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Almost Quasi $\tau^\star(\sigma_1, \sigma_2)$-Continuous and Weakly Quasi $\tau^\star(\sigma_1, \sigma_2)$-Continuous Functions

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Abstract. This paper introduces two classes of continuous functions defined between an ideal topological space and a bitopological space, called almost quasi $\tau^\star(\sigma_1, \sigma_2)$-continuous functions and weakly quasi $\tau^\star(\sigma_1, \sigma_2)$-continuous functions. Furthermore, several characterizations and some properties concerning almost quasi $\tau^\star(\sigma_1, \sigma_2)$-continuous functions and weakly quasi $\tau^\star(\sigma_1, \sigma_2)$-continuous functions are investigated. Moreover, the relationships between almost quasi $\tau^\star(\sigma_1, \sigma_2)$-continuity and weak quasi $\tau^\star(\sigma_1, \sigma_2)$-continuity are considered.

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1. Introduction

In 1961, Marcus [1] introduced the concept of quasi continuous functions. Popa [2] introduced and investigated the notion of almost quasi continuous functions. Neubrunnovaá [3] showed that quasi continuity is equivalent to semi-continuity due to Levine [4]. Popa and Stan [5] introduced and studied the notion of weakly quasi continuous functions. Weak quasi continuity is implied by quasi continuity and weak continuity [6] which are independent of each other. It is shown in [7] that weak quasi continuity is equivalent to weak semi-continuity due to Arya and Bhamini [8] and Kar and Bhattacharyya [9]. In 1990, Janković and Hamlett [10] introduced the concept of $\mathscr{I}$-open sets in ideal topological spaces. Abd El-Monsef et al. [11] introduced and studied the concepts of $\mathscr{I}$-closed sets and $\mathscr{I}$-continuous functions. Semi-$\mathscr{I}$-open sets, pre-$\mathscr{I}$-open sets, α-$\mathscr{I}$-open sets, β-$\mathscr{I}$-open sets and δ-$\mathscr{I}$-open sets play an important role in the research of generalizations of continuity in ideal topological spaces. Using these notions many authors introduced

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and studied various types of generalizations of continuity for functions. Hatir and Noiri [12] introduced and investigated the notions of weakly pre-$\mathscr{I}$-open sets and weakly pre-$\mathscr{I}$-continuous functions. Moreover, Hatir and Noiri [13] investigated further properties of semi-$\mathscr{I}$-open sets and semi-$\mathscr{I}$-continuous functions. On the other hand, the present authors introduced and investigated the concepts of $\mathfrak{pi}$-continuous functions [14], weakly $\mathfrak{pi}$-continuous functions [14], (τ_1, τ_2)-continuous functions [15], almost (τ_1, τ_2)-continuous functions [16] and weakly (τ_1, τ_2)-continuous functions [17]. Kong-ied et al. [18] introduced and studied the notion of almost quasi (τ_1, τ_2)-continuous functions. Chiangpradit et al. [19] introduced and investigated the concept of weakly quasi (τ_1, τ_2)-continuous functions. In this paper, we introduce new classes of functions between an ideal topological space and a bitopological space, namely almost quasi $\tau^\star(\sigma_1, \sigma_2)$-continuous functions and weakly quasi $\tau^\star(\sigma_1, \sigma_2)$-continuous functions. We also investigate several characterizations of almost quasi $\tau^\star(\sigma_1, \sigma_2)$-continuous functions and weakly quasi $\tau^\star(\sigma_1, \sigma_2)$-continuous functions.

2. Preliminaries

Throughout the present paper, spaces (X, τ_1, τ_2) and (Y, σ_1, σ_2) (or simply X and Y) always mean bitopological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a bitopological space (X, τ_1, τ_2). The closure of A and the interior of A with respect to τ_i are denoted by τ_i-Cl(A) and τ_i-Int(A), respectively, for $i = 1, 2$. A subset A of a bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$-*closed* [20] if $A = \tau_1\text{-Cl}(\tau_2\text{-Cl}(A))$. The complement of a $\tau_1\tau_2$-closed set is called $\tau_1\tau_2$-*open*. The intersection of all $\tau_1\tau_2$-closed sets of X containing A is called the $\tau_1\tau_2$-*closure* [20] of A and is denoted by $\tau_1\tau_2$-Cl(A). The union of all $\tau_1\tau_2$-open sets of X contained in A is called the $\tau_1\tau_2$-*interior* [20] of A and is denoted by $\tau_1\tau_2$-Int(A).

Lemma 1. [20] *Let A and B be subsets of a bitopological space (X, τ_1, τ_2). For the $\tau_1\tau_2$-closure, the following properties hold:*

(1) $A \subseteq \tau_1\tau_2\text{-}Cl(A)$ and $\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Cl(A)) = \tau_1\tau_2\text{-}Cl(A)$.

(2) If $A \subseteq B$, then $\tau_1\tau_2\text{-}Cl(A) \subseteq \tau_1\tau_2\text{-}Cl(B)$.

(3) $\tau_1\tau_2\text{-}Cl(A)$ is $\tau_1\tau_2$-closed.

(4) A is $\tau_1\tau_2$-closed if and only if $A = \tau_1\tau_2\text{-}Cl(A)$.

(5) $\tau_1\tau_2\text{-}Cl(X - A) = X - \tau_1\tau_2\text{-}Int(A)$.

A subset A of a bitopological space (X, τ_1, τ_2) is called $(\tau_1, \tau_2)r$-*open* [21] (resp. $(\tau_1, \tau_2)s$-*open* [22], $(\tau_1, \tau_2)p$-*open* [22], $(\tau_1, \tau_2)\beta$-*open* [22]) if $A = \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A))$ (resp. $A \subseteq \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(A))$, $A \subseteq \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A))$, $A \subseteq \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A)))$). The complement of a $(\tau_1, \tau_2)r$-open (resp. $(\tau_1, \tau_2)s$-open, $(\tau_1, \tau_2)p$-open, $(\tau_1, \tau_2)\beta$-open) set is said to be $(\tau_1, \tau_2)r$-*closed* (resp. $(\tau_1, \tau_2)s$-*closed*, $(\tau_1, \tau_2)p$-*closed*, $(\tau_1, \tau_2)\beta$-*closed*). A subset A of a bitopological space (X, τ_1, τ_2) is said to be $\alpha(\tau_1, \tau_2)$-*open* [23] if $A \subseteq$

$\tau_1\tau_2$-Int($\tau_1\tau_2$-Cl($\tau_1\tau_2$-Int(A))). The complement of an $\alpha(\tau_1,\tau_2)$-open set is said to be $\alpha(\tau_1,\tau_2)$*-closed*. Let A be a subset of a bitopological space (X,τ_1,τ_2). The intersection of all $(\tau_1,\tau_2)p$-closed (resp. $(\tau_1,\tau_2)s$-closed, $\alpha(\tau_1,\tau_2)$-closed) sets of X containing A is called the $(\tau_1,\tau_2)p$*-closure* [24] (resp. $(\tau_1,\tau_2)s$*-closure* [22], $\alpha(\tau_1,\tau_2)$*-closure* [25]) of A and is denoted by (τ_1,τ_2)-pCl(A) (resp. (τ_1,τ_2)-sCl(A), $\alpha(\tau_1,\tau_2)$-Cl(A)). The union of all $(\tau_1,\tau_2)p$-open (resp. $(\tau_1,\tau_2)s$-open, $\alpha(\tau_1,\tau_2)$-open) sets of X contained in A is called the $(\tau_1,\tau_2)p$*-interior* [24] (resp. $(\tau_1,\tau_2)s$*-interior* [22], $\alpha(\tau_1,\tau_2)$*-interior* [25]) of A and is denoted by (τ_1,τ_2)-pInt(A) (resp. (τ_1,τ_2)-sInt(A), $\alpha(\tau_1,\tau_2)$-Int(A)).

Lemma 2. [18] *For a subset A of a bitopological space (X,τ_1,τ_2), the following properties hold:*

(1) (τ_1,τ_2)-*sCl*$(A) = \tau_1\tau_2$-*Int*$(\tau_1\tau_2$-*Cl*$(A)) \cup A$ [17];

(2) (τ_1,τ_2)-*sInt*$(A) = \tau_1\tau_2$-*Cl*$(\tau_1\tau_2$-*Int*$(A)) \cap A$.

For a subset A of a bitopological space (X,τ_1,τ_2), a point $x \in X$ is called a $(\tau_1,\tau_2)\theta$-*cluster point* of A if $\tau_1\tau_2$-Cl$(U) \cap A \neq \emptyset$ for every $\tau_1\tau_2$-open set U containing x. The set of all $(\tau_1,\tau_2)\theta$-cluster points of A is called the $(\tau_1,\tau_2)\theta$*-closure* of A and is denoted by $(\tau_1,\tau_2)\theta$-Cl(A). A subset A of a bitopological space (X,τ_1,τ_2) is said to be $(\tau_1,\tau_2)\theta$*-closed* if $(\tau_1,\tau_2)\theta$-Cl$(A) = A$. The complement of a $(\tau_1,\tau_2)\theta$-closed set is said to be $(\tau_1,\tau_2)\theta$*-open*. The union of all $(\tau_1,\tau_2)\theta$-open sets of X contained in A is called the $(\tau_1,\tau_2)\theta$*-interior* of A and is denoted by $(\tau_1,\tau_2)\theta$-Int(A) [21].

Lemma 3. [21] *For a subset A of a bitopological space (X,τ_1,τ_2), the following properties hold:*

(1) If A is $\tau_1\tau_2$-open in X, then $\tau_1\tau_2$-Cl$(A) = (\tau_1,\tau_2)\theta$-Cl$(A)$.

(2) $(\tau_1,\tau_2)\theta$-Cl(A) is $\tau_1\tau_2$-closed in X.

An ideal $\mathscr{I}$ on a topological space (X,τ) is a nonempty collection of subsets of X satisfying the following properties: (1) $A \in \mathscr{I}$ and $B \subseteq A$ imply $B \in \mathscr{I}$; (2) $A \in \mathscr{I}$ and $B \in \mathscr{I}$ imply $A \cup B \in \mathscr{I}$. A topological space (X,τ) with an ideal $\mathscr{I}$ on X is called an ideal topological space and is denoted by $(X,\tau,\mathscr{I})$. For an ideal topological space $(X,\tau,\mathscr{I})$ and a subset A of X, $A^\star(\mathscr{I})$ is defined as follows:

$$A^\star(\mathscr{I}) = \{x \in X : U \cap A \notin \mathscr{I} \text{ for every open neighbourhood } U \text{ of } x\}.$$

In case there is no chance for confusion, $A^\star(\mathscr{I})$ is simply written as $A^\star$. In [26], $A^\star$ is called the local function of A with respect to $\mathscr{I}$ and τ and Cl$^\star(A) = A^\star \cup A$ defines a Kuratowski closure operator for a topology $\tau^\star(\mathscr{I})$ finer than τ. A subset A is said to be $\star$-closed [10] if $A^\star \subseteq A$. The interior of a subset A in $(X,\tau^\star(\mathscr{I}))$ is denoted by Int$^\star(A)$. A subset A of an ideal topological space $(X,\tau,\mathscr{I})$ is said to be *semi***-$\mathscr{I}$-open* [27] (resp. *semi-$\mathscr{I}$-open* [13]) if $A \subseteq$ Cl(Int$^\star(A)$) (resp. $A \subseteq$ Cl*(Int(A))). The complement of a semi*-$\mathscr{I}$-open (resp. semi-$\mathscr{I}$-open) set is said to be *semi***-$\mathscr{I}$-closed* [27] (resp. semi-$\mathscr{I}$-closed [13]). A subset A

of an ideal topological space $(X, \tau, \mathscr{I})$ is said to be *semi-$\mathscr{I}^\star$-open* [28] if $A \subseteq \mathrm{Cl}^\star(\mathrm{Int}^\star(A))$. The complement of a semi-$\mathscr{I}^\star$-open set is called *semi-$\mathscr{I}^\star$-closed* [28]. For a subset A of an ideal topological space $(X, \tau, \mathscr{I})$, the intersection of all semi-$\mathscr{I}$-closed sets containing A is called the *semi-$\mathscr{I}^\star$-closure* [28] of A and is denoted by $\mathrm{sCl}^\star(A)$ ($\mathrm{sCl}_{\mathscr{I}^\star}(A)$ [28]). The union of all semi-$\mathscr{I}$-open sets contained in A is called the *semi-$\mathscr{I}^\star$-interior* [28] of A and is denoted by $\mathrm{sInt}^\star(A)$ ($\mathrm{sInt}_{\mathscr{I}^\star}(A)$ [28]).

Lemma 4. [28] *For a subset A of a an ideal topological space $(X, \tau, \mathscr{I})$, the following properties hold:*

(1) $sCl^\star(A) = A \cup Int^\star(Cl^\star(A))$;

(2) $sInt^\star(A) = A \cap Cl^\star(Int^\star(A))$.

3. Almost quasi $\tau^\star(\sigma_1, \sigma_2)$-continuous functions

In this section, we introduce the concept of almost quasi $\tau^\star(\sigma_1, \sigma_2)$-continuous functions. Moreover, some characterizations of almost quasi $\tau^\star(\sigma_1, \sigma_2)$-continuous functions are considered.

Definition 1. *A function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is said to be almost quasi $\tau^\star(\sigma_1, \sigma_2)$-continuous at a point $x \in X$ if for every $\sigma_1\sigma_2$-open set V of Y containing $f(x)$ and each $\star$-open set U of X containing x, there exists a nonempty $\star$-open set G such that $G \subseteq U$ and $f(G) \subseteq (\sigma_1, \sigma_2)$-$sCl(V)$. A function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is said to be almost quasi $\tau^\star(\sigma_1, \sigma_2)$-continuous if f is almost quasi $\tau^\star(\sigma_1, \sigma_2)$-continuous at each point of X.*

Theorem 1. *For a function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) f is almost quasi $\tau^\star(\sigma_1, \sigma_2)$-continuous at $x \in X$;

(2) for every $\sigma_1\sigma_2$-open set V of Y containing $f(x)$, there exists a semi-$\mathscr{I}^\star$-open set U of X containing x such that $f(U) \subseteq (\sigma_1, \sigma_2)$-$sCl(V)$;

(3) $x \in sInt^\star(f^{-1}((\sigma_1, \sigma_2)$-$sCl(V)))$ for every $\sigma_1\sigma_2$-open set V of Y containing $f(x)$;

(4) $x \in Cl^\star(Int^\star(f^{-1}((\sigma_1, \sigma_2)$-$sCl(V))))$ for every $\sigma_1\sigma_2$-open set V of Y containing $f(x)$.

Proof. (1) $\Rightarrow$ (2): Let $\mathscr{U}(x)$ be the family of all $\star$-open sets of X containing x. Let V be any $\sigma_1\sigma_2$-open set of Y containing $f(x)$. For each $H \in \mathscr{U}(x)$, there exists a nonempty $\star$-open set G_H such that $G_H \subseteq H$, $f(G_H) \subseteq (\sigma_1, \sigma_2)$-$\mathrm{sCl}(V)$. Let $W = \cup\{G_H \mid H \in \mathscr{U}(x)\}$. Then, W is $\star$-open in X and $x \in \mathrm{Cl}^\star(W)$. Put $U = W \cup \{x\}$. Then, U is a semi-$\mathscr{I}^\star$-open set of X containing x and $f(U) \subseteq (\sigma_1, \sigma_2)$-$\mathrm{sCl}(V)$.

(2) $\Rightarrow$ (3): Let V be any $\sigma_1\sigma_2$-open set of Y containing $f(x)$. Then, there exists a semi-$\mathscr{I}^\star$-open set U of X containing x such that $f(U) \subseteq (\sigma_1, \sigma_2)$-$\mathrm{sCl}(V)$. Thus,

$$x \in U \subseteq f^{-1}((\sigma_1, \sigma_2)\text{-sCl}(V))$$

and hence $x \in U \subseteq \text{sInt}^\star(f^{-1}((\sigma_1, \sigma_2)\text{-sCl}(V)))$.

(3) $\Rightarrow$ (4): Let V be any $\sigma_1\sigma_2$-open set of Y containing $f(x)$. Thus by (3),

$$x \in \text{sInt}^\star(f^{-1}((\sigma_1, \sigma_2)\text{-sCl}(V))).$$

Now, put $U = \text{sInt}^\star(f^{-1}((\sigma_1, \sigma_2)\text{-sCl}(V)))$. Then, we have U is semi-$\mathscr{I}^\star$-open in X and by Lemma 4, $x \in U \subseteq \text{Cl}^\star(\text{Int}^\star(U)) \subseteq \text{Cl}^\star(\text{Int}^\star(f^{-1}((\sigma_1, \sigma_2)\text{-sCl}(V))))$.

(4) $\Rightarrow$ (1): Let U be any $\star$-open set of X containing x and V be any $\sigma_1\sigma_2$-open set of Y containing $f(x)$. Then, we have $x \in \text{Cl}^\star(\text{Int}^\star(f^{-1}((\sigma_1, \sigma_2)\text{-sCl}(V))))$ and hence $U \cap \text{Int}^\star(f^{-1}((\sigma_1, \sigma_2)\text{-sCl}(V))) \neq \emptyset$. Put $W = U \cap \text{Int}^\star(f^{-1}((\sigma_1, \sigma_2)\text{-sCl}(V)))$. Then, W is a nonempty $\star$-open set of X such that $W \subseteq U$, $f(W) \subseteq (\sigma_1, \sigma_2)\text{-sCl}(V)$. This shows that f is almost quasi $\tau^\star(\sigma_1, \sigma_2)$-continuous at x.

Theorem 2. *For a function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) f is almost quasi $\tau^\star(\sigma_1, \sigma_2)$-continuous;

(2) for each $x \in X$ and every $\sigma_1\sigma_2$-open set V of Y containing $f(x)$, there exists a semi-$\mathscr{I}^\star$-open set U of X containing x such that $f(U) \subseteq (\sigma_1, \sigma_2)$-sCl$(V)$;

(3) $f^{-1}(V)$ is semi-$\mathscr{I}^\star$-open in X for every $(\sigma_1, \sigma_2)r$-open set V of Y;

(4) $f^{-1}(V) \subseteq sInt^\star(f^{-1}((\sigma_1, \sigma_2)\text{-}sCl(V)))$ for every $\sigma_1\sigma_2$-open set V of Y;

(5) $sCl^\star(f^{-1}(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(B))))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-}Cl(B))$ for every subset B of Y;

(6) $f^{-1}(V) \subseteq Cl^\star(Int^\star(f^{-1}((\sigma_1, \sigma_2)\text{-}sCl(V))))$ for every $\sigma_1\sigma_2$-open set V of Y.

Proof. (1) $\Rightarrow$ (2): The proof follows from Theorem 1.

(2) $\Rightarrow$ (3): Let V be any $(\sigma_1, \sigma_2)r$-open set of Y and $x \in f^{-1}(V)$. Then, $f(x) \in V$ and there exists a semi-$\mathscr{I}^\star$-open set U of X containing x such that $f(U) \subseteq V$. Thus, $x \in U \subseteq f^{-1}(V)$ and hence $x \in \text{sInt}^\star(f^{-1}(V))$. Therefore, $f^{-1}(V) \subseteq \text{sInt}^\star(f^{-1}(V))$. This shows that $f^{-1}(V)$ is semi-$\mathscr{I}^\star$-open in X.

(3) $\Rightarrow$ (4): Let V be any $\sigma_1\sigma_2$-open set of Y and $x \in f^{-1}(V)$. Then, we have

$$f(x) \in V \subseteq (\sigma_1, \sigma_2)\text{-sCl}(V).$$

Thus, $x \in f^{-1}((\sigma_1, \sigma_2)\text{-sCl}(V))$. By Lemma 2, $(\sigma_1, \sigma_2)\text{-sCl}(V)$ is $(\sigma_1, \sigma_2)r$-open in Y. By (3), $f^{-1}((\sigma_1, \sigma_2)\text{-sCl}(V))$ is semi-$\mathscr{I}^\star$-open in X and so $x \in \text{sInt}^\star(f^{-1}((\sigma_1, \sigma_2)\text{-sCl}(V)))$. Thus, $f^{-1}(V) \subseteq \text{sInt}^\star(f^{-1}((\sigma_1, \sigma_2)\text{-sCl}(V)))$.

(4) $\Rightarrow$ (5): Let B be any subset of Y. Then, we have $Y - \sigma_1\sigma_2\text{-Cl}(B)$ is $\sigma_1\sigma_2$-open in Y. Thus by (4) and Lemma 2,

$$X - f^{-1}(\sigma_1\sigma_2\text{-Cl}(B)) = f^{-1}(Y - \sigma_1\sigma_2\text{-Cl}(B))$$

$$\begin{aligned}&\subseteq \mathrm{sInt}^\star(f^{-1}((\sigma_1,\sigma_2)\text{-sCl}(Y-\sigma_1\sigma_2\text{-Cl}(B))))\\&=\mathrm{sInt}^\star(X-f^{-1}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(B)))))\\&=X-\mathrm{sCl}^\star(f^{-1}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(B)))))\end{aligned}$$

and hence

$$\mathrm{sCl}^\star(f^{-1}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(B))))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(B)).$$

(5) $\Rightarrow$ (6): Let V be any $\sigma_1\sigma_2$-open set of Y. Then, $Y-V$ is $\sigma_1\sigma_2$-closed in Y. By (5) and Lemma 4,

$$\mathrm{Int}^\star(\mathrm{Cl}^\star(f^{-1}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(Y-V))))) \subseteq f^{-1}(Y-V)=X-f^{-1}(V).$$

Moreover, we have

$$\begin{aligned}\mathrm{Int}^\star(\mathrm{Cl}^\star(f^{-1}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(Y-V)))))&=\mathrm{Int}^\star(\mathrm{Cl}^\star(f^{-1}(Y-\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))))\\&=\mathrm{Int}^\star(\mathrm{Cl}^\star(X-f^{-1}((\sigma_1,\sigma_2)\text{-sCl}(V))))\\&=X-\mathrm{Cl}^\star(\mathrm{Int}^\star(f^{-1}((\sigma_1,\sigma_2)\text{-sCl}(V)))).\end{aligned}$$

Thus, $f^{-1}(V)\subseteq \mathrm{Cl}^\star(\mathrm{Int}^\star(f^{-1}((\sigma_1,\sigma_2)\text{-sCl}(V))))$.

(6) $\Rightarrow$ (1): Let $x\in X$ and V be any $\sigma_1\sigma_2$-open set of Y containing $f(x)$. By (6), we have $x\in f^{-1}(V)\subseteq \mathrm{Cl}^\star(\mathrm{Int}^\star(f^{-1}((\sigma_1,\sigma_2)\text{-sCl}(V))))$ and by Lemma 4,

$$x\in f^{-1}(V)\subseteq \mathrm{sInt}^\star(f^{-1}((\sigma_1,\sigma_2)\text{-sCl}(V))).$$

Put $U=\mathrm{sInt}^\star(f^{-1}((\sigma_1,\sigma_2)\text{-sCl}(V)))$. Then, U is semi-$\mathscr{I}^\star$-open set of X containing x such that $f(U)\subseteq(\sigma_1,\sigma_2)$-sCl$(V)$. This shows that f is almost quasi $\tau^\star(\sigma_1,\sigma_2)$-continuous.

Theorem 3. *For a function $f:(X,\tau,\mathscr{I})\to(Y,\sigma_1,\sigma_2)$, the following properties are equivalent:*

(1) f is almost quasi $\tau^\star(\sigma_1,\sigma_2)$-continuous;

(2) $sCl^\star(f^{-1}(V))\subseteq f^{-1}(\sigma_1\sigma_2\text{-}Cl(V))$ for every $(\sigma_1,\sigma_2)\beta$-open set V of Y;

(3) $sCl^\star(f^{-1}(V))\subseteq f^{-1}(\sigma_1\sigma_2\text{-}Cl(V))$ for every $(\sigma_1,\sigma_2)s$-open set V of Y;

(4) $f^{-1}(V)\subseteq sInt^\star(f^{-1}(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V))))$ for every $(\sigma_1,\sigma_2)p$-open set V of Y.

Proof. The proof follows from Theorem 2.

4. Weakly quasi $\tau^\star(\sigma_1, \sigma_2)$-continuous functions

In this section, we introduce the notion of weakly quasi $\tau^\star(\sigma_1, \sigma_2)$-continuous functions. Furthermore, several characterizations of weakly quasi $\tau^\star(\sigma_1, \sigma_2)$-continuous functions are discussed.

Definition 2. *A function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is said to be weakly quasi $\tau^\star(\sigma_1, \sigma_2)$-continuous at a point $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y containing $f(x)$ and each $\star$-open set U of X containing x, there exists a nonempty $\star$-open set G such that $G \subseteq U$, $f(G) \subseteq \sigma_1\sigma_2$-Cl$(V)$. A function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is said to be weakly quasi $\tau^\star(\sigma_1, \sigma_2)$-continuous if f is weakly quasi $\tau^\star(\sigma_1, \sigma_2)$-continuous at each point of X.*

Remark 1. *For a function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following implication holds:*

almost quasi $\tau^\star(\sigma_1, \sigma_2)$-continuity $\Rightarrow$ weakly quasi $\tau^\star(\sigma_1, \sigma_2)$-continuity.

The converse of the implication is not true in general. We give an example for the implication as follows.

Example 1. *Let $X = \{1, 2, 3\}$ with a topology $\tau = \{\emptyset, \{1\}, \{2, 3\}, X\}$ and an ideal $\mathscr{I} = \{\emptyset, \{1\}\}$. Let $Y = \{a, b, c\}$ with topologies $\sigma_1 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, Y\}$ and $\sigma_2 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, Y\}$. A function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$ is defined as follows: $f(1) = a$, $f(2) = b$ and $f(3) = c$. Then, f is weakly quasi $\tau^\star(\sigma_1, \sigma_2)$-continuous but f is not almost quasi $\tau^\star(\sigma_1, \sigma_2)$-continuous.*

Theorem 4. *For a function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) f is weakly quasi $\tau^\star(\sigma_1, \sigma_2)$-continuous;

(2) for each $x \in X$ and every $\sigma_1\sigma_2$-open set V of Y containing $f(x)$, there exists a semi-$\mathscr{I}^\star$-open set of X containing x such that $f(U) \subseteq \sigma_1\sigma_2$-Cl$(V)$;

(3) Int*(Cl*(f^{-1}($\sigma_1\sigma_2$-Int(K)))) $\subseteq f^{-1}(K)$ for every $\sigma_1\sigma_2$-closed set K of Y;

(4) $f^{-1}(V) \subseteq$ sInt*(f^{-1}($\sigma_1\sigma_2$-Cl(V))) for every $\sigma_1\sigma_2$-open set V of Y;

(5) sCl*($f^{-1}(V)$) $\subseteq f^{-1}$($\sigma_1\sigma_2$-Cl(V)) for every $\sigma_1\sigma_2$-open sets V of Y.

Proof. (1) $\Rightarrow$ (2): Let $\mathscr{U}(x)$ be the family of all $\star$-open sets of X containing x. Let V be any $\sigma_1\sigma_2$-open set of Y containing $f(x)$. For each $H \in \mathscr{U}(x)$, there exists a nonempty $\star$-open set G_H such that $G_H \subseteq H$ and $f(G_H) \subseteq \sigma_1\sigma_2$-Cl$(V)$. Put $W = \cup\{G_H \mid H \in \mathscr{U}(x)\}$. Then, W is $\star$-open in X and $x \in \mathrm{Cl}^\star(W)$. Let $U = W \cup \{x\}$. Then, U is a semi-$\mathscr{I}^\star$-open set of X containing x and $f(U) \subseteq \sigma_1\sigma_2$-Cl$(V)$.

(2) $\Rightarrow$ (4): Let V be any $\sigma_1\sigma_2$-open set of Y and $x \in f^{-1}(V)$. Then, $f(x) \in V$ and there exists a semi-$\mathscr{I}^\star$-open set U of X containing x such that $f(U) \subseteq \sigma_1\sigma_2$-Cl$(V)$. Thus,

$$x \in U \subseteq \mathrm{sInt}^\star(f^{-1}(\sigma_1\sigma_2\text{-}\mathrm{Cl}(V)))$$

and hence $f^{-1}(V) \subseteq \text{sInt}^\star(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)))$.

(4) ⇒ (5): Let V be any $\sigma_1\sigma_2$-open set of Y. Then by (4), we have

$$\begin{aligned} X - f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)) &= f^{-1}(Y - \sigma_1\sigma_2\text{-Cl}(V)) \\ &\subseteq \text{sInt}^\star(f^{-1}(\sigma_1\sigma_2\text{-Cl}(Y - \sigma_1\sigma_2\text{-Cl}(V)))) \\ &= \text{sInt}^\star(f^{-1}(Y - \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) \\ &\subseteq \text{sInt}^\star(f^{-1}(Y - V)) \\ &= \text{sInt}^\star(X - f^{-1}(V)) \\ &= X - \text{sCl}^\star(f^{-1}(V)) \end{aligned}$$

and so $\text{sCl}^\star(f^{-1}(V)) \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))$.

(5) ⇒ (3): Let K be any $\sigma_1\sigma_2$-closed set of Y. Thus by (5) and Lemma 4,

$$\begin{aligned} \text{Int}^\star(\text{Cl}^\star(f^{-1}(\sigma_1\sigma_2\text{-Int}(K)))) &\subseteq \text{sCl}^\star(f^{-1}(\sigma_1\sigma_2\text{-Int}(K))) \\ &\subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K))) \\ &\subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(K)) = f^{-1}(K). \end{aligned}$$

(3) ⇒ (4): Let V be any $\sigma_1\sigma_2$-open set of Y. By (3) and Lemma 4,

$$\begin{aligned} X - \text{sInt}^\star(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))) &= \text{sCl}^\star(f^{-1}(Y - \sigma_1\sigma_2\text{-Cl}(V))) \\ &\subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(Y - \sigma_1\sigma_2\text{-Cl}(V))) \\ &= f^{-1}(Y - \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))) \\ &\subseteq f^{-1}(Y - V) = X - f^{-1}(V) \end{aligned}$$

and hence $f^{-1}(V) \subseteq \text{sInt}^\star(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)))$.

(4) ⇒ (1): Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y containing $f(x)$. By (4), we have $f^{-1}(V) \subseteq \text{sInt}^\star(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)))$. Put $U = \text{sInt}^\star(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)))$. Then, U is a semi-$\mathscr{I}^\star$-open set of X containing x such that $f(U) \subseteq \sigma_1\sigma_2\text{-Cl}(V)$. This shows that f is weakly quasi $\tau^\star(\sigma_1, \sigma_2)$-continuous.

Theorem 5. *For a function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) f is weakly quasi $\tau^\star(\sigma_1, \sigma_2)$-continuous;

(2) $sCl^\star(f^{-1}(\sigma_1\sigma_2\text{-}Int((\sigma_1, \sigma_2)\theta\text{-}Cl(B)))) \subseteq f^{-1}((\sigma_1, \sigma_2)\theta\text{-}Cl(B))$ for every subset B of Y;

(3) $sCl^\star(f^{-1}(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(B)))) \subseteq f^{-1}((\sigma_1, \sigma_2)\theta\text{-}Cl(B))$ for every subset B of Y;

(4) $sCl^\star(f^{-1}(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V)))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-}Cl(V))$ for every $\sigma_1\sigma_2$-open set V of Y;

(5) $sCl^\star(f^{-1}(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V)))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-}Cl(V))$ *for every* $(\sigma_1,\sigma_2)p$*-open set* V *of* Y*;*

(6) $sCl^\star(f^{-1}(\sigma_1\sigma_2\text{-}Int(K))) \subseteq f^{-1}(K)$ *for every* $(\sigma_1,\sigma_2)r$*-closed set* K *of* Y*.*

Proof. (1) ⇒ (2): Let B be any subset of Y. Since $(\sigma_1,\sigma_2)\theta\text{-Cl}(B)$ is $\sigma_1\sigma_2$-closed in Y, by Theorem 4 we have $\text{Int}^\star(\text{Cl}^\star(f^{-1}(\sigma_1\sigma_2\text{-Int}((\sigma_1,\sigma_2)\theta\text{-Cl}(B))))) \subseteq f^{-1}((\sigma_1,\sigma_2)\theta\text{-Cl}(B))$ and by Lemma 4, $\text{sCl}^\star(f^{-1}(\sigma_1\sigma_2\text{-Int}((\sigma_1,\sigma_2)\theta\text{-Cl}(B)))) \subseteq f^{-1}((\sigma_1,\sigma_2)\theta\text{-Cl}(B))$.

(2) ⇒ (3): This is obvious since $\sigma_1\sigma_2\text{-Cl}(B) \subseteq (\sigma_1,\sigma_2)\theta\text{-Cl}(B)$ for every subset B of Y.

(3) ⇒ (4): This is obvious since $\sigma_1\sigma_2\text{-Cl}(V) = (\sigma_1,\sigma_2)\theta\text{-Cl}(V)$ for every $\sigma_1\sigma_2$-open set V of Y.

(4) ⇒ (5): Let V be any $(\sigma_1,\sigma_2)p$-open set of Y. Then, $V \subseteq \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))$ and $\sigma_1\sigma_2\text{-Cl}(V) = \sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))$. Now, put $G = \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))$, then G is $\sigma_1\sigma_2$-open in Y and $\sigma_1\sigma_2\text{-Cl}(G) = \sigma_1\sigma_2\text{-Cl}(V)$. Thus by (4),

$$\text{sCl}^\star(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)).$$

(5) ⇒ (6): Let K be any $(\sigma_1,\sigma_2)r$-closed set of Y. Since $\sigma_1\sigma_2\text{-Int}(K)$ is $(\sigma_1,\sigma_2)p$-open in Y and by (5), we have

$$\begin{aligned}\text{sCl}^\star(f^{-1}(\sigma_1\sigma_2\text{-Int}(K))) &= \text{sCl}^\star(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K))))) \\ &\subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K))) \\ &= f^{-1}(K).\end{aligned}$$

(6) ⇒ (1): Let V be any $\sigma_1\sigma_2$-open set of Y. Then, $\sigma_1\sigma_2\text{-Cl}(V)$ is $(\sigma_1,\sigma_2)r$-closed in Y and by (6), $\text{sCl}^\star(f^{-1}(V)) \subseteq \text{sCl}^\star(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))$. It follows from Theorem 4 that f is weakly quasi $\tau^\star(\sigma_1,\sigma_2)$-continuous.

Theorem 6. *For a function* $f : (X,\tau,\mathscr{I}) \to (Y,\sigma_1,\sigma_2)$*, the following properties are equivalent:*

(1) f *is weakly quasi* $\tau^\star(\sigma_1,\sigma_2)$*-continuous;*

(2) $sCl^\star(f^{-1}(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V)))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-}Cl(V))$ *for every* $(\sigma_1,\sigma_2)\beta$*-open set* V *of* Y*;*

(3) $sCl^\star(f^{-1}(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V)))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-}Cl(V))$ *for every* $(\sigma_1,\sigma_2)s$*-open set* V *of* Y*;*

(4) $sCl^\star(f^{-1}(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V)))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-}Cl(V))$ *for every* $(\sigma_1,\sigma_2)p$*-open set* V *of* Y*.*

Proof. (1) ⇒ (2): Let V be any $(\sigma_1,\sigma_2)\beta$-open set of Y. Then, we have

$$V \subseteq \sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))$$

and hence $\sigma_1\sigma_2\text{-Cl}(V) = \sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))$. Since $\sigma_1\sigma_2\text{-Cl}(V)$ is $(\sigma_1, \sigma_2)r$-closed in Y and by Theorem 5, $\text{sCl}^\star(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))$.

(2) $\Rightarrow$ (3): This is obvious since every $(\sigma_1, \sigma_2)s$-open set is $(\sigma_1, \sigma_2)\beta$-open.

(3) $\Rightarrow$ (4): Let V be any $(\sigma_1, \sigma_2)p$-open set of Y. Since $\sigma_1\sigma_2\text{-Cl}(V)$ is $(\sigma_1, \sigma_2)r$-closed, we have $\sigma_1\sigma_2\text{-Cl}(V)$ is $(\sigma_1, \sigma_2)s$-open in Y. Thus by (3),

$$\text{sCl}^\star(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)).$$

(4) $\Rightarrow$ (1): Let V be any $\sigma_1\sigma_2$-open set of Y. Then, V is $(\sigma_1, \sigma_2)p$-open in Y. By (4), $\text{sCl}^\star(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))$. It follows from Theorem 5 that f is weakly quasi $\tau^\star(\sigma_1, \sigma_2)$-continuous.

Theorem 7. *For a function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) f is weakly quasi $\tau^\star(\sigma_1, \sigma_2)$-continuous;

(2) $\mathit{Int}^\star(\mathit{Cl}^\star(f^{-1}(V))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-}\mathit{Cl}(V))$ for every $(\sigma_1, \sigma_2)p$-open set V of Y;

(3) $\mathit{sCl}^\star(f^{-1}(V)) \subseteq f^{-1}(\sigma_1\sigma_2\text{-}\mathit{Cl}(V))$ for every $(\sigma_1, \sigma_2)p$-open set V of Y;

(4) $f^{-1}(V) \subseteq \mathit{sInt}^\star(f^{-1}(\sigma_1\sigma_2\text{-}\mathit{Cl}(V)))$ for every $(\sigma_1, \sigma_2)p$-open set V of Y.

Proof. (1) $\Rightarrow$ (2): Let V be any $(\sigma_1, \sigma_2)p$-open set of Y. Since f is weakly quasi $\tau^\star(\sigma_1, \sigma_2)$-continuous, by Theorem 5 and Lemma 4

$$\text{Int}^\star(\text{Cl}^\star(f^{-1}(V))) \subseteq \text{Int}^\star(\text{Cl}^\star(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)).$$

(2) $\Rightarrow$ (3): Let V be any $(\sigma_1, \sigma_2)p$-open set of Y. By (2) and Lemma 4, we have

$$\text{sCl}^\star(f^{-1}(V)) = f^{-1}(V) \cup \text{Int}^\star(\text{Cl}^\star(f^{-1}(V))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)).$$

(3) $\Rightarrow$ (4): Let V be any $(\sigma_1, \sigma_2)p$-open set of Y. Thus by (3),

$$\begin{aligned}
X - \text{sInt}^\star(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))) &= \text{sCl}^\star(X - (f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)))) \\
&= \text{sCl}^\star(X - f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))) \\
&= \text{sCl}^\star(f^{-1}(Y - \sigma_1\sigma_2\text{-Cl}(V))) \\
&\subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(Y - \sigma_1\sigma_2\text{-Cl}(V))) \\
&= X - f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))) \\
&\subseteq X - f^{-1}(V)
\end{aligned}$$

and hence $f^{-1}(V) \subseteq \text{sInt}^\star(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)))$.

(4) $\Rightarrow$ (1): Since every $\sigma_1\sigma_2$-open set is $(\sigma_1, \sigma_2)p$-open, this follows from Theorem 4.

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