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A Natural Extension of the Banach fixed-point Theorem in a b-Metric Space with an Orthogonal Direct Sum Structure

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Abstract. This study aims to develop new versions of the Banach fixed-point theorem in generalized metric spaces endowed with a direct sum structure. Specifically, we assume a diagonal matrix A in $\mathbb{R}^{d\times d}$ and establish more appropriate contraction conditions to improve the applicability of fixed-point results within this framework. Since the condition that the matrix A must converge to zero is unnecessary, our approach yields stronger results than the Perov one. As an application of our findings, we examine the existence and uniqueness of solutions for a system of matrix equations. This version is more powerful than the Perov version. We introduced some examples and applications to illustrate our result.

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Key Words and Phrases: Fixed point, Banach contraction principle, generalized b-metric space, direct sum

1. Introduction

A fixed-point of $\mathscr{F}$ on $\mathcal{X}$ is an element q such that $\mathscr{F}(q) = q$. Under certain circumstances on the operator $\mathscr{F}$ and the space $\mathcal{X}$, the operator $\mathscr{F}$ admits one or more fixed points, according to the fixed-point theorem. For various applications of the fixed-point theorem, there are numerous outcomes Brouwer's fixed-point theorem, Schauder's fixed-point theorem, and the Banach contraction principle are the foundational ideas of fixed-point theory.

Perov generalized the notion of metric spaces in [1] by replacing the set of real numbers with vector-valued space $\mathbb{R}^d$. Thus, the classical contraction operator principle was extended to contraction operators in a vector-valued metric space, known as a generalized

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metric space. In this context, several interesting results have been introduced by various authors (see [2, 3]).

Omran presented versions of Banach fixed-point theorems in a generalized metric space endowed with the Hadamard product. Omran [4] applied the Hadamard product to solve a system of symmetric positive definite matrix equations, demonstrating that results based on the Hadamard product of a vector-valued metric are stronger than those using Perov's condition. Specifically, it eliminates the need to assume that $\mathcal{A}$ is a matrix converging to zero.

In Banach's 1922 thesis [5], the fixed-point theorem—also known as the Banach contraction principle—was first used in its final form in 1922 to prove that an integral equation had a solution. Since then, it has been found to be a useful tool for solving many practical problems in various branches of mathematical analysis. In this principle, b-metric space $\mathcal{X}$ is complete if $\mathscr{F}\colon \mathcal{X} \to \mathcal{X}$ is a contraction operator; that is, $\delta_{\mathbb{R}}(\mathscr{F}x, \mathscr{F}y) \leq \lambda \delta_{\mathbb{R}}(x, y)$ for all $x, y \in \mathcal{X}$, where $\lambda \in (0, \frac{1}{s})$ is a constant, then $\mathscr{F}$ has a fixed point.

In 1989, Bakhtin [6] introduced the concept of b-metric spaces, which Czerwik further developed in 1993 [7]. This advancement sparked significant research into generalizing the Banach fixed-point theorem for b-metric spaces. Notable contributions from Mehmet Kir [8], Boriceanu [9], Bota [10], Pacurar [11], and Czerwik [7] have extended fixed-point theorems in this context. Additionally, Czerwik explored the convergence of measurable functions with respect to measure and generalized the Banach contraction principle .

In the same year, Czerwik [7] proposed a new axiom for semimetric spaces, relaxing the classical triangle inequality to broaden the scope of the Banach contraction principle. This idea aligns with the nonlinear elastic matching (NEM) distance discussed by Fagin et al. [12], which has found applications in diverse fields, such as trademark shape analysis [13] and ice floe measurements [14]. Building on this concept, Q. Xia [15] applied semimetric distances to study optimal transport paths between probability measures, coining the term quasi-metric spaces, a designation also used in Heinonen's book [16]. Our goal is to further extend well-known fixed-point theorems within the framework of b-metric spaces.

Some articles examine various generalizations of classical metric spaces to establish new fixed-point theorems. Building on the framework of b-metric spaces presented by Jleli and Samet [17]. Albargi and Ahmad [18] developed standard $\mathcal{F}$-fuzzy fixed-point theorems under rational (β-ϕ)-contractive conditions, with applications to fuzzy integrodifferential equations via the generalized Hukuhara derivative. Furthermore, Huang and Samet [19] introduced two novel self-mapping classes on complete metric spaces: p-contractions concerning families of mappings, and (ψ, Γ, α)-contractions, which involve the Euler Gamma function. Also, [20, 21] developed fixed-point results to fuzzy normed spaces using a newly introduced triangle property and to algebra fuzzy metric spaces. These results generalized, unified, and improved many classical theorems.

The aim of this work is to present new versions of Banach fixed-point theorems in a generalized b-metric space endowed with the orthogonal direct sum, where A is a diagonal matrix in $\mathbb{R}^d$. This approach, constructed via the direct sum, ensures more suitable contraction conditions.

Now, we can summarize the content of this manuscript as follows:
Section 1 provides a brief historical overview to put the fixed-point theorems established in this paper into context. Section 2 introduces the definitions and basic concepts that lay the foundation for understanding generalized b-metric spaces endowed with the orthogonal direct sum. Section 3 presents several theorems that generalize the theorems given in [1, 4], which prove the existence and uniqueness of fixed points.

2. Preliminaries

The real vector space sum of the real vector spaces $\mathbb{R}_1, \ldots, \mathbb{R}_d$ in a nonempty finite ordered list is the real vector space whose underlying set is the Cartesian product $\mathbb{R}_1 \times \cdots \times \mathbb{R}_d = \prod_{i=1}^{d} \mathbb{R}_i$, with vector space operations defined by the following formulas:

$$(x_1, x_2, \cdots, x_d) + (y_1, y_2, \cdots, y_d) = (x_1 + y_1, x_2 + y_2, \cdots, x_d + y_d).$$
$$\alpha(x_1, x_2, \cdots, x_d) = (\alpha x_1, \alpha x_2, \cdots, \alpha x_d).$$

Definition 1. *[22] The direct sum, or external direct sum, of a family* $(\mathbb{R}_i)_{i\in I}$ *is the following sub-module of* $\prod_{i\in I} \mathbb{R}_i$*:*

$$\mathbb{R}^{\oplus^I} = \bigoplus_{i\in I} \mathbb{R}_i = \{(x_i)_{i\in I} \in \prod_{i\in I} \mathbb{R}_i \mid x_i = 0 \quad \text{for almost all } i \in I\}.$$

Immediately, this defines a sub-module. If $I = \emptyset$, then $\bigoplus_{i\in I} \mathbb{R}_i = 0$.
If $I = \{1\}$, then $\bigoplus_{i\in I} \mathbb{R}_i = \mathbb{R}_1$. If $I = \{1, 2, \cdots, d\}$, then $\bigoplus_{i\in I} \mathbb{R}_i$ is also denoted by $\mathbb{R}_1 \oplus \mathbb{R}_2 \oplus \mathbb{R}_3 \oplus \cdots \oplus \mathbb{R}_d = \bigoplus \sum_{i=1}^{d} \mathbb{R}_i$, and coincides with $\mathbb{R}_1 \times \mathbb{R}_2 \times \cdots \times \mathbb{R}_d$.
The direct sum $\bigoplus_{i\in I} \mathbb{R}_i$ comes with an injection $\iota_j \colon \mathbb{R}_j \to \bigoplus_{i\in I} \mathbb{R}_i$ for every $j \in I$, defined by its components: for all $x_j \in \mathbb{R}_j$,

$$\iota_j(x)_j = x \in \mathbb{R}_j, \quad \iota_j(x)_i = 0 \in \mathbb{R}_i \quad \text{for all } i \neq j,$$

every ι_j is an injective homomorphism.
If $\mathbb{R}_1, \ldots, \mathbb{R}_d$ are normed spaces, their vector space sum can be equipped with a norm inspired by the norm of the Euclidean n space. Specifically, a norm on the vector space sum can be defined in a way that resembles the standard Euclidean norm.

Definition 2. *Let* $\mathbb{R}_1, \cdots, \mathbb{R}_d$ *be a normed space with respective norms* $|\cdot|_{\mathbb{R}_1}, \cdots, |\cdot|_{\mathbb{R}_d}$ *is the normed space whose underlying vector space is the vector space sum of* $\mathbb{R}_1, \cdots, \mathbb{R}_d$

and whose norm is the direct sum norm given by the formula

$$\|(x_1, \cdots, x_n)\| = \left(\sum_{j=1}^{n} \mid x_j \mid_{\mathbb{R}_j}^2 \right)^{\frac{1}{2}}.$$

This normed space is dented by $\mathbb{R}_1 \oplus \cdots \oplus \mathbb{R}_d$.

Corolary 1. *For any two spaces A and B, the direct sum can be defined as a homomorphism $f\colon A \oplus B \to M_s$, where the matrix M_s of size s, given by:*

$$\mathscr{F}(A, B) = A \oplus B = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix},$$

where A and B are embedded as diagonal blocks in the matrix form, and 0 represents appropriately sized zero matrices.

Theorem 1. *[23] Let $\mathbb{R}_1, \mathbb{R}_2, \cdots, \mathbb{R}_d$ be a family of sets of all real numbers. Then the following are equivalent:*

1. *$\mathbb{R}^{\oplus^d} = \bigoplus \sum_{i=1}^{d} \mathbb{R}_i$ is a direct sum.*
2. *If $0 = \sum_{i=1}^{d} x_i$, $x_i \in \mathbb{R}_i$, then $x_i = 0$, $i = 1, 2, \cdots, d$.*
3. *$\mathbb{R}_i \cap \bigoplus \sum_{j=1,\, j \neq i}^{d} \mathbb{R}_j = \tilde{0}$, $i = 1, 2, \cdots, d$.*

Theorem 2. *[23] Let $\mathbb{R}_1, \mathbb{R}_2, \cdots, \mathbb{R}_d$ be a family of sets of all real numbers, and let $\mathbb{R}_1 \times \cdots \times \mathbb{R}_d = \prod_{i=1}^{d} \mathbb{R}_i$ be their direct product. Let $\mathbb{R}_i^* = \{(, \cdots, 0, x_i, 0, \cdots, 0) | \; x_i \in \mathbb{R}_i\}$. Then $\mathbb{R}^{\oplus^d} = \bigoplus \sum_{i=1}^{d} \mathbb{R}_i^*$ is a direct sum of $\mathbb{R}_i^*$.*

We recall some reminders and auxiliary results that will be used throughout this paper. Let $\mathbb{R}^d$ denotes the orthogonal direct sum $\mathbb{R} \oplus \mathbb{R} \oplus \cdots \oplus \mathbb{R}$ of d-copies of $\mathbb{R}$. Let $\mathbb{R}^{\oplus^d}$ be the direct sum set, and $x, y \in \mathbb{R}^{\oplus^d}$, $x = (x_1, x_2, \cdots, x_d)$, $y = (y_1, y_2, \cdots, y_d)$ by $x \precsim y$ (respectively, $x_i < y_i$) for all $i = 1, 2, \cdots d$. In addition, $\mathbb{R}_+^{\oplus^d} = \{x \in \mathbb{R}^{\oplus^d} \colon x_i \geq 0, \;\; \forall i = 1, 2, \cdots, d\}$ is the set of positive elements in $\mathbb{R}^{\oplus^d}$, and we denote $x \succsim \tilde{0}$ if $x_i \geq 0$ for all $i = 1, 2, \cdots, d$, where $\tilde{0}$ is the $d \times d$ zero matrix in $\mathbb{R}^{\oplus^d}$.
By taking the product of two square diagonal matrices, we can define $x \cdot y$ as follows:

$$x \cdot y \;\; = \;\; \begin{pmatrix} x_1 y_1 & 0 & \cdots & 0 \\ 0 & x_2 y_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_d y_d \end{pmatrix} = \operatorname{diag}\begin{pmatrix} x_1 y_1, & x_2 y_2, & \cdots, & x_d y_d \end{pmatrix} \in \mathbb{R}^{\oplus^d},$$

and the direct sum unit over $\mathbb{R}^{\oplus^d}$ will be denoted by

$$I_+^{\mathbb{R}^{\oplus^d}} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} = \operatorname{diag}\begin{pmatrix} 1, & 1, & \cdots, & 1 \end{pmatrix}.$$

An element $x = (x_1, x_2, \cdots, x_d) \in \mathbb{R}^{\oplus^d}$ has an invers if $x_i \neq 0$, for all $i = 1, 2, \cdots, d$ and is denoted by

$$x^{-1} = \operatorname{diag}\left(x_1^{-1}, \quad x_2^{-1}, \quad \cdots \quad x_d^{-1}\right) = \operatorname{diag}\left(\tfrac{1}{x_1}, \quad \tfrac{1}{x_2}, \quad \cdots \quad \tfrac{1}{x_d}\right),$$

and x is called invertible if it has an inverse.

Definition 3. *Let $\mathcal{X}$ be a nonempty set, $s \geq 1$ be a constant. Let $\mathbb{R}^d$ denote the orthogonal direct sum $\mathbb{R} \oplus \mathbb{R} \oplus \cdots \oplus \mathbb{R}$ of d-copies of $\mathbb{R}$. An operator $\delta^b_{\mathbb{R}^{\oplus d}} : \mathcal{X} \times \mathcal{X} \to \mathbb{R}^{\oplus^d}$ is called a generalized b-metric endowed with the orthogonal direct sum if the following properties are satisfied:*

(B_1) $\delta^b_{\mathbb{R}^{\oplus d}}(x, y) \succsim \widetilde{0}$, *for all $x, y \in \mathcal{X}$ and $\delta^b_{\mathbb{R}^{\oplus d}}(x, y) = \widetilde{0}$ if and only if $x = y$;*

(B_2) $\delta^b_{\mathbb{R}^{\oplus d}}(x, y) = \delta^b_{\mathbb{R}^{\oplus d}}(y, x)$, *for all $x, y \in \mathcal{X}$;*

(B_3) $\delta^b_{\mathbb{R}^{\oplus d}}(x, z) \precsim s\left[\delta^b_{\mathbb{R}^{\oplus d}}(x, y) + \delta^b_{\mathbb{R}^{\oplus d}}(y, z)\right]$, *for all $x, y, z \in \mathcal{X}$.*

Then, we call $(\mathcal{X}, \mathbb{R}^{\oplus^d}, \delta^b_{\mathbb{R}^{\oplus d}})$ a generalized b-metric space on $\mathcal{X}$.

Example 1. *Let $(\mathcal{X}, \delta_{\mathbb{R}})$ be a metric space, and $\delta^b_{\mathbb{R}^{\oplus d}} : \mathcal{X} \times \mathcal{X} \to \mathbb{R}^{\oplus^d}$ be defined by*

$$\delta^b_{\mathbb{R}^{\oplus d}}(x, y) = \operatorname{diag}\left([d(x, y)]^p, \quad [d(x, y)]^p, \quad \cdots, \quad [d(x, y)]^p\right) \in \mathbb{R}^{\oplus^d},$$

where $p > 1$ is a fixed real number. Then $\delta^b_{\mathbb{R}^{\oplus d}}$ is the generalized b-metric space endowed with the orthogonal direct sum, where $s = 2^{p-1}$. Indeed, conditions (B_1) and (B_2) in Definition 3 are satisfied and thus we only need to show that condition (B_3) holds for $\delta^b_{\mathbb{R}^{\oplus d}}$. It is easy to see that if $1 < p < +\infty$, then the convexity of the function $\mathscr{F}(x) = x^p$, where $x \geq 0$, implies

$$\left(\frac{a + c}{2}\right)^p \leq \frac{1}{2}(a^p + c^p),$$

and hence

$$(a + c)^p \leq 2^{p-1}(a^p + c^p).$$

Therefore, for each $x, y, z \in X$, we get

$$\begin{aligned}
\delta^b_{\mathbb{R}^{\oplus d}}(x, y) &= \operatorname{diag}\left([d(x, y)]^p, \quad [d(x, y)]^p, \quad \cdots, \quad [d(x, y)]^p\right) \\
&\leq \operatorname{diag}\left([d(x, z) + d(z, y)]^p, \quad [d(x, z) + d(z, y)]^p, \quad \cdots, \quad [d(x, z) + d(z, y)]^p\right) \\
&\leq 2^{p-1} \operatorname{diag}\left(d(x, z)^p + d(z, y)^p, \quad d(x, z)^p + d(z, y)^p, \quad \cdots, \quad d(x, z)^p + d(z, y)^p\right) \\
&= 2^{p-1} \Bigg[\operatorname{diag}\left(d(x, z)^p, \quad d(x, z)^p, \quad \cdots, \quad d(x, z)^p\right)
\end{aligned}$$

$$+ \operatorname{diag}\left(d(z,y)^p, \quad d(z,y)^p, \quad \cdots, \quad d(z,y)^p\right)\Big]$$

$$= \quad 2^{p-1}\left[\delta^b_{\mathbb{R}^{\oplus^d}}(x,z) + \delta^b_{\mathbb{R}^{\oplus^d}}(z,y)\right].$$

So condition (B_3) *in Definition 3 holds and then* $\delta^b_{\mathbb{R}^{\oplus^d}}$ *is a b-metric coefficient* $s = 2^{p-1} > 1$.

Definition 4. *Suppose that* $(\mathcal{X}, \mathbb{R}^{\oplus^d}, \delta^b_{\mathbb{R}^{\oplus^d}})$ *is a generalized b-metric space endowed with the orthogonal direct sum. Then a sequence* $\{x_n\}$ *in* $\mathcal{X}$ *is called:*

1. *b-convergent sequence with respect to* $\mathbb{R}^{\oplus^d}$. *If, for every* $\widetilde{\epsilon} \in \mathbb{R}^{\oplus^d}$, *with* $\widetilde{\epsilon} \succ \widetilde{0}$, *if there exists* $x \in \mathcal{X}$ *and there is* $N \in \mathbb{N}$ *such that for all* $n > N$, $\delta^b_{\mathbb{R}^{\oplus^d}}(x_n, x) \prec \widetilde{\epsilon}$.
2. *b-Cauchy sequence with respect to* $\mathbb{R}^{\oplus^d}$. *If, for any* $\widetilde{\epsilon} \succ \widetilde{0}$ *there exists* $N \in \mathbb{N}$ *such that for all* $n, m > N$, $\delta^b_{\mathbb{R}^{\oplus^d}}(x_n, x_m) \prec \widetilde{\epsilon}$.

Example 2. *Assume that* $\mathcal{X} = l_p(\mathbb{R})$ *with* $p \in (0,1)$, *where*

$$l_p(\mathbb{R}) := \left\{\{x_n\} \subseteq \mathbb{R} \mid \sum_{n=1}^{+\infty} \mid x_n \mid^p < +\infty\right\},$$

together with the operator $\delta^b_{\mathbb{R}^{\oplus^d}} \colon l_p(\mathbb{R}) \times l_p(\mathbb{R}) \to \mathbb{R}_+^{\oplus^d}$ *defined by*

$$\delta^b_{\mathbb{R}^{\oplus^d}}(x,y) = \operatorname{diag}\left(\left(\sum_{n=1}^{+\infty} \mid x_n - y_n \mid^p\right)^{\frac{1}{p}}, \quad \cdots, \quad \left(\sum_{n=1}^{+\infty} \mid x_n - y_n \mid^p\right)^{\frac{1}{p}}\right) \in \mathbb{R}^{\oplus^d},$$

where $x = \{x_n\}, \quad y = \{y_n\} \in l_p(\mathbb{R})$, *is a generalized b-metric space endowed with the orthogonal direct sum, where coefficient* $s = 2^{\frac{1}{p}} > 1$. *Then* $(l_p, \mathbb{R}^d, d)$ *is a complete generalized b-metric space endowed with the orthogonal direct sum.*

Example 3. *Assume that* $\mathcal{X} = L^p(\mathbb{R})$ *with* $p \in (0,1)$, *where*

$$L^p(\mathbb{R}) := \left\{x(t) \subseteq C([0,1],\mathbb{R}) \middle| \int_0^1 \mid x(t) \mid^p < +\infty\right\},$$

together with the operator $d \colon L^p(\mathbb{R}) \times L^p(\mathbb{R}) \to \mathbb{R}_+^{\oplus^d}$ *defined by*

$$\delta^b_{\mathbb{R}^{\oplus^d}}(x,y) = \operatorname{diag}\left(\left(\int_0^1 \mid x(t) - y(t) \mid^p \; dt\right)^{\frac{1}{p}}, \quad \cdots, \quad \left(\int_0^1 \mid x(t) - y(t) \mid^p \; dt\right)^{\frac{1}{p}}\right) \in \mathbb{R}^{\oplus^d},$$

where $x = \{x_n\}, \;\; y = \{y_n\} \in l_p(\mathbb{R})$, *is a b-metric space with coefficient* $s = 2^{\frac{1}{p}} > 1$. *Then* $(l_p, \mathbb{R}^{\oplus^d}, \delta^b_{\mathbb{R}^{\oplus^d}})$ *is a complete generalized b-metric endowed with the orthogonal direct sum space.*

Example 4. *Let* $\mathcal{X} = \{0, 1, 2\}$ *and the operator* $\delta^b \colon \mathcal{X} \times \mathcal{X} \to \mathbb{R}_+$ *defined by*

$$\delta^b(0,0) = \delta^b(1,1) = \delta^b(2,2) = 0,$$
$$\delta^b(0,1) = \delta^b(1,0) = \delta^b(1,2) = \delta^b(2,1) = 1,$$

and

$$\delta^b(2,0) = \delta^b(0,2) = m,$$

where m *is given real number such that* $m \geq 2$.

$$\delta^b_{\mathbb{R}^{\oplus d}}(x,y) = \operatorname{diag}\left(\delta^b(x,y), \quad \delta^b(x,y), \quad \cdots, \quad \delta^b(x,y)\right) \in \mathbb{R}_+^{\oplus^d}.$$

It is easy to see that

$$\delta^b_{\mathbb{R}^{\oplus d}}(x,y) \leq \frac{m}{2}\left[\delta^b_{\mathbb{R}^{\oplus d}}(x,z) + \delta^b_{\mathbb{R}^{\oplus d}}(z,y)\right],$$

for all $x, y, z \in \mathcal{X}$*. Therefore,* $(\mathcal{X}, \mathbb{R}_+^{\oplus^d}, \delta^b_{\mathbb{R}^{\oplus d}})$ *is a generalized b-metric space with coefficient* $s = \frac{m}{2}$*. We obtain that the ordinary triangle inequality does not hold if* $m > 2$*, and then* $(\mathcal{X}, d)$ *is not a metric space.*

Corolary 2. *Let* $(\mathcal{X}, \mathbb{R}_+^{\oplus^d}, \delta^b_{\mathbb{R}^{\oplus d}})$ *be a generalized b-metric space. Then, the following assertions hold:*

1. *A b-convergent sequence with respect to* $\mathbb{R}^{\oplus^d}$ *has a unique limit.*

2. *Each b-convergent sequence with respect to* $\mathbb{R}^{\oplus^d}$ *is a b-Cauchy sequence.*

Definition 5. *Suppose that* $(\mathcal{X}, \mathbb{R}^{\oplus^d}, \delta^b_{\mathbb{R}^{\oplus d}})$ *and* $(\mathcal{Y}, \mathbb{R}^{\oplus^d}, \delta^b_{\mathbb{R}^{\oplus d}})$ *are two generalized b-metric spaces endowed with the orthogonal direct sum.*

1. *The space* $(\mathcal{X}, \mathbb{R}^{\oplus^d}, \delta^b_{\mathbb{R}^{\oplus d}})$ *is a complete if every Cauchy sequence with respect to* $\mathbb{R}^d$ *is converges.*

2. *A function* $\mathscr{F} : \mathcal{X} \to \mathcal{Y}$ *is said to be continuous at a point* $x \in \mathcal{X}$ *if, for every sequence* $\{x_n\} \subseteq \mathcal{X}$ *that converges to* x*, the sequence* $\mathscr{F}(x_n)$ *converges to* $\mathscr{F}(x)$*.*

3. Main Results

In this section, we will present some Banach fixed-point theorems combining the direct sum.

Definition 6. *Suppose that* $(\mathcal{X}, \mathbb{R}^{\oplus^d}, \delta^b_{\mathbb{R}^{\oplus d}})$ *is a complete generalized b-metric space endowed with the orthogonal direct sum. An operator* $\mathscr{F} \colon \mathcal{X} \to \mathcal{X}$ *is said to be a contraction and endowed with the orthogonal direct sum if there exists a diagonal matrix* $\Lambda = \operatorname{diag}(\lambda_1, \lambda_2, \cdots, \lambda_d) \in \mathbb{R}_+^d$ *with* $\tilde{0} \precsim \lambda_i \prec \frac{1}{s}$ *such that*

$$\delta^b_{\mathbb{R}^{\oplus d}}(\mathscr{F}(x),\ \mathscr{F}(y)) \precsim \Lambda\ \delta^b_{\mathbb{R}^{\oplus d}}(x,y),$$

for all $x,\ y \in \mathcal{X}$

Example 5. *Let $\mathcal{X} = \mathbb{R}$ and $\delta^b_{\mathbb{R}^{\oplus d}} : \mathcal{X} \times \mathcal{X} \to \mathbb{R}^{\oplus^d}$ be a generalized b-metric space endowed with the orthogonal direct sum defined by*

$$\delta^b_{\mathbb{R}^{\oplus d}}(x, y) = \operatorname{diag}\left(| x - y |^2, \quad | x - y |^2, \quad \cdots, \quad | x - y |^2\right) \in \mathbb{R}^{\oplus^d},$$

and let $\Lambda = \operatorname{diag}\left(\frac{1}{4}, \quad \frac{1}{4}, \quad \cdots, \quad \frac{1}{4}\right)$ and $\mathscr{F} : \mathcal{X} \to \mathcal{X}$ such that $\mathscr{F}(x) = \frac{x}{2}$. Then,

$$\delta^b_{\mathbb{R}^{\oplus d}}(\mathscr{F}(x), \mathscr{F}(y)) = \operatorname{diag}\left(\frac{1}{4} \cdot | x - y |^2, \quad \frac{1}{4} \cdot | x - y |^2, \quad \cdots, \quad \frac{1}{4} \cdot | x - y |^2\right) \in \mathbb{R}^{\oplus^d}.$$

it is clear that

$$\delta^b_{\mathbb{R}^{\oplus d}}(\mathscr{F}(x), \mathscr{F}(y)) \precsim \ \Lambda \cdot \delta^b_{\mathbb{R}^{\oplus d}}(x, y),$$

where $\Lambda \prec I_+^{\mathbb{R}^{\oplus^d}}$.

Therefore, $\mathscr{F}$ is a contraction and endowed with the orthogonal direct sum on a generalized b-metric space endowed with direct sum $(\mathcal{X}, \mathbb{R}^{\oplus^d}, \delta^b_{\mathbb{R}^{\oplus d}})$.

Example 6. *(Numerical example)*
Consider all hypotheses of Example 4 are true. Let $\mathscr{F} : \mathcal{X} \to \mathcal{X}$ such that $\mathscr{F}(x) = \frac{x}{4}$. Then,

$$\begin{aligned}
\delta^b_{\mathbb{R}^{\oplus d}}(\mathscr{F}(x), \mathscr{F}(y)) &= \frac{1}{4} \cdot I_+^{\mathbb{R}^{\oplus^d}} \operatorname{diag}\left(\delta^b(x, y), \quad \delta^b(x, y), \quad \cdots, \quad \delta^b(x, y)\right) \\
&= \operatorname{diag}\left(\frac{1}{4}, \quad \frac{1}{4}, \quad \cdots, \quad \frac{1}{4}\right) \cdot \operatorname{diag}\left(\delta^b(x, y), \quad \delta^b(x, y), \quad \cdots, \quad \delta^b(x, y)\right).
\end{aligned}$$

It is clear that

$$\delta^b_{\mathbb{R}^{\oplus d}}(\mathscr{F}(x), \mathscr{F}(y)) \precsim \ \Lambda \cdot \delta^b_{\mathbb{R}^{\oplus d}}(x, y),$$

where $\Lambda \prec I_+^{\mathbb{R}^{\oplus^d}}$.
Therefore, $\mathscr{F}$ is a contraction endowed the direct sum for the a generalized b-metric space endowed with the orthogonal direct sum $(\mathcal{X}, \mathbb{R}^{\oplus^d}, \quad \delta^b_{\mathbb{R}^{\oplus d}})$.

Theorem 3 (Banach contraction Type)**.** *Suppose that $(\mathcal{X}, \mathbb{R}^{\oplus^d}, \delta^b_{\mathbb{R}^{\oplus d}})$ is a complete generalized b-metric space endowed with the orthogonal direct sum, and let $\mathscr{F} : \mathcal{X} \to \mathcal{X}$ be a contraction operator endowed with the direct sum. That is,*

$$\delta^b_{\mathbb{R}^{\oplus d}}(\mathscr{F}(x), \mathscr{F}(y)) \precsim \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_d)\, \delta^b_{\mathbb{R}^{\oplus d}}(x, y),$$

for all $x, y \in \mathcal{X}$, where $\lambda_i \in (0, 1)$ for $i = 1, 2, \ldots, d$. Then $\mathscr{F}$ has a unique fixed-point in $\mathcal{X}$.

Proof. Let x_0 be any point in $\mathcal{X}$, that is $x_0 \in \mathcal{X}$. Let us define a sequence $\{x_n\}$ in $\mathcal{X}$ as given below.

$$x_{n+1} = \mathscr{F}(x_n) = \mathscr{F}^{n+1}(x_0) \ \forall \ n \geq 0.$$

By the contraction condition, we get

$$\begin{aligned}
\delta^b_{\mathbb{R}^{\oplus d}}(x_{n-1}, x_n) &= \delta^b_{\mathbb{R}^{\oplus d}}(\mathscr{F}(x_{n-2}), \mathscr{F}(x_{n-1})) \\
&\precsim \operatorname{diag}\begin{pmatrix}\lambda_1, & \lambda_2, & \cdots, & \lambda_d\end{pmatrix} [\delta^b_{\mathbb{R}^{\oplus d}}(x_{n-2}, x_{n-1})] \\
&\vdots \\
&\precsim \operatorname{diag}\begin{pmatrix}\lambda_1^{n-1}, & \lambda_2^{n-1}, & \cdots, \lambda_d^{n-1}\end{pmatrix} \delta^b_{\mathbb{R}^{\oplus d}}(x_0, x_1).
\end{aligned}$$

Let us prove that $\{x_n\}$ is a Cauchy sequence.
Suppose that $n > m$ from the contraction and triangle inequality property, we can write as:

$$\begin{aligned}
& \delta^b_{\mathbb{R}^{\oplus d}}(x_m, x_n) \\
\precsim \ & s\delta^b_{\mathbb{R}^{\oplus d}}(x_m, x_{m+1}) + s^2 \ \delta^b_{\mathbb{R}^{\oplus d}}(x_{m+1}, x_{m+2}) + s^3 \left[\delta^b_{\mathbb{R}^{\oplus d}}(x_{m+2}, x_{m+3}) + \delta^b_{\mathbb{R}^{\oplus d}}(x_{m+3}, x_n)\right] \\
\vdots \\
\precsim \ & s \ \delta^b_{\mathbb{R}^{\oplus d}}(x_m, x_{m+1}) + s^2 \ \delta^b_{\mathbb{R}^{\oplus d}}(x_{m+1}, x_{m+2}) + s^3 \ \delta^b_{\mathbb{R}^{\oplus d}}(x_{m+2}, x_{m+3}) \\
& + \cdots + s^{n-m} \ \delta^b_{\mathbb{R}^{\oplus d}}(x_{n-1}, x_n) \\
\precsim \ & s \ \operatorname{diag}(\lambda_1^m, \lambda_2^m, \cdots, \lambda_d^m)\delta^b_{\mathbb{R}^{\oplus d}}(x_0, x_1) + s^2 \ \operatorname{diag}(\lambda_1^{m+1}, \lambda_2^{m+1}, \cdots, \lambda_d^{m+1})\delta^b_{\mathbb{R}^{\oplus d}}(x_0, x_1) \\
& \quad + \cdots + s^{n-m} \ \operatorname{diag}(\lambda_1^{n-1}, \lambda_2^{n-1}, \cdots, \lambda_d^{n-1})\delta^b_{\mathbb{R}^{\oplus d}}(x_0, x_1) \\
= \ & \Big((s \ \lambda_1^m + s^2 \ \lambda_1^{m+1} + \cdots + s^{n-m} \ \lambda_1^{n-1})\delta^b_{\mathbb{R}^{\oplus d}}(x_0, x_1), (s \ \lambda_2^m + s^2 \ \lambda_2^{m+1} + \cdots \\
& +s^{n-m} \ \lambda_2^{n-1})\delta^b_{\mathbb{R}^{\oplus d}}(x_0, x_1), \cdots, (s\lambda_d^m + s^2 \ \lambda_d^{m+1} + \cdots + s^{n-m} \ \lambda_d^{n-1})\delta^b_{\mathbb{R}^{\oplus d}}(x_0, x_1)\Big) \\
= \ & \operatorname{diag}\left(\sum_{k=1}^{n-m}\sum_{j=m}^{n-1} s^k \lambda_1^j \delta^b_{\mathbb{R}^{\oplus d}}(x_0, x_1), \sum_{k=1}^{n-m}\sum_{j=m}^{n-1} s^k \lambda_2^j \delta^b_{\mathbb{R}^{\oplus d}}(x_0, x_1), \cdots, \right. \\
& \left. \sum_{k=1}^{n-m}\sum_{j=m}^{n-1} s^k \ \lambda_d^j \delta^b_{\mathbb{R}^{\oplus d}}(x_0, x_1)\right) \\
= \ & \operatorname{diag}\left(\sum_{k=1}^{n-m}\sum_{j=m}^{n-1} s^k \lambda_1^j, \sum_{k=1}^{n-m}\sum_{j=m}^{n-1} s^k \lambda_2^j, \cdots, \sum_{k=1}^{n-m}\sum_{j=m}^{n-1} s^k \lambda_d^j\right) \delta^b_{\mathbb{R}^{\oplus d}}(x_0, x_1)
\end{aligned}$$

$$\begin{aligned}
&= \operatorname{diag}\left(s\lambda_1^m \sum_{j=0}^{n-m-1} (s\lambda_1)^j, s\lambda_2^m \sum_{j=0}^{n-m-1} (s\lambda_2)^j, \cdots, s\lambda_d^m \sum_{j=0}^{n-m-1} (s\lambda_d)^j\right)\delta^b_{\mathbb{R}^{\oplus d}}(x_0, x_1)\\
&\precsim \operatorname{diag}\left(s\lambda_1^m \sum_{j=0}^{+\infty} (s\lambda_1)^j + s\lambda_2^m \sum_{j=0}^{+\infty} (s\lambda_2)^j + \cdots + s\lambda_d^m \sum_{j=0}^{+\infty} (s\lambda_d)^j\right)\delta^b_{\mathbb{R}^{\oplus d}}(x_0, x_1)\\
&= \operatorname{diag}\left(s\lambda_1^m \left[\frac{1}{1-s\lambda_1}\right], s\lambda_2^m \left[\frac{1}{1-s\lambda_2}\right], \cdots, s\lambda_d^m \left[\frac{1}{1-s\lambda_d}\right]\right)\delta^b_{\mathbb{R}^{\oplus d}}(x_0, x_1)\\
&\precsim \operatorname{diag}(s\lambda_1^m, \cdots, s\lambda_d^m)\operatorname{diag}\left(\left[\frac{1}{1-s\lambda_1}\right], \cdots, \left[\frac{1}{1-s\lambda_d}\right]\right)\delta^b_{\mathbb{R}^{\oplus d}}(x_0, x_1) \to 0,
\end{aligned}$$

as $n, m \to +\infty$. Since $\tilde{0} \prec \operatorname{diag}(s\lambda_1^j, s\lambda_2^j, \cdots, s\lambda_d^j) \prec I_+^{\mathbb{R}^{\oplus d}}$ and $\delta^b_{\mathbb{R}^{\oplus d}}(x_0, x_1)$ are fixed, it is evident that by selecting m sufficiently large (with $n > m$), we can make $\delta^b_{\mathbb{R}^{\oplus d}}(x_n, x_m)$ arbitrarily small. This shows that $\{x_n\}$ is a Cauchy sequence. Finally, because $(\mathcal{X}, \mathbb{R}^{\oplus^d}, \delta^b_{\mathbb{R}^{\oplus d}})$ is complete, there exists some $z \in \mathcal{X}$ such that $x_n \to z$.
To show that $\mathcal{X}$ has a fixed point, we consider the distance $\delta^b_{\mathbb{R}^{\oplus d}}(z, \mathscr{F}(z))$. From the triangle inequality and contraction condition, we get

$$\begin{aligned}
\delta^b_{\mathbb{R}^{\oplus d}}(z, \mathscr{F}(z)) &\precsim s\left[\delta^b_{\mathbb{R}^{\oplus d}}(z, x_n) + \delta^b_{\mathbb{R}^{\oplus d}}(x_n, \mathscr{F}(z))\right]\\
&= s\left[\delta^b_{\mathbb{R}^{\oplus d}}(z, x_n) + \delta^b_{\mathbb{R}^{\oplus d}}(\mathscr{F}(x_{n-1}), \mathscr{F}(z))\right]\\
&\precsim s\left[\delta^b_{\mathbb{R}^{\oplus d}}(z, x_n) + \operatorname{diag}\left(\lambda_1, \quad \lambda_2, \quad \cdots, \quad \lambda_d\right)\delta^b_{\mathbb{R}^{\oplus d}}(x_{n-1}, z)\right],
\end{aligned}$$

and since $x_n \to z$, it is clear that we can make this distance as small as we please by choosing n sufficiently large. We conclude that

$$\delta^b_{\mathbb{R}^{\oplus d}}(z, \mathscr{F}(z)) = 0 \implies \mathscr{F}(z) = z,$$

so $x \in \mathcal{X}$ is a fixed-point of $\mathscr{F}$.
To prove uniqueness, suppose there are two fixed points $x = \mathscr{F}(x)$ and $y = \mathscr{F}(x)$. Then, from the contraction condition, we have

$$\delta^b_{\mathbb{R}^{\oplus d}}(x, y) = \delta^b_{\mathbb{R}^{\oplus d}}(\mathscr{F}(x), \mathscr{F}(y)) \precsim \operatorname{diag}\left(\lambda_1, \quad \lambda_2, \quad \cdots, \quad \lambda_d\right)\ \delta^b_{\mathbb{R}^{\oplus d}}(x, y)$$

this implies $\delta^b_{\mathbb{R}^{\oplus d}}(x, y) = 0$ since $\tilde{0} \prec \Lambda \prec I_+^{\mathbb{R}^{\oplus d}}$. Hence $x = y$, and the fixed-point $\mathcal{X}$ of $\mathscr{F}$ is unique.

Example 7 (A Numerical Iterative Example)**.** *Let $\mathcal{X} = \mathbb{R}$ with the generalized metric $\delta^b_{\mathbb{R}^{\oplus d}}$ defined as:*

$$\delta^b_{\mathbb{R}^{\oplus d}}(x, y) = diag(|x-y|^2, |x-y|^2, \ldots, |x-y|^2).$$

Consider the function:

$$\mathscr{F}(x) = \frac{x}{3}.$$

We choose the diagonal contraction matrix:

$$\Lambda = diag\left(\frac{1}{9}, \frac{1}{9}, \ldots, \frac{1}{9}\right),$$

which satisfies:

$$\delta^b_{\mathbb{R}^{\oplus d}}(\mathscr{F}(x), \mathscr{F}(y)) = \frac{1}{9} \cdot \delta^b_{\mathbb{R}^{\oplus d}}(x, y).$$

Since $0 < \frac{1}{9} < 1$*, the operator* $\mathscr{F}$ *satisfies the contraction condition required by Theorem 3, we can find the fixed-point by using the iterative sequence:*

$$x_{n+1} = \mathscr{F}(x_n),$$

starting from an arbitrary initial point $x_0 = 6$*, we get:*

$$\begin{aligned} x_1 &= \frac{6}{3} = 2, \\ x_2 &= \frac{2}{3} \approx 0.67, \\ x_3 &= \frac{0.67}{3} \approx 0.22, \\ &\vdots \\ x_n &= \frac{6}{3^n}, \\ &\vdots \end{aligned}$$

Taking the limit as $n \to +\infty$*:*

$$\lim_{n \to +\infty} x_n = \lim_{n \to +\infty} \frac{6}{3^n} = 0.$$

Thus, the iterative process confirms that the fixed-point of $\mathscr{F}$ *is indeed* $z = 0$*.*

Example 8 (A Fredholm Integral Equation)**.** *Let* $\mathcal{X} = C([0,1], \mathbb{R})$*, the space of continuous functions on* $[0,1]$*, equipped with the norm:*

$$\|f\| = \sup_{x \in [0,1]} |f(x)|.$$

Define the generalized metric:

$$\delta^b_{\mathbb{R}^{\oplus d}}(f, g) = diag(\|f - g\|^2, \|f - g\|^2, \ldots, \|f - g\|^2).$$

Consider the Fredholm integral equation:

$$f(x) = \int_0^1 k(x,t)f(t)dt,$$

where $k(x,t)$ is a continuous kernel.

Define the operator $\mathfrak{F}\colon C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ by:

$$(\mathfrak{F}f)(x) = \int_0^1 k(x,t)f(t)dt.$$

Suppose there exists $\lambda \in (0, \frac{1}{s})$ such that:

$$\sup_{x\in[0,1]} \int_0^1 |k(x,t)|dt \leq \lambda.$$

For any two functions $f, g \in C([0,1],\mathbb{R})$:

$$\|\mathfrak{F}f - \mathfrak{F}g\|^2 = \sup_{x\in[0,1]} \left| \int_0^1 k(x,t)(f(t)-g(t))dt \right|^2 \leq \lambda \sup_{t\in[0,1]} |f(t)-g(t)|^2 = \lambda \|f-g\|^2.$$

Thus, $\delta^b_{\mathbb{R}^{\oplus d}}(f,g)$ is a contraction with $\Lambda = diag(\lambda, \lambda, \ldots, \lambda)$, where $0 < \lambda < \frac{1}{s}$ and

$$\delta^b_{\mathbb{R}^{\oplus d}}(\mathfrak{F}f, \mathfrak{F}g) \precsim \Lambda \delta^b_{\mathbb{R}^{\oplus d}}(f,g).$$

By Theorem 3, F has a unique fixed point, meaning that the integral equation has a unique solution.

If $d = 1$ and $s = 1$, then the theorem 3 can be reduced to the standard Banach contraction principle in a metric space.

Corolary 3 (Banach contraction principle, [5])**.** *Suppose that $(\mathcal{X}, d)$ is a complete metric space and $\mathscr{F} : \mathcal{X} \to \mathcal{X}$ is a contraction operator with Lipschitz constant $\lambda < 1$. Then $\mathscr{F}$ has a unique fixed-point $z \in \mathcal{X}$.*

Theorem 4. ***(Kannan Type)*** *Suppose that $(\mathcal{X}, \mathbb{R}^{\oplus^d}, \delta^b_{\mathbb{R}^{\oplus d}})$ is a complete generalized b-metric space endowed with the orthogonal direct sum and $\mathscr{F} \colon \mathcal{X} \to \mathcal{X}$ is an operator satisfying the following condition*

$$\delta^b_{\mathbb{R}^{\oplus d}}(\mathscr{F}(x), \mathscr{F}(y)) \precsim \operatorname{diag}\begin{pmatrix}\alpha_1, & \alpha_2, & \cdots, & \alpha_d\end{pmatrix} [\delta^b_{\mathbb{R}^{\oplus d}}(x, \mathscr{F}(x)) + \delta^b_{\mathbb{R}^{\oplus d}}(y, \mathscr{F}(y))], \tag{1}$$

where $\alpha_i \in (0, \frac{1}{2})$ for all $i = 1, 2, \cdots, d$. Then, $\mathscr{F}$ has a unique fixed-point in $\mathcal{X}$.

Proof. Let x_0 be any point in $\mathcal{X}$, that is $x_0 \in \mathcal{X}$. Let us define a sequence $\{x_n\}$ in $\mathcal{X}$ as given below.

$$x_{n+1} = \mathscr{F}(x_n) = \mathscr{F}^{n+1}(x_0) \;\; \forall \;\; n \geq 0.$$

We have

$$\begin{aligned}
\delta^b_{\mathbb{R}^{\oplus d}}(x_n, x_{n+1}) &= \delta^b_{\mathbb{R}^{\oplus d}}(\mathscr{F}(x_{n-1}), \mathscr{F}(x_n)) \\
&\precsim \operatorname{diag}\left(\alpha_1, \quad \alpha_2, \quad \cdots, \quad \alpha_d\right) \left[\delta^b_{\mathbb{R}^{\oplus d}}(x_{n-1}, \mathscr{F}(x_{n-1})) + \delta^b_{\mathbb{R}^{\oplus d}}(x_n, \mathscr{F}(x_n))\right] \\
&= \operatorname{diag}\left(\alpha_1, \quad \alpha_2, \quad \cdots, \quad \alpha_d\right) \delta^b_{\mathbb{R}^{\oplus d}}(x_{n-1}, x_n) \\
&\quad + \operatorname{diag}\left(\alpha_1, \quad \alpha_2, \quad \cdots, \quad \alpha_d\right) \delta^b_{\mathbb{R}^{\oplus d}}(x_n, x_{n+1}),
\end{aligned}$$

and

$$\begin{aligned}
&\operatorname{diag}\left((1-\alpha_1), \quad (1-\alpha_2), \quad \cdots, \quad (1-\alpha_d)\right) \delta^b_{\mathbb{R}^{\oplus d}}(x_n, x_{n+1}) \\
\precsim\; &\operatorname{diag}\left(\alpha_1, \quad \alpha_2, \quad \cdots, \quad \cdots \quad \alpha_d\right) \delta^b_{\mathbb{R}^{\oplus d}}(x_{n-1}, x_n),
\end{aligned}$$

since the inverse of the diagonal matrix is $\operatorname{diag}\left((1-\alpha_1)^{-1}, \quad (1-\alpha_2)^{-1}, \quad \cdots, \quad (1-\alpha_d)^{-1}\right)$. Thus,

$$\begin{aligned}
\delta^b_{\mathbb{R}^{\oplus d}}(x_n, x_{n+1}) &\precsim \operatorname{diag}\left(\frac{\alpha_1}{(1-\alpha_1)}, \cdots\cdots, \frac{\alpha_d}{(1-\alpha_d)}\right) \delta^b_{\mathbb{R}^{\oplus d}}(x_{n-1}, x_n) \\
&\precsim \operatorname{diag}\left(\left(\frac{\alpha_1}{(1-\alpha_1)}\right)^2, \cdots\cdots, \left(\frac{\alpha_d}{(1-\alpha_d)}\right)^2\right) \delta^b_{\mathbb{R}^{\oplus d}}(x_{n-2}, x_{n-1}) \\
&\precsim \operatorname{diag}\left(\left(\frac{\alpha_1}{(1-\alpha_1)}\right)^3, \cdots, \left(\frac{\alpha_d}{(1-\alpha_d)}\right)^3\right) \delta^b_{\mathbb{R}^{\oplus d}}(x_{n-3}, x_{n-2}) \\
&\vdots \\
&\precsim \operatorname{diag}\left(\left(\frac{\alpha_1}{(1-\alpha_1)}\right)^n, \cdots\cdots, \left(\frac{\alpha_d}{(1-\alpha_d)}\right)^n\right) \delta^b_{\mathbb{R}^{\oplus d}}(x_0, x_1),
\end{aligned}$$

If we let $\beta_i = \dfrac{\alpha_i}{1-\alpha_i}$, then

$$\delta^b_{\mathbb{R}^{\oplus d}}(x_n, x_{n+1}) \precsim \operatorname{diag}\left(\beta_1^n, \quad \beta_2^n, \quad \cdots, \quad \beta_d^n\right) \delta^b_{\mathbb{R}^{\oplus d}}(x_0, x_1).$$

Let us prove that $\{x_n\}$ is a Cauchy sequence.

Suppose that $n > m$, then from (1) and the triangle inequality property, we can write as:

$$\begin{aligned}
\delta^b_{\mathbb{R}^{\oplus d}}(x_m, x_n) &\precsim s\left[\delta^b_{\mathbb{R}^{\oplus d}}(x_m, x_{m+1}) + \delta^b_{\mathbb{R}^{\oplus d}}(x_{m+1}, x_n)\right] \\
&\precsim s\delta^b_{\mathbb{R}^{\oplus d}}(x_m, x_{m+1}) + s^2 \left[\delta^b_{\mathbb{R}^{\oplus d}}(x_{m+1}, x_{m+2}) + \delta^b_{\mathbb{R}^{\oplus d}}(x_{m+2}, x_n)\right]
\end{aligned}$$

$$\begin{aligned}
&\precsim s\ \delta^b_{\mathbb{R}^{\oplus d}}(x_m, x_{m+1}) + s^2\ \delta^b_{\mathbb{R}^{\oplus d}}(x_{m+1}, x_{m+2}) \\
&\quad + s^3\ \left[\delta^b_{\mathbb{R}^{\oplus d}}(x_{m+2}, x_{m+3}) + \delta^b_{\mathbb{R}^{\oplus d}}(x_{m+3}, x_n)\right] \\
&\vdots \\
&\precsim s\ \delta^b_{\mathbb{R}^{\oplus d}}(x_m, x_{m+1}) + s^2\ \delta^b_{\mathbb{R}^{\oplus d}}(x_{m+1}, x_{m+2}) \\
&\quad + s^3\ \delta^b_{\mathbb{R}^{\oplus d}}(x_{m+2}, x_{m+3}) + \cdots + s^{n-m}\delta^b_{\mathbb{R}^{\oplus d}}(x_{n-1}, x_n)
\end{aligned}$$

$$\begin{aligned}
&= \operatorname{diag}(s\beta_1^m, \cdots,\ s\beta_d^m)\, \delta^b_{\mathbb{R}^{\oplus d}}(x_0, x_1) + \operatorname{diag}(s^2\beta_1^{m+1}, \cdots, s^2\beta_d^{m+1})\delta^b_{\mathbb{R}^{\oplus d}}(x_0, x_1) \\
&\quad + \cdots + \operatorname{diag}\left(s^{n-m}\beta_1^{n-1}, \cdots,\ s^{n-m}\beta_d^{n-1}\right) \delta^b_{\mathbb{R}^{\oplus d}}(x_0, x_1) \\
&= \operatorname{diag}\Bigg(\left[s\beta_1^m + s^2\beta_1^{m+1} + \cdots + s^{n-m}\beta_1^{n-1}\right],\ \cdots, \\
&\qquad \left[s\beta_d^m + s^2\beta_d^{m+1} + \cdots + s^{n-m}\beta_d^{n-1}\right] \Bigg)\delta^b_{\mathbb{R}^{\oplus d}}(x_0, x_1) \\
&= \operatorname{diag}\left(\sum_{i=1}^{n-m}\sum_{j=m}^{n-1} s^i\beta_1^j\ , \cdots, \sum_{i=1}^{n-m}\sum_{j=m}^{n-1} s^i\beta_d^j \right) \delta^b_{\mathbb{R}^{\oplus d}}(x_0, x_1) \\
&= \operatorname{diag}\left(s\beta_1^m \sum_{j=0}^{n-m-1} (s\beta_1)^j\ , \cdots,\ s\beta_d^m \sum_{j=0}^{n-m-1} (s\beta_d)^j \right)\delta^b_{\mathbb{R}^{\oplus d}}(x_0, x_1) \\
&\precsim \operatorname{diag}\left(s\beta_1^m \sum_{j=0}^{+\infty} (s\beta_1)^j\ ,\ \cdots,\ s\beta_d^m \sum_{j=0}^{+\infty} (s\beta_d)^j \right)\delta^b_{\mathbb{R}^{\oplus d}}(x_0, x_1) \\
&= \operatorname{diag}\left(s\beta_1^m,\ \cdots, s\beta_d^m \right) \operatorname{diag}\left(\sum_{j=0}^{+\infty} (s\beta_1)^j, \cdots \sum_{j=0}^{+\infty} (s\beta_d)^j \right)\delta^b_{\mathbb{R}^{\oplus d}}(x_0, x_1) \\
&= \operatorname{diag}\left(s\beta_1^m,\ \cdots, s\beta_d^m \right) \operatorname{diag}\left(\left[\frac{1}{1 - s\beta_1}\right], \cdots, \left[\frac{1}{1 - s\beta_d}\right] \right)\delta^b_{\mathbb{R}^{\oplus d}}(x_0, x_1).
\end{aligned}$$

Since $s\beta_i \in (0,1)$ for all $i = 1, 2, \cdots, d$ and $\delta^b_{\mathbb{R}^{\oplus d}}(x_0, x_1)$ are fixed, it is evident that by selecting m sufficiently large (with $n > m$), we can make $\delta^b_{\mathbb{R}^{\oplus d}}(x_n, x_m)$ arbitrarily small. This shows that $\{x_n\}$ is a Cauchy sequence. Finally, because $(\mathcal{X}, \mathbb{R}^{\oplus^d}, \delta^b_{\mathbb{R}^{\oplus d}})$ is complete, there exists some $z \in \mathcal{X}$ such that $x_n \to z$.
To show that $\mathcal{X}$ has a fixed point, we consider the distance $\delta^b_{\mathbb{R}^{\oplus d}}(z, \mathscr{F}(z))$. From the triangle inequality and contraction condition, we get

$$\delta^b_{\mathbb{R}^{\oplus d}}(z, \mathscr{F}(z)) \precsim s\left[\delta^b_{\mathbb{R}^{\oplus d}}(z, x_n) + \delta^b_{\mathbb{R}^{\oplus d}}(x_n, \mathscr{F}(z))\right]$$

$$
\begin{aligned}
&= s\left[\delta^b_{\mathbb{R}^{\oplus d}}(z,x_n)+\delta^b_{\mathbb{R}^{\oplus d}}(\mathscr{F}(x_{n-1}),\mathscr{F}(z))\right]\\
&\precsim s\left[\delta^b_{\mathbb{R}^{\oplus d}}(z,x_n)+\operatorname{diag}\left(\alpha_1,\ \cdots,\ \alpha_d\right)\delta^b_{\mathbb{R}^{\oplus d}}(x_{n-1},\mathscr{F}(x_{n-1}))+\delta^b_{\mathbb{R}^{\oplus d}}(z,\mathscr{F}(z))\right]\\
&\precsim \operatorname{diag}\left(s\alpha_1,\ \cdots,\ s\alpha_d\right)\delta^b_{\mathbb{R}^{\oplus d}}(z,\mathscr{F}(z)),
\end{aligned}
$$

since $x_n \to z$ and $s\alpha_i \in (0,1)$, $\forall i = \{1,2,\cdots,d\}$ it is clear that we can make this distance as small as we please by choosing n sufficiently large. We conclude that

$$\delta^b_{\mathbb{R}^{\oplus d}}(z,\mathscr{F}(z)) = 0 \implies \mathscr{F}(z) = z,$$

so $z \in \mathcal{X}$ is a fixed-point of $\mathscr{F}$.
To prove uniqueness, suppose there are two fixed points $x = \mathscr{F}(x)$ and $y = \mathscr{F}(x)$. Then, from the contraction condition, we have

$$\delta^b_{\mathbb{R}^{\oplus d}}(x,y) = \delta^b_{\mathbb{R}^{\oplus d}}(\mathscr{F}(x),\mathscr{F}(y)) \precsim \operatorname{diag}\left(s\alpha_1,\ \cdots,\ s\alpha_d\right)\ [\delta^b_{\mathbb{R}^{\oplus d}}(x,\mathscr{F}(x))+\delta^b_{\mathbb{R}^{\oplus d}}(y,\mathscr{F}(y))],$$

this implies $\delta^b_{\mathbb{R}^{\oplus d}}(x,y) = 0$. Hence $x = y$, and the fixed-point $\mathcal{X}$ of $\mathscr{F}$ is unique.

If $d = 1$ and $s = 1$, then the theorem 4 can be reduced to the standard Kannan in the metric space.

Corolary 4 (Kannan, [24])**.** *Suppose that $(\mathcal{X}, d)$ is a complete metric space and $\mathscr{F}: \mathcal{X} \to \mathcal{X}$ is an operator such that*

$$\delta(\mathscr{F}x,\mathscr{F}y) \le \alpha\left[\delta(\mathscr{F}x,x)+\delta(\mathscr{F}y,y)\right],$$

for constant $\alpha \in (0,\frac{1}{2})$ and for every $x, y \in \mathcal{X}$. Then $\mathscr{F}$ has a unique fixed-point $z \in \mathcal{X}$.

Theorem 5. *(**Chatterjee Type**) Suppose that $(\mathcal{X}, \mathbb{R}^{\oplus^d}, \delta^b_{\mathbb{R}^{\oplus d}})$ is a complete generalized b-metric space endowed with the orthogonal direct sum and $\mathscr{F}: \mathcal{X} \to \mathcal{X}$ is an operator satisfying the following condition*

$$\delta^b_{\mathbb{R}^{\oplus d}}(\mathscr{F}(x),\mathscr{F}(y)) \precsim \operatorname{diag}\left(\alpha_1,\ \cdots,\ \alpha_d\right)[\delta^b_{\mathbb{R}^{\oplus d}}(\mathscr{F}(x),y)+\delta^b_{\mathbb{R}^{\oplus d}}(\mathscr{F}(y),x)], \tag{2}$$

for $s\alpha_i \in (0,\frac{1}{2})$ for all $i = 1,2,\cdots,d$, and for each $x, y \in \mathcal{X}$. Then, $\mathscr{F}$ has a unique fixed-point in $\mathcal{X}$.

Proof. Let x_0 be any point in $\mathcal{X}$, that is $x_0 \in \mathcal{X}$. Let us define a sequence $\{x_n\}$ in $\mathcal{X}$ as given below.

$$x_{n+1} = \mathscr{F}(x_n) = \mathscr{F}^{n+1}(x_0)\ \ \forall\ \ n \ge 0.$$

We have

$$
\begin{aligned}
&\delta^b_{\mathbb{R}^{\oplus d}}(x_n,x_{n+1})\\
=\ &\delta^b_{\mathbb{R}^{\oplus d}}(\mathscr{F}(x_{n-1}),\mathscr{F}(x_n))
\end{aligned}
$$

$$
\begin{aligned}
&\precsim \ \operatorname{diag}\left(\alpha_1, \ \cdots, \ \alpha_d\right)\left[\delta^b_{\mathbb{R}\oplus^d}(\mathscr{F}(x_{n-1}), x_n) + \delta^b_{\mathbb{R}\oplus^d}(\mathscr{F}(x_n), x_{n-1})\right] \\
&= \ \operatorname{diag}\left(\alpha_1, \ \cdots, \ \alpha_d\right)[\delta^b_{\mathbb{R}\oplus^d}(x_n, x_n) + \delta^b_{\mathbb{R}\oplus^d}(x_{n+1}, x_{n-1})] \\
&= \ \operatorname{diag}\left(\alpha_1, \ \cdots, \ \alpha_d\right)\delta^b_{\mathbb{R}\oplus^d}(\mathscr{F}(x_n), \mathscr{F}(x_{n-2})) \\
&\precsim \ \operatorname{diag}\left(s\alpha_1, \ \cdots, \ s\alpha_d\right)\left[\delta^b_{\mathbb{R}\oplus^d}(\mathscr{F}(x_n), \mathscr{F}(x_{n-1})) + \delta^b_{\mathbb{R}\oplus^d}(\mathscr{F}(x_{n-1}), \mathscr{F}(x_{n-2}))\right] \\
&= \ \operatorname{diag}\left(s\alpha_1, \ \cdots, \ s\alpha_d\right)\left[\delta^b_{\mathbb{R}\oplus^d}(x_{n+1}, x_n) + \delta^b_{\mathbb{R}\oplus^d}(x_n, x_{n-1})\right].
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\delta^b_{\mathbb{R}\oplus^d}(x_n, x_{n+1}) \\
\precsim \ & \operatorname{diag}\left(\tfrac{s\alpha_1}{(1-s\alpha_1)}, \ \cdots, \ \tfrac{s\alpha_d}{(1-s\alpha_d)}\right)\delta^b_{\mathbb{R}\oplus^d}(x_{n-1}, x_n), \quad \text{put } \left(\frac{s\alpha_i}{(1-s\alpha_i)} = \lambda_i\right) \\
= \ & \operatorname{diag}(\lambda_1, \cdots, \lambda_d)\ \delta^b_{\mathbb{R}\oplus^d}(x_{n-1}, x_n), \quad \text{where } (\lambda_1, , \cdots, \lambda_d) \prec I_+^{\mathbb{R}\oplus^d} \\
\precsim \ & \operatorname{diag}(\lambda_1^2, \cdots, \lambda_d^2)\ \delta^b_{\mathbb{R}\oplus^d}(x_{n-2}, x_{n-1}) \\
\precsim \ & \operatorname{diag}(\lambda_1^3, \cdots, \lambda_d^3)\ \delta^b_{\mathbb{R}\oplus^d}(x_{n-3}, x_{n-2}) \\
\vdots \ & \\
\precsim \ & \operatorname{diag}(\lambda_1^n, \cdots, \lambda_d^n)\delta^b_{\mathbb{R}\oplus^d}(x_0, x_1).
\end{aligned}
$$

Let us prove that $\{x_n\}$ is a Cauchy sequence. Suppose that $n > m$, then from (2) and the triangle inequality property, we can write as:

$$
\begin{aligned}
&\delta^b_{\mathbb{R}\oplus^d}(x_m, x_n) \\
\precsim \ & s\left[\delta^b_{\mathbb{R}\oplus^d}(x_m, x_{m+1}) + \delta^b_{\mathbb{R}\oplus^d}(x_{m+1}, x_n)\right] \\
\precsim \ & s\delta^b_{\mathbb{R}\oplus^d}(x_m, x_{m+1}) + s^2\left[\delta^b_{\mathbb{R}\oplus^d}(x_{m+1}, x_{m+2}) + \delta^b_{\mathbb{R}\oplus^d}(x_{m+2}, x_n)\right] \\
\precsim \ & s\delta^b_{\mathbb{R}\oplus^d}(x_m, x_{m+1}) + s^2\delta^b_{\mathbb{R}\oplus^d}(x_{m+1}, x_{m+2}) + s^3\left[\delta^b_{\mathbb{R}\oplus^d}(x_{m+2}, x_{m+3}) + \delta^b_{\mathbb{R}\oplus^d}(x_{m+3}, x_n)\right] \\
\vdots \ & \\
\precsim \ & s\delta^b_{\mathbb{R}\oplus^d}(x_m, x_{m+1}) + s^2\delta^b_{\mathbb{R}\oplus^d}(x_{m+1}, x_{m+2}) + \cdots + s^{n-m}\delta^b_{\mathbb{R}\oplus^d}(x_{n-1}, x_n) \\
= \ & \operatorname{diag}(s\lambda_1^m, s\lambda_2^m, \cdots, s\lambda_d^m)\delta^b_{\mathbb{R}\oplus^d}(x_0, x_1) + \operatorname{diag}(s^2\lambda_1^{m+1}, s^2\lambda_2^{m+1}, \cdots, s^2\lambda_d^{m+1})\delta^b_{\mathbb{R}\oplus^d}(x_0, x_1) \\
& \quad + \cdots + \operatorname{diag}(s^{n-m}\lambda_1^{n-1}, s^{n-m}\lambda_2^{n-1}, \cdots, s^{n-m}\lambda_d^{n-1})\delta^b_{\mathbb{R}\oplus^d}(x_0, x_1) \\
= \ & \Big((s\lambda_1^m + s^2\lambda_1^{m+1} + \cdots + s^{n-m}\lambda_1^{n-1})\delta^b_{\mathbb{R}\oplus^d}(x_0, x_1), (s\lambda_2^m + s^2\lambda_2^{m+1} \\
& + \cdots + s^{n-m}\lambda_2^{n-1})\delta^b_{\mathbb{R}\oplus^d}(x_0, x_1) \\
& \quad , \cdots, (s\lambda_d^m + s^2\lambda_d^{m+1} + \cdots + s^{n-m}\lambda_d^{n-1})\delta^b_{\mathbb{R}\oplus^d}(x_0, x_1)\Big)
\end{aligned}
$$

$$
\begin{aligned}
&= \operatorname{diag}\left(\sum_{i=1}^{n-m}\sum_{j=m}^{n-1} s^i\lambda_1^j\delta^b_{\mathbb{R}^{\oplus d}}(x_0,x_1), \sum_{i=1}^{n-m}\sum_{j=m}^{n-1} s^i\lambda_2^j\delta^b_{\mathbb{R}^{\oplus d}}(x_0,x_1),\cdots,\sum_{i=1}^{n-m}\sum_{j=m}^{n-1} s^i\lambda_d^j\delta^b_{\mathbb{R}^{\oplus d}}(x_0,x_1)\right)\\
&= \operatorname{diag}\left(\sum_{i=1}^{n-m}\sum_{j=m}^{n-1} s^i\lambda_1^j,\ \sum_{i=1}^{n-m}\sum_{j=m}^{n-1} s^i\lambda_2^j,\cdots,\sum_{i=1}^{n-m}\sum_{j=m}^{n-1} s^i\lambda_d^j\right)\delta^b_{\mathbb{R}^{\oplus d}}(x_0,x_1)\\
&= \operatorname{diag}\left(s\lambda_1^m\sum_{j=0}^{n-m-1}(s\lambda_1)^j, s\lambda_2^m\sum_{j=0}^{n-m-1}(s\lambda_2)^j,\cdots,s\lambda_d^m\sum_{j=0}^{n-m-1}(s\lambda_d)^j\right)\delta^b_{\mathbb{R}^{\oplus d}}(x_0,x_1)\\
&\precsim \operatorname{diag}\left(s\lambda_1^m\sum_{j=0}^{+\infty}(s\lambda_1)^j,\ s\lambda_2^m\sum_{j=0}^{+\infty}(s\lambda_2)^j,\ \cdots,\ s\lambda_1^m\sum_{j=0}^{+\infty}(s\lambda_d)^j\right)\delta^b_{\mathbb{R}^{\oplus d}}(x_0,x_1)\\
&= \operatorname{diag}\left(s\lambda_1^m\left[\frac{1}{1-s\lambda_1}\right], s\lambda_2^m\left[\frac{1}{1-s\lambda_2}\right],\cdots,s\lambda_1^m\left[\frac{1}{1-s\lambda_d}\right]\right)\delta^b_{\mathbb{R}^{\oplus d}}(x_0,x_1)\\
&\precsim \operatorname{diag}(s\lambda_1^m,s\lambda_2^m,\cdots,s\lambda_d^m)\operatorname{diag}\left(\left[\frac{1}{1-s\lambda_1}\right],\left[\frac{1}{1-s\lambda_2}\right],\cdots,\left[\frac{1}{1-s\lambda_d}\right]\right)\delta^b_{\mathbb{R}^{\oplus d}}(x_0,x_1)\to 0,
\end{aligned}
$$

as $n,m\to+\infty$. Since $\tilde{0}\prec\operatorname{diag}(s\lambda_1^j,s\lambda_2^j,\cdots,s\lambda_d^j)\prec I_+^{\mathbb{R}^{\oplus d}}$ and $\delta^b_{\mathbb{R}^{\oplus d}}(x_0,x_1)$ are fixed, it is evident that by selecting m sufficiently large (with $n>m$), we can make $\delta^b_{\mathbb{R}^{\oplus d}}(x_n,x_m)$ arbitrarily small. This shows that $\{x_n\}$ is a Cauchy sequence. Finally, because $(\mathcal{X},\mathbb{R}^{\oplus^d},\delta^b_{\mathbb{R}^{\oplus d}})$ is complete, there exists some $z\in\mathcal{X}$ such that $x_n\to z$.
To show that $\mathcal{X}$ has a fixed point, we consider the distance $\delta^b_{\mathbb{R}^{\oplus d}}(z,\mathscr{F}(z))$. From the triangle inequality and contraction condition, we get

$$
\begin{aligned}
&\delta^b_{\mathbb{R}^{\oplus d}}(z,\mathscr{F}(z))\\
\precsim\ & s\left[\delta^b_{\mathbb{R}^{\oplus d}}(z,x_n)+\delta^b_{\mathbb{R}^{\oplus d}}(x_n,\mathscr{F}(z))\right]\\
=\ & s\left[\delta^b_{\mathbb{R}^{\oplus d}}(z,x_n)+\delta^b_{\mathbb{R}^{\oplus d}}(\mathscr{F}(x_{n-1}),\mathscr{F}(z))\right]\\
\precsim\ & s\left[\delta^b_{\mathbb{R}^{\oplus d}}(z,x_n)+\operatorname{diag}\left(\alpha_1,\quad\alpha_2,\quad\cdots,\quad\alpha_d\right)(\delta^b_{\mathbb{R}^{\oplus d}}(\mathscr{F}(x_{n-1}),z)+\delta^b_{\mathbb{R}^{\oplus d}}(\mathscr{F}(z),x_{n-1}))\right]\\
=\ & \operatorname{diag}\left(s(1+\alpha_1),\quad s(1+\alpha_2),\quad\cdots,\quad s(1+\alpha_d)\right)\delta^b_{\mathbb{R}^{\oplus d}}(x_n,z)\\
&\quad+\operatorname{diag}\left(s\alpha_1,\quad s\alpha_2,\quad\cdots,\quad s\alpha_d\right)\delta^b_{\mathbb{R}^{\oplus d}}(\mathscr{F}(z),x_{n-1}),
\end{aligned}
$$

since $x_n\to z$ it is clear that

$$\delta^b_{\mathbb{R}^{\oplus d}}(z,\mathscr{F}(z))=0\implies\mathscr{F}(z)=z,$$

so $z\in\mathcal{X}$ is a fixed-point of $\mathscr{F}$.
To prove uniqueness, suppose there are two fixed points $x=\mathscr{F}(x)$ and $y=\mathscr{F}(y)$. Then, from the contraction condition, we have

$$\delta^b_{\mathbb{R}^{\oplus d}}(x,y)=\delta^b_{\mathbb{R}^{\oplus d}}(\mathscr{F}(x),\mathscr{F}(y))$$

$$
\begin{aligned}
&\precsim \operatorname{diag}\left(s\alpha_1, \quad \cdots, \quad s\alpha_d\right)\left[\delta^b_{\mathbb{R}^{\oplus d}}(\mathscr{F}(x), x) + \delta^b_{\mathbb{R}^{\oplus d}}(x, y) + \delta^b_{\mathbb{R}^{\oplus d}}(\mathscr{F}(y), y) + \delta^b_{\mathbb{R}^{\oplus d}}(y, x)\right] \\
&= 2\operatorname{diag}\left(s\alpha_1, \quad \cdots, \quad s\alpha_d\right)\ \delta^b_{\mathbb{R}^{\oplus d}}(x, y) \prec \delta^b_{\mathbb{R}^{\oplus d}}(x, y),
\end{aligned}
$$

this implies $\delta^b_{\mathbb{R}^{\oplus d}}(x, y) = 0$. Hence $x = y$, and the fixed-point $\mathcal{X}$ of $\mathscr{F}$ is unique.

If $d = 1$ and $s = 1$, then the theorem 5 can be reduced to the standard Chatterjee in the metric space.

Corolary 5 (Chatterjee, [25])**.** *Suppose that $(\mathcal{X}, d)$ is a complete metric space and $\mathscr{F}: \mathcal{X} \to \mathcal{X}$ is an operator such that*

$$
\delta(\mathscr{F}x, \mathscr{F}y) \leq \alpha\left[\delta(\mathscr{F}x, y) + \delta(\mathscr{F}y, x)\right],
$$

for constant $\alpha \in (0, \frac{1}{2})$ and for every $x, y \in \mathcal{X}$. Then $\mathscr{F}$ has a unique fixed-point $z \in \mathcal{X}$.

Theorem 6. *Suppose that $(\mathcal{X}, \mathbb{R}^{\oplus^d}, \delta^b_{\mathbb{R}^{\oplus d}})$ is a complete generalized b-metric space endowed with the orthogonal direct sum and $\mathscr{F}: \mathcal{X} \to \mathcal{X}$ is an operator satisfying the following condition*

$$
\begin{aligned}
\delta^b_{\mathbb{R}^{\oplus d}}(\mathscr{F}x, \mathscr{F}y) \quad &\precsim \quad \operatorname{diag}\left(\alpha_1, \quad \alpha_2, \quad \cdots, \quad \alpha_d\right)\ \delta^b_{\mathbb{R}^{\oplus d}}(x, y) \\
&\quad + \operatorname{diag}\left(\beta_1, \quad \beta_2, \quad \cdots, \quad \beta_d\right)\left[\delta^b_{\mathbb{R}^{\oplus d}}(x, \mathscr{F}(x)) + \delta^b_{\mathbb{R}^{\oplus d}}(y, \mathscr{F}(y))\right],
\end{aligned}
$$

for $\alpha_i,\ \beta_i \in \mathbb{R}^d_+$ and $\alpha_i + 2\beta_i \in [0, 1)$, $i = 1, 2, \cdots, d$ for each $x, y \in \mathcal{X}$. Then, $\mathscr{F}$ has a unique fixed-point in $\mathcal{X}$.

Proof. Let x_0 be any point in $\mathcal{X}$, that is $x_0 \in \mathcal{X}$. Let us define a sequence $\{x_n\}$ in $\mathcal{X}$ as given below.

$$
x_{n+1} = \mathscr{F}(x_n) = \mathscr{F}^{n+1}(x_0) \ \ \forall \ \ n \geq 0.
$$

We have

$$
\begin{aligned}
&\delta^b_{\mathbb{R}^{\oplus d}}(x_n, x_{n+1}) = \delta^b_{\mathbb{R}^{\oplus d}}(\mathscr{F}(x_{n-1}), \mathscr{F}(x_n)) \\
\precsim \quad &\operatorname{diag}\left(\alpha_1, \quad \alpha_2, \quad \cdots, \quad \alpha_d\right)\ \delta^b_{\mathbb{R}^{\oplus d}}(x_{n-1}, x_n) \\
&\quad + \operatorname{diag}\left(\beta_1, \quad \beta_2, \quad \cdots, \quad \beta_d\right)\left[\delta^b_{\mathbb{R}^{\oplus d}}(x_{n-1}, \mathscr{F}(x_{n-1})) + \delta^b_{\mathbb{R}^{\oplus d}}(x_n, \mathscr{F}(x_n))\right] \\
= \quad &\operatorname{diag}\left(\alpha_1, \quad \alpha_2, \quad \cdots, \quad \alpha_d\right)\ \delta^b_{\mathbb{R}^{\oplus d}}(x_{n-1}, x_n) \\
&\quad + \operatorname{diag}\left(\beta_1, \quad \beta_2, \quad \cdots, \quad \beta_d\right)\left[\delta^b_{\mathbb{R}^{\oplus d}}(x_{n-1}, x_n) + \delta^b_{\mathbb{R}^{\oplus d}}(x_n, x_{n+1})\right] \\
= \quad &\operatorname{diag}\left((\alpha_1 + \beta_1), \quad (\alpha_2 + \beta_2), \quad \cdots, \quad (\alpha_d + \beta_d)\right)\ \delta^b_{\mathbb{R}^{\oplus d}}(\mathscr{F}(x_{n-2}), \mathscr{F}(x_{n-1})) \\
&\quad + \operatorname{diag}\left(\beta_1 \quad \beta_2, \quad \cdots, \quad \beta_d\right)\ \delta^b_{\mathbb{R}^{\oplus d}}(x_n, x_{n+1}).
\end{aligned}
$$

Thus,

$$
\operatorname{diag}\left((1 - \beta_1), \quad (1 - \beta_2), \quad \cdots, \quad (1 - \beta_d)\right)\ \delta^b_{\mathbb{R}^{\oplus d}}(x_n, x_{n+1})
$$

$$\precsim \quad \operatorname{diag}\big((\alpha_1+\beta_1), \quad (\alpha_2+\beta_2), \quad \cdots, \quad (\alpha_d+\beta_d)\big)\, \delta^b_{\mathbb{R}^{\oplus d}}(\mathscr{F}(x_{n-2}), \mathscr{F}(x_{n-1})).$$

This implies that

$$\delta^b_{\mathbb{R}^{\oplus d}}(x_n, x_{n+1}) \quad \precsim \quad \operatorname{diag}\left(\frac{(\alpha_1+\beta_1)}{(1-\beta_1)}, \quad \cdots, \quad \frac{(\alpha_d+\beta_d)}{(1-\beta_d)}\right) \delta^b_{\mathbb{R}^{\oplus d}}(x_{n-1}, x_n),$$

we use $\lambda_i = \dfrac{\alpha_i+\beta_i}{1-\beta_i} \in (0,1)$, $i = 1, 2, \cdots, d$ we obtain

$$\delta^b_{\mathbb{R}^{\oplus d}}(x_n, x_{n+1}) \quad \precsim \quad \operatorname{diag}\big(\lambda_1, \quad \lambda_2, \quad \cdots, \quad \lambda_d\big)\, \delta^b_{\mathbb{R}^{\oplus d}}(x_n, x_{n-1}).$$

So

$$\begin{aligned}
& \delta^b_{\mathbb{R}^{\oplus d}}(x_n, x_{n+1}) \\
\precsim \;& \operatorname{diag}\big(\lambda_1, \quad \lambda_2, \quad \cdots, \quad \lambda_d\big)\; \delta^b_{\mathbb{R}^{\oplus d}}(x_{n-1}, x_n) \\
\precsim \;& \operatorname{diag}\big(\lambda_1^2, \quad \lambda_2^2, \quad \cdots, \quad \lambda_d^2\big)\; \delta^b_{\mathbb{R}^{\oplus d}}(x_{n-2}, x_{n-1}) \\
\precsim \;& \operatorname{diag}\big(\lambda_1^3, \quad \lambda_2^3, \quad \cdots, \quad \lambda_d^3\big)\; \delta^b_{\mathbb{R}^{\oplus d}}(x_{n-3}, x_{n-2}) \\
\precsim \;& \\
\vdots \;& \\
\precsim \;& \operatorname{diag}\big(\lambda_1^n, \quad \lambda_2^n, \quad \cdots, \quad \lambda_d^n\big)\; \delta^b_{\mathbb{R}^{\oplus d}}(x_0, x_1).
\end{aligned}$$

Let us prove that $\{x_n\}$ is a Cauchy sequence. Suppose that $m < n$, then from (3) and the triangle inequality property, we can write as:

$$\begin{aligned}
& \delta^b_{\mathbb{R}^{\oplus d}}(x_m, x_n) \precsim s\delta^b_{\mathbb{R}^{\oplus d}}(x_m, x_{m+1}) + s\delta^b_{\mathbb{R}^{\oplus d}}(x_{m+1}, x_n) \\
\precsim \quad & s\delta^b_{\mathbb{R}^{\oplus d}}(x_m, x_{m+1}) + s^2\left[\delta^b_{\mathbb{R}^{\oplus d}}(x_{m+1}, x_{m+2}) + \delta^b_{\mathbb{R}^{\oplus d}}(x_{m+2}, x_n)\right] \\
\precsim \quad & s\delta^b_{\mathbb{R}^{\oplus d}}(x_m, x_{m+1}) + s^2\delta^b_{\mathbb{R}^{\oplus d}}(x_{m+1}, x_{m+2}) + s^3\left[\delta^b_{\mathbb{R}^{\oplus d}}(x_{m+2}, x_{m+3}) + \delta^b_{\mathbb{R}^{\oplus d}}(x_{m+3}, x_n)\right] \\
\vdots \quad & \\
\precsim \quad & s\delta^b_{\mathbb{R}^{\oplus d}}(x_m, x_{m+1}) + s^2\delta^b_{\mathbb{R}^{\oplus d}}(x_{m+1}, x_{m+2}) + \cdots + s^{n-m}\delta^b_{\mathbb{R}^{\oplus d}}(x_{n-1}, x_n) \\
= \quad & \operatorname{diag}\big(s\lambda_1^m, \quad s\lambda_2^m, \quad \cdots, \quad s\lambda_d^m\big)\, \delta^b_{\mathbb{R}^{\oplus d}}(x_0, x_1) \\
& + \operatorname{diag}\big(s^2\lambda_1^{m+1}, \quad s^2\lambda_2^{m+1}, \quad \cdots, \quad s^2\lambda_d^{m+1}\big)\, \delta^b_{\mathbb{R}^{\oplus d}}(x_0, x_1) \\
& \quad + \operatorname{diag}\big(s^3\lambda_1^{m+2}, \quad s^3\lambda_2^{m+2}, \quad \cdots, \quad s^3\lambda_d^{m+2}\big)\, \delta^b_{\mathbb{R}^{\oplus d}}(x_0, x_1) + \cdots \\
& + \operatorname{diag}\big(s^{n-m}\lambda_1^{n-1}, \quad s^{n-m}\lambda_2^{n-1}, \quad \cdots, \quad s^{n-m}\lambda_d^{n-1}\big)\, \delta^b_{\mathbb{R}^{\oplus d}}(x_0, x_1) \\
= \quad & \Big((s\lambda_1^m + s^2\lambda_1^{m+1} + \cdots + s^{n-m}\lambda_1^{n-1})\delta^b_{\mathbb{R}^{\oplus d}}(x_0, x_1),
\end{aligned}$$

$$
\begin{aligned}
&(s\lambda_2^m + s^2\lambda_2^{m+1} + \cdots + s^{n-m}\lambda_2^{n-1})\delta^b_{\mathbb{R}^{\oplus d}}(x_0, x_1)\\
&, \cdots, (s\lambda_d^m + s^2\lambda_d^{m+1} + \cdots + s^{n-m}\lambda_d^{n-1})\delta^b_{\mathbb{R}^{\oplus d}}(x_0, x_1)\Big)\\
= \;& \operatorname{diag}\left(\textstyle\sum_{i=1}^{n-m}\sum_{j=m}^{n-1} s^i\lambda_1^j, \quad \cdots, \quad \sum_{i=1}^{n-m}\sum_{j=m}^{n-1} s^i\lambda_d^j\right)\delta^b_{\mathbb{R}^{\oplus d}}(x_0, x_1)\\
= \;& \operatorname{diag}\left(s\lambda_1^m \textstyle\sum_{j=0}^{n-m-1}(s\lambda_1)^j, \ \cdots, s\lambda_1^m \sum_{j=0}^{n-m-1}(s\lambda_d)^j\right)\delta^b_{\mathbb{R}^{\oplus d}}(x_0, x_1)\\
\precsim \;& \operatorname{diag}\left(s\lambda_1^m \textstyle\sum_{j=0}^{+\infty}(s\lambda_1)^j, \ \cdots, \ s\lambda_1^m \sum_{j=0}^{+\infty}(s\lambda_d)^j\right)\delta^b_{\mathbb{R}^{\oplus d}}(x_0, x_1)\\
= \;& \operatorname{diag}\left(s\lambda_1^m\left[\frac{1}{1-s\lambda_1}\right], \cdots, s\lambda_d^m\left[\frac{1}{1-s\lambda_d}\right]\right)\delta^b_{\mathbb{R}^{\oplus d}}(x_0, x_1)\\
\precsim \;& \operatorname{diag}\left(s\lambda_1^m, \cdots, s\lambda_d^m\right)\operatorname{diag}\left(\left[\frac{1}{1-s\lambda_1}\right], \cdots, \left[\frac{1}{1-s\lambda_d}\right]\right)\delta^b_{\mathbb{R}^{\oplus d}}(x_0, x_1) \to 0,
\end{aligned}
$$

as $n, m \to +\infty$. Since $s\lambda_i \in (0,1)$ for all $i = 1, 2, \cdots, d$ and $\delta^b_{\mathbb{R}^{\oplus d}}(x_0, x_1)$ are fixed, it is evident that by selecting m sufficiently large (with $n > m$), we can make $\delta^b_{\mathbb{R}^{\oplus d}}(x_n, x_m)$ arbitrarily small. This shows that $\{x_n\}$ is a Cauchy sequence. Finally, because $(\mathcal{X}, \mathbb{R}^{\oplus^d}, \delta^b_{\mathbb{R}^{\oplus d}})$ is complete, there exists some $z \in \mathcal{X}$ such that $x_n \to z$.
To show that z is a fixed point, we consider the distance $\delta^b_{\mathbb{R}^{\oplus d}}(z, \mathscr{F}(z))$. From the triangle inequality and contraction condition, we get

$$
\begin{aligned}
\delta^b_{\mathbb{R}^{\oplus d}}(z, \mathscr{F}(z)) &\precsim s[\delta^b_{\mathbb{R}^{\oplus d}}(z, x_n) + \delta^b_{\mathbb{R}^{\oplus d}}(x_n, \mathscr{F}(z))]\\
&= s[\delta^b_{\mathbb{R}^{\oplus d}}(z, x_n) + \delta^b_{\mathbb{R}^{\oplus d}}(\mathscr{F}(x_{n-1}), \mathscr{F}(z))]\\
&\precsim s\Big[\delta^b_{\mathbb{R}^{\oplus d}}(z, x_n) + \operatorname{diag}\left(\alpha_1, \quad \cdots, \quad \alpha_d\right)\delta^b_{\mathbb{R}^{\oplus d}}(x_{n-1}, z)\\
&+ \operatorname{diag}\left(\beta_1, \quad \cdots, \quad \beta_d\right)\Big[\delta^b_{\mathbb{R}^{\oplus d}}(x_{n-1}, \mathscr{F}(x_{n-1})) + \delta^b_{\mathbb{R}^{\oplus d}}(z, \mathscr{F}(z))\Big]\Big],
\end{aligned}
$$

therefore

$$
\begin{aligned}
&\left(1 - s\beta_1, \quad \cdots, \quad 1 - s\beta_d\right)\delta^b_{\mathbb{R}^{\oplus d}}(z, \mathscr{F}(z))\\
&\precsim s\delta^b_{\mathbb{R}^{\oplus d}}(z, x_n) + \operatorname{diag}\left(s\alpha_1, \quad \cdots, \quad s\alpha_d\right)\delta^b_{\mathbb{R}^{\oplus d}}(x_{n-1}, z)\\
&+ \operatorname{diag}\left(s\beta_1, \quad \cdots, \quad s\beta_d\right)\Big[\delta^b_{\mathbb{R}^{\oplus d}}(x_{n-1}, \mathscr{F}(x_{n-1}))\Big] \to 0,
\end{aligned}
$$

and since $x_n \to z$ it is clear that we can make this distance as small as we please by choosing n sufficiently large. We conclude that

$$
\delta^b_{\mathbb{R}^{\oplus d}}(z, \mathscr{F}(z)) = 0 \implies \mathscr{F}(z) = z,
$$

so $z \in \mathcal{X}$ is a fixed-point of $\mathscr{F}$.
To prove uniqueness, suppose there are two fixed points $x = \mathscr{F}(x)$ and $y = \mathscr{F}(y)$. Then, from the contraction condition, we have

$$
\delta^b_{\mathbb{R}^{\oplus d}}(x, y) = \delta^b_{\mathbb{R}^{\oplus d}}(\mathscr{F}(x), \mathscr{F}(y))
$$

$$\begin{aligned}
&\precsim \ \operatorname{diag}\left(\alpha_1, \quad \alpha_2, \quad \cdots, \quad \alpha_d\right) \ \delta^b_{\mathbb{R}^{\oplus d}}(x,y)\\
&\quad + \operatorname{diag}\left(\beta_1, \quad \beta_2, \quad \cdots, \quad \beta_d\right) \ [\delta^b_{\mathbb{R}^{\oplus d}}(x,\mathscr{F}(x)) + \delta^b_{\mathbb{R}^{\oplus d}}(y,\mathscr{F}(y))]\\
&= \ \operatorname{diag}\left(\alpha_1, \quad \alpha_2, \quad \cdots, \quad \alpha_d\right) \ \delta^b_{\mathbb{R}^{\oplus d}}(x,y) \prec \delta^b_{\mathbb{R}^{\oplus d}}(x,y),
\end{aligned}$$

this implies $\delta^b_{\mathbb{R}^{\oplus d}}(x,y) = 0$. Hence $x = y$, and the fixed-point x of $\mathscr{F}$ is unique.

Theorem 7. *Suppose that $(\mathcal{X}, \mathbb{R}^{\oplus^d}, \delta^b_{\mathbb{R}^{\oplus d}})$ is a complete generalized b-metric space endowed with the orthogonal direct sum and $\mathscr{F}\colon \mathcal{X} \to \mathcal{X}$ is a continuous operator satisfying the following condition*

$$\begin{aligned}
\delta^b_{\mathbb{R}^{\oplus d}}(\mathscr{F}x, \mathscr{F}y) \ &\precsim \ \operatorname{diag}\left(\alpha_1, \quad \alpha_2, \quad \cdots, \quad \alpha_d\right) \ \delta^b_{\mathbb{R}^{\oplus d}}(x,y) &(3)\\
&\quad + \operatorname{diag}\left(\beta_1, \quad \beta_2, \quad \cdots, \quad \beta_d\right) [\delta^b_{\mathbb{R}^{\oplus d}}(x,\mathscr{F}(x)) + \delta^b_{\mathbb{R}^{\oplus d}}(y,\mathscr{F}(y))]\\
&\quad + \operatorname{diag}\left(\gamma_1, \quad \gamma_2, \quad \cdots, \quad \gamma_d\right) \ [\delta^b_{\mathbb{R}^{\oplus d}}(x,\mathscr{F}(y)) + \delta^b_{\mathbb{R}^{\oplus d}}(y,\mathscr{F}(x))] &(4)
\end{aligned}$$

for $\alpha_i, \ \beta_i, \ \gamma_i \in \mathbb{R}_+ \ \ \forall \ i = 1,2,3,\cdots,d$ and $\alpha_i + 2(\beta_i + s\gamma_i) \in [0,1)$, for each $x, y \in \mathcal{X}$. Then, $\mathscr{F}$ has a unique fixed-point in $\mathcal{X}$.

Proof. Let x_0 be any point in $\mathcal{X}$, that is $x_0 \in \mathcal{X}$. Let us define a sequence $\{x_n\}$ in $\mathcal{X}$ as given below.

$$x_{n+1} = \mathscr{F}(x_n) = \mathscr{F}^{n+1}(x_0) \ \ \forall \ \ n \geq 0.$$

We have

$$\begin{aligned}
&\delta^b_{\mathbb{R}^{\oplus d}}(x_n, x_{n+1})\\
= \ &\delta^b_{\mathbb{R}^{\oplus d}}(\mathscr{F}(x_{n-1}), \mathscr{F}(x_n))\\
\precsim \ &\left(\alpha_1, \quad \alpha_2, \quad \cdots, \quad \alpha_d\right) \delta^b_{\mathbb{R}^{\oplus d}}(x_{n-1}, x_n)\\
&\quad + \operatorname{diag}\left(\beta_1, \quad \beta_2, \quad \cdots, \quad \beta_d\right) [\delta^b_{\mathbb{R}^{\oplus d}}(x_{n-1}, \mathscr{F}(x_{n-1})) + \delta^b_{\mathbb{R}^{\oplus d}}(x_n, \mathscr{F}(x_n))]\\
&\quad + \operatorname{diag}\left(\gamma_1, \quad \gamma_2, \quad \cdots, \quad \gamma_d\right) \ [\delta^b_{\mathbb{R}^{\oplus d}}(x_{n-1}, \mathscr{F}(x_n)) + \delta^b_{\mathbb{R}^{\oplus d}}(x_n, \mathscr{F}(x_{n-1}))]\\
= \ &\operatorname{diag}\left(\alpha_1, \quad \alpha_2, \quad \cdots, \quad \alpha_d\right) \delta^b_{\mathbb{R}^{\oplus d}}(x_{n-1}, x_n)\\
&\quad + \operatorname{diag}\left(\beta_1, \quad \beta_2, \quad \cdots, \quad \beta_d\right) \ [\delta^b_{\mathbb{R}^{\oplus d}}(x_{n-1}, x_n) + \delta^b_{\mathbb{R}^{\oplus d}}(x_n, x_{n+1})]\\
&\quad + \operatorname{diag}\left(\gamma_1, \quad \gamma_2, \quad \cdots, \quad \gamma_d\right) [\delta^b_{\mathbb{R}^{\oplus d}}(x_{n-1}, x_{n+1}))]\\
\precsim \ &\operatorname{diag}\left(\alpha_1 + \beta_1, \quad \alpha_2 + \beta_2, \quad \cdots, \quad \alpha_d + \beta_d\right) \delta^b_{\mathbb{R}^{\oplus d}}(x_{n-1}, x_n)\\
&\quad + \operatorname{diag}\left(\beta_1, \quad \beta_2, \quad \cdots, \quad \beta_d\right) \ \delta^b_{\mathbb{R}^{\oplus d}}(x_n, x_{n+1})\\
&\quad + \operatorname{diag}\left(s\gamma_1, \quad s\gamma_2, \quad \cdots, \quad s\gamma_d\right) [\delta^b_{\mathbb{R}^{\oplus d}}(x_{n-1}, x_n) + \delta^b_{\mathbb{R}^{\oplus d}}(x_n, x_{n+1}))]\\
= \ &\operatorname{diag}\left(\alpha_1 + \beta_1 + s\gamma_1, \quad \alpha_2 + \beta_2 + s\gamma_2, \quad \cdots, \quad \alpha_d + \beta_d + s\gamma_d\right) \delta^b_{\mathbb{R}^{\oplus d}}(x_{n-1}, x_n)
\end{aligned}$$

$$+\operatorname{diag}\left(\beta_1+s\gamma_1,\quad \beta_2+s\gamma_2,\quad \cdots,\quad \beta_d+s\gamma_d\right)\ \delta^b_{\mathbb{R}\oplus^d}(x_n,x_{n+1}).$$

Thus,

$$\begin{aligned}
& \operatorname{diag}\left((1-(\beta_1+s\gamma_1)),\quad (1-(\beta_2+s\gamma_2)),\quad \cdots,\quad (1-(\beta_d+s\gamma_d))\right)\delta^b_{\mathbb{R}\oplus^d}(x_n,x_{n+1})\\
\precsim\ & \operatorname{diag}\left(\alpha_1+\beta_1+s\gamma_1,\quad \alpha_2+\beta_2+s\gamma_2,\quad \cdots,\quad \alpha_d+\beta_d+s\gamma_d\right)\delta^b_{\mathbb{R}\oplus^d}(x_{n-1},x_n).
\end{aligned}$$

This implies that

$$\begin{aligned}
& \delta^b_{\mathbb{R}\oplus^d}(x_n,x_{n+1})\\
\precsim\ & \operatorname{diag}\left(\frac{(\alpha_1+\beta_1+s\gamma_1)}{(1-(\beta_1+s\gamma_1))},\quad \frac{(\alpha_2+\beta_2+s\gamma_2)}{(1-(\beta_2+s\gamma_2))},\quad \cdots,\quad \frac{(\alpha_d+\beta_d+s\gamma_d)}{(1-(\beta_d+s\gamma_d))}\right)\delta^b_{\mathbb{R}\oplus^d}(x_{n-1},x_n).
\end{aligned}$$

we use $\lambda_i=\dfrac{\alpha_i+\beta_i+s\gamma_i}{1-(\beta_i+s\gamma_i)}\in(0,1)$ for all $i=1,2,\cdots,d$, we obtain

$$\delta^b_{\mathbb{R}\oplus^d}(x_n,x_{n+1})\ \precsim\ \operatorname{diag}\left(\lambda_1,\quad \lambda_2,\quad \cdots,\quad \lambda_d\right)\ \delta^b_{\mathbb{R}\oplus^d}(x_{n-1},x_n).$$

So

$$\begin{aligned}
\delta^b_{\mathbb{R}\oplus^d}(x_n,x_{n+1}) &\precsim \operatorname{diag}\left(\lambda_1,\quad \lambda_2,\quad \cdots,\quad \lambda_d\right)\ \delta^b_{\mathbb{R}\oplus^d}(x_{n-1},x_n)\\
&\precsim \operatorname{diag}\left(\lambda_1^2,\quad \lambda_2^2,\quad \cdots,\quad \lambda_d^2\right)\ \delta^b_{\mathbb{R}\oplus^d}(x_{n-2},x_{n-1})\\
&\precsim \operatorname{diag}\left(\lambda_1^3,\quad \lambda_2^3,\quad \cdots,\quad \lambda_d^3\right)\ \delta^b_{\mathbb{R}\oplus^d}(x_{n-3},x_{n-2})\\
&\precsim \cdots\\
&\precsim \operatorname{diag}\left(\lambda_1^n,\quad \lambda_2^n,\quad \cdots,\quad \lambda_d^n\right)\ \delta^b_{\mathbb{R}\oplus^d}(x_0,x_1). \qquad (5)
\end{aligned}$$

Let us prove that $\{x_n\}$ is a Cauchy sequence. Suppose that $m<n$, then from (5) and the triangle inequality property, we can write as:

$$\begin{aligned}
& \delta^b_{\mathbb{R}\oplus^d}(x_m,x_n)\\
\precsim\ & s\left[\delta^b_{\mathbb{R}\oplus^d}(x_m,x_{m+1})+\delta^b_{\mathbb{R}\oplus^d}(x_{m+1},x_n)\right]\\
\precsim\ & s\delta^b_{\mathbb{R}\oplus^d}(x_m,x_{m+1})+s^2\left[\delta^b_{\mathbb{R}\oplus^d}(x_{m+1},x_{m+2})+\delta^b_{\mathbb{R}\oplus^d}(x_{m+2},x_n)\right]\\
\precsim\ & s\delta^b_{\mathbb{R}\oplus^d}(x_m,x_{m+1})+s^2\delta^b_{\mathbb{R}\oplus^d}(x_{m+1},x_{m+2})+s^3\left[\delta^b_{\mathbb{R}\oplus^d}(x_{m+2},x_{m+3})+\delta^b_{\mathbb{R}\oplus^d}(x_{m+3},x_n)\right]\\
\vdots\ & \\
\precsim\ & s\delta^b_{\mathbb{R}\oplus^d}(x_m,x_{m+1})+s^2\delta^b_{\mathbb{R}\oplus^d}(x_{m+1},x_{m+2})+s^3\delta^b_{\mathbb{R}\oplus^d}(x_{m+2},x_{m+3})+\cdots+s^{n-m}\delta^b_{\mathbb{R}\oplus^d}(x_{n-1},x_n)\\
=\ & \operatorname{diag}\left(s\lambda_1^m,\quad s\lambda_2^m,\quad \cdots,\quad s\lambda_d^m\right)\delta^b_{\mathbb{R}\oplus^d}(x_0,x_1)\\
& \quad +\operatorname{diag}\left(s^2\lambda_1^{m+1},\quad s^2\lambda_2^{m+1},\quad \cdots,\quad s^2\lambda_d^{m+1}\right)\delta^b_{\mathbb{R}\oplus^d}(x_0,x_1)
\end{aligned}$$

$$
\begin{aligned}
&+\operatorname{diag}\left(s^3\lambda_1^{m+2},\quad s^3\lambda_2^{m+2},\quad \cdots,\quad s^3\lambda_d^{m+2}\right)\delta^b_{\mathbb{R}^{\oplus d}}(x_0,x_1)\\
&+\cdots+\operatorname{diag}\left(s^{n-m}\lambda_1^{n-1},\quad s^{n-m}\lambda_2^{n-1},\quad \cdots,\quad s^{n-m}\lambda_d^{n-1}\right)\delta^b_{\mathbb{R}^{\oplus d}}(x_0,x_1)\\
=\ &\operatorname{diag}\Big((s\lambda_1^m+s^2\lambda_1^{m+1}+\cdots+s^{n-m}\lambda_1^{n-1}),(s\lambda_2^m+s^2\lambda_2^{m+1}+\cdots+s^{n-m}\lambda_2^{n-1}),\\
&\quad\cdots,(s\lambda_d^m+s^2\lambda_d^{m+1}+\cdots+s^{n-m}\lambda_d^{n-1})\Big)\delta^b_{\mathbb{R}^{\oplus d}}(x_0,x_1)\\
=\ &\operatorname{diag}\left(\sum_{i=1}^{n-m}\sum_{j=m}^{n-1}s^i\lambda_1^j,\ \sum_{i=1}^{n-m}\sum_{j=m}^{n-1}s^i\lambda_2^j,\quad\cdots,\ \sum_{i=1}^{n-m}\sum_{j=m}^{n-1}s^i\lambda_d^j\right)\delta^b_{\mathbb{R}^{\oplus d}}(x_0,x_1)\\
=\ &\operatorname{diag}\left(s\lambda_1^m\sum_{j=0}^{n-m-1}(s\lambda_1)^j,\quad s\lambda_2^m\sum_{j=0}^{n-m-1}(s\lambda_2)^j,\quad\cdots,\quad s\lambda_1^m\sum_{j=0}^{n-m-1}(s\lambda_d)^j\right)\delta^b_{\mathbb{R}^{\oplus d}}(x_0,x_1)\\
\precsim\ &\operatorname{diag}\left(s\lambda_1^m\sum_{j=0}^{+\infty}(s\lambda_1)^j,\quad s\lambda_2^m\sum_{j=0}^{+\infty}(s\lambda_2)^j,\quad\cdots,\quad s\lambda_1^m\sum_{j=0}^{+\infty}(s\lambda_d)^j\right)\delta^b_{\mathbb{R}^{\oplus d}}(x_0,x_1)\\
=\ &\operatorname{diag}\left(s\lambda_1^m\left[\frac{1}{1-s\lambda_1}\right],\quad s\lambda_2^m\left[\frac{1}{1-s\lambda_2}\right],\quad\cdots,\quad s\lambda_d^m\left[\frac{1}{1-s\lambda_d}\right]\right)\delta^b_{\mathbb{R}^{\oplus d}}(x_0,x_1)\\
\precsim\ &\operatorname{diag}(s\lambda_1^m,\quad\cdots,\quad s\lambda_d^m)\operatorname{diag}\left(\left[\frac{1}{1-s\lambda_1}\right],\quad\cdots,\quad\left[\frac{1}{1-s\lambda_d}\right]\right)\delta^b_{\mathbb{R}^{\oplus d}}(x_0,x_1)\\
&\to 0,
\end{aligned}
$$

as $n,m\to+\infty$. Since $\lambda_i\in(0,1)\quad\forall i$ and $\delta^b_{\mathbb{R}^{\oplus d}}(x_0,x_1)$ are fixed, it is evident that by selecting m sufficiently large (with $n>m$), we can make $\delta^b_{\mathbb{R}^{\oplus d}}(x_n,x_m)$ arbitrarily small. This shows that $\{x_n\}$ is a Cauchy sequence. Finally, because $(\mathcal{X},\mathbb{R}^{\oplus^d},\delta^b_{\mathbb{R}^{\oplus d}})$ is complete, there exists some $z\in\mathcal{X}$ such that $x_n\to z$.
To show that x is a fixed point, we consider the distance $\delta^b_{\mathbb{R}^{\oplus d}}(z,\mathscr{F}(z))$. From the triangle

inequality and contraction condition, we get

$$\begin{aligned}
&\delta^b_{\mathbb{R}\oplus^d}(z,\mathscr{F}(z))\\
&\precsim s\delta^b_{\mathbb{R}\oplus^d}(z,x_n)+s\delta^b_{\mathbb{R}\oplus^d}(x_n,\mathscr{F}(z))\\
&= s\delta^b_{\mathbb{R}\oplus^d}(z,x_n)+\ s\delta^b_{\mathbb{R}\oplus^d}(\mathscr{F}(x_{n-1}),\mathscr{F}(z))\\
&\precsim s\delta^b_{\mathbb{R}\oplus^d}(z,x_n)+s\,\mathrm{diag}\begin{pmatrix}\alpha_1, & \alpha_2, & \cdots, & \alpha_d\end{pmatrix}\ \delta^b_{\mathbb{R}\oplus^d}(x_{n-1},z)\\
&\quad + s\,\mathrm{diag}\begin{pmatrix}\beta_1, & \beta_2, & \cdots, & \beta_d\end{pmatrix}[\delta^b_{\mathbb{R}\oplus^d}(x_{n-1},\mathscr{F}(x_{n-1}))+\delta^b_{\mathbb{R}\oplus^d}(z,\mathscr{F}(z))]\\
&\quad + s\,\mathrm{diag}\begin{pmatrix}\gamma_1, & \gamma_2, & \cdots, & \gamma_d\end{pmatrix}\ [\delta^b_{\mathbb{R}\oplus^d}(x_{n-1},\mathscr{F}(z))+\delta^b_{\mathbb{R}\oplus^d}(z,\mathscr{F}(x_{n-1}))]\\
&= s\,\mathrm{diag}\begin{pmatrix}1+\gamma_1, & 1+\gamma_2, & \cdots, & 1+\gamma_d\end{pmatrix}\delta^b_{\mathbb{R}\oplus^d}(z,x_n)\\
&+ s\,\mathrm{diag}\begin{pmatrix}\alpha_1, & \alpha_2, & \cdots, & \alpha_d\end{pmatrix}\ \delta^b_{\mathbb{R}\oplus^d}(x_{n-1},z)\\
&\quad + s\,\mathrm{diag}\begin{pmatrix}\beta_1, & \beta_2, & \cdots, & \beta_d\end{pmatrix}[\delta^b_{\mathbb{R}\oplus^d}(x_{n-1},x_n)+\delta^b_{\mathbb{R}\oplus^d}(z,\mathscr{F}(z))]\\
&\quad + s\,\mathrm{diag}\begin{pmatrix}\gamma_1, & \gamma_2, & \cdots, & \gamma_d\end{pmatrix}\ \delta^b_{\mathbb{R}\oplus^d}(x_{n-1},\mathscr{F}(z)),
\end{aligned}$$

hence

$$\begin{aligned}
&\mathrm{diag}\begin{pmatrix}1+s\beta_1, & \cdots, & 1+s\beta_d\end{pmatrix}\delta^b_{\mathbb{R}\oplus^d}(z,\mathscr{F}(z))\\
&\precsim s\,\mathrm{diag}\begin{pmatrix}1+\gamma_1, & \cdots, & 1+\gamma_d\end{pmatrix}\delta^b_{\mathbb{R}\oplus^d}(z,x_n)+s\,\mathrm{diag}\begin{pmatrix}\alpha_1, & \cdots, & \alpha_d\end{pmatrix}\ \delta^b_{\mathbb{R}\oplus^d}(x_{n-1},z)\\
&\quad + s\,\mathrm{diag}\begin{pmatrix}\beta_1, & \cdots, & \beta_d\end{pmatrix}\delta^b_{\mathbb{R}\oplus^d}(x_{n-1},x_n)\ + s\,\mathrm{diag}\begin{pmatrix}\gamma_1, & \cdots, & \gamma_d\end{pmatrix}\ \delta^b_{\mathbb{R}\oplus^d}(\mathscr{F}(x_{n-2}),\mathscr{F}(z)),
\end{aligned}$$

and since $x_n \to z$ it is clear that we can make this distance as small as we please by choosing n sufficiently large. We conclude that

$$\delta^b_{\mathbb{R}\oplus^d}(z,\mathscr{F}(z))=0 \implies \mathscr{F}(z)=z,$$

so $z\in\mathcal{X}$ is a fixed-point of $\mathscr{F}$.
To prove uniqueness, suppose there are two fixed points $x=\mathscr{F}(x)$ and $y=\mathscr{F}(y)$. Then, from the contraction condition, we have

$$\begin{aligned}
&\delta^b_{\mathbb{R}\oplus^d}(x,y)=\delta^b_{\mathbb{R}\oplus^d}(\mathscr{F}(x),\mathscr{F}(y))\\
\precsim\ &\mathrm{diag}\begin{pmatrix}\alpha_1, & \alpha_2, & \cdots, & \alpha_d\end{pmatrix}\ \delta^b_{\mathbb{R}\oplus^d}(x,y)\\
&\quad +\mathrm{diag}\begin{pmatrix}\beta_1, & \beta_2, & \cdots, & \beta_d\end{pmatrix}\ [\delta^b_{\mathbb{R}\oplus^d}(x,\mathscr{F}(x))+\delta^b_{\mathbb{R}\oplus^d}(y,\mathscr{F}(y))]\\
&\quad +\mathrm{diag}\begin{pmatrix}\gamma_1, & \gamma_2, & \cdots, & \gamma_d\end{pmatrix}\ [\delta^b_{\mathbb{R}\oplus^d}(x,\mathscr{F}(y))+\delta^b_{\mathbb{R}\oplus^d}(y,\mathscr{F}(x))]\\
=\ &\mathrm{diag}\begin{pmatrix}\alpha_1, & \alpha_2, & \cdots, & \alpha_d\end{pmatrix}\ \delta^b_{\mathbb{R}\oplus^d}(x,y)\\
&\quad +\mathrm{diag}\begin{pmatrix}2\gamma_1, & 2\gamma_2, & \cdots, & 2\gamma_d\end{pmatrix}\delta^b_{\mathbb{R}\oplus^d}(x,y)\\
=\ &\mathrm{diag}\begin{pmatrix}\alpha_1+\ 2\gamma_1, & \alpha_2+2\gamma_2, & \cdots, & \alpha_d+2\gamma_d\end{pmatrix}\delta^b_{\mathbb{R}\oplus^d}(x,y)
\end{aligned}$$

$$\prec \quad \delta^b_{\mathbb{R}\oplus^d}(x,y),$$

this implies $\delta^b_{\mathbb{R}\oplus^d}(x,y) = 0$. Hence $x = y$, and the fixed-point x of $\mathscr{F}$ is unique.

4. Conclusions and Future Work

This research presents novel extensions of the Banach fixed-point theorem within generalized metric spaces that include a direct sum structure. By introducing a diagonal matrix A in $\mathbb{R}^d$, we establish evolved contraction conditions that enhance the applicability of fixed-point results in this framework. Our method provides a more structured and flexible methodology for analyzing contraction operators, thereby contributing to the promotion of fixed-point theory in vector-valued metric spaces. The results obtained extend classical fixed-point theorems and offer new understandings of the interplay between metric structures and direct sum operations. Future studies may expand these results to more comprehensive classes of generalised metric spaces, such as G-metric or partial metric spaces. Studying non-diagonal or operator-valued matrix systems within the direct sum framework could yield more profound insights. Applications to vector-valued integral and differential equations, as well as iterative schemes for approximating fixed points under the new conditions, also certify further study.

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Declarations

- **Conflicts of interest** The authors declare that they have no conflicts of interest.
- **Author's contributions** Conceptualization, Writing original draft (G. Albeladi); Methodology, Data curation (S. Omran); Formal analysis, Revisions (S. Omran); Project administration (G. Albeladi); Supervision (S. Omran).

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