)))$ and hence $F^-(V) \subseteq \text{Int}^\star(\text{Cl}^\star(F^-(V)))$. Therefore, $F^-(V)$ is $\mathscr{I}^\star$-preopen in X. By Theorem 12, F is lower $\tau^\star(\sigma_1, \sigma_2)$-precontinuous.

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