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Almost Quasi Continuity and Weak Quasi Continuity for Multifunctions between an Ideal Topological Space and a Bitopological Space

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Abstract. This paper introduces four classes of continuous multifunctions defined from an ideal topological space into a bitopological space, namely upper almost quasi $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions, lower almost quasi $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions, upper weakly quasi $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions and lower weakly quasi $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions. Moreover, several characterizations of upper almost quasi $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions, lower almost quasi $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions, upper weakly quasi $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions and lower weakly quasi $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions are established. Furthermore, the relationships between almost quasi $\tau^\star(\sigma_1, \sigma_2)$-continuity and weak quasi $\tau^\star(\sigma_1, \sigma_2)$-continuity are discussed.

2020 Mathematics Subject Classifications: 54C08, 54C60

Key Words and Phrases: Upper almost quasi $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunction, lower almost quasi $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunction, upper weakly quasi $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunction, lower weakly quasi $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunction

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1. Introduction

The concept of quasi continuous functions was introduced by Marcus [1]. Popa [2] introduced and investigated the notion of almost quasi continuous functions. Neubrunnovaá [3] showed that quasi continuity is equivalent to semi-continuity due to Levine [4]. Popa and Stan [5] introduced and studied the notion of weakly quasi continuous functions. Weak quasi continuity is implied by quasi continuity and weak continuity [6] which are independent of each other. It is shown in [7] that weak quasi continuity

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is equivalent to weak semi-continuity due to Arya and Bhamini [8] and Kar and Bhattacharyya [9]. Janković and Hamlett [10] introduced the concept of $\mathscr{I}$-open sets in ideal topological spaces. Abd El-Monsef et al. [11] introduced and studied the notions of $\mathscr{I}$-closed sets and $\mathscr{I}$-continuous functions. Semi-$\mathscr{I}$-open sets, pre-$\mathscr{I}$-open sets, α-$\mathscr{I}$-open sets, β-$\mathscr{I}$-open sets and δ-$\mathscr{I}$-open sets play an important role in the research of generalizations of continuity in ideal topological spaces. Using these notions many authors introduced and studied various types of generalizations of continuity for functions and multifunctions. Hatir and Noiri [12] introduced and investigated the notions of weakly pre-$\mathscr{I}$-open sets and weakly pre-$\mathscr{I}$-continuous functions. Moreover, Hatir and Noiri [13] investigated further properties of semi-$\mathscr{I}$-open sets and semi-$\mathscr{I}$-continuous functions. On the other hand, the present author introduced and investigated the concepts of almost quasi $\star$-continuous multifunctions [14], weakly quasi $\star$-continuous multifunctions [14], $p\iota$-continuous multifunctions [15] and weakly $p\iota$-continuous multifunctions [15]. Pue-on et al. [16] introduced and studied the concepts of upper (τ_1, τ_2)-continuous multifunctions and lower (τ_1, τ_2)-continuous multifunctions. Klanarong et al. [17] introduced and investigated the notions of upper almost (τ_1, τ_2)-continuous multifunctions and lower almost (τ_1, τ_2)-continuous multifunctions. Thongmoon et al. [18] introduced and studied the concepts of upper weakly (τ_1, τ_2)-continuous multifunctions and lower weakly (τ_1, τ_2)-continuous multifunctions. Chiangpradit et al. [19] introduced and investigated the notions of upper almost quasi (τ_1, τ_2)-continuous multifunctions and lower almost quasi (τ_1, τ_2)-continuous multifunctions. Quite recently, Pue-on et al. [20] introduced and studied the concepts of upper weakly quasi (τ_1, τ_2)-continuous multifunctions and lower weakly quasi (τ_1, τ_2)-continuous multifunctions. In this paper, we introduce new classes of continuous multifunctions between an ideal topological space and a bitopological space, namely upper almost quasi $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions, lower almost quasi $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions, upper weakly quasi $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions and lower weakly quasi $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions. We also investigate several characterizations of upper almost quasi $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions, lower almost quasi $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions, upper weakly quasi $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions and lower weakly quasi $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions.

2. Preliminaries

Throughout the present paper, spaces (X, τ_1, τ_2) and (Y, σ_1, σ_2) (or simply X and Y) always mean bitopological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a bitopological space (X, τ_1, τ_2). The closure of A and the interior of A with respect to τ_i are denoted by τ_i-Cl(A) and τ_i-Int(A), respectively, for $i = 1, 2$. A subset A of a bitopological space (X, τ_1, τ_2) is called *$\tau_1\tau_2$-closed* [21] if $A = \tau_1\text{-Cl}(\tau_2\text{-Cl}(A))$. The complement of a $\tau_1\tau_2$-closed set is called *$\tau_1\tau_2$-open*. The intersection of all $\tau_1\tau_2$-closed sets of X containing A is called the *$\tau_1\tau_2$-closure* [21] of A and is denoted by $\tau_1\tau_2$-Cl(A). The union of all $\tau_1\tau_2$-open sets of X contained in A is called the *$\tau_1\tau_2$-interior* [21] of A and is denoted by $\tau_1\tau_2$-Int(A).

Lemma 1. [21] *Let A and B be subsets of a bitopological space (X, τ_1, τ_2). For the $\tau_1\tau_2$-*

closure, the following properties hold:

(1) $A \subseteq \tau_1\tau_2\text{-}Cl(A)$ *and* $\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Cl(A)) = \tau_1\tau_2\text{-}Cl(A)$.

(2) If $A \subseteq B$*, then* $\tau_1\tau_2\text{-}Cl(A) \subseteq \tau_1\tau_2\text{-}Cl(B)$.

(3) $\tau_1\tau_2\text{-}Cl(A)$ *is* $\tau_1\tau_2$*-closed.*

(4) A *is* $\tau_1\tau_2$*-closed if and only if* $A = \tau_1\tau_2\text{-}Cl(A)$.

(5) $\tau_1\tau_2\text{-}Cl(X - A) = X - \tau_1\tau_2\text{-}Int(A)$.

A subset A of a bitopological space (X, τ_1, τ_2) is said to be $(\tau_1, \tau_2)r$*-open* [22] (resp. $(\tau_1, \tau_2)s$*-open* [23], $(\tau_1, \tau_2)p$*-open* [23], $(\tau_1, \tau_2)\beta$*-open* [23]) if $A = \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A))$ (resp. $A \subseteq \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(A))$, $A \subseteq \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A))$, $A \subseteq \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A)))$). The complement of a $(\tau_1, \tau_2)r$-open (resp. $(\tau_1, \tau_2)s$-open, $(\tau_1, \tau_2)p$-open, $(\tau_1, \tau_2)\beta$-open) set is said to be $(\tau_1, \tau_2)r$*-closed* (resp. $(\tau_1, \tau_2)s$*-closed*, $(\tau_1, \tau_2)p$*-closed*, $(\tau_1, \tau_2)\beta$*-closed*). A subset A of a bitopological space (X, τ_1, τ_2) is said to be $\alpha(\tau_1, \tau_2)$*-open* [24] if $A \subseteq \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(A)))$. The complement of an $\alpha(\tau_1, \tau_2)$-open set is said to be $\alpha(\tau_1, \tau_2)$*-closed*. Let A be a subset of a bitopological space (X, τ_1, τ_2). The intersection of all $(\tau_1, \tau_2)p$-closed (resp. $(\tau_1, \tau_2)s$-closed, $\alpha(\tau_1, \tau_2)$-closed) sets of X containing A is called the $(\tau_1, \tau_2)p$*-closure* [25] (resp. $(\tau_1, \tau_2)s$*-closure* [23], $\alpha(\tau_1, \tau_2)$*-closure* [26]) of A and is denoted by $(\tau_1, \tau_2)\text{-pCl}(A)$ (resp. $(\tau_1, \tau_2)\text{-sCl}(A)$, $\alpha(\tau_1, \tau_2)\text{-Cl}(A)$). The union of all $(\tau_1, \tau_2)p$-open (resp. $(\tau_1, \tau_2)s$-open, $\alpha(\tau_1, \tau_2)$-open) sets of X contained in A is called the $(\tau_1, \tau_2)p$*-interior* [25] (resp. $(\tau_1, \tau_2)s$*-interior* [23], $\alpha(\tau_1, \tau_2)$*-interior* [26]) of A and is denoted by $(\tau_1, \tau_2)\text{-pInt}(A)$ (resp. $(\tau_1, \tau_2)\text{-sInt}(A)$, $\alpha(\tau_1, \tau_2)\text{-Int}(A)$). For a subset A of a bitopological space (X, τ_1, τ_2), a point $x \in X$ is called a $(\tau_1, \tau_2)\theta$*-cluster point* [22] of A if $\tau_1\tau_2\text{-Cl}(U) \cap A \neq \emptyset$ for every $\tau_1\tau_2$-open set U containing x. The set of all $(\tau_1, \tau_2)\theta$-cluster points of A is called the $(\tau_1, \tau_2)\theta$*-closure* [22] of A and is denoted by $(\tau_1, \tau_2)\theta\text{-Cl}(A)$. A subset A of a bitopological space (X, τ_1, τ_2) is said to be $(\tau_1, \tau_2)\theta$*-closed* [22] if $(\tau_1, \tau_2)\theta\text{-Cl}(A) = A$. The complement of a $(\tau_1, \tau_2)\theta$-closed set is said to be $(\tau_1, \tau_2)\theta$*-open*. The union of all $(\tau_1, \tau_2)\theta$-open sets of X contained in A is called the $(\tau_1, \tau_2)\theta$*-interior* [22] of A and is denoted by $(\tau_1, \tau_2)\theta\text{-Int}(A)$.

Lemma 2. [27] *For a subset* A *of a bitopological space* (X, τ_1, τ_2)*, the following properties hold:*

(1) $(\tau_1, \tau_2)\text{-}sCl(A) = \tau_1\tau_2\text{-}Int(\tau_1\tau_2\text{-}Cl(A)) \cup A$ [28];

(2) $(\tau_1, \tau_2)\text{-}sInt(A) = \tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Int(A)) \cap A$.

An ideal $\mathscr{I}$ on a topological space (X, τ) is a nonempty collection of subsets of X satisfying the following properties: (1) $A \in \mathscr{I}$ and $B \subseteq A$ imply $B \in \mathscr{I}$; (2) $A \in \mathscr{I}$ and $B \in \mathscr{I}$ imply $A \cup B \in \mathscr{I}$. A topological space (X, τ) with an ideal $\mathscr{I}$ on X is called an ideal topological space and is denoted by $(X, \tau, \mathscr{I})$. For an ideal topological space $(X, \tau, \mathscr{I})$ and a subset A of X, $A^\star(\mathscr{I})$ is defined as follows:

$$A^\star(\mathscr{I}) = \{x \in X : U \cap A \notin \mathscr{I} \text{ for every open neighbourhood } U \text{ of } x\}.$$

In case there is no chance for confusion, $A^\star(\mathscr{I})$ is simply written as $A^\star$. In [29], $A^\star$ is called the local function of A with respect to $\mathscr{I}$ and τ and $\mathrm{Cl}^\star(A) = A^\star \cup A$ defines a Kuratowski closure operator for a topology $\tau^\star(\mathscr{I})$ finer than τ. A subset A is said to be $\star$-closed [10] if $A^\star \subseteq A$. The interior of a subset A in $(X, \tau^\star(\mathscr{I}))$ is denoted by $\mathrm{Int}^\star(A)$. A subset A of an ideal topological space $(X, \tau, \mathscr{I})$ is said to be *semi***-*$\mathscr{I}$*-open* [30] (resp. *semi-*$\mathscr{I}$*-open* [13]) if $A \subseteq \mathrm{Cl}(\mathrm{Int}^\star(A))$ (resp. $A \subseteq \mathrm{Cl}^\star(\mathrm{Int}(A))$). The complement of a semi*-$\mathscr{I}$-open (resp. semi-$\mathscr{I}$-open) set is said to be *semi***-*$\mathscr{I}$*-closed* [30] (resp. semi-$\mathscr{I}$-closed [13]). A subset A of an ideal topological space $(X, \tau, \mathscr{I})$ is said to be *semi-*$\mathscr{I}^\star$*-open* [14] if $A \subseteq \mathrm{Cl}^\star(\mathrm{Int}^\star(A))$. The complement of a semi-$\mathscr{I}^\star$-open set is called *semi-*$\mathscr{I}^\star$*-closed* [14]. For a subset A of an ideal topological space $(X, \tau, \mathscr{I})$, the intersection of all semi-$\mathscr{I}$-closed sets containing A is called the *semi-*$\mathscr{I}^\star$*-closure* [14] of A and is denoted by $\mathrm{sCl}^\star(A)$ ($\mathrm{sCl}_{\mathscr{I}^\star}(A)$ [14]). The union of all semi-$\mathscr{I}$-open sets contained in A is called the *semi-*$\mathscr{I}^\star$*-interior* [14] of A and is denoted by $\mathrm{sInt}^\star(A)$ ($\mathrm{sInt}_{\mathscr{I}^\star}(A)$ [14]).

Lemma 3. [14] *For a subset A of a an ideal topological space $(X, \tau, \mathscr{I})$, the following properties hold:*

(1) $sCl^\star(A) = A \cup Int^\star(Cl^\star(A))$;

(2) $sInt^\star(A) = A \cap Cl^\star(Int^\star(A))$.

By a multifunction $F : X \to Y$, we mean a point-to-set correspondence from X into Y, and we always assume that $F(x) \neq \emptyset$ for all $x \in X$. For a multifunction $F : X \to Y$, we shall denote the upper and lower inverse of a set B of Y by $F^+(B)$ and $F^-(B)$, respectively, that is, $F^+(B) = \{x \in X \mid F(x) \subseteq B\}$ and $F^-(B) = \{x \in X \mid F(x) \cap B \neq \emptyset\}$. In particular, $F^-(y) = \{x \in X \mid y \in F(x)\}$ for each point $y \in Y$. For each $A \subseteq X$, $F(A) = \cup_{x \in A} F(x)$.

3. Upper and lower almost quasi $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions

In this section, we introduce the concepts of upper almost quasi $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions and lower almost quasi $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions. Furthermore, several characterizations of upper almost quasi $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions and lower almost quasi $\tau^\star(\sigma_1, \sigma_2)$-continuous multifunctions are discussed.

Definition 1. *A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is said to be upper almost quasi $\tau^\star(\sigma_1, \sigma_2)$-continuous at a point $x \in X$ if for every $\sigma_1\sigma_2$-open set V of Y such that $F(x) \subseteq V$ and each $\star$-open set U of X containing x, there exists a nonempty $\star$-open set G such that $G \subseteq U$ and $F(G) \subseteq (\sigma_1, \sigma_2)$-$sCl(V)$. A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is said to be upper almost quasi $\tau^\star(\sigma_1, \sigma_2)$-continuous if F is upper almost quasi $\tau^\star(\sigma_1, \sigma_2)$-continuous at each point x of X.*

Theorem 1. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is upper almost quasi $\tau^\star(\sigma_1, \sigma_2)$-continuous at $x \in X$;

(2) for every $\sigma_1\sigma_2$-open set V of Y such that $F(x) \subseteq V$, there exists a semi-$\mathscr{I}^\star$-open set U of X containing x such that $F(U) \subseteq (\sigma_1, \sigma_2)$-sCl$(V)$;

(3) $x \in$ sInt$^\star(F^+((\sigma_1, \sigma_2)$-sCl$(V)))$ for every $\sigma_1\sigma_2$-open set V of Y such that $F(x) \subseteq V$;

(4) $x \in$ Cl*(Int$^\star(F^+((\sigma_1, \sigma_2)$-sCl$(V))))$ for every $\sigma_1\sigma_2$-open set V of Y such that $F(x) \subseteq V$.

Proof. (1) ⇒ (2): Let $\mathscr{U}(x)$ be the family of all $\star$-open sets of X containing x. Let V be any $\sigma_1\sigma_2$-open set of Y such that $F(x) \subseteq V$. For each $H \in \mathscr{U}(x)$, there exists a nonempty $\star$-open set G_H such that $G_H \subseteq H$ and $F(G_H) \subseteq (\sigma_1, \sigma_2)$-sCl$(V)$. Put $W = \cup\{G_H \mid H \in \mathscr{U}(x)\}$. Then, W is $\star$-open in X, $x \in \text{Cl}^\star(W)$ and $F(W) \subseteq (\sigma_1, \sigma_2)$-sCl$(V)$. Put $U = W \cup \{x\}$, then $W \subseteq U \subseteq \text{Cl}^\star(W)$. Thus, U is a semi-$\mathscr{I}^\star$-open set of X containing x such that $F(U) \subseteq (\sigma_1, \sigma_2)$-sCl$(V)$.

(2) ⇒ (3): Let V be any $\sigma_1\sigma_2$-open set of Y and $F(x) \subseteq V$. Then, there exists a semi-$\mathscr{I}^\star$-open set U of X containing x such that $F(U) \subseteq (\sigma_1, \sigma_2)$-sCl$(V)$. Thus, $x \in U \subseteq F^+((\sigma_1, \sigma_2)$-sCl$(V))$ and hence $x \in U \subseteq \text{sInt}^\star(F^+((\sigma_1, \sigma_2)$-sCl$(V)))$.

(3) ⇒ (4): Let V be any $\sigma_1\sigma_2$-open set of Y such that $F(x) \subseteq V$. By (3), we have $x \in \text{sInt}^\star(F^+((\sigma_1, \sigma_2)$-sCl$(V)))$. Now, put $U = \text{sInt}^\star(F^+((\sigma_1, \sigma_2)$-sCl$(V)))$. Then, U is semi-$\mathscr{I}^\star$-open in X and by Lemma 3,

$$x \in U \subseteq \text{Cl}^\star(\text{Int}^\star(U)) \subseteq \text{Cl}^\star(\text{Int}^\star(F^+((\sigma_1, \sigma_2)\text{-sCl}(V)))).$$

(4) ⇒ (1): Let U be any $\star$-open set of X containing x and V be any $\sigma_1\sigma_2$-open set of Y such that $F(x) \subseteq V$. Thus by (4), $x \in \text{Cl}^\star(\text{Int}^\star(F^+((\sigma_1, \sigma_2)$-sCl$(V))))$ and hence $\text{Int}^\star(F^+((\sigma_1, \sigma_2)$-sCl$(V))) \cap U \neq \emptyset$. Put $W = \text{Int}^\star(F^+((\sigma_1, \sigma_2)$-sCl$(V))) \cap U$. Then, we have W is a nonempty $\star$-open set of X such that $W \subseteq U$ and $F(W) \subseteq (\sigma_1, \sigma_2)$-sCl$(V)$. This shows that F is upper almost quasi $\tau^\star(\sigma_1, \sigma_2)$-continuous at x.

Definition 2. *A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is said to be lower almost quasi $\tau^\star(\sigma_1, \sigma_2)$-continuous at a point $x \in X$ if for every $\sigma_1\sigma_2$-open set V of Y such that $F(x) \cap V \neq \emptyset$ and each $\star$-open set U of X containing x, there exists a nonempty $\star$-open set G such that $G \subseteq U$ and (σ_1, σ_2)-sCl$(V) \cap F(z) \neq \emptyset$ for each $z \in G$. A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is said to be lower almost quasi $\tau^\star(\sigma_1, \sigma_2)$-continuous if F is lower almost quasi $\tau^\star(\sigma_1, \sigma_2)$-continuous at each point x of X.*

Theorem 2. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is lower almost quasi $\tau^\star(\sigma_1, \sigma_2)$-continuous at $x \in X$;

(2) for every $\sigma_1\sigma_2$-open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists a semi-$\mathscr{I}^\star$-open set U of X containing x such that (σ_1, σ_2)-sCl$(V) \cap F(z) \neq \emptyset$ for every $z \in U$;

(3) $x \in$ sInt$^\star(F^-((\sigma_1, \sigma_2)$-sCl$(V)))$ for every $\sigma_1\sigma_2$-open set V of Y such that

$$F(x) \cap V \neq \emptyset;$$

(4) $x \in Cl^\star(Int^\star(F^-((\sigma_1, \sigma_2)\text{-}sCl(V))))$ *for every* $\sigma_1\sigma_2$*-open set* V *of* Y *such that*

$$F(x) \cap V \neq \emptyset.$$

Proof. The proof is similar to that of Theorem 1.

Theorem 3. *For a multifunction* $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$*, the following properties are equivalent:*

(1) F *is upper almost quasi* $\tau^\star(\sigma_1, \sigma_2)$*-continuous;*

(2) for each $x \in X$ *and every* $\sigma_1\sigma_2$*-open set* V *of* Y *such that* $F(x) \subseteq V$*, there exists a semi-*$\mathscr{I}^\star$*-open set* U *of* X *containing* x *such that* $F(U) \subseteq (\sigma_1, \sigma_2)\text{-}sCl(V)$*;*

(3) $F^+(V)$ *is semi-*$\mathscr{I}^\star$*-open in* X *for every* $(\sigma_1, \sigma_2)r$*-open set* V *of* Y*;*

(4) $F^+(V) \subseteq sInt^\star(F^+((\sigma_1, \sigma_2)\text{-}sCl(V)))$ *for every* $\sigma_1\sigma_2$*-open set* V *of* Y*;*

(5) $sCl^\star(F^-(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(B))))) \subseteq F^-(\sigma_1\sigma_2\text{-}Cl(B))$ *for every subset* B *of* Y*;*

(6) $F^+(V) \subseteq Cl^\star(Int^\star(F^+((\sigma_1, \sigma_2)\text{-}sCl(V))))$ *for every* $\sigma_1\sigma_2$*-open set* V *of* Y*.*

Proof. (1) $\Rightarrow$ (2): It follows from Theorem 1.

(2) $\Rightarrow$ (3): Let V be any $(\sigma_1, \sigma_2)r$-open set of Y and $x \in F^+(V)$. Then, $F(x) \subseteq V$ and there exists a semi-$\mathscr{I}^\star$-open set U of X containing x such that $F(U) \subseteq V$. Thus, $x \in U \subseteq F^+(V)$ and hence $x \in \text{sInt}^\star(F^+(V))$. Therefore, $F^+(V) \subseteq \text{sInt}^\star(F^+(V))$. This shows that $F^+(V)$ is semi-$\mathscr{I}^\star$-open in X.

(3) $\Rightarrow$ (4): Let V be any $\sigma_1\sigma_2$-open set of Y and $x \in F^+(V)$. Then, we have $F(x) \subseteq V \subseteq (\sigma_1, \sigma_2)\text{-sCl}(V)$. Thus, $x \in F^+((\sigma_1, \sigma_2)\text{-sCl}(V))$. By Lemma 2, we have $(\sigma_1, \sigma_2)\text{-sCl}(V)$ is $(\sigma_1, \sigma_2)r$-open in Y and by (3), $F^+((\sigma_1, \sigma_2)\text{-sCl}(V))$ is semi-$\mathscr{I}^\star$-open in X and $x \in \text{sInt}^\star(F^+((\sigma_1, \sigma_2)\text{-sCl}(V)))$. Thus, $F^+(V) \subseteq \text{sInt}^\star(F^+((\sigma_1, \sigma_2)\text{-sCl}(V)))$.

(4) $\Rightarrow$ (5): Let B be any subset of Y. Then, we have $Y - \sigma_1\sigma_2\text{-Cl}(B)$ is $\sigma_1\sigma_2$-open in Y. Thus by (4) and Lemma 2, we have

$$\begin{aligned} X - F^-(\sigma_1\sigma_2\text{-Cl}(B)) &= F^+(Y - \sigma_1\sigma_2\text{-Cl}(B)) \\ &\subseteq \text{sInt}^\star(F^+((\sigma_1, \sigma_2)\text{-sCl}(Y - \sigma_1\sigma_2\text{-Cl}(B)))) \\ &= \text{sInt}^\star(X - F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_1\text{-Int}(\sigma_1\sigma_2\text{-Cl}(B))))) \\ &= X - \text{sCl}^\star(F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(B))))) \end{aligned}$$

and hence $\text{sCl}^\star(F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(B))))) \subseteq F^-(\sigma_1\sigma_2\text{-Cl}(B))$.

(5) $\Rightarrow$ (6): Let V be any $\sigma_1\sigma_2$-open set of Y. Then, $Y - V$ is $\sigma_1\sigma_2$-closed in Y. By (5) and Lemma 3, $\text{Int}^\star(\text{Cl}^\star(F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(Y - V))))) \subseteq F^-(Y - V) = X - F^+(V)$. Moreover, we have

$$\text{Int}^\star(\text{Cl}^\star(F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(Y - V))))) = \text{Int}^\star(\text{Cl}^\star(F^-(Y - \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))))$$

$$= \mathrm{Int}^\star(\mathrm{Cl}^\star(X - F^+((\sigma_1, \sigma_2)\text{-sCl}(V))))$$
$$= X - \mathrm{Cl}^\star(\mathrm{Int}^\star(F^+((\sigma_1, \sigma_2)\text{-sCl}(V)))).$$

Thus, $F^+(V) \subseteq \mathrm{Cl}^\star(\mathrm{Int}^\star(F^+((\sigma_1, \sigma_2)\text{-sCl}(V))))$.

(6) $\Rightarrow$ (1): Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y such that $F(x) \subseteq V$. By (6), we have $x \in F^+(V) \subseteq \mathrm{Cl}^\star(\mathrm{Int}^\star(F^+((\sigma_1, \sigma_2)\text{-sCl}(V))))$ and by Lemma 3,

$$x \in F^+(V) \subseteq \mathrm{sInt}^\star(F^+((\sigma_1, \sigma_2)\text{-sCl}(V))).$$

Put $U = \mathrm{sInt}^\star(F^+((\sigma_1, \sigma_2)\text{-sCl}(V)))$. Then, U is a semi-$\mathscr{I}^\star$-open set of X containing x such that $F(U) \subseteq (\sigma_1, \sigma_2)$-sCl$(V)$. This shows that F is upper almost quasi $\tau^\star(\sigma_1, \sigma_2)$-continuous.

Theorem 4. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is lower almost quasi $\tau^\star(\sigma_1, \sigma_2)$-continuous;

(2) for each $x \in X$ and every $\sigma_1\sigma_2$-open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists a semi-$\mathscr{I}^\star$-open set U of X containing x such that (σ_1, σ_2)-sCl$(V) \cap F(z) \neq \emptyset$ for every $z \in U$;

(3) $F^-(V)$ is semi-$\mathscr{I}^\star$-open in X for every $(\sigma_1, \sigma_2)r$-open set V of Y;

(4) $F^-(V) \subseteq \mathrm{sInt}^\star(F^-((\sigma_1, \sigma_2)\text{-sCl}(V)))$ for every $\sigma_1\sigma_2$-open set V of Y;

(5) $\mathrm{sCl}^\star(F^+(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(B))))) \subseteq F^+(\sigma_1\sigma_2\text{-Cl}(B))$ for every subset B of Y;

(6) $F^-(V) \subseteq \mathrm{Cl}^\star(\mathrm{Int}^\star(F^-((\sigma_1, \sigma_2)\text{-sCl}(V))))$ for every $\sigma_1\sigma_2$-open set V of Y.

Proof. The proof is similar to that of Theorem 3.

Theorem 5. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is upper almost quasi $\tau^\star(\sigma_1, \sigma_2)$-continuous;

(2) $\mathrm{sCl}^\star(F^-(V)) \subseteq F^-(\sigma_1\sigma_2\text{-Cl}(V))$ for every $(\sigma_1, \sigma_2)\beta$-open set V of Y;

(3) $\mathrm{sCl}^\star(F^-(V)) \subseteq F^-(\sigma_1\sigma_2\text{-Cl}(V))$ for every $(\sigma_1, \sigma_2)s$-open set V of Y;

(4) $F^+(V) \subseteq \mathrm{sInt}^\star(F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))))$ for every $(\sigma_1, \sigma_2)p$-open set V of Y.

Proof. The proof is similar to that of Theorem 3.

Theorem 6. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F *is lower almost quasi* $\tau^\star(\sigma_1,\sigma_2)$*-continuous;*

(2) $sCl^\star(F^+(V)) \subseteq F^+(\sigma_1\sigma_2\text{-}Cl(V))$ *for every* $(\sigma_1,\sigma_2)\beta$*-open set* V *of* Y*;*

(3) $sCl^\star(F^+(V)) \subseteq F^+(\sigma_1\sigma_2\text{-}Cl(V))$ *for every* $(\sigma_1,\sigma_2)s$*-open set* V *of* Y*;*

(4) $F^-(V) \subseteq sInt^\star(F^-(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V))))$ *for every* $(\sigma_1,\sigma_2)p$*-open set* V *of* Y*.*

Proof. The proof is similar to that of Theorem 5.

4. Upper and lower weakly quasi $\tau^\star(\sigma_1,\sigma_2)$-continuous multifunctions

In this section, we introduce the concepts of upper weakly quasi $\tau^\star(\sigma_1,\sigma_2)$-continuous multifunctions and lower weakly quasi $\tau^\star(\sigma_1,\sigma_2)$-continuous multifunctions. Moreover, several characterizations of upper weakly quasi $\tau^\star(\sigma_1,\sigma_2)$-continuous multifunctions and lower weakly quasi $\tau^\star(\sigma_1,\sigma_2)$-continuous multifunctions are discussed.

Definition 3. *A multifunction* $F : (X,\tau,\mathscr{I}) \to (Y,\sigma_1,\sigma_2)$ *is said to be upper weakly quasi* $\tau^\star(\sigma_1,\sigma_2)$*-continuous at a point* $x \in X$ *if for each* $\sigma_1\sigma_2$*-open set* V *of* Y *such that* $F(x) \subseteq V$ *and each* $\star$*-open set* U *of* X *containing* x*, there exists a nonempty* $\star$*-open set* G *such that* $G \subseteq U$ *and* $F(G) \subseteq \sigma_1\sigma_2\text{-}Cl(V)$*. A multifunction* $F : (X,\tau,\mathscr{I}) \to (Y,\sigma_1,\sigma_2)$ *is said to be upper weakly quasi* $\tau^\star(\sigma_1,\sigma_2)$*-continuous if* F *is upper weakly quasi* $\tau^\star(\sigma_1,\sigma_2)$*-continuous at each point* x *of* X*.*

Remark 1. *For a multifunction* $F : (X,\tau,\mathscr{I}) \to (Y,\sigma_1,\sigma_2)$*, the following implication holds:*

upper almost quasi $\tau^\star(\sigma_1,\sigma_2)$*-continuity* $\Rightarrow$ *upper weakly quasi* $\tau^\star(\sigma_1,\sigma_2)$*-continuity.*

The converse of the implication is not true in general. We give an example for the implication as follows.

Example 1. *Let* $X = \{1,2,3\}$ *with a topology* $\tau = \{\emptyset,\{1\},\{2,3\},X\}$ *and an ideal* $\mathscr{I} = \{\emptyset,\{1\}\}$*. Let* $Y = \{a,b,c\}$ *with topologies* $\sigma_1 = \{\emptyset,\{a\},\{b\},\{a,b\},\{a,c\},Y\}$ *and* $\sigma_2 = \{\emptyset,\{a\},\{b\},\{a,b\},Y\}$*. A multifunction* $F : (X,\tau,\mathscr{I}) \to (Y,\sigma_1,\sigma_2)$ *is defined as follows:* $F(1) = \{a\}$*,* $F(2) = \{b\}$ *and* $F(3) = \{b,c\}$*. Then,* F *is upper weakly quasi* $\tau^\star(\sigma_1,\sigma_2)$*-continuous but* F *is not upper almost quasi* $\tau^\star(\sigma_1,\sigma_2)$*-continuous.*

Theorem 7. *For a multifunction* $F : (X,\tau,\mathscr{I}) \to (Y,\sigma_1,\sigma_2)$*, the following properties are equivalent:*

(1) F *is upper weakly quasi* $\tau^\star(\sigma_1,\sigma_2)$*-continuous;*

(2) for each $x \in X$ *and every* $\sigma_1\sigma_2$*-open set* V *of* Y *such that* $F(x) \subseteq V$*, there exists a semi-*$\mathscr{I}^\star$*-open set* U *of* X *containing* x *such that* $F(U) \subseteq \sigma_1\sigma_2\text{-}Cl(V)$*;*

(3) $Int^\star(Cl^\star(F^-(\sigma_1\sigma_2\text{-}Int(K)))) \subseteq F^-(K)$ *for every* $\sigma_1\sigma_2$*-closed set* K *of* Y*;*

(4) $F^+(V) \subseteq sInt^\star(F^+(\sigma_1\sigma_2\text{-}Cl(V)))$ *for every* $\sigma_1\sigma_2$*-open set* V *of* Y*;*

(5) $sCl^\star(F^-(V)) \subseteq F^-(\sigma_1\sigma_2\text{-}Cl(V))$ *for every* $\sigma_1\sigma_2$*-open set* V *of* Y*.*

Proof. (1) $\Rightarrow$ (2): Let $\mathscr{U}(x)$ be the family of all $\star$-open sets of X containing x. Let V be any $\sigma_1\sigma_2$-open set of Y such that $F(x) \subseteq V$. For each $H \in \mathscr{U}(x)$, there exists a nonempty $\star$-open set G_H such that $G_H \subseteq H$ and $F(G_H) \subseteq \sigma_1\sigma_2\text{-Cl}(V)$. Let $W = \cup\{G_H \mid H \in \mathscr{U}(x)\}$. Then, W is $\star$-open in X, $x \in \text{Cl}^\star(W)$ and $F(W) \subseteq \sigma_1\sigma_2\text{-Cl}(V)$. Put $U = W \cup \{x\}$, then $W \subseteq U \subseteq \text{Cl}^\star(W)$. Thus, U is a semi-$\mathscr{I}^\star$-open set of X containing x such that $F(U) \subseteq \sigma_1\sigma_2\text{-Cl}(V)$.

(2) $\Rightarrow$ (4): Let V be any $\sigma_1\sigma_2$-open set of Y and $x \in F^+(V)$. Then, $F(x) \subseteq V$ and there exists a semi-$\mathscr{I}^\star$-open set U of X containing x such that $F(U) \subseteq \sigma_1\sigma_2\text{-Cl}(V)$. Thus, $x \in U \subseteq \text{sInt}^\star(F^+(\sigma_1\sigma_2\text{-Cl}(V)))$ and so $F^+(V) \subseteq \text{sInt}^\star(F^+(\sigma_1\sigma_2\text{-Cl}(V)))$.

(4) $\Rightarrow$ (5): Let V be any $\sigma_1\sigma_2$-open set of Y. Then by (4), we have

$$\begin{aligned} X - F^-(\sigma_1\sigma_2\text{-Cl}(V)) &= F^+(Y - \sigma_1\sigma_2\text{-Cl}(V)) \\ &\subseteq \text{sInt}^\star(F^+(\sigma_1\sigma_2\text{-Cl}(Y - \sigma_1\sigma_2\text{-Cl}(V)))) \\ &= \text{sInt}^\star(F^+(Y - \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) \\ &\subseteq \text{sInt}^\star(F^+(Y - V)) \\ &= \text{sInt}^\star(X - F^-(V)) \\ &= X - \text{sCl}^\star(F^-(V)) \end{aligned}$$

and hence $\text{sCl}^\star(F^-(V)) \subseteq F^-(\sigma_1\sigma_2\text{-Cl}(V))$.

(5) $\Rightarrow$ (3): Let K be any $\sigma_1\sigma_2$-closed set of Y. By (5) and Lemma 3, we have

$$\begin{aligned} \text{Int}^\star(\text{Cl}^\star(F^-(\sigma_1\sigma_2\text{-Int}(K)))) &\subseteq \text{sCl}^\star(F^-(\sigma_1\sigma_2\text{-Int}(K))) \\ &\subseteq F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K))) \\ &\subseteq F^-(\sigma_1\sigma_2\text{-Cl}(K)) = F^-(K). \end{aligned}$$

(3) $\Rightarrow$ (4): Let V be any $\sigma_1\sigma_2$-open set of Y. By (3) and Lemma 3,

$$\begin{aligned} X - \text{sInt}^\star(F^+(\sigma_1\sigma_2\text{-Cl}(V))) &= \text{sCl}^\star(F^-(Y - \sigma_1\sigma_2\text{-Cl}(V))) \\ &\subseteq F^-(\sigma_1\sigma_2\text{-Cl}(Y - \sigma_1\sigma_2\text{-Cl}(V))) \\ &= F^-(Y - \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))) \\ &\subseteq F^-(Y - V) = X - F^+(V) \end{aligned}$$

and so $F^+(V) \subseteq \text{sInt}^\star(F^+(\sigma_1\sigma_2\text{-Cl}(V)))$.

(4) $\Rightarrow$ (1): Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y such that $F(x) \subseteq V$. Thus by (4), we have $F^+(V) \subseteq \text{sInt}^\star(F^+(\sigma_1\sigma_2\text{-Cl}(V)))$. Put $U = \text{sInt}^\star(F^+(\sigma_1\sigma_2\text{-Cl}(V)))$, then U is a semi-$\mathscr{I}^\star$-open set of X containing x such that $F(U) \subseteq \sigma_1\sigma_2\text{-Cl}(V)$. This shows that F is upper weakly quasi $\tau^\star(\sigma_1, \sigma_2)$-continuous.

Definition 4. *A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is said to be lower weakly quasi $\tau^\star(\sigma_1, \sigma_2)$-continuous at a point $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y such that $F(x) \cap V \neq \emptyset$ and each $\star$-open set U of X containing x, there exists a nonempty $\star$-open set G such that $G \subseteq U$ and $\sigma_1\sigma_2$-$Cl(V) \cap F(z) \neq \emptyset$ for each $z \in G$. A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is said to be lower weakly quasi $\tau^\star(\sigma_1, \sigma_2)$-continuous if F is lower weakly quasi $\tau^\star(\sigma_1, \sigma_2)$-continuous at each point x of X.*

Theorem 8. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is lower weakly quasi $\tau^\star(\sigma_1, \sigma_2)$-continuous;

(2) for each $x \in X$ and every $\sigma_1\sigma_2$-open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists a semi-$\mathscr{I}^\star$-open set U of X containing x such that $\sigma_1\sigma_2$-$Cl(V) \cap F(z) \neq \emptyset$ for every $z \in U$;

(3) $Int^\star(Cl^\star(F^+(\sigma_1\sigma_2\text{-}Int(K)))) \subseteq F^+(K)$ for every $\sigma_1\sigma_2$-closed set K of Y;

(4) $F^-(V) \subseteq sInt^\star(F^-(\sigma_1\sigma_2\text{-}Cl(V)))$ for every $\sigma_1\sigma_2$-open set V of Y;

(5) $sCl^\star(F^+(V)) \subseteq F^+(\sigma_1\sigma_2\text{-}Cl(V))$ for every $\sigma_1\sigma_2$-open set V of Y.

Proof. The proof is similar to that of Theorem 7.

Theorem 9. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is upper weakly quasi $\tau^\star(\sigma_1, \sigma_2)$-continuous;

(2) $sCl^\star(F^-(\sigma_1\sigma_2\text{-}Int((\sigma_1, \sigma_2)\theta\text{-}Cl(B)))) \subseteq F^-((\sigma_1, \sigma_2)\theta\text{-}Cl(B))$ for every subset B of Y;

(3) $sCl^\star(F^-(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(B)))) \subseteq F^-((\sigma_1, \sigma_2)\theta\text{-}Cl(B))$ for every subset B of Y;

(4) $sCl^\star(F^-(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V)))) \subseteq F^-(\sigma_1\sigma_2\text{-}Cl(V))$ for every $\sigma_1\sigma_2$-open set V of Y;

(5) $sCl^\star(F^-(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V)))) \subseteq F^-(\sigma_1\sigma_2\text{-}Cl(V))$ for every $(\sigma_1, \sigma_2)p$-open set V of Y;

(6) $sCl^\star(F^-(\sigma_1\sigma_2\text{-}Int(K))) \subseteq F^-(K)$ for every $(\sigma_1, \sigma_2)r$-closed set K of Y.

Proof. (1) $\Rightarrow$ (2): Let B be any subset of Y. Since $(\sigma_1, \sigma_2)\theta$-Cl$(B)$ is $\sigma_1\sigma_2$-closed in Y, by Theorem 7 we have $\text{Int}^\star(\text{Cl}^\star(F^-(\sigma_1\sigma_2\text{-Int}((\sigma_1, \sigma_2)\theta\text{-Cl}(B))))) \subseteq F^-((\sigma_1, \sigma_2)\theta\text{-Cl}(B))$ and by Lemma 3, $\text{sCl}^\star(F^-(\sigma_1\sigma_2\text{-Int}((\sigma_1, \sigma_2)\theta\text{-Cl}(B)))) \subseteq F^-((\sigma_1, \sigma_2)\theta\text{-Cl}(B))$.

(2) $\Rightarrow$ (3): This is obvious since $\sigma_1\sigma_2$-Cl$(B) \subseteq (\sigma_1, \sigma_2)\theta$-Cl$(B)$ for every subset B of Y.

(3) $\Rightarrow$ (4): This is obvious since $\sigma_1\sigma_2$-Cl$(V) = (\sigma_1, \sigma_2)\theta$-Cl$(V)$ for every $\sigma_1\sigma_2$-open set V of Y.

(4) $\Rightarrow$ (5): Let V be any $(\sigma_1, \sigma_2)p$-open set of Y. Then, $V \subseteq \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))$ and $\sigma_1\sigma_2\text{-Cl}(V) = \sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))$. Now, put $G = \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))$, then G is $\sigma_1\sigma_2$-open in Y and $\sigma_1\sigma_2\text{-Cl}(G) = \sigma_1\sigma_2\text{-Cl}(V)$. Thus by (4),

$$\text{sCl}^\star(F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) \subseteq F^-(\sigma_1\sigma_2\text{-Cl}(V)).$$

(5) $\Rightarrow$ (6): Let K be any $(\sigma_1, \sigma_2)r$-closed set of Y. Since $\sigma_1\sigma_2\text{-Int}(K)$ is $(\sigma_1, \sigma_2)p$-open in Y, by (5) we have

$$\begin{aligned}\text{sCl}^\star(F^-(\sigma_1\sigma_2\text{-Int}(K))) &= \text{sCl}^\star(F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K))))) \\ &\subseteq F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K))) = F^-(K).\end{aligned}$$

(6) $\Rightarrow$ (1): Let V be any $\sigma_1\sigma_2$-open set of Y. Then, $\sigma_1\sigma_2\text{-Cl}(V)$ is $(\sigma_1, \sigma_2)r$-closed in Y and by (6), $\text{sCl}^\star(F^-(V)) \subseteq \text{sCl}^\star(F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) \subseteq F^-(\sigma_1\sigma_2\text{-Cl}(V))$. It follows from Theorem 7 that F is upper weakly quasi $\tau^\star(\sigma_1, \sigma_2)$-continuous.

Theorem 10. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is lower weakly quasi $\tau^\star(\sigma_1, \sigma_2)$-continuous;

(2) $sCl^\star(F^+(\sigma_1\sigma_2\text{-}Int((\sigma_1, \sigma_2)\theta\text{-}Cl(B)))) \subseteq F^+((\sigma_1, \sigma_2)\theta\text{-}Cl(B))$ for every subset B of Y;

(3) $sCl^\star(F^+(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(B)))) \subseteq F^+((\sigma_1, \sigma_2)\theta\text{-}Cl(B))$ for every subset B of Y;

(4) $sCl^\star(F^+(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V)))) \subseteq F^+(\sigma_1\sigma_2\text{-}Cl(V))$ for every $\sigma_1\sigma_2$-open set V of Y;

(5) $sCl^\star(F^+(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V)))) \subseteq F^+(\sigma_1\sigma_2\text{-}Cl(V))$ for every $(\sigma_1, \sigma_2)p$-open set V of Y;

(6) $sCl^\star(F^+(\sigma_1\sigma_2\text{-}Int(K))) \subseteq F^+(K)$ for every $(\sigma_1, \sigma_2)r$-closed set K of Y.

Proof. The proof is similar to that of Theorem 9.

Theorem 11. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is upper weakly quasi $\tau^\star(\sigma_1, \sigma_2)$-continuous;

(2) $sCl^\star(F^-(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V)))) \subseteq F^-(\sigma_1\sigma_2\text{-}Cl(V))$ for every $(\sigma_1, \sigma_2)\beta$-open set V of Y;

(3) $sCl^\star(F^-(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V)))) \subseteq F^-(\sigma_1\sigma_2\text{-}Cl(V))$ for every $(\sigma_1, \sigma_2)s$-open set V of Y;

(4) $sCl^\star(F^-(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V)))) \subseteq F^-(\sigma_1\sigma_2\text{-}Cl(V))$ for every $(\sigma_1, \sigma_2)p$-open set V of Y.

Proof. (1) $\Rightarrow$ (2): Let V be any $(\sigma_1, \sigma_2)\beta$-open set of Y. Then, we have

$$V \subseteq \sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))$$

and hence $\sigma_1\sigma_2\text{-Cl}(V) = \sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))$. Since $\sigma_1\sigma_2\text{-Cl}(V)$ is $(\sigma_1, \sigma_2)r$-closed in Y and by Theorem 9, $\text{sCl}^\star(F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) \subseteq F^-(\sigma_1\sigma_2\text{-Cl}(V))$.

(2) $\Rightarrow$ (3): This is obvious since every $(\sigma_1, \sigma_2)s$-open set is $(\sigma_1, \sigma_2)\beta$-open.

(3) $\Rightarrow$ (4): For any $(\sigma_1, \sigma_2)p$-open set V of Y, $\sigma_1\sigma_2\text{-Cl}(V)$ is $(\sigma_1, \sigma_2)r$-closed and $\sigma_1\sigma_2\text{-Cl}(V)$ is $(\sigma_1, \sigma_2)s$-open in Y.

(4) $\Rightarrow$ (1): Let V be any $\sigma_1\sigma_2$-open set of Y. Then, V is $(\sigma_1, \sigma_2)p$-open in Y. By (4), we have $\text{sCl}^\star(F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) \subseteq F^-(\sigma_1\sigma_2\text{-Cl}(V))$. It follows from Theorem 9 that F is upper weakly quasi $\tau^\star(\sigma_1, \sigma_2)$-continuous.

Theorem 12. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is lower weakly quasi $\tau^\star(\sigma_1, \sigma_2)$-continuous;

(2) $sCl^\star(F^+(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V)))) \subseteq F^+(\sigma_1\sigma_2\text{-}Cl(V))$ for every $(\sigma_1, \sigma_2)\beta$-open set V of Y;

(3) $sCl^\star(F^+(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V)))) \subseteq F^+(\sigma_1\sigma_2\text{-}Cl(V))$ for every $(\sigma_1, \sigma_2)s$-open set V of Y;

(4) $sCl^\star(F^+(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V)))) \subseteq F^+(\sigma_1\sigma_2\text{-}Cl(V))$ for every $(\sigma_1, \sigma_2)p$-open set V of Y.

Proof. The proof is similar to that of Theorem 11.

Theorem 13. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is upper weakly quasi $\tau^\star(\sigma_1, \sigma_2)$-continuous;

(2) $Int^\star(Cl^\star(F^-(V))) \subseteq F^-(\sigma_1\sigma_2\text{-}Cl(V))$ for every $(\sigma_1, \sigma_2)p$-open set V of Y;

(3) $sCl^\star(F^-(V)) \subseteq F^-(\sigma_1\sigma_2\text{-}Cl(V))$ for every $(\sigma_1, \sigma_2)p$-open set V of Y;

(4) $F^+(V) \subseteq sInt^\star(F^+(\sigma_1\sigma_2\text{-}Cl(V)))$ for every $(\sigma_1, \sigma_2)p$-open set V of Y.

Proof. (1) $\Rightarrow$ (2): Let V be any $(\sigma_1, \sigma_2)p$-open set of Y. Since F is upper weakly quasi $\tau^\star(\sigma_1, \sigma_2)$-continuous, by Lemma 3 and Theorem 9,

$$\text{Int}^\star(\text{Cl}^\star(F^-(V))) \subseteq \text{Int}^\star(\text{Cl}^\star(F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))))) \subseteq F^-(\sigma_1\sigma_2\text{-Cl}(V)).$$

(2) $\Rightarrow$ (3): Let V be any $(\sigma_1, \sigma_2)p$-open set of Y. By (2) and Lemma 3, we have

$$\text{sCl}^\star(F^-(V)) = F^-(V) \cup \text{Int}^\star(\text{Cl}^\star(F^-(V))) \subseteq F^-(\sigma_1\sigma_2\text{-Cl}(V)).$$

(3) ⇒ (4): Let V be any $(\sigma_1, \sigma_2)p$-open set of Y. Then by (3), we have

$$\begin{aligned} X - \mathrm{sInt}^\star(F^+(\sigma_1\sigma_2\text{-}\mathrm{Cl}(V))) &= \mathrm{sCl}^\star(X - F^+(\sigma_1\sigma_2\text{-}\mathrm{Cl}(V))) \\ &= \mathrm{sCl}^\star(F^-(Y - \sigma_1\sigma_2\text{-}\mathrm{Cl}(V))) \\ &\subseteq F^-(\sigma_1\sigma_2\text{-}\mathrm{Cl}(Y - \sigma_1\sigma_2\text{-}\mathrm{Cl}(V))) \\ &= X - F^+(\sigma_1\sigma_2\text{-}\mathrm{Int}(\sigma_1\sigma_2\text{-}\mathrm{Cl}(V))) \\ &\subseteq X - F^+(V) \end{aligned}$$

and hence $F^+(V) \subseteq \mathrm{sInt}^\star(F^+(\sigma_1\sigma_2\text{-}\mathrm{Cl}(V)))$.

(4) ⇒ (1): Since every $\sigma_1\sigma_2$-open set is $(\sigma_1, \sigma_2)p$-open, this follows from Theorem 7.

Theorem 14. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is lower weakly quasi $\tau^\star(\sigma_1, \sigma_2)$-continuous;

(2) $Int^\star(Cl^\star(F^+(V))) \subseteq F^+(\sigma_1\sigma_2\text{-}Cl(V))$ for every $(\sigma_1, \sigma_2)p$-open set V of Y;

(3) $sCl^\star(F^+(V)) \subseteq F^+(\sigma_1\sigma_2\text{-}Cl(V))$ for every $(\sigma_1, \sigma_2)p$-open set V of Y;

(4) $F^-(V) \subseteq sInt^\star(F^-(\sigma_1\sigma_2\text{-}Cl(V)))$ for every $(\sigma_1, \sigma_2)p$-open set V of Y.

Proof. The proof is similar to that of Theorem 13.

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