EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 3, Article Number 6575
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

On the Evaluation of Certain Unsolved Definite Integrals

Irshad Ayoob

[1] *Department of Mathematics and Sciences, Prince Sultan University, P.O. Box 66833, Riyadh 11586, Saudi Arabia*

Abstract. We study the following three definite integrals, previously posed as open problems by another researcher: $I(\alpha) = \int_0^\infty x^{-1/2} \ln(1 + x^{-\alpha})\, dx$, $I_n(\alpha) = \int_0^\infty \frac{1}{\sqrt{x(x^2+4\alpha^2)^n}}\, dx$, and $I(\beta) = \int_0^\infty x^{-3/2}\big[f(2\beta/x) - f(2/x)\big]\, dx$. We establish sufficient conditions for the convergence of these integrals and evaluate them in closed form using special functions. In particular, the third integral $I(\beta)$ turns out to be similar to Frullani integral, and we obtain two interesting formulas for this integral. These types of integrals have been used to establish logarithmic Hardy-Hilbert-type inequalities.

2020 Mathematics Subject Classifications: 26A42, 33B15, 26A06, 44A20

Key Words and Phrases: Definite integral, Beta function, Leibniz integral rule, Mellin transform

---

Definite integrals appear in various contexts across both pure and applied branches of mathematics. While many definite integrals can be solved with elementary techniques such as substitution or integration by parts, a significant number of them are non-elementary that cannot be expressed in terms of basic functions. These integrals demand more advanced methods for exact evaluation, including transformations (e.g., Laplace or Mellin), complex analysis (e.g., contour integration and residue theory), and special functions (such as the Gamma, Beta, and hypergeometric functions).

Finding exact solutions, when possible, is of high practical and theoretical importance as it allows for precise predictions, deeper analytic understanding, and verification of numerical methods. As such, the study of advanced techniques for evaluating definite integrals remains an active area of research. An extensive compilation of definite integrals, ranging from elementary to non-elementary forms, is available in [1] and the references cited therein. Recent studies, such as those presented in [2–7], highlight ongoing developments and underscore the sustained interest in this field.

The evaluation of the following definite integrals are posed as open problems in [8].

DOI: https://doi.org/10.29020/nybg.ejpam.v18i3.6575

Email address: iayoub@psu.edu.sa (I. Ayoob)

Problem 1. Evaluate the integral

$$I(\alpha) = \int_0^\infty x^{-1/2} \ln\left(1 + x^{-\alpha}\right) dx, \qquad \text{for } \alpha > \tfrac{1}{2}.$$

Problem 2. Determine a closed-form expression for

$$I_n(\alpha) = \int_0^\infty \frac{1}{\sqrt{x}\,(x^2 + 4\alpha^2)^n}\, dx, \qquad \text{for } \alpha > 0,\ n \in \mathbb{N}.$$

Problem 3. Evaluate

$$I(\beta) = \int_0^\infty x^{-3/2} \left[f\left(\frac{2\beta}{x}\right) - f\left(\frac{2}{x}\right) \right] dx, \qquad \text{for a suitable function } f \text{ and } \beta > 0.$$

The author in [8] has evaluated the special cases of the mentioned integrals. We list the special cases as following.

Corollary 1. *Proposition 2.8 in [8] is a special case of Problem 1 with* $\alpha = 2$.

Corollary 2. *Proposition 2.2, 2.3, and 2.4 in [8] are special cases of Problem 2 with* $n = 1, 2$ *and* $n = 3$ *respectively.*

Corollary 3. *Proposition 2.5 in [8] is a special case of Problem 3 with* $f(x) = arctan(x)$.

The further implications of these corollaries yield some integral formulas for π (see [8]). As a further applications, the author in [8] has applied the special cases of the integrals given in Problem 1,2 and 3 to obtain the logarithmic Hardy-Hilbert-type inequalities [9] (for example see Proposition 2.11 etc).

Our goal is to establish the convergence results for the integrals given in Problem 1, 2 and 3, and evaluate them in closed form for general parameters n, α and β.

1. Main Results

1.1. Convergence and evaluation of the first integral.

Here is the convergence result for the first integral.

Proposition 1. *Let*

$$I(\alpha) = \int_0^\infty x^{-\frac{1}{2}} \ln\left(1 + x^{-\alpha}\right) dx.$$

If $\alpha > \frac{1}{2}$, *then the improper integral* $I(\alpha)$ *converges.*

Proof. We split the integral at $x = 1$,

$$I(\alpha) = \int_0^1 x^{-\frac{1}{2}} \ln\left(1 + x^{-\alpha}\right) dx \ + \ \int_1^\infty x^{-\frac{1}{2}} \ln\left(1 + x^{-\alpha}\right) dx = I_1 + I_2.$$

(i) Convergence of I_2: For $x \geq 1$, $0 \leq x^{-\alpha} \leq 1$, and since $\ln(1+u) \leq u$ for $u \geq 0$, we have

$$0 \ \leq \ \ln\bigl(1+x^{-\alpha}\bigr) \ \leq \ x^{-\alpha}.$$

Hence

$$0 \ \leq \ I_2 \ \leq \ \int_1^\infty x^{-\frac{1}{2}} x^{-\alpha}\, dx = \int_1^\infty x^{-(\alpha+\frac{1}{2})}\, dx.$$

Because $\alpha + \frac{1}{2} > 1$, this last integral converges, so $I_2 < \infty$.

(ii) Convergence of I_1: On $(0,1]$, $x^{-\alpha} \geq 1$. For $u \geq 1$, $\ln(1+u) = \ln u + \ln(1+u^{-1}) \leq \ln u + \ln 2$. Setting $u = x^{-\alpha}$ gives

$$\ln\bigl(1+x^{-\alpha}\bigr) \ \leq \ \ln 2 + \alpha(-\ln x).$$

Therefore for $0 < x \leq 1$,

$$0 \ \leq \ x^{-\frac{1}{2}} \ln\bigl(1+x^{-\alpha}\bigr) \ \leq \ (\ln 2)\, x^{-\frac{1}{2}} \ + \ \alpha\, x^{-\frac{1}{2}}(-\ln x).$$

We check each term:

$$\int_0^1 x^{-\frac{1}{2}}\, dx = 2, \qquad \int_0^1 x^{-\frac{1}{2}}(-\ln x)\, dx = 4.$$

Hence

$$I_1 \leq (\ln 2) \cdot 2 \ + \ \alpha \cdot 4 < \infty.$$

Combining (i) and (ii) shows $I_1 < \infty$ and $I_2 < \infty$. Thus $I(\alpha)$ converges for all $\alpha > \frac{1}{2}$.

Now we solve the first integral.

Theorem 1. *For every $\alpha > \frac{1}{2}$, we have*

$$I(\alpha) = \int_0^\infty x^{-\frac{1}{2}} \ \ln\bigl(1+x^{-\alpha}\bigr)\, dx \ = \ 2\pi \ \csc\Bigl(\frac{\pi}{2\alpha}\Bigr).$$

Proof. We write

$$I(\alpha) = \int_0^\infty x^{-\frac{1}{2}} \ \ln\bigl(1+x^{-\alpha}\bigr)\, dx.$$

Since

$$\frac{\partial}{\partial\alpha} \ln\bigl(1+x^{-\alpha}\bigr) = -\frac{x^{-\alpha}\ln x}{1+x^{-\alpha}},$$

we may differentiate under the integral sign to obtain

$$\frac{dI}{d\alpha} = -\int_0^\infty x^{-\frac{1}{2}} \frac{x^{-\alpha}\ln x}{1+x^{-\alpha}}\, dx = -\int_0^\infty \frac{x^{-\frac{1}{2}} \ \ln x}{x^\alpha+1}\, dx.$$

Set $u = x^\alpha$, so that

$$x = u^{1/\alpha}, \quad dx = \frac{1}{\alpha} u^{\frac{1}{\alpha}-1} du, \quad \ln x = \frac{1}{\alpha} \ln u, \quad x^{-\frac{1}{2}} = u^{-\frac{1}{2\alpha}}.$$

Then

$$\frac{dI}{d\alpha} = -\frac{1}{\alpha^2} \int_0^\infty \frac{u^{-\frac{1}{2\alpha}+\frac{1}{\alpha}-1} \ln u}{u+1} \, du = -\frac{1}{\alpha^2} \int_0^\infty \frac{u^{s-1} \ln u}{1+u} \, du,$$

where $s = \frac{1}{2\alpha} \in (0, 1)$. The following integral is well-known (see [1]),

$$\int_0^\infty \frac{u^{s-1}}{1+u} \, du = \frac{\pi}{\sin(\pi s)},$$

and differentiating in s yields

$$\int_0^\infty \frac{u^{s-1} \ln u}{1+u} \, du = -\frac{\pi^2 \cos(\pi s)}{\sin^2(\pi s)}.$$

Hence

$$\frac{dI}{d\alpha} = \frac{\pi^2}{\alpha^2} \frac{\cos\left(\frac{\pi}{2\alpha}\right)}{\sin^2\left(\frac{\pi}{2\alpha}\right)}.$$

Next, we set $t = \frac{\pi}{2\alpha}$, so that $\alpha = \frac{\pi}{2t}$ and

$$d\alpha = -\frac{\pi}{2t^2} \, dt, \quad \frac{\pi^2}{\alpha^2} = 4t^2.$$

Thus

$$\frac{dI}{d\alpha} \, d\alpha = 4t^2 \frac{\cos t}{\sin^2 t} \left(-\frac{\pi}{2t^2} \, dt\right) = -2\pi \frac{\cos t}{\sin^2 t} \, dt = 2\pi \, d\left(\csc t\right).$$

Integrating shows

$$I(\alpha) = 2\pi \csc\left(\frac{\pi}{2\alpha}\right) + C.$$

Finally,

$$I(2) = 2\pi \csc\left(\frac{\pi}{2\cdot 2}\right) + C = 2\pi \csc\left(\frac{\pi}{4}\right) + C = 2\pi\sqrt{2} + C.$$

Since we know [8] that $I(2) = 2\pi\sqrt{2}$, it follows that

$$2\pi\sqrt{2} = 2\pi\sqrt{2} + C \quad \Longrightarrow \quad C = 0.$$

1.2. Convergence and evaluation of the second integral.

Here is the convergence result for the second integral.

Proposition 2. *Let* $\alpha > 0$ *and* $n \in \mathbb{N}$, $n \geq 1$. *Then*

$$I_n(\alpha) \;=\; \int_0^{\infty} \frac{1}{\sqrt{x}\,\left(x^2+4\alpha^2\right)^n}\,dx$$

converges.

Proof. We split the integral at $x = 1$,

$$I_n(\alpha) = \int_0^1 \frac{dx}{\sqrt{x}\,(x^2+4\alpha^2)^n} \;+\; \int_1^{\infty} \frac{dx}{\sqrt{x}\,(x^2+4\alpha^2)^n} = I_1 + I_2.$$

1. Convergence on $[0, 1]$:
For $0 < x \leq 1$, we have

$$x^2 + 4\alpha^2 \;\geq\; 4\alpha^2,$$

so

$$\frac{1}{\sqrt{x}\,(x^2+4\alpha^2)^n} \;\leq\; \frac{1}{\sqrt{x}\,(4\alpha^2)^n} = (4\alpha^2)^{-n}\,x^{-\frac{1}{2}}.$$

Since

$$\int_0^1 x^{-\frac{1}{2}}\,dx = 2 < \infty,$$

it follows by the comparison test that $I_1 < \infty$.

2. Convergence on $[1, \infty)$:
For $x \geq 1$, we have

$$x^2 + 4\alpha^2 \;\geq\; x^2,$$

so

$$\frac{1}{\sqrt{x}\,(x^2+4\alpha^2)^n} \;\leq\; \frac{1}{\sqrt{x}\,x^{2n}} \;=\; x^{-\left(2n+\frac{1}{2}\right)}.$$

Since $2n + \frac{1}{2} > 1$, the p-integral $\displaystyle\int_1^{\infty} x^{-\,(2n+\frac{1}{2})}dx$ converges. Hence $I_2 < \infty$ by comparison.

Combining these two estimates shows $I_n(\alpha) = I_1 + I_2 < \infty$, as claimed.

Next, we solve the second integral.

Theorem 2. *For every* $\alpha > 0$ *and integer* $n \geq 1$, *we have the following integral*

$$I_n(\alpha) = \int_0^{\infty} \frac{dx}{\sqrt{x}\,(x^2+4\alpha^2)^n} = \frac{1}{2}\,(4\alpha^2)^{-\left(n-\frac{1}{4}\right)}\,\frac{\Gamma\!\left(\frac{1}{4}\right)\Gamma\!\left(n-\frac{1}{4}\right)}{\Gamma(n)}.$$

Proof.
We set

$$x = 2\alpha\, t, \quad dx = 2\alpha\, dt, \quad \sqrt{x} = (2\alpha)^{\frac{1}{2}} t^{\frac{1}{2}}, \quad x^2 + 4\alpha^2 = 4\alpha^2(1+t^2).$$

Therefore

$$I_n(\alpha) = \int_0^\infty \frac{1}{(2\alpha)^{\frac{1}{2}} t^{\frac{1}{2}} \, [4\alpha^2(1+t^2)]^n} \,(2\alpha\, dt) = (2\alpha)^{\frac{1}{2}} (4\alpha^2)^{-n} \int_0^\infty \frac{dt}{t^{\frac{1}{2}}(1+t^2)^n}.$$

Since

$$(2\alpha)^{\frac{1}{2}} (4\alpha^2)^{-n} = 2^{\frac{1}{2}-2n}\, \alpha^{\frac{1}{2}-2n},$$

we have

$$I_n(\alpha) = 2^{\frac{1}{2}-2n}\, \alpha^{\frac{1}{2}-2n} \int_0^\infty t^{-\frac{1}{2}}(1+t^2)^{-n}\, dt.$$

Set

$$J_n = \int_0^\infty t^{-\frac{1}{2}}(1+t^2)^{-n}\, dt.$$

With the substitution $u = t^2$, $du = 2t\, dt$, $t^{-\frac{1}{2}} = u^{-\frac{1}{4}}$, we get

$$J_n = \int_0^\infty u^{-\frac{1}{4}}(1+u)^{-n} \frac{du}{2\, u^{\frac{1}{2}}} = \frac{1}{2}\int_0^\infty u^{\frac{1}{4}-1}(1+u)^{-n}\, du = \frac{1}{2} B\big(\tfrac{1}{4},\ n-\tfrac{1}{4}\big).$$

For $s > 0$, recall the definition of gamma function $\Gamma(s) = \int_0^\infty t^{s-1}e^{-t}\, dt$, and the relationship with the beta function, $B(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$, it follows that

$$J_n = \frac{1}{2}\, \frac{\Gamma(\frac{1}{4})\, \Gamma(n-\frac{1}{4})}{\Gamma(n)}.$$

Hence

$$I_n(\alpha) = 2^{\frac{1}{2}-2n}\, \alpha^{\frac{1}{2}-2n}\, \frac{1}{2}\, \frac{\Gamma(\frac{1}{4})\, \Gamma(n-\frac{1}{4})}{\Gamma(n)} = \frac{1}{2}\, (4\alpha^2)^{-(n-\frac{1}{4})}\, \frac{\Gamma(\frac{1}{4})\, \Gamma(n-\frac{1}{4})}{\Gamma(n)},$$

as claimed.

1.3. Convergence and evaluation of the third integral.

Here is the convergence result for the third integral.

Proposition 3. *Let $f\colon (0,\infty) \to \mathbb{R}$ be a continuously differentiable function such that*

$$\lim_{t\to 0^+} f(t) = f(0) \in \mathbb{R}, \qquad \lim_{t\to\infty} f(t) = f(\infty) \in \mathbb{R},$$

and

$$\int_0^\infty t^{\frac{1}{2}} \left|f'(t)\right| dt < \infty.$$

Then for each fixed $\beta > 0$ *the integral*

$$I(\beta) = \int_0^\infty x^{-\frac{3}{2}} \left[f\big(\tfrac{2\beta}{x}\big) - f\big(\tfrac{2}{x}\big) \right] dx$$

converges absolutely.

Proof. Fix $\beta > 0$. We may write

$$I(\beta) = \int_0^\infty x^{-3/2}\big[f(2\beta/x) - f(2/x)\big]\, dx = \int_0^{x_0} + \int_{x_0}^{X_1} + \int_{X_1}^\infty x^{-3/2}\big[f(2\beta/x) - f(2/x)\big]\, dx,$$

where we choose $0 < x_0 < X_1 < \infty$ so that

$$\frac{2\beta}{x_0} \geq T, \quad \frac{2}{X_1} \leq S,$$

with T large and S small enough that $\int_T^\infty t^{\frac{1}{2}} |f'(t)|\, dt < \varepsilon$ and $\int_0^S t^{\frac{1}{2}} |f'(t)|\, dt < \varepsilon$.

(i) Tail as $x \to 0^+$**:** For $0 < x \leq x_0$, both $\frac{2\beta}{x}$ and $\frac{2}{x}$ lie in $[T, \infty)$. By the Mean-Value Theorem there exists $\xi \in [\frac{2}{x}, \frac{2\beta}{x}]$ such that

$$f\big(\tfrac{2\beta}{x}\big) - f\big(\tfrac{2}{x}\big) = f'(\xi) \left(\tfrac{2\beta}{x} - \tfrac{2}{x} \right) = (\beta - 1)\, \frac{2}{x}\, f'(\xi).$$

Hence

$$\left|x^{-3/2}[\, f(2\beta/x) - f(2/x)]\right| \; = \; 2|\beta - 1|\, x^{-5/2} \left|f'(\xi)\right|.$$

Since $\xi \geq T$ and $t \mapsto t^{1/2}|f'(t)|$ is integrable on $[T, \infty)$, the change of variable $\xi = 2\theta/x$ shows

$$\int_0^{x_0} x^{-3/2} \big|f(2\beta/x) - f(2/x)\big|\, dx \; = \; 2|\beta - 1| \int_T^\infty t^{\frac{1}{2}} \left|f'(t)\right| dt < 2|\beta - 1|\, \varepsilon.$$

(ii) Tail as $x \to \infty$**:** For $x \geq X_1$, both $\frac{2\beta}{x}$ and $\frac{2}{x}$ lie in $(0, S]$. Again by the Mean-Value Theorem there is $\eta \in [\frac{2}{x}, \frac{2\beta}{x}] \subset (0, S]$ such that

$$f\big(\tfrac{2\beta}{x}\big) - f\big(\tfrac{2}{x}\big) = f'(\eta) \left(\tfrac{2\beta}{x} - \tfrac{2}{x} \right) = (\beta - 1)\, \frac{2}{x}\, f'(\eta),$$

and hence

$$\int_{X_1}^\infty x^{-3/2} \big|f(2\beta/x) - f(2/x)\big|\, dx = 2|\beta - 1| \int_0^S t^{\frac{1}{2}} \left|f'(t)\right| dt < 2|\beta - 1|\, \varepsilon.$$

(iii) Middle region: On the compact interval $[x_0, X_1]$, the function

$$x \mapsto x^{-3/2}\big[f(2\beta/x) - f(2/x)\big]$$

is continuous and hence bounded, so its integral is finite.

Combining (i), (ii), and (iii) shows $I(\beta) = \int_0^\infty x^{-\frac{3}{2}}\Big[f\big(\frac{2\beta}{x}\big) - f\big(\frac{2}{x}\big)\Big]\,dx < \infty$. Therefore $I(\beta)$ converges absolutely.

We have the following result which is similar to Frullani type integral.

Theorem 3. *Let $f\colon (0,\infty) \to \mathbb{R}$ be a continuously differentiable function such that*

$$\lim_{t\to 0^+} f(t) = f(0) \in \mathbb{R}, \qquad \lim_{t\to\infty} f(t) = f(\infty) \in \mathbb{R},$$

and

$$A = \int_0^\infty t^{\frac{1}{2}}\,\big|f'(t)\big|\,dt < \infty.$$

Then for every $\beta > 0$,

$$\int_0^\infty x^{-\frac{3}{2}}\Big[f\big(\tfrac{2\beta}{x}\big) \;-\; f\big(\tfrac{2}{x}\big)\Big]\,dx = \sqrt{2}\,A\,\big(1 - \beta^{-\frac{1}{2}}\big).$$

Proof. Define

$$I(\beta) = \int_0^\infty x^{-\frac{3}{2}}\Big[f\big(\tfrac{2\beta}{x}\big) - f\big(\tfrac{2}{x}\big)\Big]\,dx.$$

Since f is C^1 and the subtraction makes the integrand absolutely convergent at both ends, we may differentiate under the integral sign:

$$\frac{dI}{d\beta} = \int_0^\infty x^{-\frac{3}{2}}\,\frac{\partial}{\partial\beta}\big[f\big(\tfrac{2\beta}{x}\big)\big]\,dx = 2\int_0^\infty x^{-\frac{5}{2}}\,f'\big(\tfrac{2\beta}{x}\big)\,dx.$$

Perform the substitution $t = \frac{2\beta}{x}$, so $x = \frac{2\beta}{t}$ and $dx = -\,\frac{2\beta}{t^2}\,dt$. Then

$$x^{-\frac{5}{2}} = \Big(\tfrac{2\beta}{t}\Big)^{-\frac{5}{2}} = (2\beta)^{-\frac{5}{2}}\,t^{\frac{5}{2}},$$

and

$$x^{-\frac{5}{2}}\,dx = (2\beta)^{-\frac{5}{2}}\,t^{\frac{5}{2}}\,\big(-\tfrac{2\beta}{t^2}\big)\,dt = -(2\beta)^{-\frac{3}{2}}\,t^{\frac{1}{2}}\,dt.$$

Hence

$$\int_0^\infty x^{-\frac{5}{2}} f'\big(\tfrac{2\beta}{x}\big)\,dx = (2\beta)^{-\frac{3}{2}}\int_\infty^0 t^{\frac{1}{2}} f'(t)\,(-dt) = (2\beta)^{-\frac{3}{2}}\int_0^\infty t^{\frac{1}{2}} f'(t)\,dt = (2\beta)^{-\frac{3}{2}}\,A,$$

where $A = \int_0^\infty t^{\frac{1}{2}}\,\big|f'(t)\big|\,dt < \infty$.
Thus

$$\frac{dI}{d\beta} = 2\,(2\beta)^{-\frac{3}{2}}\,A = 2^{-\frac{1}{2}}\,A\,\beta^{-\frac{3}{2}}.$$

Integrating from $\beta = 1$ (where clearly $I(1) = 0$) to a general $\beta > 0$ gives

$$I(\beta) = \int_1^\beta \frac{dI}{db}\, db = 2^{-\frac{1}{2}} A \int_1^\beta b^{-\frac{3}{2}}\, db = 2^{-\frac{1}{2}} A \big[-2\, b^{-\frac{1}{2}}\big]_1^\beta = \sqrt{2}\, A\big(1 - \beta^{-\frac{1}{2}}\big).$$

This proves the general formula.

The following special case in [8] is deduced from Theorem 3.

Remark 1. *In the special case, $f(x) = \arctan x$, we have $f'(t) = 1/(1+t^2)$, and the constant*

$$A = \int_0^\infty \frac{t^{\frac{1}{2}}}{1+t^2}\, dt = \tfrac{1}{2}\, B\big(\tfrac{3}{4}, \tfrac{1}{4}\big) = \frac{\pi\sqrt{2}}{2},$$

yielding

$$\int_0^\infty \frac{\arctan\big(\frac{2\beta}{x}\big) - \arctan\big(\frac{2}{x}\big)}{x\sqrt{x}}\, dx = \pi\big(1 - \beta^{-\frac{1}{2}}\big).$$

Note that Theorem 3 requires f to be continuously differentiable. The following result gives one more closed form solution of the integral $I(\beta)$ for more general class of functions in terms of Mellin transform.

Proposition 4. *Let $f : (0, \infty) \to \mathbb{R}$ be measurable and suppose its Mellin transform*

$$M_f(s) \;=\; \int_0^\infty t^{s-1} f(t)\, dt$$

converges at $s = \frac{1}{2}$, i.e.

$$\int_0^\infty t^{-1/2}\, \big|f(t)\big|\, dt < \infty.$$

Then for each fixed $\beta > 0$ the integral

$$I(\beta) = \int_0^\infty \frac{1}{x\sqrt{x}} \Big[f\big(\tfrac{2\beta}{x}\big) - f\big(\tfrac{2}{x}\big) \Big]\, dx$$

converges absolutely.

Proof. We set

$$g(x) = \frac{1}{x\sqrt{x}} \Big[f\big(\tfrac{2\beta}{x}\big) - f\big(\tfrac{2}{x}\big) \Big].$$

Make the substitution

$$t = \frac{2}{x}, \qquad x = \frac{2}{t}, \qquad dx = -\frac{2}{t^2}\, dt.$$

Then

$$\frac{dx}{x\sqrt{x}} = \frac{-2}{t^2}\, \frac{1}{(2/t)\sqrt{2/t}} = -2^{-\frac{1}{2}}\, t^{-\frac{1}{2}}\, dt,$$

and

$$\frac{2\beta}{x} = \beta t, \quad \frac{2}{x} = t.$$

Hence

$$I(\beta) = \int_0^\infty g(x)\,dx = \int_\infty^0 f(\beta t) - f(t)\;(-2^{-\frac{1}{2}} t^{-\frac{1}{2}})\,dt$$
$$= 2^{-\frac{1}{2}} \int_0^\infty t^{-\frac{1}{2}} \big[f(\beta t) - f(t)\big]\,dt.$$

It suffices to show

$$\int_0^\infty t^{-\frac{1}{2}} \big|f(\beta t) - f(t)\big|\,dt < \infty.$$

By the triangle inequality,

$$\big|f(\beta t) - f(t)\big| \le |f(\beta t)| + |f(t)|,$$

so

$$\int_0^\infty t^{-\frac{1}{2}} \big|f(\beta t) - f(t)\big|\,dt \le \int_0^\infty t^{-\frac{1}{2}} |f(\beta t)|\,dt + \int_0^\infty t^{-\frac{1}{2}} |f(t)|\,dt.$$

For the first term, substitute $u = \beta t$, $du = \beta\,dt$:

$$\int_0^\infty t^{-\frac{1}{2}} |f(\beta t)|\,dt = \beta^{-\frac{1}{2}} \int_0^\infty u^{-\frac{1}{2}} |f(u)|\,du.$$

Hence

$$\int_0^\infty t^{-\frac{1}{2}} \big(|f(\beta t)| + |f(t)|\big)\,dt = \big(1 + \beta^{-\frac{1}{2}}\big) \int_0^\infty t^{-\frac{1}{2}} |f(t)|\,dt,$$

which is finite by hypothesis. Therefore $I(\beta)$ converges absolutely.

Theorem 4. *Let $f : (0,\infty) \to \mathbb{R}$ be measurable and suppose its Mellin transform*

$$M_f(s) \;=\; \int_0^\infty t^{s-1} f(t)\,dt$$

converges at $s = \frac{1}{2}$, i.e.

$$\int_0^\infty t^{-1/2}\,\big|f(t)\big|\,dt < \infty.$$

Then we have,

$$I(\beta) = \int_0^\infty \frac{1}{x\sqrt{x}} \Big[f\big(\tfrac{2\beta}{x}\big) - f\big(\tfrac{2}{x}\big)\Big]\,dx = \frac{1}{\sqrt{2}}\,\big(\beta^{-1/2} - 1\big)\,M_f\big(\tfrac{1}{2}\big).$$

Proof.

Starting from the following convergent integral (Proposition 4 ensures convergence),

$$I(\beta) = \int_0^\infty \frac{1}{x\sqrt{x}} \big[f(2\beta/x) - f(2/x)\big]\,dx,$$

we perform the change of variable $t = 2/x$, so that $x = 2/t$ and $dx = -2\,t^{-2}dt$. As in the proof of Proposition 4,

$$\frac{dx}{x\sqrt{x}} = -2^{-\frac{1}{2}}\,t^{-\frac{1}{2}}\,dt, \quad \frac{2\beta}{x} = \beta t, \quad \frac{2}{x} = t.$$

Hence

$$\begin{aligned} I(\beta) &= \int_{\infty}^{0} \big[f(\beta t) - f(t)\big]\,\big(-2^{-\frac{1}{2}} t^{-\frac{1}{2}}\big)\,dt = 2^{-\frac{1}{2}} \int_0^{\infty} t^{-\frac{1}{2}} \big[f(\beta t) - f(t)\big]\,dt \\ &= 2^{-\frac{1}{2}} \Big[\int_0^{\infty} t^{-\frac{1}{2}} f(\beta t)\,dt - \int_0^{\infty} t^{-\frac{1}{2}} f(t)\,dt\Big]. \end{aligned}$$

In the first integral substitute $u = \beta t$, $du = \beta\,dt$, giving

$$\int_0^{\infty} t^{-\frac{1}{2}} f(\beta t)\,dt = \beta^{-\frac{1}{2}} \int_0^{\infty} u^{-\frac{1}{2}} f(u)\,du = \beta^{-\frac{1}{2}}\,M_f\big(\tfrac{1}{2}\big).$$

The second integral is exactly $M_f(\frac{1}{2})$. Therefore

$$I(\beta) = 2^{-\frac{1}{2}} \Big[\beta^{-\frac{1}{2}} M_f(\tfrac{1}{2}) \; - \; M_f(\tfrac{1}{2})\Big] = \frac{M_f(\frac{1}{2})}{\sqrt{2}}\,\big(\beta^{-\frac{1}{2}} - 1\big),$$

as claimed.

2. Conclusion

In this work, we have evaluated three classes of definite integrals that were originally posed as open problems in the literature. Our approach involved rigorously establishing sufficient conditions under which these integrals converge, followed by the derivation of explicit closed-form expressions. These results are presented in Theorems 1, 2, 3 and 4 of the paper. Beyond their intrinsic analytical interest, these integrals serve as foundational kernels in constructing a new class of generalized logarithmic Hardy–Hilbert-type inequalities, extending those previously established in [8]. The identification of optimal constants, further generalization to multidimensional or operator-theoretic settings, and exploration of applications in functional inequalities represent promising directions for future research.

Acknowledgements

The author would like to thank the Prince Sultan University for paying the publication fees for this work through TAS LAB.

References

[1] S. Gradshteyn, I. and M. Ryzhik, I.˙ *Table of Integrals, Series, and Products.* Academic Press, New York, 7th edition, 2007.

[2] Christophe Chesneau. On a new one-parameter arctangent-power integral. *International Journal of Open Problems in Computer Science and Mathematics*, 17(4):1–8, 2024.

[3] Christophe Chesneau. New integral formulas inspired by an old integral result. *International Journal of Open Problems in Computer Science and Mathematics*, 18(2):53–71, 2025.

[4] R. Reynolds and A. Stauffer. A definite integral involving the logarithmic function in terms of the lerch function. *Mathematics*, 7(1):1–5, 2019.

[5] R. Reynolds and A. Stauffer. Definite integral of arctangent and polylogarithmic functions expressed as a series. *Mathematics*, 7(7):1–7, 2019.

[6] R. Reynolds and A. Stauffer. Derivation of logarithmic and logarithmic hyperbolic tangent integrals expressed in terms of special functions. *Mathematics*, 8:1–6, 2020.

[7] R. Reynolds and A. Stauffer. A quadruple definite integral expressed in terms of the lerch function. *Symmetry*, 13(1):1–8, 2021.

[8] Christophe Chesneau. On some new integral formulas with applications. *International Journal of Open Problems in Computer Science and Mathematics*, 18(3):1–21, September 2025.

[9] H. Hardy, G. E. Littlewood, J. and G. Pólya. *Inequalities.* Cambridge University Press, Cambridge, 1934.