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# On Weak Forms of Open and Closed Functions between Bitopological Spaces

Nipaporn Chutiman[1], Areeyuth Sama-Ae[2], Chawalit Boonpok[1,*]

[1] *Mathematics and Applied Mathematics Research Unit, Department of Mathematics, Faculty of Science, Mahasarakham University, Maha Sarakham, 44150, Thailand*
[2] *Department of Mathematics and Computer Science, Faculty of Science and Technology, Prince of Songkla University, Pattani Campus, Pattani, 94000, Thailand*

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**Abstract.** This paper presents new classes of open and closed functions defined between bitopological spaces, called weakly $\theta(\tau_1, \tau_2)b$-open functions and weakly $\theta(\tau_1, \tau_2)b$-closed functions. Moreover, several characterizations and some properties concerning weakly $\theta(\tau_1, \tau_2)b$-open functions and weakly $\theta(\tau_1, \tau_2)b$-closed functions are established.

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**Key Words and Phrases**: Weakly $\theta(\tau_1, \tau_2)b$-open function, weakly $\theta(\tau_1, \tau_2)b$-closed function

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## 1. Introduction

Topology is concerned with all questions directly or indirectly related to openness and closedness. Semi-open sets, preopen sets, $\alpha$-open sets, $\beta$-open sets, $b$-open sets, $\delta$-open sets, $\theta$-open sets and $b$-$\theta$-open sets play an important role in the researches of generalizations of open functions and closed functions. By using these sets, many authors introduced and studied various types of open functions and closed functions. In 1983, Rose [1] introduced and studied the notions of weakly open functions and almost open functions. In 1987, Rose and Janković [2] investigated some of the fundamental properties of weakly closed functions. In 2006, Caldas et al. [3] introduced and studied the concepts of $\theta$-preopen functions and $\theta$-preclosed functions by using the notions of pre-$\theta$-interior and pre-$\theta$-closure. Moreover, Caldas et al. [4] introduced and investigated the concepts of weakly semi-$\theta$-open functions and weakly semi-$\theta$-closed functions. In 2009, Noiri et al. [5] introduced and studied two new classes of functions called weakly $b$-$\theta$-open functions and weakly $b$-$\theta$-open functions by utilizing the notions of $b$-$\theta$-open sets and the $b$-$\theta$-closure operator. Weak $b$-$\theta$-openness (resp. $b$-$\theta$-closedness) is a generalization of both $\theta$-preopenness

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*Email addresses:* nipaporn.c@msu.ac.th (N. Chutiman), areeyuth.s@psu.ac.th (A. Sama-Ae), chawalit.b@msu.ac.th (C. Boonpok)

 

and weak semi-$\theta$-openness (resp. $\theta$-preclosedness and weak semi-$\theta$-closedness). Quite recently, Klanarong and Boonpok [6] studied the notions of weakly $s(\Lambda, p)$-open functions and weakly $s(\Lambda, p)$-closed functions by utilizing $s(\Lambda, p)$-open sets and the $s(\Lambda, p)$-closure operator. On the other hand, the present authors introduced and studied the concepts of $\theta p(\Lambda, p)$-open functions [7], $\theta p(\Lambda, p)$-closed functions [7], semi-$(\mathscr{I}, \mathscr{J})$-open functions [8], semi-$(\mathscr{I}, \mathscr{J})$-closed functions [8], weakly $\delta(\Lambda, p)$-open functions [9], weakly $\delta(\Lambda, p)$-closed functions [10], weakly $\theta s(\Lambda, p)$-open functions [11], weakly $\theta s(\Lambda, p)$-closed functions [11] and weakly $b(\Lambda, p)$-open functions [12]. In this paper, we introduce the notions of weakly $\theta(\tau_1, \tau_2)b$-open functions and $\theta(\tau_1, \tau_2)b$-closed functions. Furthermore, several characterizations of weakly $\theta(\tau_1, \tau_2)b$-open functions and $\theta(\tau_1, \tau_2)b$-closed functions are investigated.

## 2. Preliminaries

Throughout the present paper, spaces $(X, \tau_1, \tau_2)$ and $(Y, \sigma_1, \sigma_2)$ (or simply $X$ and $Y$) always mean bitopological spaces on which no separation axioms are assumed unless explicitly stated. Let $A$ be a subset of a bitopological space $(X, \tau_1, \tau_2)$. The closure of $A$ and the interior of $A$ with respect to $\tau_i$ are denoted by $\tau_i$-Cl$(A)$ and $\tau_i$-Int$(A)$, respectively, for $i = 1, 2$. A subset $A$ of a bitopological space $(X, \tau_1, \tau_2)$ is called $\tau_1\tau_2$-*closed* [13] if $A = \tau_1$-Cl$(\tau_2$-Cl$(A))$. The complement of a $\tau_1\tau_2$-closed set is called $\tau_1\tau_2$-*open*. Let $A$ be a subset of a bitopological space $(X, \tau_1, \tau_2)$. The intersection of all $\tau_1\tau_2$-closed sets of $X$ containing $A$ is called the $\tau_1\tau_2$-*closure* [13] of $A$ and is denoted by $\tau_1\tau_2$-Cl$(A)$. The union of all $\tau_1\tau_2$-open sets of $X$ contained in $A$ is called the $\tau_1\tau_2$-*interior* [13] of $A$ and is denoted by $\tau_1\tau_2$-Int$(A)$.

**Lemma 1.** [13] *Let $A$ and $B$ be subsets of a bitopological space $(X, \tau_1, \tau_2)$. For the $\tau_1\tau_2$-closure, the following properties hold:*

*(1) $A \subseteq \tau_1\tau_2$-Cl$(A)$ and $\tau_1\tau_2$-Cl$(\tau_1\tau_2$-Cl$(A)) = \tau_1\tau_2$-Cl$(A)$.*

*(2) If $A \subseteq B$, then $\tau_1\tau_2$-Cl$(A) \subseteq \tau_1\tau_2$-Cl$(B)$.*

*(3) $\tau_1\tau_2$-Cl$(A)$ is $\tau_1\tau_2$-closed.*

*(4) $A$ is $\tau_1\tau_2$-closed if and only if $A = \tau_1\tau_2$-Cl$(A)$.*

*(5) $\tau_1\tau_2$-Cl$(X - A) = X - \tau_1\tau_2$-Int$(A)$.*

A subset $A$ of a bitopological space $(X, \tau_1, \tau_2)$ is said to be $(\tau_1, \tau_2)r$-*open* [14] (resp. $(\tau_1, \tau_2)s$-*open* [15], $(\tau_1, \tau_2)p$-*open* [15], $(\tau_1, \tau_2)\beta$-*open* [15]) if $A = \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))$ (resp. $A \subseteq \tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(A))$, $A \subseteq \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))$, $A \subseteq \tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))))$. The complement of a $(\tau_1, \tau_2)r$-open (resp. $(\tau_1, \tau_2)s$-open, $(\tau_1, \tau_2)p$-open, $(\tau_1, \tau_2)\beta$-open) set is called $(\tau_1, \tau_2)r$-*closed* (resp. $(\tau_1, \tau_2)s$-*closed*, $(\tau_1, \tau_2)p$-*closed*, $(\tau_1, \tau_2)\beta$-*closed*). A subset $A$ of a bitopological space $(X, \tau_1, \tau_2)$ is said to be $\alpha(\tau_1, \tau_2)$-*open* [16] if $A \subseteq \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(A)))$. The complement of an $\alpha(\tau_1, \tau_2)$-open set is said to be $\alpha(\tau_1, \tau_2)$-*closed*. Let $A$ be a subset of a bitopological space $(X, \tau_1, \tau_2)$. The intersection of all $(\tau_1, \tau_2)p$-closed (resp. $(\tau_1, \tau_2)s$-closed, $\alpha(\tau_1, \tau_2)$-closed) sets of $X$ containing $A$ is called the $(\tau_1, \tau_2)$*p-closure* [17] (resp. $(\tau_1, \tau_2)$*s-closure* [15], $\alpha(\tau_1, \tau_2)$*-closure* [18]) of $A$ and is denoted by $(\tau_1, \tau_2)$-pCl$(A)$ (resp. $(\tau_1, \tau_2)$-sCl$(A)$, $\alpha(\tau_1, \tau_2)$-Cl$(A)$). The union of all $(\tau_1, \tau_2)p$-open (resp. $(\tau_1, \tau_2)s$-open, $\alpha(\tau_1, \tau_2)$-open) sets of $X$ contained in $A$ is called the $(\tau_1, \tau_2)$*p-interior* [17] (resp. $(\tau_1, \tau_2)$*s-interior* [15], $\alpha(\tau_1, \tau_2)$*-interior* [18]) of $A$ and is denoted by $(\tau_1, \tau_2)$-pInt$(A)$ (resp. $(\tau_1, \tau_2)$-sInt$(A)$, $\alpha(\tau_1, \tau_2)$-Int$(A)$).

**Lemma 2.** *For subsets $A$ and $B$ of a bitopological space $(X, \tau_1, \tau_2)$, the following properties hold:*

*(1)* $\alpha(\tau_1, \tau_2)\text{-}Int(A) = \tau_1\tau_2\text{-}Int(\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Int(A))) \cap A$;

*(2)* $(\tau_1, \tau_2)\text{-}sInt(A) = \tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Int(A)) \cap A$ [19];

*(3)* $(\tau_1, \tau_2)\text{-}pInt(A) = \tau_1\tau_2\text{-}Int(\tau_1\tau_2\text{-}Cl(A)) \cap A$ [20].

For a subset $A$ of a bitopological space $(X, \tau_1, \tau_2)$, a point $x \in X$ is called $(\tau_1, \tau_2)\theta$-*cluster point* [14] of $A$ if $\tau_1\tau_2$-Cl$(U) \cap A \neq \emptyset$ for every $\tau_1\tau_2$-open set $U$ containing $x$. The set of all $(\tau_1, \tau_2)\theta$-cluster points of $A$ is called the $(\tau_1, \tau_2)\theta$*-closure* [14] of $A$ and is denoted by $(\tau_1, \tau_2)\theta$-Cl$(A)$. A subset $A$ of a bitopological space $(X, \tau_1, \tau_2)$ is said to be $(\tau_1, \tau_2)\theta$*-closed* [14] if $A = (\tau_1, \tau_2)\theta$-Cl$(A)$. The complement of a $(\tau_1, \tau_2)\theta$-closed set is said to be $(\tau_1, \tau_2)\theta$-*open*. The union of all $(\tau_1, \tau_2)\theta$-open sets contained in $A$ is called the $(\tau_1, \tau_2)\theta$*-interior* [14] of $A$ and is denoted by $(\tau_1, \tau_2)\theta$-Int$(A)$.

**Lemma 3.** [14] *For a subset $A$ of a bitopological space $(X, \tau_1, \tau_2)$, the following properties hold:*

*(1) If $A$ is $\tau_2\tau_2$-open in $X$, then $\tau_1\tau_2$-Cl$(A) = (\tau_1, \tau_2)\theta$-Cl$(A)$.*

*(2) $(\tau_1, \tau_2)\theta$-Cl$(A)$ is $\tau_1\tau_2$-closed in $X$.*

A subset $A$ of a bitopological space $(X, \tau_1, \tau_2)$ is said to be $(\tau_1, \tau_2)$*b-open* if $A \subseteq \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A)) \cup \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(A))$. The complement of a $(\tau_1, \tau_2)b$-open set is called $(\tau_1, \tau_2)$*b-closed*. Let $A$ be a subset of a bitopological space $(X, \tau_1, \tau_2)$. The union of all $(\tau_1, \tau_2)b$-open sets of $X$ contained in $A$ is called the $(\tau_1, \tau_2)$*b-interior* of $A$ and is denoted by $(\tau_1, \tau_2)$-bInt$(A)$. The intersection of all $(\tau_1, \tau_2)b$-closed sets of $X$ containing $A$ is called the $(\tau_1, \tau_2)$*b-closure* of $A$ and is denoted by $(\tau_1, \tau_2)$-bCl$(A)$.

**Lemma 4.** *For subsets $A$ and $B$ of a bitopological space $(X, \tau_1, \tau_2)$, the following properties hold:*

*(1)* $(\tau_1, \tau_2)\text{-}bInt(A) = (\tau_1, \tau_2)\text{-}sInt(A) \cup (\tau_1, \tau_2)\text{-}pInt(A)$;

*(2)* $(\tau_1, \tau_2)\text{-}bCl(A) = (\tau_1, \tau_2)\text{-}sCl(A) \cap (\tau_1, \tau_2)\text{-}pCl(A)$;

*(3)* $(\tau_1, \tau_2)\text{-}bCl(X - A) = X - (\tau_1, \tau_2)\text{-}bInt(A)$;

*(4)* $x \in (\tau_1, \tau_2)\text{-}bInt(A)$ *if and only if* $A \cap U \neq \emptyset$ *for every* $(\tau_1, \tau_2)b$*-open set* $U$ *of* $X$ *containing* $x$;

*(5) $A$ is $(\tau_1, \tau_2)b$-closed if and only if $A = (\tau_1, \tau_2)$-$bCl(A)$;*

*(6) $(\tau_1, \tau_2)$-$pInt((\tau_1, \tau_2)$-$bCl(A)) = (\tau_1, \tau_2)$-$bCl((\tau_1, \tau_2)$-$pInt(A))$.*

For a subset $A$ of a bitopological space $(X, \tau_1, \tau_2)$, a point $x \in X$ is called a $\theta(\tau_1, \tau_2)b$-*cluster point* of $A$ if $(\tau_1, \tau_2)$-bCl$(U) \cap A \neq \emptyset$ for every $(\tau_1, \tau_2)b$-open set $U$ of $X$ containing $x$. The set of all $\theta(\tau_1, \tau_2)b$-cluster points of $A$ is called the $\theta(\tau_1, \tau_2)b$-*closure* of $A$ and is denoted by $\theta(\tau_1, \tau_2)b$-Cl$(A)$. If $A = \theta(\tau_1, \tau_2)b$-Cl$(A)$, then $A$ is called $\theta(\tau_1, \tau_2)b$-*closed*. The complement of a $\theta(\tau_1, \tau_2)b$-closed set is called $\theta(\tau_1, \tau_2)b$-*open*. The $\theta(\tau_1, \tau_2)b$-*interior* of $A$ is defined by the union of all $\theta(\tau_2, \tau_1)b$-open sets of $X$ contained in $A$ and is denoted by $\theta(\tau_1, \tau_2)b$-Int$(A)$.

**Lemma 5.** *For subsets $A$ and $C_\gamma(\gamma \in \nabla)$ of a bitopological space $(X, \tau_1, \tau_2)$, the following properties hold:*

*(1) If $C_\gamma$ is $\theta(\tau_1, \tau_2)b$-open for each $\gamma \in \nabla$, then $\cup_{\gamma \in \nabla} C_\gamma$ is $\theta(\tau_1, \tau_2)b$-open.*

*(2) If $A$ is $(\tau_1, \tau_2)b$-closed, then $(\tau_1, \tau_2)$-$bInt(A) = \theta(\tau_1, \tau_2)b$-$Int(A)$.*

*(3) $\theta(\tau_1, \tau_2)b$-$Cl(A)$ is $\theta(\tau_1, \tau_2)b$-closed.*

## 3. Weakly $\theta(\tau_1, \tau_2)b$-open functions

In this section, we introduce the concept of weakly $\theta(\tau_1, \tau_2)b$-open functions. Moreover, some characterizations of weakly $\theta(\tau_1, \tau_2)b$-open functions are discussed.

**Definition 1.** *A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be weakly $\theta(\tau_1, \tau_2)b$-open if $f(U) \subseteq \theta(\sigma_1, \sigma_2)b$-$Int(f(\tau_1\tau_2$-$Cl(U)))$ for every $\tau_1\tau_2$-open set $U$ of $X$.*

**Theorem 1.** *For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

*(1) $f$ is weakly $\theta(\tau_1, \tau_2)b$-open;*

*(2) $f((\tau_1, \tau_2)\theta$-$Int(A)) \subseteq \theta(\sigma_1, \sigma_2)b$-$Int(f(A))$ for every subset $A$ of $X$;*

*(3) $(\tau_1, \tau_2)\theta$-$Int(f^{-1}(B)) \subseteq f^{-1}(\theta(\sigma_1, \sigma_2)b$-$Int(B))$ for every subset $B$ of $Y$;*

*(4) $f^{-1}(\theta(\sigma_1, \sigma_2)b$-$Cl(B)) \subseteq (\tau_1, \tau_2)\theta$-$Cl(f^{-1}(B))$ for every subset $B$ of $Y$.*

*Proof.* (1) $\Rightarrow$ (2): Let $A$ be any subset of $X$ and $x \in (\tau_1, \tau_2)\theta$-Int$(A)$. Then, there exists a $\tau_1\tau_2$-open set $U$ of $X$ such that $x \in U \subseteq \tau_1\tau_2$-Cl$(U) \subseteq A$. Therefore,

$$f(x) \in f(U) \subseteq f(\tau_1\tau_2\text{-Cl}(U)) \subseteq f(A).$$

Since $f$ is weakly $\theta(\tau_1, \tau_2)b$-open, we have

$$f(U) \subseteq \theta(\sigma_1, \sigma_2)b\text{-Int}(f(\tau_1\tau_2\text{-Cl}(U))) \subseteq \theta(\sigma_1, \sigma_2)b\text{-Int}(f(A)).$$

This implies that $f(x) \in \theta(\sigma_1, \sigma_2)b\text{-Int}(f(A))$. Thus, $x \in f^{-1}(\theta(\sigma_1, \sigma_2)b\text{-Int}(f(A)))$ and hence $(\tau_1, \tau_2)\theta\text{-Int}(A) \subseteq f^{-1}(\theta(\sigma_1, \sigma_2)b\text{-Int}(f(A)))$. This shows that

$$f((\tau_1, \tau_2)\theta\text{-Int}(A)) \subseteq \theta(\sigma_1, \sigma_2)b\text{-Int}(f(A)).$$

(2) ⇒ (3): Let $B$ be any subset of $Y$. Thus by (2),

$$f((\tau_1, \tau_2)\theta\text{-Int}(f^{-1}(B))) \subseteq \theta(\sigma_1, \sigma_2)b\text{-Int}(B)$$

and so $(\tau_1, \tau_2)\theta\text{-Int}(f^{-1}(B)) \subseteq f^{-1}(\theta(\sigma_1, \sigma_2)b\text{-Int}(B))$.

(3) ⇒ (4): Let $B$ be any subset of $Y$. Using (3), we have

$$\begin{aligned} X - (\tau_1, \tau_2)\theta\text{-Cl}(f^{-1}(B)) &= (\tau_1, \tau_2)\theta\text{-Int}(X - f^{-1}(B)) \\ &= (\tau_1, \tau_2)\theta\text{-Int}(f^{-1}(Y - B)) \\ &\subseteq f^{-1}(\theta(\sigma_1, \sigma_2)b\text{-Int}(Y - B)) \\ &= f^{-1}(Y - \theta(\sigma_1, \sigma_2)b\text{-Cl}(B)) \\ &= X - f^{-1}(\theta(\sigma_1, \sigma_2)b\text{-Cl}(B)). \end{aligned}$$

Thus, $f^{-1}(\theta(\sigma_1, \sigma_2)b\text{-Cl}(B)) \subseteq (\tau_1, \tau_2)\theta\text{-Cl}(f^{-1}(B))$.

(4) ⇒ (1): Let $U$ be any $\tau_1\tau_2$-open set of $X$. By (4),

$$f^{-1}(\theta(\sigma_1, \sigma_2)b\text{-Cl}(Y - f(\tau_1\tau_2\text{-Cl}(U)))) \subseteq (\tau_1, \tau_2)\theta\text{-Cl}(f^{-1}(Y - f(\tau_1\tau_2\text{-Cl}(U)))).$$

Thus,

$$\begin{aligned} f^{-1}(Y - \theta(\sigma_1, \sigma_2)b\text{-Int}(f(\tau_1\tau_2\text{-Cl}(U)))) &\subseteq (\tau_1, \tau_2)\theta\text{-Cl}(X - f^{-1}(f(\tau_1\tau_2\text{-Cl}(U)))) \\ &\subseteq (\tau_1, \tau_2)\theta\text{-Cl}(X - \tau_1\tau_2\text{-Cl}(U)) \end{aligned}$$

and hence $U \subseteq (\tau_1, \tau_2)\theta\text{-Int}(\tau_1\tau_2\text{-Cl}(U)) \subseteq f^{-1}(\theta(\sigma_1, \sigma_2)b\text{-Int}(f(\tau_1\tau_2\text{-Cl}(U))))$. Therefore, $f(U) \subseteq \theta(\sigma_1, \sigma_2)b\text{-Int}(f(\tau_1\tau_2\text{-Cl}(U)))$. This shows that $f$ is weakly $\theta(\tau_1, \tau_2)b$-open.

**Theorem 2.** *For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

*(1) $f$ is weakly $\theta(\tau_1, \tau_2)b$-open;*

*(2) for each $x \in X$ and each $\tau_1\tau_2$-open set $U$ of $X$ containing $x$, there exists a $\theta(\sigma_1, \sigma_2)b$-open set $V$ of $Y$ containing $f(x)$ such that $V \subseteq f(\tau_1\tau_2\text{-}Cl(U))$.*

*Proof.* (1) ⇒ (2): Let $x \in X$ and $U$ be any $\tau_1\tau_2$-open set of $X$ containing $x$. Since $f$ is weakly $\theta(\tau_1, \tau_2)b$-open, $f(x) \in f(U) \subseteq \theta(\sigma_1, \sigma_2)b\text{-Int}(f(\tau_1\tau_2\text{-Cl}(U)))$. Let $V = \theta(\sigma_1, \sigma_2)b\text{-Int}(f(\tau_1\tau_2\text{-Cl}(U)))$. Then, $V$ is $\theta(\sigma_1, \sigma_2)b$-open in $Y$ and

$$f(x) \in V \subseteq f(\tau_1\tau_2\text{-Cl}(U)).$$

(2) $\Rightarrow$ (1): Let $U$ be any $\tau_1\tau_2$-open set of $X$ and $y \in f(U)$. It follows from (2) that $V \subseteq f(\tau_1\tau_2\text{-Cl}(U))$ for some $\theta(\sigma_1, \sigma_2)b$-open set $V$ of $Y$ containing $y$. Thus,

$$y \in V \subseteq \theta(\sigma_1, \sigma_2)b\text{-Int}(f(\tau_1\tau_2\text{-Cl}(U)))$$

and hence $f(U) \subseteq \theta(\sigma_1, \sigma_2)b\text{-Int}(f(\tau_1\tau_2\text{-Cl}(U)))$. This shows that $f$ is weakly $\theta(\tau_1, \tau_2)b$-open.

**Theorem 3.** *For a bijective function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

*(1) $f$ is weakly $\theta(\tau_1, \tau_2)b$-open;*

*(2) $\theta(\sigma_1, \sigma_2)b$-Cl$(f(\tau_1\tau_2\text{-Int}(K))) \subseteq f(K)$ for each $\tau_1\tau_2$-closed set $K$ of $X$;*

*(3) $\theta(\sigma_1, \sigma_2)b$-Cl$(f(U)) \subseteq f(\tau_1\tau_2\text{-Cl}(U))$ for each $\tau_1\tau_2$-open set $U$ of $X$.*

*Proof.* (1) $\Rightarrow$ (2): Let $K$ be any $\tau_1\tau_2$-closed set of $X$. Then, we have

$$f(X - K) = Y - f(K) \subseteq \theta(\sigma_1, \sigma_2)b\text{-Int}(f(\tau_1\tau_2\text{-Cl}(X - K)))$$

and hence $Y - f(K) \subseteq Y - \theta(\sigma_1, \sigma_2)b\text{-Cl}(f(\tau_1\tau_2\text{-Int}(K)))$. Thus,

$$\theta(\sigma_1, \sigma_2)b\text{-Cl}(f(\tau_1\tau_2\text{-Int}(K))) \subseteq f(K).$$

(2) $\Rightarrow$ (3): Let $U$ be any $\tau_1\tau_2$-open set of $X$. Since $\tau_1\tau_2$-Cl$(U)$ is a $\tau_1\tau_2$-closed set and $U \subseteq \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(U))$, by (2) we have

$$\begin{aligned}\theta(\sigma_1, \sigma_2)b\text{-Cl}(f(U)) &\subseteq \theta(\sigma_1, \sigma_2)b\text{-Cl}(f(\tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(U)))) \\ &\subseteq f(\tau_1\tau_2\text{-Cl}(U)).\end{aligned}$$

(3) $\Rightarrow$ (1): Let $U$ be any $\tau_1\tau_2$-open set of $X$. Using (3), we have

$$\begin{aligned}Y - \theta(\sigma_1, \sigma_2)b\text{-Int}(f(\tau_1\tau_2\text{-Cl}(U))) &= \theta(\sigma_1, \sigma_2)b\text{-Cl}(Y - f(\tau_1\tau_2\text{-Cl}(U))) \\ &= \theta(\sigma_1, \sigma_2)b\text{-Cl}(f(X - \tau_1\tau_2\text{-Cl}(U))) \\ &\subseteq f(\tau_1\tau_2\text{-Cl}(X - \tau_1\tau_2\text{-Cl}(U))) \\ &= f(X - \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(U))) \\ &\subseteq f(X - U) = Y - f(U)\end{aligned}$$

and so $f(U) \subseteq \theta(\sigma_1, \sigma_2)b\text{-Int}(f(\tau_1\tau_2\text{-Cl}(U)))$. This shows that $f$ is weakly $\theta(\tau_1, \tau_2)b$-open.

The proof of the following theorem is straightforward and thus is omitted.

**Theorem 4.** *For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

*(1) $f$ is weakly $\theta(\tau_1, \tau_2)b$-open;*

*(2)* $f(U) \subseteq \theta(\sigma_1, \sigma_2)b\text{-}Int(f(\tau_1\tau_2\text{-}Cl(U)))$ *for each* $(\tau_1, \tau_2)p$*-open set* $U$ *of* $X$*;*

*(3)* $f(U) \subseteq \theta(\sigma_1, \sigma_2)b\text{-}Int(f(\tau_1\tau_2\text{-}Cl(U)))$ *for each* $\alpha(\tau_1, \tau_2)$*-open set* $U$ *of* $X$*;*

*(4)* $f(\tau_1\tau_2\text{-}Int(\tau_1\tau_2\text{-}Cl(U))) \subseteq \theta(\sigma_1, \sigma_2)b\text{-}Int(f(\tau_1\tau_2\text{-}Cl(U)))$ *for each* $\tau_1\tau_2$*-open set* $U$ *of* $X$*;*

*(5)* $f(\tau_1\tau_2\text{-}Int(K)) \subseteq \theta(\sigma_1, \sigma_2)b\text{-}Int(f(K))$ *for each* $\tau_1\tau_2$*-closed set* $K$ *of* $X$*.*

## 4. Weakly $\theta(\tau_1, \tau_2)b$-closed functions

In this section, we introduce the concept of weakly $\theta(\tau_1, \tau_2)b$-closed functions. Furthermore, some characterizations of weakly $\theta(\tau_1, \tau_2)b$-closed functions are investigated.

**Definition 2.** *A function* $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *is said to be weakly* $\theta(\tau_1, \tau_2)b$*-closed if* $\theta(\sigma_1, \sigma_2)b\text{-}Cl(f(\tau_1\tau_2\text{-}Int(K))) \subseteq f(K)$ *for every* $\tau_1\tau_2$*-closed set* $K$ *of* $X$*.*

**Theorem 5.** *For a function* $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$*, the following properties are equivalent:*

*(1)* $f$ *is weakly* $\theta(\tau_1, \tau_2)b$*-closed;*

*(2)* $\theta(\sigma_1, \sigma_2)b\text{-}Cl(f(U)) \subseteq f(\tau_1\tau_2\text{-}Cl(U))$ *for every* $\tau_1\tau_2$*-open set* $U$ *of* $X$*.*

*Proof.* (1) $\Rightarrow$ (2): Let $U$ be any $\tau_1\tau_2$-open set of $X$. Since $\tau_1\tau_2$-Cl$(U)$ is a $\tau_1\tau_2$-closed set and $U \subseteq \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(U))$, we have

$$\begin{aligned} \theta(\sigma_1, \sigma_2)b\text{-Cl}(f(U)) &\subseteq \theta(\sigma_1, \sigma_2)b\text{-Cl}(f(\tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(U)))) \\ &\subseteq f(\tau_1\tau_2\text{-Cl}(U)). \end{aligned}$$

(2) $\Rightarrow$ (1): Let $K$ be any $\tau_1\tau_2$-closed set of $X$. Then, we have

$$\begin{aligned} \theta(\sigma_1, \sigma_2)b\text{-Cl}(f(\tau_1\tau_2\text{-Int}(K))) &\subseteq f(\tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(K))) \\ &\subseteq f(\tau_1\tau_2\text{-Cl}(K)) \\ &= f(K) \end{aligned}$$

and hence $f$ is weakly $\theta(\tau_1, \tau_2)b$-closed.

**Corollary 1.** *A bijective function* $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *is weakly* $\theta(\tau_1, \tau_2)b$*-open if and only if* $f$ *is weakly* $\theta(\tau_1, \tau_2)b$*-closed.*

*Proof.* This is an immediate consequence of Theorem 3 and 5.

The proof of the following theorem is straightforward and thus is omitted.

**Theorem 6.** *For a function* $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$*, the following properties are equivalent:*

*(1)* $f$ *is weakly* $\theta(\tau_1, \tau_2)b$*-closed;*

*(2)* $\theta(\sigma_1, \sigma_2)b$*-Cl*$(f(\tau_1\tau_2$*-Int*$(K))) \subseteq f(K)$ *for every* $(\tau_1, \tau_2)p$*-closed set* $K$ *of* $X$*;*

*(3)* $\theta(\sigma_1, \sigma_2)b$*-Cl*$(f(\tau_1\tau_2$*-Int*$(K))) \subseteq f(K)$ *for every* $\alpha(\tau_1, \tau_2)$*-closed set* $K$ *of* $X$*;*

*(4)* $\theta(\sigma_1, \sigma_2)b$*-Cl*$(f(\tau_1\tau_2$*-Int*$(\tau_1\tau_2$*-Cl*$(A)))) \subseteq f(\tau_1\tau_2$*-Cl*$(A))$ *for every subset* $A$ *of* $X$*;*

*(5)* $\theta(\sigma_1, \sigma_2)b$*-Cl*$(f(U)) \subseteq f(\tau_1\tau_2$*-Cl*$(U))$ *for every* $(\tau_1, \tau_2)p$*-open set* $U$ *of* $X$*.*

**Theorem 7.** *For a function* $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$*, the following properties are equivalent:*

*(1)* $f$ *is weakly* $\theta(\tau_1, \tau_2)b$*-closed;*

*(2)* $\theta(\sigma_1, \sigma_2)b$*-Cl*$(f(U)) \subseteq f(\tau_1\tau_2$*-Cl*$(U))$ *for every* $(\tau_1, \tau_2)r$*-open set* $U$ *of* $X$*;*

*(3) for each subset* $B$ *of* $Y$ *and each* $\tau_1\tau_2$*-open set* $U$ *of* $X$ *with* $f^{-1}(B) \subseteq U$*, there exists a* $\theta(\sigma_1, \sigma_2)b$*-open set* $V$ *of* $Y$ *such that* $B \subseteq V$ *and* $f^{-1}(V) \subseteq \tau_1\tau_2$*-Cl*$(U)$*;*

*(4) for each point* $y \in Y$ *and each* $\tau_1\tau_2$*-open set* $U$ *of* $X$ *with* $f^{-1}(y) \subseteq U$*, there exists a* $\theta(\sigma_1, \sigma_2)b$*-open set* $V$ *of* $Y$ *containing* $y$ *and* $f^{-1}(V) \subseteq \tau_1\tau_2$*-Cl*$(U)$*.*

*Proof.* (1) ⇒ (2): It follows from Theorem 5.

(2) ⇒ (3): Let $B$ be any subset of $Y$ and $U$ be any $\tau_1\tau_2$-open set of $X$ with $f^{-1}(B) \subseteq U$. Then, we have $f^{-1}(B) \cap \tau_1\tau_2\text{-Cl}(X - \tau_1\tau_2\text{-Cl}(U)) = \emptyset$ and hence

$$B \cap f(\tau_1\tau_2\text{-Cl}(X - \tau_1\tau_2\text{-Cl}(U))) = \emptyset.$$

Since $X - \tau_1\tau_2\text{-Cl}(U)$ is $(\tau_1, \tau_2)r$-open, $B \cap \theta(\sigma_1, \sigma_2)b\text{-Cl}(f(X - \tau_1\tau_2\text{-Cl}(U))) = \emptyset$. Let $V = Y - \theta(\sigma_1, \sigma_2)b\text{-Cl}(f(X - \tau_1\tau_2\text{-Cl}(U)))$. Then, $V$ is a $\theta(\sigma_1, \sigma_2)b$-open set with $B \subseteq V$ and

$$\begin{aligned} f^{-1}(V) &\subseteq X - f^{-1}(\theta(\sigma_1, \sigma_2)b\text{-Cl}(f(X - \tau_1\tau_2\text{-Cl}(U)))) \\ &\subseteq X - f^{-1}(f(X - \tau_1\tau_2\text{-Cl}(U))) \\ &\subseteq \tau_1\tau_2\text{-Cl}(U). \end{aligned}$$

(3) ⇒ (4): The proof is obvious.

(4) ⇒ (1): Let $K$ be any $\tau_1\tau_2$-closed set of $Y$ and $y \in Y - f(K)$. Since $f^{-1}(y) \subseteq X - K$, by (4) there exists a $\theta(\sigma_1, \sigma_2)b$-open set $V$ of $Y$ such that $y \in V$ and

$$f^{-1}(V) \subseteq \tau_1\tau_2\text{-Cl}(X - K) = X - \tau_1\tau_2\text{-Int}(K).$$

Thus, $V \cap f(\tau_1\tau_2\text{-Int}(K)) = \emptyset$ and hence $y \notin \theta(\sigma_1, \sigma_2)b\text{-Cl}(f(\tau_1\tau_2\text{-Int}(K)))$. Therefore, $\theta(\sigma_1, \sigma_2)b\text{-Cl}(f(\tau_1\tau_2\text{-Int}(K))) \subseteq f(K)$. This shows that $f$ is weakly $\theta(\tau_1, \tau_2)b$-closed.

**Theorem 8.** *If* $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *is a bijective weakly* $\theta(\tau_1, \tau_2)b$*-closed function, then for every subset* $B$ *of* $Y$ *and every* $\tau_1\tau_2$*-open set* $U$ *of* $X$ *with* $f^{-1}(B) \subseteq U$*, there exists a* $\theta(\sigma_1, \sigma_2)b$*-closed set* $K$ *of* $Y$ *such that* $B \subseteq K$ *and* $f^{-1}(K) \subseteq \tau_1\tau_2$*-Cl*$(U)$*.*

*Proof.* Let $B$ be any subset of $Y$ and $U$ be any $\tau_1\tau_2$-open set of $X$ with $f^{-1}(B) \subseteq U$. Put $K = \theta(\sigma_1, \sigma_2)b\text{-Cl}(f(\tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(U))))$. Then, $K$ is a $\theta(\sigma_1, \sigma_2)b$-closed set of $Y$ such that $B \subseteq K$, since

$$B \subseteq f(U) \subseteq f(\tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(U))) \subseteq \theta(\sigma_1, \sigma_2)b\text{-Cl}(f(\tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(U)))) = K.$$

Since $f$ is weakly $\theta(\tau_1, \tau_2)b$-closed, by Theorem 6 we have $f^{-1}(K) \subseteq \tau_1\tau_2\text{-Cl}(U)$.

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