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On Weak Forms of Open and Closed Functions via $(\tau_1, \tau_2)\beta$-Open Sets

Monchaya Chiangpradit[1], Areeyuth Sama-Ae[2], Chawalit Boonpok[1,*]

[1] *Mathematics and Applied Mathematics Research Unit, Department of Mathematics, Faculty of Science, Mahasarakham University, Maha Sarakham, 44150, Thailand*
[2] *Department of Mathematics and Computer Science, Faculty of Science and Technology, Prince of Songkla University, Pattani Campus, Pattani, 94000, Thailand*

Abstract. This paper is concerned with the concepts of weakly $(\tau_1, \tau_2)\beta$-open functions and weakly $(\tau_1, \tau_2)\beta$-closed functions. Furthermore, several characterizations of weakly $(\tau_1, \tau_2)\beta$-open functions and weakly $(\tau_1, \tau_2)\beta$-closed functions are investigated.

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1. Introduction

It is well-known that the branch of mathematics called topology is related to all questions directly or indirectly concerned with openness and closedness. Semi-open sets, preopen sets, α-open sets, β-open sets, b-open sets, δ-open sets and θ-open sets play an important role in the researches of generalizations of open functions and closed functions. By using these sets, many authors introduced and studied various types of open functions and closed functions. The concept of weakly open functions was first introduced by Rose [1]. Rose and Janković [2] investigated some of the fundamental properties of weakly closed functions. Caldas and Navalagi [3] introduced two new classes of functions called weakly preopen functions and weakly preclosed functions as generalization of weak openness and weak closedness due to [1] and [2], respectively. Moreover, Caldas and Navalagi [4] introduced and investigated the concepts of weakly semi-open functions and weakly semi-closed functions as a new generalization of weakly open functions and weakly closed functions, respectively. Noiri et al. [5] introduced and studied two new classes of functions called weakly b-θ-open functions and weakly b-θ-open functions by utilizing the notions of b-θ-open sets and the b-θ-closure operator. Weak b-θ-openness (resp. b-θ-closedness) is

*Corresponding author.
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Email addresses: monchaya.c@msu.ac.th (M. Chiangpradit), areeyuth.s@psu.ac.th (A. Sama-Ae), chawalit.b@msu.ac.th (C. Boonpok)

a generalization of both θ-preopenness and weak semi-θ-openness (resp. θ-preclosedness and weak semi-θ-closedness). Caldas and Navalagi [6] introduced and investigated the notions of weakly β-open functions and weakly β-closed functions. Quite recently, Chutiman and Boonpok [7] studied some properties of weakly $b(\Lambda, p)$-open functions. On the other hand, the present authors introduced and studied the notions of semi-$(\mathscr{I}, \mathscr{J})$-open functions [8], semi-$(\mathscr{I}, \mathscr{J})$-closed functions [8], weakly $s(\Lambda, p)$-open functions [9], weakly $s(\Lambda, p)$-closed functions [9], weakly $\delta(\Lambda, p)$-open functions [10], weakly $\delta(\Lambda, p)$-closed functions [11], weakly $\beta(\Lambda, p)$-open functions [12], weakly $\beta(\Lambda, p)$-closed functions [12], weakly $p(\Lambda, p)$-open functions [13], weakly $p(\Lambda, p)$-closed functions [13], weakly $\theta s(\Lambda, p)$-open functions [14] and weakly $\theta s(\Lambda, p)$-closed functions [14]. In this paper, we introduce the notions of weakly $(\tau_1, \tau_2)\beta$-open functions and weakly $(\tau_1, \tau_2)\beta$-closed functions. Furthermore, several characterizations of weakly $(\tau_1, \tau_2)\beta$-open functions and weakly $(\tau_1, \tau_2)\beta$-closed functions are investigated.

2. Preliminaries

Throughout the present paper, spaces (X, τ_1, τ_2) and (Y, σ_1, σ_2) (or simply X and Y) always mean bitopological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a bitopological space (X, τ_1, τ_2). The closure of A and the interior of A with respect to τ_i are denoted by τ_i-Cl(A) and τ_i-Int(A), respectively, for $i = 1, 2$. A subset A of a bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$-*closed* [15] if $A = \tau_1\text{-Cl}(\tau_2\text{-Cl}(A))$. The complement of a $\tau_1\tau_2$-closed set is called $\tau_1\tau_2$-*open*. Let A be a subset of a bitopological space (X, τ_1, τ_2). The intersection of all $\tau_1\tau_2$-closed sets of X containing A is called the $\tau_1\tau_2$-*closure* [15] of A and is denoted by $\tau_1\tau_2$-Cl(A). The union of all $\tau_1\tau_2$-open sets of X contained in A is called the $\tau_1\tau_2$-*interior* [15] of A and is denoted by $\tau_1\tau_2$-Int(A).

Lemma 1. [15] *Let A and B be subsets of a bitopological space (X, τ_1, τ_2). For the $\tau_1\tau_2$-closure, the following properties hold:*

(1) $A \subseteq \tau_1\tau_2\text{-}Cl(A)$ and $\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Cl(A)) = \tau_1\tau_2\text{-}Cl(A)$.

(2) If $A \subseteq B$, then $\tau_1\tau_2\text{-}Cl(A) \subseteq \tau_1\tau_2\text{-}Cl(B)$.

(3) $\tau_1\tau_2\text{-}Cl(A)$ is $\tau_1\tau_2$-closed.

(4) A is $\tau_1\tau_2$-closed if and only if $A = \tau_1\tau_2\text{-}Cl(A)$.

(5) $\tau_1\tau_2\text{-}Cl(X - A) = X - \tau_1\tau_2\text{-}Int(A)$.

A subset A of a bitopological space (X, τ_1, τ_2) is said to be $(\tau_1, \tau_2)r$-*open* [16] (resp. $(\tau_1, \tau_2)s$-*open* [17], $(\tau_1, \tau_2)p$-*open* [17], $(\tau_1, \tau_2)\beta$-*open* [17]) if $A = \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A))$ (resp. $A \subseteq \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(A))$, $A \subseteq \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A))$, $A \subseteq \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A)))$). The complement of a $(\tau_1, \tau_2)r$-open (resp. $(\tau_1, \tau_2)s$-open, $(\tau_1, \tau_2)p$-open, $(\tau_1, \tau_2)\beta$-open) set is called $(\tau_1, \tau_2)r$-*closed* (resp. $(\tau_1, \tau_2)s$-*closed*, $(\tau_1, \tau_2)p$-*closed*, $(\tau_1, \tau_2)\beta$-*closed*). A subset A of a bitopological space (X, τ_1, τ_2) is said to be $\alpha(\tau_1, \tau_2)$-*open* [18] if $A \subseteq$

$\tau_1\tau_2$-Int($\tau_1\tau_2$-Cl($\tau_1\tau_2$-Int(A))). The complement of an $\alpha(\tau_1, \tau_2)$-open set is said to be $\alpha(\tau_1, \tau_2)$-*closed*. Let A be a subset of a bitopological space (X, τ_1, τ_2). The intersection of all $(\tau_1, \tau_2)\beta$-closed sets of X containing A is called the $(\tau_1, \tau_2)\beta$-*closure* [19] of A and is denoted by $(\tau_1, \tau_2)\beta$-Cl(A). The union of all $(\tau_1, \tau_2)\beta$-open sets of X contained in A is called the $(\tau_1, \tau_2)\beta$-*interior* [19] of A and is denoted by $(\tau_1, \tau_2)\beta$-Int(A).

Lemma 2. [19] *For subsets A and B of a bitopological space (X, τ_1, τ_2), the following properties hold:*

(1) $A \subseteq (\tau_1, \tau_2)\beta$-Cl($A$) and $(\tau_1, \tau_2)\beta$-Cl($(\tau_1, \tau_2)\beta$-Cl(A)) $= (\tau_1, \tau_2)\beta$-Cl(A).

(2) If $A \subseteq B$, then $(\tau_1, \tau_2)\beta$-Cl(A) $\subseteq (\tau_1, \tau_2)\beta$-Cl($B$).

(3) $(\tau_1, \tau_2)\beta$-Cl(A) is $(\tau_1, \tau_2)\beta$-closed.

(4) A is $(\tau_1, \tau_2)\beta$-closed if and only if $A = (\tau_1, \tau_2)\beta$-Cl($A$).

(5) $(\tau_1, \tau_2)\beta$-Cl($X - A$) $= X - (\tau_1, \tau_2)\beta$-Int($A$).

For a subset A of a bitopological space (X, τ_1, τ_2), a point $x \in X$ is called $(\tau_1, \tau_2)\theta$-*cluster point* of A if $\tau_1\tau_2$-Cl(U) $\cap A \neq \emptyset$ for every $\tau_1\tau_2$-open set U containing x. The set of all $(\tau_1, \tau_2)\theta$-cluster points of A is called the $(\tau_1, \tau_2)\theta$-*closure* of A and is denoted by $(\tau_1, \tau_2)\theta$-Cl(A). A subset A of a bitopological space (X, τ_1, τ_2) is said to be $(\tau_1, \tau_2)\theta$-*closed* if $A = (\tau_1, \tau_2)\theta$-Cl($A$). The complement of a $(\tau_1, \tau_2)\theta$-closed set is said to be $(\tau_1, \tau_2)\theta$-*open*. The union of all $(\tau_1, \tau_2)\theta$-open sets contained in A is called the $(\tau_1, \tau_2)\theta$-*interior* of A and is denoted by $(\tau_1, \tau_2)\theta$-Int(A) [16].

Lemma 3. [16] *For a subset A of a bitopological space (X, τ_1, τ_2), the following properties hold:*

(1) If A is $\tau_2\tau_2$-open in X, then $\tau_1\tau_2$-Cl(A) $= (\tau_1, \tau_2)\theta$-Cl(A).

(2) $(\tau_1, \tau_2)\theta$-Cl(A) is $\tau_1\tau_2$-closed in X.

3. Characterizations of weakly $(\tau_1, \tau_2)\beta$-open functions

In this section, we introduce the concept of weakly $(\tau_1, \tau_2)\beta$-open functions. Moreover, some characterizations of weakly $(\tau_1, \tau_2)\beta$-open functions are discussed.

Definition 1. *A functions $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be weakly $(\tau_1, \tau_2)\beta$-open if $f(U) \subseteq (\sigma_1, \sigma_2)\beta$-Int($f(\tau_1\tau_2$-Cl($U$))) for every $\tau_1\tau_2$-open set U of X.*

Theorem 1. *For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1\sigma_2)$, the following properties are equivalent:*

(1) f is weakly $(\tau_1, \tau_2)\beta$-open;

(2) $f((\tau_1, \tau_2)\theta$-Int(A)) $\subseteq (\sigma_1, \sigma_2)\beta$-Int($f(A)$) for every subset A of X;

(3) $(\tau_1, \tau_2)\theta$*-Int*$(f^{-1}(B)) \subseteq f^{-1}((\sigma_1, \sigma_2)\beta$*-Int*$(B))$ *for every subset* B *of* Y*;*

(4) $f^{-1}((\sigma_1, \sigma_2)\beta$*-Cl*$(B)) \subseteq (\tau_1, \tau_2)\theta$*-Cl*$(f^{-1}(B))$ *for every subset* B *of* Y*;*

(5) for each $x \in X$ *and each* $\tau_1\tau_2$*-open set* U *of* X *containing* x*, there exists a* $(\sigma_1, \sigma_2)\beta$*-open set* V *of* Y *containing* $f(x)$ *such that* $V \subseteq f(\tau_1\tau_2$*-Cl*$(U))$*;*

(6) $f(\tau_1\tau_2$*-Int*$(K)) \subseteq (\sigma_1, \sigma_2)\beta$*-Int*$(f(K))$ *for every* $\tau_1\tau_2$*-closed set* K *of* X*;*

(7) $f(\tau_1\tau_2$*-Int*$(\tau_1\tau_2$*-Cl*$(U))) \subseteq (\sigma_1, \sigma_2)\beta$*-Int*$(f(\tau_1\tau_2$*-Cl*$(U)))$ *for every* $\tau_1\tau_2$*-open set* U *of* X*;*

(8) $f(U) \subseteq (\sigma_1, \sigma_2)\beta$*-Int*$(f(\tau_1\tau_2$*-Cl*$(U)))$ *for every* $(\tau_1, \tau_2)p$*-open set* U *of* X*;*

(9) $f(U) \subseteq (\sigma_1, \sigma_2)\beta$*-Int*$(f(\tau_1\tau_2$*-Cl*$(U)))$ *for every* $\alpha(\tau_1, \tau_2)$*-open set* U *of* X*.*

Proof. (1) $\Rightarrow$ (2): Let A be any subset of X and $x \in (\tau_1, \tau_2)\theta\text{-Int}(A)$. Then, there exists a $\tau_1\tau_2$-open set U of X such that $x \in U \subseteq \tau_1\tau_2\text{-Cl}(U) \subseteq A$. Therefore,

$$f(x) \in f(U) \subseteq f(\tau_1\tau_2\text{-Cl}(U)) \subseteq f(A).$$

Since f is weakly $(\tau_1, \tau_2)\beta$-open, $f(U) \subseteq (\sigma_1, \sigma_2)\beta\text{-Int}(f(\tau_1\tau_2\text{-Cl}(U))) \subseteq (\sigma_1, \sigma_2)\beta\text{-Int}(f(A))$. It implies that $f(x) \in (\sigma_1, \sigma_2)\beta\text{-Int}(f(A))$. Thus, $x \in f^{-1}((\sigma_1, \sigma_2)\beta\text{-Int}(f(A)))$ and hence $(\tau_1, \tau_2)\theta\text{-Int}(A) \subseteq f^{-1}((\sigma_1, \sigma_2)\beta\text{-Int}(f(A)))$. This shows that

$$f((\tau_1, \tau_2)\theta\text{-Int}(A)) \subseteq (\sigma_1, \sigma_2)\beta\text{-Int}(f(A)).$$

(2) $\Rightarrow$ (1): Let U be any $\tau_1\tau_2$-open set of X. As $U \subseteq (\tau_1, \tau_2)\theta\text{-Int}(\tau_1\tau_2\text{-Cl}(U))$ implies $f(U) \subseteq f((\tau_1, \tau_2)\theta\text{-Int}(\tau_1\tau_2\text{-Cl}(U))) \subseteq (\sigma_1, \sigma_2)\beta\text{-Int}(f(\tau_1\tau_2\text{-Cl}(U)))$. Thus, f is weakly $(\tau_1, \tau_2)\beta$-open.

(2) $\Rightarrow$ (3): Let B be any subset of Y. Thus by (2), we have

$$f((\tau_1, \tau_2)\theta\text{-Int}(f^{-1}(B)) \subseteq (\sigma_1, \sigma_2)\beta\text{-Int}(B)$$

and hence $(\tau_1, \tau_2)\theta\text{-Int}(f^{-1}(B)) \subseteq f^{-1}((\sigma_1, \sigma_2)\beta\text{-Int}(B))$.

(3) $\Rightarrow$ (2): The proof is obvious.

(3) $\Rightarrow$ (4): Let B be any subset of Y. Using (3), we have

$$\begin{aligned} X - (\tau_1, \tau_2)\theta\text{-Cl}(f^{-1}(B)) &= (\tau_1, \tau_2)\theta\text{-Int}(X - f^{-1}(B)) \\ &= (\tau_1, \tau_2)\theta\text{-Int}(f^{-1}(Y - B)) \\ &\subseteq f^{-1}((\sigma_1, \sigma_2)\beta\text{-Int}(Y - B)) \\ &= f^{-1}(Y - (\sigma_1, \sigma_2)\beta\text{-Cl}(B)) \\ &= X - f^{-1}((\sigma_1, \sigma_2)\beta\text{-Cl}(B)) \end{aligned}$$

and so $f^{-1}((\sigma_1, \sigma_2)\beta\text{-Cl}(B)) \subseteq (\tau_1, \tau_2)\theta\text{-Cl}(f^{-1}(B))$.

(4) ⇒ (3): Let B be any subset of Y. Then by (4),

$$X - f^{-1}((\sigma_1, \sigma_2)\beta\text{-Int}(B)) \subseteq X - (\tau_1, \tau_2)\theta\text{-Int}(f^{-1}(B)).$$

Thus, $(\tau_1, \tau_2)\theta\text{-Int}(f^{-1}(B)) \subseteq f^{-1}((\sigma_1, \sigma_2)\beta\text{-Int}(B))$.

(1) ⇒ (5): Let $x \in X$ and U be any $\tau_1\tau_2$-open set of X containing x. Since f is weakly $(\tau_1, \tau_2)\beta$-open, $f(x) \in f(U) \subseteq (\sigma_1, \sigma_2)\beta\text{-Int}(f(\tau_1\tau_2\text{-Cl}(U)))$. Put

$$V = (\sigma_1, \sigma_2)\beta\text{-Int}(f(\tau_1\tau_2\text{-Cl}(U))).$$

Then, V is a $(\sigma_1, \sigma_2)\beta$-open set of Y containing $f(x)$ such that $V \subseteq f(\tau_1\tau_2\text{-Cl}(U))$.

(5) ⇒ (1): Let U be any $\tau_1\tau_2$-open set of X and $y \in f(U)$. It follows from (5) that $V \subseteq f(\tau_1\tau_2\text{-Cl}(U))$ for some $(\sigma_1, \sigma_2)\beta$-open set V of Y containing y. Thus,

$$y \in V \subseteq (\sigma_1, \sigma_2)\beta\text{-Int}(f(\tau_1\tau_2\text{-Cl}(U)))$$

and hence $f(U) \subseteq (\sigma_1, \sigma_2)\beta\text{-Int}(f(\tau_1\tau_2\text{-Cl}(U)))$. This shows that f is weakly $(\tau_1, \tau_2)\beta$-open.

(1) ⇒ (6) ⇒ (7) ⇒ (8) ⇒ (9) ⇒ (1): This is obvious.

Theorem 2. *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a bijective function. Then, the following properties are equivalent:*

(1) f is weakly $(\tau_1, \tau_2)\beta$-open;

(2) $(\sigma_1, \sigma_2)\beta\text{-}Cl(f(U)) \subseteq f(\tau_1\tau_2\text{-}Cl(U))$ for every $\tau_1\tau_2$-open set U of X;

(3) $(\sigma_1, \sigma_2)\beta\text{-}Cl(f(\tau_1\tau_2\text{-}Int(K))) \subseteq f(K)$ for every $\tau_1\tau_2$-closed set K of X.

Proof. (1) ⇒ (3): Let K be any $\tau_1\tau_2$-closed set of X. Then, we have

$$f(X - K) = Y - f(K) \subseteq (\sigma_1, \sigma_2)\beta\text{-Int}(f(\tau_1\tau_2\text{-Cl}(X - K)))$$

and hence $Y - f(K) \subseteq Y - (\sigma_1, \sigma_2)\beta\text{-Cl}(f(\tau_1\tau_2\text{-Int}(K)))$. Thus,

$$(\sigma_1, \sigma_2)\beta\text{-Cl}(f(\tau_1\tau_2\text{-Int}(K))) \subseteq f(K).$$

(3) ⇒ (2): Let U be any $\tau_1\tau_2$-open set of X. Since $\tau_1\tau_2\text{-Cl}(U)$ is $\tau_1\tau_2$-closed and $U \subseteq \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(U))$, by (3) we have

$$(\sigma_1, \sigma_2)\beta\text{-Cl}(f(U)) \subseteq (\sigma_1, \sigma_2)\beta\text{-Cl}(f(\tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(U)))) \subseteq f(\tau_1\tau_2\text{-Cl}(U)).$$

(2) ⇒ (1): Let U be any $\tau_1\tau_2$-open set of X. By (2), we have

$$(\sigma_1, \sigma_2)\beta\text{-Cl}(f(X - \tau_1\tau_2\text{-Cl}(U))) \subseteq f(\tau_1\tau_2\text{-Cl}(X - \tau_1\tau_2\text{-Cl}(U))).$$

Since f is bijective, $(\sigma_1, \sigma_2)\beta\text{-Cl}(f(X - \tau_1\tau_2\text{-Cl}(U))) = Y - (\sigma_1, \sigma_2)\beta\text{-Int}(f(\tau_1\tau_2\text{-Cl}(U)))$ and

$$\begin{aligned} f(\tau_1\tau_2\text{-Cl}(X - \tau_1\tau_2\text{-Cl}(U))) &= f(X - \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(U))) \\ &\subseteq f(X - U) = Y - f(U). \end{aligned}$$

Thus, $f(U) \subseteq (\sigma_1, \sigma_2)\beta\text{-Int}(f(\tau_1\tau_2\text{-Cl}(U)))$ and hence f is weakly $(\tau_1, \tau_2)\beta$-open.

4. Characterizations of weakly $(\tau_1, \tau_2)\beta$-closed functions

In this section, we introduce the concept of weakly $(\tau_1, \tau_2)\beta$-closed functions. Furthermore, some characterizations of weakly $(\tau_1, \tau_2)\beta$-closed functions are discussed.

Definition 2. *A functions $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be weakly $(\tau_1, \tau_2)\beta$-closed if $(\sigma_1, \sigma_2)\beta\text{-}Cl(f(\tau_1\tau_2\text{-}Int(K))) \subseteq f(K)$ for every $\tau_1\tau_2$-closed set K of X.*

Theorem 3. *For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) f is weakly $(\tau_1, \tau_2)\beta$-closed;

(2) $(\sigma_1, \sigma_2)\beta\text{-}Cl(f(U)) \subseteq f(\tau_1\tau_2\text{-}Cl(U))$ for every $\tau_1\tau_2$-open set U of X;

(3) $(\sigma_1, \sigma_2)\beta\text{-}Cl(f(U)) \subseteq f(\tau_1\tau_2\text{-}Cl(U))$ for every $(\tau_1, \tau_2)r$-open set U of X;

(4) for each subset B of Y and each $\tau_1\tau_2$-open set U of X with $f^{-1}(B) \subseteq U$, there exists a $(\sigma_1, \sigma_2)\beta$-open set V of Y such that $B \subseteq V$ and $f^{-1}(V) \subseteq \tau_1\tau_2\text{-}Cl(U)$;

(5) for each point $y \in Y$ and each $\tau_1\tau_2$-open set U of X with $f^{-1}(y) \subseteq U$, there exists a $(\sigma_1, \sigma_2)\beta$-open set V of Y containing y such that $f^{-1}(V) \subseteq \tau_1\tau_2\text{-}Cl(U)$;

(6) $(\sigma_1, \sigma_2)\beta\text{-}Cl(f(\tau_1\tau_2\text{-}Int(\tau_1\tau_2\text{-}Cl(U)))) \subseteq f(\tau_1\tau_2\text{-}Cl(U))$ for every $\tau_1\tau_2$-open set U of X;

(7) $(\sigma_1, \sigma_2)\beta\text{-}Cl(f(\tau_1\tau_2\text{-}Int((\tau_1, \tau_2)\theta\text{-}Cl(U)))) \subseteq f((\tau_1, \tau_2)\theta\text{-}Cl(U))$ for every $\tau_1\tau_2$-open set U of X;

(8) $(\sigma_1, \sigma_2)\beta\text{-}Cl(f(U)) \subseteq f(\tau_1\tau_2\text{-}Cl(U))$ for every $(\tau_1, \tau_2)\beta$-open set U of X.

Proof. (1) ⇒ (2): Let U be any $\tau_1\tau_2$-open set of X. Then by (1),

$$\begin{aligned}(\sigma_1, \sigma_2)\beta\text{-Cl}(f(U)) &= (\sigma_1, \sigma_2)\beta\text{-Cl}(f(\tau_1\tau_2\text{-Int}(U)))\\ &\subseteq (\sigma_1, \sigma_2)\beta\text{-Cl}(f(\tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(U))))\\ &\subseteq f(\tau_1\tau_2\text{-Cl}(U)).\end{aligned}$$

(2) ⇒ (1): Let K be any $\tau_1\tau_2$-closed set of X. Using (2), we have

$$\begin{aligned}(\sigma_1, \sigma_2)\beta\text{-Cl}(f(\tau_1\tau_2\text{-Int}(K))) &\subseteq f(\tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(K)))\\ &\subseteq f(\tau_1\tau_2\text{-Cl}(K)) = f(K).\end{aligned}$$

This shows that f is weakly $(\tau_1, \tau_2)\beta$-closed.

It is clear that (1) ⇒ (7), (4) ⇒ (5) and (1) ⇒ (6) ⇒ (8) ⇒ (3) ⇒ (1). To show that (3) ⇒ (4): Let B be any subset of Y and U be any $\tau_1\tau_2$-open set of X with $f^{-1}(B) \subseteq U$. Then, $f^{-1}(B) \cap \tau_1\tau_2\text{-Cl}(X - \tau_1\tau_2\text{-Cl}(U)) = \emptyset$ and $B \cap f(\tau_1\tau_2\text{-Cl}(X - \tau_1\tau_2\text{-Cl}(U))) = \emptyset$. Since $X - \tau_1\tau_2\text{-Cl}(U)$ is $(\tau_1, \tau_2)r$-open, $B \cap (\sigma_1, \sigma_2)\beta\text{-Cl}(f(X - \tau_1\tau_2\text{-Cl}(U))) = \emptyset$ by (3).

Put $V = Y - (\sigma_1, \sigma_2)\beta\text{-Cl}(f(X - \tau_1\tau_2\text{-Cl}(U)))$. Then, V is a $(\sigma_1, \sigma_2)\beta$-open set of Y such that $B \subseteq V$ and

$$\begin{aligned} f^{-1}(V) &\subseteq X - f^{-1}((\sigma_1, \sigma_2)\beta\text{-Cl}(f(X - \tau_1\tau_2\text{-Cl}(U)))) \\ &\subseteq X - f^{-1}(f(X - \tau_1\tau_2\text{-Cl}(U))) \\ &\subseteq \tau_1\tau_2\text{-Cl}(U). \end{aligned}$$

(7) $\Rightarrow$ (1): It is suffices see that $(\tau_1, \tau_2)\theta\text{-Cl}(U) = \tau_1\tau_2\text{-Cl}(U)$ for every $\tau_1\tau_2$-open set U of X.

(5) $\Rightarrow$ (1): Let K be any $\tau_1\tau_2$-closed set U of X and $y \in Y - f(K)$. Since $f^{-1}(y) \subseteq X - K$, there exists a $(\sigma_1, \sigma_2)\beta$-open set V of Y such that $y \in V$ and

$$f^{-1}(V) \subseteq \tau_1\tau_2\text{-Cl}(X - K) = X - \tau_1\tau_2\text{-Int}(K)$$

by (5). Thus, $V \cap f(\tau_1\tau_2\text{-Int}(K)) = \emptyset$ and so $y \in Y - (\sigma_1, \sigma_2)\beta\text{-Cl}(f(\tau_1\tau_2\text{-Int}(K)))$. Therefore, $(\sigma_1, \sigma_2)\beta\text{-Cl}(f(\tau_1\tau_2\text{-Int}(K))) \subseteq f(K)$. This shows that f is weakly $(\tau_1, \tau_2)\beta$-closed.

(7) $\Rightarrow$ (8): This is obvious since $(\tau_1, \tau_2)\theta\text{-Cl}(U) = \tau_1\tau_2\text{-Cl}(U)$ for every $(\tau_1, \tau_2)\beta$-open set U of X.

The following theorem the proof is mostly straightforward and is omitted.

Theorem 4. *For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) f is weakly $(\tau_1, \tau_2)\beta$-closed;

(2) $(\sigma_1, \sigma_2)\beta\text{-}Cl(f(\tau_1\tau_2\text{-}Int(K))) \subseteq f(K)$ for every $(\tau_1, \tau_2)\beta$-closed set K of X;

(3) $(\sigma_1, \sigma_2)\beta\text{-}Cl(f(\tau_1\tau_2\text{-}Int(K))) \subseteq f(K)$ for every $\alpha(\tau_1, \tau_2)$-closed set K of X.

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