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A Kurosh-Amitsur Completely Prime Radical for Near-rings

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Abstract. Two generalizations of the completely prime radical of rings to near-rings, namely the completely prime radical of near-rings and the completely equiprime radical of near-rings were introduced and studied. First one is not a Kurosh-Amitsur radical but the second one is a special radical in near-rings.
In this article another generalization of the completely prime radical of rings is introduced in near-rings using right modules of near-rings. For this completely prime right N-groups of type-$r(1)$ are introduced in near-rings, N is a near-ring. Making use of these right N-groups of type-$r(1)$, the completely prime radical of near-rings of type-$r(1)$ is introduced. It is observed that the completely prime radical of type-$r(1)$ is a Kurosh-Amitsur radical.

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Key Words and Phrases: Near-ring, right N-group, right$N$-groups of type $r(1)$, completely Prime radical of type $r(1)$

1. Introduction

In this article, we consider right zero-symmetric near-rings. The concept of the completely prime radical, originally developed for rings, was extended to near-rings by N. J. Groenewald [1]. He demonstrated that, analogous to the prime radical in near-rings, the completely prime radical of near-rings does not satisfy the properties of a Kurosh-Amitsur radical. Based on the notion of equiprime ideals in near-rings, a further generalization of

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completely prime radical of rings called the completely equiprime radical was introduced in [2]. It was shown that this radical forms a special class of radicals within the framework of near-rings.

Completely prime modules of rings were introduced in [3]. In [4], completely prime modules of rings were extended to (left) N-groups, N is a near-ring and the corresponding radical is the completely prime radical of near-ring which is not a Kurosh-Amitsur radical of near-rings.

Right module theoretic characterization of radicals of near-rings was studied in [5]. Prime right N-groups were introduced and their correspoding radicals were studied in [6], [7] and [8].

In this article, the concept of completely prime module is extended to (right) N-groups which leads to another completely prime radical of near-rings and is a Kurosh-Amitsur radical of near-rings.

A group $(T, +)$ is a right N-group if there is a mapping $(t, a) \to ta$ of $T \times N$ into T such that:

(i) $t(ab) = (ta)b$;

(ii) $t(a + b) = ta + tb$ for all $t \in T, a, b \in N$.

I is a right N-group for any right ideal I of N under the multiplication in N. Moreover, for a right ideal I of N, N/I is a right N-group under $(a + I)b = ab + I$, $a, b \in N$.

A subgroup (normal subgroup) D of the right N-group T is a right N-subgroup (ideal) of T if $da \in D$ for all $d \in D, a \in N$.

$t \in T$ is called a distributive element of the right N-group T if $t(a + b) = ta + tb$ for all $a, b \in N$.

2. Completely Prime Right N-groups of type $r(1)$

Unless or otherwise specified near-rings considered are zero-symmetric right near-rings.

Definition 1. *A right N-group G with $GN \neq \{0\}$ and $G0 = \{0\}$ is called a* completely prime N-group of type $r(1)$ *if*

(i) every non-zero right N-subgroup of G contains a distributive element $g_0(\neq 0)$ and;

(ii) $gr = 0$ implies either $g = 0$ or $Gr = \{0\}$ for all $g \in G, r \in R$.

Remark 1. *Let R be a ring and M be a completely prime (right) R-module. Then M is also a completely prime right R-group of type $r(1)$, when R is considered as a near-ring.*

Example 1. *Let N be a near-field. It is clear that N is a completely prime right N-group of type $r(1)$.*

The following Proposition is obvious in view of the above definition.

Proposition 1. *Let G be right N-group with $GN \neq \{0\}$ and each non-zero N-subgroup of G has a non-zero distributive element. Then G is a completely prime N-group of type $r(1)$ if and only if $(g:0) = (G:0)$ for all $0 \neq g \in G$.*

Proof. Suppose that G completely prime right N-group of type $r(1)$. Let $0 \neq g \in G$ and $r \in N$ and $gr = 0$. By definition of completely prime right N-group of type $r(1)$, $Gr = \{0\}$. So $(g:0) \subseteq (G:0)$. Since $(G:0) \subseteq (g:0)$, we have $(g:0) = (G:0)$. Conversely suppose that $(g:0) = (G:0)$ for all $0 \neq g \in G$. Let $0 \neq g \in G, r \in N$ and $gr = 0$. By assumption, $Gr = \{0\}$. So G completely prime right N-group of type $r(1)$.

Proposition 2. *Let G be a completely prime right N-group of type $r(1)$ and H be N-subgroup of G. Then H is a completely prime right N-group of type $r(1)$.*

Proof. This follows from the definition of completely prime right N-group.

Proposition 3. *Let G be a completely prime right N-group of type $r(1)$. If N is a ring, then G is a completely prime ring N-module.*

Proof. Obvious from the definition of the Completely prime right N-group.

Proposition 4. *Let G be a completely prime right N-group of type $r(1)$ and I be an ideal of N and $GI = \{0\}$.Then G has a right N-subgroup H which is a Completely prime N/I-group of type $r(1)$.*

Proof. Let g_0 be a distributive element of the right N-group G. Consider $g_0N := \{g_0a \mid a \in N\}$. For g_1, $g_2 \in g_0N$, $g_1 = g_0b$ and $g_2 = g_0c$ for some $b, c \in N$. Now $g_1 - g_2 = g_0(b-c) \in g_0N$. So g_0N is a subgroup of $(G, +)$. Since $(g_0N)N \subseteq g_0(NN) \subseteq g_0N$, g_0N is a right N-subgroup of G. We claim that g_0N is a right N/I-group, under $g(x+I) = gx$, $g \in g_0N$ and $x + I \in N/I$. Let $g := g_0a \in g_0N$ and $x+I$, $y+I \in N/I$ and $x+I = y+I$. We have $x - y \in I$. $gx = (g_0a)x = g_0(ax) = g_0(a((x-y)+y)) - g_0(ay) + g_0(ay) = g_0(a((x-y)+y) - ay) + g_0(ay) = 0 + (g_0a)y) = gy$. So the operation is well defined and g_0N is a right N/I-group. It is clear that $g^{'} \in g_0N$ is a distributive element of the right N/I-group g_0N if and only if $g^{'}$ is a distributive element of the right N-group G. Since every right N/I-subgroup of g_0N is an N subgroup of G, every non-zero N/I-subgroup of g_0N contains a non-zero distributive element. Let $0 \neq g_3 \in g_0N$. We have $(g_3:0)_{N/I} = \{n+I \in N/I \mid g_3(n+I) = 0\} = \{n+I \in N/I \mid g_3n = 0\} = (g_3:0)_N/I = (g_0N:0)_N/I = (g_0N:0)_{N/I}$. Hence g_0N is a completely prime right N/I-group of type $r(1)$.

Proposition 5. *Let N be a near-ring and I is an ideal of N and G be completely prime right N/I-group type $r(1)$. Then G is a completely prime right N-group of type $r(1)$.*

Proof. N is a near-ring and I is an ideal of N and G is a completely prime right N/I-group type $r(1)$. Define $gx := g(x+I)$ for all $g \in G$, $x \in N$. This makes G a right N-group. Let H be a non-zero right N-subgroup of G. It is clear that H is also a non-zero right N/I-subgroup of G and it contains a non-zero distributive element h_0. We have $h_0(x+y) = h_0(x+y+I) = h_0((x+I)+(y+I)) = h_0(x+I)+h_0(y+I) = h_0x+h_0y$. So h_0 is also a non-zero distributive element of the right N - subgroup H of G. Let $0 \neq g_1, 0 \neq g_2 \in G$. Now $(g_1:0)_N = \{x \in N \mid g_1x = 0\} = \{x \in N \mid g_1(x+I) = 0\} = \{x \in N \mid g_2(x+I) = 0\} = \{x \in N \mid g_2x = 0\} = (g_2:0)_N$. Hence G is a completely prime right N-group of type$r(1)$.

Proposition 6. *Let G be a Completely prime right N-group of type* 1 *and* $(G:0) = \{x \in N \mid gx = 0 \ for \ all \ g \in G\}$ *be the annihilator of G. Then there is a largest ideal of N contained in* $(G:0)$*.*

Proof. We have $0 \in (G:0)$. Let I, J be ideals of N contained in $(G:0)$. By definition, G has a non zero distributive element g_0. We have $(g_0:0) = (G:0)$ and $g_0(y+z) = g_0y+g_0z = 0+0 = 0$ for all $y \in I$ and $z \in J$. So $g_0(I+J) = 0$. Therefore $I+J \subseteq (g_0:0) = (G:0)$. Hence there is a largest ideal of N contained in $(G:0)$.

$\underline{(G:0)}$ denotes the largest ideal of N contained in $(G:0)$.

Definition 2. *Let G be a completely prime right N-group of type* 1 *and* $P := \underline{(G:0)}$. *Then P is called a* completely prime ideal of N of type $r(1)$*.*

Definition 3. *A near-ring N is called* a completely prime near-ring of type $r(1)$ *if it's zero ideal is a completely prime ideal of type* $r(1)$*.*

Corollary 1. *Let N be a near-ring and P be a completely prime ideal of N of type $r(1)$. Then N/P completely prime near-ring of type $r(1)$.*

Proof. Since P is a completely prime ideal of N of type $r(1)$, there is a completely prime right N-group G of type $r(1)$ and P is the largest ideal of N contained in $(G:0)$. By Propostion 4, there is a right N-subgroup H of G which is a completely prime N/P group of type $r(1)$, where $h(x+P) := hx$ for all $h \in H$, $x \in N$ and $(H:0)_{N/P} = (G:0)_N/P$. Therefore the zero ideal, (0), is the largest ideal of N/P contained in $(H:0)_{N/P}$. So (0) is a completely prime ideal of N/P of type $r(1)$. Hence N/P is a completely prime near-ring of type $r(1)$.

Corollary 2. *Let N be a near-ring and P be an ideal of N and N/P be a completely prime near ring of type $r(1)$. Then P is a completely prime ideal of N of type $r(1)$.*

Proof. P is an ideal of a near-ring N and N/P is a completely prime near-ring of type $r(1)$. So the zero ideal, (0), is a completely prime ideal of N/P of type $r(1)$.Therefore there is a completely prime right N/P - group G of type $r(1)$ such that the zero ideal of N/P is the largest ideal of N/P contained in $(G:0)_{N/P}$. By Propostion 5, G is also a completely prime right N-group of type $r(1)$, where $gx := g(x+I)$ for all $g \in G$ and $x \in N$ and $(G:0)_{N/P} = (G:0)_N/P$. Therefore P is the largest ideal of N contained in $(G:0)_N$. Hence P is a completely prime ideal of N of type $r(1)$.

Definition 4. *Let N be nearring. Define $P_{cr(1)}(N) := \cap\{I \mid I$ is a completely prime ideal of N of type $r(1)\}$ and $P_{cr(1)}(N) := N$ if N has no completely prime ideals of type $r(1)$. $P_{cr(1)}$ is the* completely prime radical of type $r(1)$.

Proposition 7. *$P_{cr(1)}$ is a H-radical corresponding to the class of all completely prime zero-symmetric near rings of type $r(1)$.*

Proof. Let $\mathcal{M}_{cr(1)} := \{N \mid N$ is completely prime zero-symmetric near rings of type $r(1)\}$. For a near-ring N, we have $(N)\mathcal{M}_{cr(1)} := \cap\{J \mid J$ is an ideal of N and $N/J \in \mathcal{M}_{cr(1)}\}$. Now Q defined by $Q(N) := (N)\mathcal{M}_{cr(1)}$, is the H-radical corresponding to the class of all completely prime zero-symmetric near rings of type $r(1)$. From the Corollaries 1 and 2, $P_{cr(1)} = Q$. Hence $P_{cr(1)}$ is the H radical corresponding to the class of all completely prime zero-symmetric near rings of type $r(1)$.

Lemma 1. *Let G be a Completely prime right N-group of type (1) and I be an ideal of N and $GI \neq \{0\}$. Then G is a Completely prime right I-group of type 1 and $\underline{(G:0)_I} \supseteq \underline{(G:0)_N} \cap I$.*

Proof. Since G is a right N-group and $GI \neq \{0\}$, G is also a non-trivial I-group under restriction. Let H be a non-zero right I-subgroup of G and $0 \neq h \in H$. Now $hI \neq \{0\}$ as $GI \neq \{0\}$. Let T be the subgroup of G generated by $hI := \{ha \mid a \in I\}$. T is a right I-subgroup of G as well as right N-subgroup of G. Being right N-subgroup of G, T has a distributive element t_0. So $t_0(x+y) = t_0x + t_0y$ for all $x, y \in N$. Hence t_0 is a distributive element of the right I-group $T \subseteq H$. For $0 \neq g_1, 0 \neq g_2 \in G, (g_1 : 0)_I = (g_1 : 0)_N \cap I = (g_2 : 0)_N \cap I = (g_2 : 0)_I$. Also $(G : 0)_I = (g_1 : 0)_I = (g_1 : 0)_N \cap I = (G : 0)_N \cap I$. Therefore $\underline{(G:0)_I} \supseteq \underline{(G:0)_N} \cap I$.

Lemma 2. *Let G be a Completely prime right I-group of type 1 and I, an ideal of a near-ring N. Then $H := g_0 I$ is a completely prime right N-group of type $r(1)$ and $\underline{(G:0)_I} = \underline{(H:0)_I} \supseteq \underline{(H:0)_N} \cap I$, g_0 is a distributive element of the right I-group G.*

Proof. It is clear that $H = g_0 I := \{g_0 a \mid a \in I\}$ is a subgroup of $(G, +)$ and hence H is a right I-subgroup of G. For $g_0 a \in H$ define $(g_0 a)x := g_0(ax)$ for all $x \in N$. We claim that this operation is well defined. Let $g_0 a = g_0 b, a, b \in I$ and $x \in N$ and $c \in I$. $[g_0(ax) - g_0(bx)]c = g_0(ax)c - g_0(bx)c = (g_0 a)(xc) - (g_0 b)(xc) = (g_0 a - g_0 b)(xc) = 0(xc) = 0$. If $g_0(ax) - g_0(bx) \neq 0$ then $GI = 0$, a contraction. So the operation is well defined and H is a right N-group with $HN \neq (0)$. Let $\triangle$ be a non zero right N-subgroup of H. It is clear that $\triangle$ is also a right I-subgroup of H and hence there is a $h_0 \in \triangle$ such that $h_0(a+b) = h_0 a + h_0 b$ for all $a, b \in I$. Now $h_0 := g_0 d$ for some $d \in I$. Let $x, y \in N$. $[h_0(x+y) - (h_0 x + h_0 y)]c = (g_0 d)(x+y)c - ((g_0 d)xc + (g_0 d)yc) = (g_0 d)(xc + yc) - ((g_0 d)(xc) + (g_0 d)(yc)) = h_0(xc) + h_0(yc) - (h_0(xc) + h_0(yc)) = 0$ for all $c \in I$. Therefore $h_0(x+y) = h_0 x + h_0 y$ and hence a distributive element of the right N-subgroup $\triangle$. Let $0 \neq g_0 a, 0 \neq g_0 b \in H$, $a, b \in I$. $(g_0 a : 0)_N = (g_0 a : 0)_I \cap N = (g_0 b : 0)_I \cap N = (g_0 b : 0)_N$. Therefore H is a completely prime N-group of type $r(1)$. We have $(H : 0)_I = (H : 0)_N \cap I$. So $\underline{(G:0)_I} = \underline{(H:0)_I} \supseteq \underline{(H:0)_N} \cap I$.

Theorem 1. *$P_{cr(1)}$ is a complete H-radical.*

Proof. Suppose that $P_{cr(1)}(J) = J$, J is an ideal of N. We claim that $J \subseteq P_{cr(1)}(N)$. On the contrary suppose that $J \not\subseteq P_{cr(1)}(N)$. So there is completely prime right N-group G of type $r(1)$ such that $GJ \neq \{0\}$. By Lemma 1, G is a completely prime right J-group of type $r(1)$. This is a contradiction to $P_{cr(1)}(J) = J$. Therefore $J \subseteq P_{cr(1)}(N)$. Hence $P_{cr(1)}$ is a complete H-radical.

Theorem 2. *The H-radical $P_{cr(1)}$ is r-hereditary.*

Proof. If J is an ideal of a near-ring N and G is a Completely prime right J-group of type 1 then by Lemma 2, there is a right J-subgroup H of G such that H is a completely prime right N-group of type $r(1)$ and $\underline{(G:0)_J} = \underline{(H:0)_J} \supseteq \underline{(H:0)_N} \cap J$. Therefore $P_{cr(1)}(J) \supseteq P_{cr(1)}(N) \cap J$. Hence $P_{cr(1)}$ is r-hereditary.

Theorem 3. *$P_{cr(1)}$ is an idempotent H-radical.*

Proof. Let N be a near-ring and $J := P_{cr(1)}(N)$. From Theorem 2, $P_{cr(1)}(J) \supseteq P_{cr(1)}(N) \cap J$. Taking $P_{cr(1)}(N)$ for J in the above inclusion, we have $P_{cr(1)}(P_{cr(1)}(N)) = P_{cr(1)}(N)$. Therefore $P_{cr(1)}$ is an idempotent H-radical.

From Proposition 7 and Theorems 3 and 1, we have the following:

Theorem 4. *The H-radical $P_{cr(1)}$ is a Kurosh-Amitsur radical in the class of all zero-symmetric near-rings.*

Proof. Since an idempotent, complete H-radical is a Kurosh-Amitsur radical, from Proposition 7 and Theorems 1 and 3, $P_{cr(1)}$ is a Kurosh-Amitsur radical.

3. Conclusions

Some important Kurosh-Amitsur prime radicals of rings fails to be Kurosh-Amitsur radicals, when they are generalized to near-rings using left N-groups. To see whether the situation will be different with right N-groups was studied in [6], [5] and [8].
As proved in [6], right N-groups play significant role in generalizing the prime radicals of rings to near-rings. This fact was completely established in [5] in general. In [8], the prime radical of rings was generalized to near-rings using right N-groups which is also a Kurosh-Amitsur radical of near-rings.
In this article, the completely prime radical of rings which is a Kurosh-Amitsur radical of rings is generalized to near-rings using right N-groups which is also a Kurosh-Amitsur radical of near-rings.

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