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On the Fractional q-Differintegral Operator for Subclasses of Bi-univalent functions Subordinate to q-Ultraspherical Polynomials

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Abstract. In this paper, we introduce a novel class of bi-univalent functions using the fractional q-differintegral operator and q-Ultraspherical polynomials. We examine the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions in this new class. We also establish Fekete-Szegö functional inequalities relevant to this subclass. By varying the parameters in our main results, we derive several new findings that contribute to the theoretical development of the field.

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Orthogonal polynomials (OP) are a fundamental concept in mathematics, playing a crucial role in various fields due to their unique properties. These polynomials, renowned for their mutual orthogonality, have become indispensable tools for solving a wide range of mathematical problems. Their applications are particularly prominent in the context of ordinary differential equations that satisfy specific modelling criteria. The significance of orthogonal polynomials extends beyond differential equations, finding utility in approximation theory and other areas of study. The discovery of orthogonal polynomials can be traced back to 1784 when Legendre [1] made a groundbreaking contribution to mathematics by identifying and exploring these polynomials. Since then, the importance and application of orthogonal polynomials have continuously expanded, solidifying their place as indispensable instruments in mathematical research and problem-solving [2–4].

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Orthogonal polynomials, represented as Π_d and Π_t with degrees d and t respectively, establish their orthogonality through the integral equation:

$$\int_{\alpha}^{\beta} \Pi_d(y)\Pi_t(y)\xi(y)dy = 0, \quad \text{for} \quad d \neq t,$$

where $\xi(y)$ is a well-defined weight function within the interval (α, β).

Among the various types of orthogonal polynomials, Ultraspherical polynomials (UP) hold significant importance. As elucidated in [5–8], a notable symbolic correlation known as T_R has been found between the generating function of orthogonal polynomials (OP) and the integral representation of real functions. This correlation, which becomes apparent through classical algebraic methods, has resulted in the identification of numerous significant inequalities within the domain of orthogonal polynomials.

Fractional calculus operators are widely used in various disciplines within the applied sciences, especially in the study of geometric functions [3, 9–11]. A significant development in traditional fractional calculus is the fractional q-calculus, which finds applications in different domains. These applications are built upon the fundamental principles established in ordinary fractional calculus. For a more comprehensive understanding of this subject, readers are advised to consult resources, including [12–17].

Fractional derivatives and integral operators have extensive applications in various scientific and engineering fields. They are used to represent anomalous diffusion and transport phenomena, characterize stress-strain relationships in viscoelastic materials, and design controllers and filters in control theory. In signal processing, they enhance edge detection and feature extraction, while in electrical engineering, they provide precise descriptions of non-integer order dynamics in circuits. Integral operators are crucial in solving differential equations, potential theory, image reconstruction, probability and statistics, and financial mathematics. These tools offer advanced models and solutions that effectively capture the complexities of real-world behaviors and systems.

Examining the implications of this discovery on the theory of q-Ultraspherical polynomials (q-UP) within this context is of significant significance. In this investigation, our efforts have been directed toward this direction, leveraging the aforementioned findings to formulate novel nonlinear connection equations for q-UP and drawing comparisons with their classical counterparts.

This research explores the characteristics of a specific class by establishing correlations between selected bi-univalent functions and q-Ultraspherical polynomials (q-UP). The next section presents important mathematical notations and definitions, providing a comprehensive framework for further analysis. In this study, we thoroughly examine the properties and behaviors of a particular category. By connecting carefully chosen bi-univalent functions and q-UP, our goal is to clarify their interactions and implications.

The following section explains essential mathematical concepts and terminologies that are necessary for understanding the subsequent discussions.

1. Preliminaries

A class of polynomials, known as q-analog of the (UP), was discovered by Askey and Ismail in 1983 [6]. These polynomials are defined as follows:

$$\mathscr{B}^{(\varkappa)}(\varepsilon, z; q) = \sum_{n=0}^{\infty} \mathscr{C}_n^{(\varkappa)}(\varepsilon; q) z^n, \qquad (\varepsilon \in [-1, 1], z \in \mathbb{U}), \tag{1}$$

where $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ is the the open unit disk in the complex plane $\mathbb{C}$.

In 2006, Chakrabarti et al. made a significant discovery regarding a set of polynomials [18]. These polynomials can be regarded as the q-analog of the Ultraspherical polynomials (UP). The first few terms of these polynomials are

$$\begin{aligned} &\mathscr{C}_0^{(\varkappa)}(\varepsilon; q) = 1, \qquad \mathscr{C}_1^{(\varkappa)}(\varepsilon; q) = 2\langle \varkappa; q \rangle \varepsilon, \\ &\mathscr{C}_2^{(\varkappa)}(\varepsilon; q) = 2\left(\langle \varkappa; q^2 \rangle + \langle \varkappa; q \rangle^2\right) \varepsilon^2 - \langle \varkappa; q^2 \rangle, \end{aligned} \tag{2}$$

the q-bracket, denoted as $\langle \varkappa; q \rangle$, is defined explicitly for $0 < q < 1$ (as shown in [19]), by

$$\langle \varkappa; q \rangle = \begin{cases} \frac{1-q^{\varkappa}}{1-q} & , \quad \text{if } \varkappa \in \mathbb{C}\backslash\{0\} \\ q^{n-1} + q^{n-2} + \cdots + q + 1 = \sum\limits_{\jmath=0}^{n-1} q^{\jmath} & , \quad \text{if } \varkappa = n \in \mathbb{N} \\ 1 & , \quad \text{if } \quad q \to 0^+, \ \varkappa \in \mathbb{C}\backslash\{0\} \\ \varkappa & , \quad \text{if } \quad q \to 1^-, \ \varkappa \in \mathbb{C}\backslash\{0\} \end{cases}. \tag{3}$$

Let $f(z)$ be an analytic (regular, holomorphic or monogenic) function defined in the open unit disk $\mathbb{U}$. If $f(z)$ can be expressed as:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{4}$$

then we say that $f(z)$ belongs to a specific class $\mathcal{A}$. The subclass $\mathcal{S}$ of $\mathcal{A}$ contains the analytic functions that are univalent and satisfy the normalization conditions:

$$f(0) = 0 \qquad \text{and} \qquad f'(0) = 1. \tag{5}$$

A regular function η that satisfying

$$|\eta(z)| < 1 \qquad \text{and} \qquad \eta(0) = 0, \qquad (z \in \mathbb{U}),$$

is commonly referred to as a Schwartz function. If we have two functions, f_1 and f_2, defined on $\mathcal{A}$, we state that f_1 is subordinated to f_2, denoted as $f_1 \prec f_2$, if there exists a

Schwartz function η such that $f_1(z) = f_2(\eta(z))$ for all $z \in \mathbb{U}$ (for more details, see [20]). The class P is connected to Carathéodory's functions functions, as defined by Miller [21]. These functions satisfy the following criteria:

$$\varphi(0) = 1 \quad \text{and} \quad \mathrm{Re}\{\varphi(z)\} > 0, \quad (\forall z \in \mathbb{U})\,.$$

A Taylor series expansion can provide an exact representation for any polynomial function $\varphi(z) \in \mathrm{P}$. This expansion is given by:

$$\varphi(z) = 1 + \sum_{n=1}^{\infty} \varphi_n z^n, \quad (z \in \mathbb{U}). \tag{6}$$

where

$$|\varphi_n| \leq 2, \quad \text{for all } n \geq 1. \tag{7}$$

This is in accordance with Carathéodory's Lemma (see [21]). Essentially, $\varphi \in \mathrm{P}$ if and only if

$$\varphi(z) \prec (1+z)(1-z)^{-1}, \quad (z \in \mathbb{U}).$$

Within the subfamily $\mathcal{S}$, each function f has an inverse function denoted as f^{-1}. This inverse function is characterized by the equations

$$z = f^{-1}(f(z)) \quad \text{and} \quad \xi = f(f^{-1}(\xi)),$$

where

$$r_0(f) \geq \frac{1}{4} \quad , \quad |\xi| < r_0(f) \quad , \quad \text{and} \quad (z \in \mathbb{U}).$$

Additionally, the expression for the inverse function $f^{-1}(\xi)$ can be represented by the equation

$$\hbar(\xi) = f^{-1}(\xi) = \xi\left(1 - a2\xi + \xi^2(-a3 + 2a_2^2) - \xi^3(a_4 + 5a2^3 - 5a3a_2) + \cdots\right). \tag{8}$$

A function $f(z)$ belongs to the subclass $\mathcal{S}$ and is called bi-univalent if its inverse function $f^{-1}(\xi)$ also belongs to $\mathcal{S}$. The subclass Σ of $\mathcal{S}$ contains all bi-univalent functions within $\mathbb{U}$. The table below provides examples of certain functions in the class Σ and their respective inverse functions.

Table 1: Lists some of the functions in class Σ along with their inverses.

The function	**The corresponding inverse**
$f_1(z) = \frac{z}{1-z}$	$f_1^{-1}(\xi) = \frac{\xi}{1+\xi}$
$f_2(z) = -\log(1-z)$	$f_2^{-1}(\xi) = \frac{e^{2\xi}-1}{e^{2\xi}+1}$
$f_3(z) = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$	$f_3^{-1}(\xi) = e^{-\xi}\left(e^{\xi}-1\right)$

We begin by revisiting the foundational concepts of q-difference calculus, a field that underpins the analytic structure and behavior of q-difference equations and their multifaceted applications. The mathematical framework of q-calculus has garnered sustained interest due to its extensive utility across disciplines such as physics, quantum mechanics, and, more prominently, geometric function theory. Central to this framework is the q-differential operator $\eth_q$, which serves as a critical tool in the study and classification of various subclasses of univalent and multivalent functions. A seminal contribution in this area was made by Ismail et al. in 1990 [22], who introduced a q-extension of starlike functions defined on the open unit disk, thereby initiating a new line of inquiry within the theory of geometric function spaces. More recently, Srivastava [23] provided a thorough exposition on the theoretical underpinnings and geometric ramifications of q-analogues of fractional derivative operators. To support continued exploration of q-calculus within geometric function theory, a substantial body of literature is available, beginning with foundational works [24, 25] and progressing through more recent advancements such as [26–34].

Definition 1. *[19] The q-difference operator, also known as the q-derivative, is defined for a function f when $0 < q < 1$ as follows:*

$$\eth_q f(z) = \begin{cases} \frac{f(z)-f(qz)}{z-qz}, & \text{if } z \neq 0, \\ f'(0), & \text{if } \quad z = 0, \\ f'(z), & \text{if } \quad q \to 1^-, \ z \neq 0. \end{cases} .$$

Remark 1. *For $f \in \mathcal{A}$ of the form (4), it can easily be seen that:*

$$\eth_q f(z) = \eth_q \left\{ z + \sum_{n=2}^{\infty} a_n z^n \right\} = 1 + \sum_{n=2}^{\infty} \langle n; q\rangle a_n \eta^{n-1}, \qquad (z \in \mathbb{U}),$$

and for f^{-1} of the form (8), we have

$$\eth_q \left(f^{-1}(\xi)\right) = 1 - \langle 2; q\rangle \ a_2 \ \xi + \langle 3; q\rangle \ (2a_2^3 - a_3) \ \xi^2 + \cdots , \qquad (\xi \in \mathbb{U}).$$

Ravikumar [35] introduced the fractional q-differintegral operator denoted by $\psi_q^\delta f(z)$ for a function $f(z)$ of the form (4). The operator is defined as follows:

$$\psi_q^\delta f(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma_q(2-\delta)\Gamma_q(n+1)}{\Gamma_q(n+1-\delta)} a_n z^n, \qquad (\delta \leq 2, \ z \in \mathbb{U}) \tag{9}$$

where Γ_q represents the q-gamma function defined by

$$\Gamma_q(n) = \frac{(q;q)_\infty (1-q)^{1-n}}{(q^n;q)_\infty}, \qquad q \in (0,1), \tag{10}$$

and $(\kappa; q)_n$ denotes the q-shifted factorial defined for $n \in \mathbb{C}$ (see [36]). The recurrence relation for the q-gamma function is given by

$$\Gamma_q(n+1) = \langle n; q\rangle \Gamma_q(n) \tag{11}$$

where $\langle n; q\rangle$ represents the bracket of the form (3).

2. Definition and Examples

In this section, we will introduce novel classifications of bi-univalent functions that exhibit a hierarchical relationship with the q-Ultraspherical polynomials (q-UP). These newly introduced subclasses provide a more sophisticated insight into the connection between bi-univalent functions and q-UP, enhancing our comprehension of their interrelationship and offering valuable insights into their properties and behaviors.

Definition 2. *Let $F \in \mathbb{C} \setminus \{0\}$ and $0 \leq \lambda \leq 1$. A bi-univalent function f, defined in equation (4), belongs to the class $\mathcal{B}_\Sigma(F, \lambda, \delta, \mathscr{B}^{(\varkappa)}(\varepsilon, z; q))$ if it satisfies the following conditions:*

$$1 + \frac{1}{F}\left(\eth_q\left(\psi_q^\delta f(z)\right) + \lambda z \eth_q^2\left(\psi_q^\delta f(z)\right) - 1\right) \prec \mathscr{B}^{(\varkappa)}(\varepsilon, z; q), \tag{12}$$

and

$$1 + \frac{1}{F}\left(\eth_q\left(\psi_q^\delta \hbar(\xi)\right) + \lambda \xi \eth_q^2\left(\psi_q^\delta \hbar(\xi)\right) - 1\right) \prec \mathscr{B}^{(\varkappa)}(\varepsilon, \xi; q). \tag{13}$$

For $\varepsilon \in (\frac{1}{2}, 1]$ and $\delta \leq 2$, the function $\hbar(\xi)$ is defined as the inverse of $f(\xi)$, which is given by equation (8). The generating function of the q-analog of the (UP), denoted as $\mathscr{B}^{(\varkappa)}(\varepsilon, z; q)$, is defined by equation (1).

Example 1. *Let $\lambda = 1$, $F \in \mathbb{C} \setminus \{0\}$. A bi-univalent function f, defined in equation (4), belongs to the class $\mathcal{B}_\Sigma(F, 1, \delta, \mathscr{B}^{(\varkappa)}(\varepsilon, z; q))$ if it satisfies the following conditions:*

$$1 + \frac{1}{F}\left(\eth_q\left(\psi_q^\delta f(z)\right) + z \eth_q^2\left(\psi_q^\delta f(z)\right) - 1\right) \prec \mathscr{B}^{(\varkappa)}(\varepsilon, z; q),$$

and

$$1 + \frac{1}{F}\left(\eth_q\left(\psi_q^\delta \hbar(\xi)\right) + \xi \eth_q^2\left(\psi_q^\delta \hbar(\xi)\right) - 1\right) \prec \mathscr{B}^{(\varkappa)}(\varepsilon, \xi; q).$$

For $\varepsilon \in (\frac{1}{2}, 1]$ and $\delta \leq 2$, the function $\hbar(\xi)$ is defined as the inverse of $f(\xi)$, which is given by equation (8). The generating function of the q-analog of the (UP), denoted as $\mathscr{B}^{(\varkappa)}(\varepsilon, z; q)$, is defined by equation (1).

Example 2. *Let $\lambda = 0$, $F \in \mathbb{C} \setminus \{0\}$. A bi-univalent function f, defined in equation (4), belongs to the class $\mathcal{B}_\Sigma(F, 0, \delta, \mathscr{B}^{(\varkappa)}(\varepsilon, z; q))$ if it satisfies the following conditions:*

$$1 + \frac{1}{F}\left(\eth_q\left(\psi_q^\delta f(z)\right) - 1\right) \prec \mathscr{B}^{(\varkappa)}(\varepsilon, z; q),$$

and

$$1+\frac{1}{F}\left(\eth_q\left(\psi_q^{\delta}\hbar(\xi)\right)-1\right)\prec \mathscr{B}^{(\varkappa)}(\varepsilon,\xi;q).$$

For $\varepsilon\in(\frac{1}{2},1]$ *and* $\delta\le 2$*, the function* $\hbar(\xi)$ *is defined as the inverse of* $f(\xi)$*, which is given by equation* (8)*. The generating function of the q-analog of the (UP), denoted as* $\mathscr{B}^{(\varkappa)}(\varepsilon,z;q)$*, is defined by equation* (1)*.*

Recently, Amourah et al. [37] and Alsoboh et al. [38–40] have introduced new categories of bi-univalent functions that possess analyticity using q-Ultraspherical polynomials (q-UP). These contributions primarily focused on clarifying Fekete-Szegö inequalities and constraints related to the coefficients $|a_2|$ and $|a_3|$ for functions in these newly established subclasses. The study of bi-univalent functions associated with q-UP has attracted significant attention from scholars, as evident in previous works such as [35, 41–44].

The primary objective of our research is to examine the characteristics exhibited by bi-univalent functions associated with q-Ultraspherical polynomials (q-UP). To achieve this aim, we consider the following definitions as fundamental aspects of our exploratory analysis.

3. Initial Bounds and the Fekete and Szegö Functional

In this section, we provide estimates for the coefficients pertaining to the class defined in Definition 2, denoted by $\mathcal{B}\Sigma(F,\lambda,\delta,\mathscr{B}^{(\varkappa)}(\varepsilon,z;q))$. In 1933, Fekete and Szegö [45] derived a rigorous upper limit for the functional $\eta a_2^2-a_3$, where η belongs to the interval $(0,1)$. This bound, known as the traditional Fekete-Szegö inequality, remains a challenging task to determine precisely for any compact family of functions f within the set $\mathcal{A}$ with arbitrary complex values of η. Building upon the findings of Zaprawa [46], we delve into the subsequent Szegö inequality concerning functions belonging to the class $\mathcal{B}_{\Sigma}(F,\lambda,\delta,\mathscr{B}^{(\varkappa)}(\varepsilon,z;q))$.

Theorem 1. *Suppose* $f\in\Sigma$ *as defined by equation (4) belongs to the class. Then, the inequalities*

$$|a_2|\le\frac{2\varepsilon\langle\varkappa;q\rangle\langle 2-\delta;q\rangle\left|F\right|\sqrt{\frac{2\varepsilon}{\Gamma_q(3)}\langle 3-\delta;q\rangle\,\langle\varkappa;q\rangle}}{\sqrt{Q_1}},$$

and

$$|a_3|\le\frac{4F^2\,\langle 2-\delta;q\rangle^2\,\langle\varkappa;q\rangle^2\varepsilon^2}{\langle 2;q\rangle^2\,\left(\Gamma_q(3)\right)^2(1+\lambda)^2}+\frac{2\langle\varkappa;q\rangle\,\langle 3-\delta;q\rangle\langle 2-\delta;q\rangle\,\varepsilon\left|F\right|}{\langle 3;q\rangle\,\Gamma_q(4)\left|1+\lambda\langle 2;q\rangle\right|}.$$

apply, where

$$Q_1=\left(\begin{array}{c}4\langle 3;q\rangle^2\langle 2-\delta;q\rangle\,\left(1+\lambda\langle 2;q\rangle\right)F\,\langle\varkappa;q\rangle^2-\\ 2\langle 2;q\rangle^3(1+\lambda)^2\,\langle 3-\delta;q\rangle\left(\langle\varkappa;q^2\rangle+\langle\varkappa;q\rangle^2\right)\varepsilon^2+\langle 2;q\rangle^3(1+\lambda)^2\,\langle 3-\delta;q\rangle\,\langle\varkappa;q^2\rangle\end{array}\right)$$

.

Proof. Let $f \in \mathcal{B}_{\Sigma}(\digamma, \lambda, \delta, \mathscr{B}^{(\varkappa)}(\varepsilon, z; q))$. Then, from Definition 2, for some regular functions ψ and ϑ, $|\psi(z)| < 1$ and $|\vartheta(\xi)| < 1$ for all $z, \xi \in \mathbb{U}$, where $\psi(0) = 0 = \vartheta(0)$. We have

$$1 + \frac{1}{\digamma}\Big(\eth_q\left(\psi_q^{\delta} f(z)\right) + \lambda z \eth_q^2\left(\psi_q^{\delta} f(z)\right) - 1\Big) = \mathscr{B}^{(\varkappa)}(\varepsilon, \psi(z); q)\,, \tag{14}$$

and

$$1 + \frac{1}{\digamma}\Big(\eth_q\left(\psi_q^{\delta} \hbar(\xi)\right) + \lambda \xi \eth_q^2\left(\psi_q^{\delta} \hbar(\xi)\right) - 1\Big) = \mathscr{B}^{(\varkappa)}(\varepsilon, \vartheta(\xi); q)\,. \tag{15}$$

From the equalities (14) and (15)

$$1 + \frac{1}{\digamma}\Big(\eth_q\left(\psi_q^{\delta} f(z)\right) + \lambda z \eth_q^2\left(\psi_q^{\delta} f(z)\right) - 1\Big) = \begin{pmatrix} 1 + C_1^{(\varkappa)}(\varepsilon; q) c_1 z + \\ \left[C_1^{(\varkappa)}(\varepsilon; q) c_2 + C_2^{(\varkappa)}(\varepsilon; q) c_1^2\right] z^2 + \cdots \end{pmatrix}, \tag{16}$$

and

$$1 + \frac{1}{\digamma}\Big(\eth_q\left(\psi_q^{\delta} \hbar(\xi)\right) + \lambda \xi \eth_q^2\left(\psi_q^{\delta} \hbar(\xi)\right) - 1\Big) = \begin{pmatrix} 1 + C_1^{(\varkappa)}(\varepsilon; q) d_1 \xi + \\ \left[C_1^{(\varkappa)}(\varepsilon; q) d_2 + C_2^{(\varkappa)}(\varepsilon; q) d_1^2\right] \xi^2 + \cdots \end{pmatrix}. \tag{17}$$

It is a widely accepted fact that if

$$|\psi(z)| = \left|c_1 z + c_2 z^2 + c_3 z^3 + \cdots\right| < 1, \quad (z \in \mathbb{U})$$

and

$$|\vartheta(\xi)| = \left|d_1 \xi + d_2 \xi^2 + d_3 \xi^3 + \cdots\right| < 1, \quad (\xi \in \mathbb{U}),$$

then

$$|c_j| \leq 1 \text{ and } |d_j| \leq 1 \text{ for all } j \in \mathbb{N}. \tag{18}$$

In view of (4), (8), from (16) and (17), we obtain

$$\begin{aligned} 1 + \frac{1+\lambda}{\digamma}\left(\frac{\langle 2; q\rangle\, \Gamma_q(3)\Gamma_q(2-\delta)}{\Gamma_q(3-\delta)}\right) a_2 z + \frac{1+\langle 2; q\rangle\lambda}{\digamma}\left(\frac{\langle 3; q\rangle\, \Gamma_q(4)\Gamma_q(2-\delta)}{\Gamma_q(4-\delta)}\right) a_3 z^2 + \cdots \\ = 1 + C_1^{(\varkappa)}(\varepsilon; q) c_1 z + \left[C_1^{(\varkappa)}(\varepsilon; q) c_2 + C_2^{(\varkappa)}(\varepsilon; q) c_1^2\right] z^2 + \cdots, \end{aligned}$$

and

$$\begin{aligned} 1 - \frac{1+\lambda}{\digamma}\left(\frac{\langle 2; q\rangle\, \Gamma_q(3)\Gamma_q(2-\delta)}{\Gamma_q(3-\delta)}\right) a_2 \xi + \frac{1+\lambda\langle 2; q\rangle}{\digamma}\left(\frac{\langle 3; q\rangle\, \Gamma_q(4)\Gamma_q(2-\delta)}{\Gamma_q(4-\delta)}\right)(2a_2^2 - a_3)\xi^2 \\ + \cdots = 1 + C_1^{(\varkappa)}(\varepsilon; q) d_1 \xi + \left[C_1^{(\varkappa)}(\varepsilon; q) d_2 + C_2^{(\varkappa)}(\varepsilon; q) d_1^2\right] \xi^2 + \cdots. \end{aligned}$$

Upon carefully examining the relevant coefficients presented in (16) and (17), the following findings emerge:

$$\frac{1+\lambda}{\digamma}\left(\frac{\langle 2; q\rangle\, \Gamma_q(3)\Gamma_q(2-\delta)}{\Gamma_q(3-\delta)}\right) a_2 = C_1^{(\varkappa)}(\varepsilon; q) c_1, \tag{19}$$

$$\frac{1+\lambda\langle 2;q\rangle}{\digamma}\left(\frac{\langle 3;q\rangle\,\Gamma_q(4)\Gamma_q(2-\delta)}{\Gamma_q(4-\delta)}\right)a_3 = C_1^{(\varkappa)}(\varepsilon;q)c_2 + C_2^{(\varkappa)}(\varepsilon;q)c_1^2, \tag{20}$$

and

$$-\frac{1+\lambda}{\digamma}\left(\frac{\langle 2;q\rangle\,\Gamma_q(3)\Gamma_q(2-\delta)}{\Gamma_q(3-\delta)}\right)a_2 = C_1^{(\varkappa)}(\varepsilon;q)d_1, \tag{21}$$

$$\frac{1+\lambda\langle 2;q\rangle}{\digamma}\left(\frac{\langle 3;q\rangle\,\Gamma_q(4)\Gamma_q(2-\delta)}{\Gamma_q(4-\delta)}\right)(2a_2^2-a_3) = C_1^{(\varkappa)}(\varepsilon;q)d_2 + C_2^{(\varkappa)}(\varepsilon;q)d_1^2. \tag{22}$$

It is clear from equations (19) and (21) that

$$c_1 = -d_1 \tag{23}$$

and

$$\begin{aligned}
&\frac{2(1+\lambda)^2}{\digamma^2}\left(\frac{\langle 2;q\rangle\,\Gamma_q(3)\Gamma_q(2-\delta)}{\Gamma_q(3-\delta)}\right)^2 a_2^2 = \left[C_1^{(\varkappa)}(\varepsilon;q)\right]^2\left(c_1^2+d_1^2\right)\\
&a_2^2 = \frac{\digamma^2\,\langle 2-\delta;q\rangle^2\left[C_1^{(\varkappa)}(\varepsilon;q)\right]^2}{2\langle 2;q\rangle^2\,\left(\Gamma_q(3)\right)^2(1+\lambda)^2}\left(c_1^2+d_1^2\right)
\end{aligned}. \tag{24}$$

By adding equation (20) to equation (22), we obtain

$$\frac{1+\lambda\langle 2;q\rangle}{\digamma}\left(\frac{2\langle 3;q\rangle\,\Gamma_q(4)\Gamma_q(2-\delta)}{\Gamma_q(4-\delta)}\right)a_2^2 = C_1^{(\varkappa)}(\varepsilon;q)(c_2+d_2) + C_2^{(\varkappa)}(\varepsilon;q)(c_1^2+d_1^2). \tag{25}$$

By substituting the value of $(c_1^2+d_1^2)$ from (24) and performing some calculations, we obtain:

$$a_2^2 = \frac{[2-\delta]_q^2\langle 3-\delta;q\rangle\,\digamma^2\left[C_1^{(\varkappa)}(\varepsilon;q)\right]^3(c_2+d_2)}{2\Gamma_q(3)\left(\langle 3;q\rangle^2\langle 2-\delta;q\rangle\;(1+\lambda\langle 2;q\rangle)\,\digamma\;\left[C_1^{(\varkappa)}(\varepsilon;q)\right]^2 - \langle 2;q\rangle^3(1+\lambda)^2\,\langle 3-\delta;q\rangle\,C_2^{(\varkappa)}(\varepsilon;q)\right)}.$$

By applying for the coefficients c_2 and d_2 and utilizing equations (11) and (2), we can obtain.

$$|a_2| \le \frac{2\varepsilon\langle\varkappa;q\rangle\langle 2-\delta;q\rangle\left|\digamma\right|\sqrt{\frac{2\,\varepsilon}{\Gamma_q(3)}\langle 3-\delta;q\rangle\,\langle\varkappa;q\rangle}}{\sqrt{\begin{array}{c}\Big(4\langle 3;q\rangle^2\langle 2-\delta;q\rangle\;(1+\lambda\langle 2;q\rangle)\,\digamma\,\langle\varkappa;q\rangle^2 - 2\langle 2;q\rangle^3(1+\lambda)^2\,\langle 3-\delta;q\rangle\,\times\\ \left(\langle\varkappa;q^2\rangle+\langle\varkappa;q\rangle^2\right)\,\varepsilon^2 + \langle 2;q\rangle^3(1+\lambda)^2\,\langle 3-\delta;q\rangle\,\langle\varkappa;q^2\rangle\Big)\end{array}}}.$$

To construct the second assertion, we can subtract (22) from (20). This will yield

$$\frac{2\big(1+\lambda\langle 2;q\rangle\big)}{\digamma}\left(\frac{\langle 3;q\rangle\,\Gamma_q(4)\Gamma_q(2-\delta)}{\Gamma_q(4-\delta)}\right)(a_3-a_2^2) = C_1^{(\varkappa)}(\varepsilon;q)\,(c_2-d_2) + C_2^{(\varkappa)}(\varepsilon;q)\,\left(c_1^2-d_1^2\right). \tag{26}$$

Then, considering (23) and (24), Equation (26) can be rewritten as

$$a_3 = \frac{\digamma^2 \langle 2-\delta; q\rangle^2 \left[C_1^{(\varkappa)}(\varepsilon;q)\right]^2}{2\langle 2;q\rangle^2 \left(\Gamma_q(3)\right)^2 (1+\lambda)^2}\left(c_1^2+d_1^2\right) + \frac{\digamma \langle 3-\delta;q\rangle\langle 2-\delta;q\rangle C_1^{(\varkappa)}(\varepsilon;q)}{2\langle 3;q\rangle\,\Gamma_q(4)\big(1+\lambda\langle 2;q\rangle\big)}\left(c_2-d_2\right).$$

Using (11) and (2), we can say that

$$|a_3| \leq \frac{4\digamma^2 \langle 2-\delta;q\rangle^2 \langle\varkappa;q\rangle^2\varepsilon^2}{\langle 2;q\rangle^2 \left(\Gamma_q(3)\right)^2 (1+\lambda)^2} + \frac{2\langle\varkappa;q\rangle\,\langle 3-\delta;q\rangle\langle 2-\delta;q\rangle\,\varepsilon\big|\digamma\big|}{\langle 3;q\rangle\,\Gamma_q(4)\big|1+\lambda\langle 2;q\rangle\big|}.$$

The proof of the theorem is considered comprehensive and conclusive.

Theorem 2. *Let $f \in \Sigma$ given by (4) belongs to the class $\mathcal{B}_\Sigma(\digamma, \lambda, \delta, \mathscr{B}^{(\varkappa)}(\varepsilon, z; q))$ and $\eta \in \mathbb{R}$. Then, we have*

$$\left|a_3 - \eta a_2^2\right| \leq \begin{cases} \dfrac{2\left|\digamma\langle\varkappa;q\rangle\right|\varepsilon}{\langle 3;q\rangle\,\Gamma_q(4)\big(1+\lambda\langle 2;q\rangle\big)}, & |1-\eta| \leq \left|1 - \dfrac{(1+\lambda)^2\,\langle 2;q\rangle^3\,\langle 3-\delta;q\rangle\,\Big(2\big(\langle\varkappa;q^2\rangle+\langle\varkappa;q\rangle^2\big)\varepsilon^2-\langle\varkappa;q^2\rangle\Big)}{8\langle\varkappa;q\rangle^2\langle 3;q\rangle^2\big(1+\lambda\langle 2;q\rangle\big)\,\langle 2-\delta;q\rangle\,\digamma\,\varepsilon}\right|, \\[2ex] 4\left|\langle\varkappa;q\rangle\right|\varepsilon\left|\mathcal{H}(\eta)\right|, & |1-\eta| \geq \left|1 - \dfrac{\langle 2;q\rangle^3(1+\lambda)^2\,\langle 3-\delta;q\rangle\,\Big(2\big(\langle\varkappa;q^2\rangle+\langle\varkappa;q\rangle^2\big)\varepsilon^2-\langle\varkappa;q^2\rangle\Big)}{8\langle\varkappa;q\rangle^2\langle 3;q\rangle^2\big(1+\lambda\langle 2;q\rangle\big)\,\langle 2-\delta;q\rangle\,\digamma\,\varepsilon}\right|. \end{cases}$$

where

$$\mathcal{H}(\eta) = \frac{(1-\eta)\,[2-\delta]_q^2\langle 3-\delta;q\rangle\,\digamma^2\left[C_1^{(\varkappa)}(\varepsilon;q)\right]^2}{2\Gamma_q(3)\left(\langle 2-\delta;q\rangle\,\left(1+\lambda\langle 2;q\rangle\right)\digamma\left[C_1^{(\varkappa)}(\varepsilon;q)\right]^2\langle 3;q\rangle^2 - \langle 2;q\rangle^3(1+\lambda)^2\,\langle 3-\delta;q\rangle\,C_2^{(\varkappa)}(\varepsilon;q)\right)}.$$

Proof. If f belongs to the set $\mathcal{B}_\Sigma(\digamma, \lambda, \delta, \mathscr{B}^{(\varkappa)}(\varepsilon, z; q))$, as defined in equation (4), then we can derive from equations (25) and (26) the following result:

$$\begin{aligned} a_3 - \eta a_2^2 &= \\ &\frac{\digamma\langle 3-\delta;q\rangle\langle 2-\delta;q\rangle C_1^{(\varkappa)}(\varepsilon;q)}{2\langle 3;q\rangle\,\Gamma_q(4)\big(1+\lambda\langle 2;q\rangle\big)}\left(c_2-d_2\right) \\ &+ \frac{(1-\eta)\,[2-\delta]_q^2\langle 3-\delta;q\rangle\,\digamma^2\left[C_1^{(\varkappa)}(\varepsilon;q)\right]^3\left(c_2+d_2\right)}{2\Gamma_q(3)\left(\langle 3;q\rangle^2\langle 2-\delta;q\rangle\,\left(1+\lambda\langle 2;q\rangle\right)\digamma\left[C_1^{(\varkappa)}(\varepsilon;q)\right]^2 - \langle 2;q\rangle^3(1+\lambda)^2\,\langle 3-\delta;q\rangle\,C_2^{(\varkappa)}(\varepsilon;q)\right)} \\ &= C_1^{(\varkappa)}(\varepsilon;q)\left[\left(\mathcal{H}(\eta) + \frac{\digamma\langle 3-\delta;q\rangle\langle 2-\delta;q\rangle}{2\langle 3;q\rangle\,\Gamma_q(4)\big(1+\lambda\langle 2;q\rangle\big)}\right)c_2 + \left(\mathcal{H}(\eta) - \frac{\digamma\langle 3-\delta;q\rangle\langle 2-\delta;q\rangle}{2\langle 3;q\rangle\,\Gamma_q(4)\big(1+\lambda\langle 2;q\rangle\big)}\right)d_2\right], \end{aligned}$$

where

$$\mathcal{H}(\eta) = \frac{(1-\eta)\,[2-\delta]_q^2\langle 3-\delta;q\rangle\,\digamma^2\left[C_1^{(\varkappa)}(\varepsilon;q)\right]^2}{2\Gamma_q(3)\left(\langle 3;q\rangle^2\langle 2-\delta;q\rangle\,\left(1+\lambda\langle 2;q\rangle\right)\digamma\left[C_1^{(\varkappa)}(\varepsilon;q)\right]^2 - \langle 2;q\rangle^3(1+\lambda)^2\,\langle 3-\delta;q\rangle\,C_2^{(\varkappa)}(\varepsilon;q)\right)}$$

Then, it can be inferred that

$$\left|a_3-\eta a_2^2\right| \leq \begin{cases} \frac{|\digamma|\left|C_1^{(\varkappa)}(\varepsilon;q)\right|}{\langle 3;q\rangle\,\Gamma_q(4)\left(1+\lambda\langle 2;q\rangle\right)}, & |\mathcal{H}(\eta)| \leq \frac{\digamma\langle 3-\delta;q\rangle\langle 2-\delta;q\rangle}{2\langle 3;q\rangle\,\Gamma_q(4)\left(1+\lambda\langle 2;q\rangle\right)}, \\ 2\left|C_1^{(\varkappa)}(\varepsilon;q)\right|\left|\mathcal{H}(\eta)\right|, & |\mathcal{H}(\eta)| \geq \frac{\digamma\langle 3-\delta;q\rangle\langle 2-\delta;q\rangle}{2\langle 3;q\rangle\,\Gamma_q(4)\left(1+\lambda\langle 2;q\rangle\right)}. \end{cases}$$

The proof of Theorem 2 is now complete.

4. Corollaries

The subsequent corollaries, which approximately correlate with Examples 1 and 2, are deduced from Theorems 1 and 2.

Corollary 1. *If $f \in \Sigma$ given by (4) belongs to the class $\mathcal{B}_\Sigma(\digamma, 1, \delta, \mathscr{B}^{(\varkappa)}(\varepsilon, z; q))$. Then*

$$|a_2| \leq \frac{\langle 2-\delta;q\rangle\left|\digamma\langle\varkappa;q\rangle\right|\varepsilon\sqrt{2\,\langle 3-\delta;q\rangle\,\langle\varkappa;q\rangle\varepsilon}}{\sqrt{\begin{array}{c}\Gamma_q(3)\,\Big(\big(\langle 3;q\rangle^2\langle 2-\delta;q\rangle\,\left(1+\langle 2;q\rangle\right)\digamma\,\langle\varkappa;q\rangle^2-2\langle 2;q\rangle^3\,\langle 3-\delta;q\rangle\,\left(\langle\varkappa;q^2\rangle+\langle\varkappa;q\rangle^2\right)\,\varepsilon^2\\ +\langle 2;q\rangle^3\,\langle 3-\delta;q\rangle\,\langle\varkappa;q^2\rangle\big)\Big)\end{array}}},$$

$$|a_3| \leq \frac{\digamma^2\,\langle 2-\delta;q\rangle^2\,\langle\varkappa;q\rangle^2\varepsilon^2}{\langle 2;q\rangle^2\,\left(\Gamma_q(3)\right)^2}+\frac{2\left|\digamma\,[\varkappa]_q\right|\,\langle 3-\delta;q\rangle\langle 2-\delta;q\rangle\,\varepsilon}{\langle 3;q\rangle\,\Gamma_q(4)\left|1+\langle 2;q\rangle\right|},$$

and

$$\left|a_3-\eta a_2^2\right| \leq \begin{cases} \frac{2\left|\digamma\langle\varkappa;q\rangle\right|\varepsilon}{\langle 3;q\rangle\,\Gamma_q(4)(1+\langle 2;q\rangle)}, & |1-\eta| \leq \left|1-\frac{4\langle 2;q\rangle^3\,\langle 3-\delta;q\rangle\,\left(2\left(\langle\varkappa;q^2\rangle+\langle\varkappa;q\rangle^2\right)\varepsilon^2-\langle\varkappa;q^2\rangle\right)}{8\langle\varkappa;q\rangle^2\langle 3;q\rangle^2(1+\langle 2;q\rangle)\,\langle 2-\delta;q\rangle\,\digamma\,\varepsilon}\right|, \\ 4\left|\langle\varkappa;q\rangle\right|\varepsilon\left|\mathcal{H}(\eta)\right|, & |1-\eta| \geq \left|1-\frac{4\langle 2;q\rangle^3\,\langle 3-\delta;q\rangle\,\left(2\left(\langle\varkappa;q^2\rangle+\langle\varkappa;q\rangle^2\right)\varepsilon^2-\langle\varkappa;q^2\rangle\right)}{8\langle\varkappa;q\rangle^2\langle 3;q\rangle^2(1+\langle 2;q\rangle)\,\langle 2-\delta;q\rangle\,\digamma\,\varepsilon}\right|. \end{cases}$$

where

$$\mathcal{H}(\eta)=\frac{(1-\eta)\,[2-\delta]_q^2\langle 3-\delta;q\rangle\,\digamma^2\left[C_1^{(\varkappa)}(\varepsilon;q)\right]^2}{2\Gamma_q(3)\left(\langle 3;q\rangle^2\langle 2-\delta;q\rangle\,\left(1+\langle 2;q\rangle\right)\digamma\,\left[C_1^{(\varkappa)}(\varepsilon;q)\right]^2-4\langle 2;q\rangle^3\,\langle 3-\delta;q\rangle\,C_2^{(\varkappa)}(\varepsilon;q)\right)}.$$

Corollary 2. *Let $f \in \Sigma$ given by (4) belongs to the class $\mathcal{B}_\Sigma(\digamma, 0, \delta, \mathscr{B}^{(\varkappa)}(\varepsilon, z; q))$. Then*

$$|a_2| \leq \frac{2\,\langle 2-\delta;q\rangle\left|\digamma\langle\varkappa;q\rangle\right|\varepsilon\sqrt{2\langle 3-\delta;q\rangle\langle\varkappa;q\rangle\varepsilon}}{\sqrt{\begin{array}{c}\Gamma_q(3)\,\left(4\langle 3;q\rangle^2\langle 2-\delta;q\rangle\,\digamma\,\langle\varkappa;q\rangle^2-2\langle 2;q\rangle^3\,\langle 3-\delta;q\rangle\,\left(\langle\varkappa;q^2\rangle+\langle\varkappa;q\rangle^2\right)\,\varepsilon^2\right.\\ \left.+\langle 2;q\rangle^3\,\langle 3-\delta;q\rangle\,\langle\varkappa;q^2\rangle\right)\end{array}}},$$

$$|a_3| \leq \frac{4F^2\langle \varkappa; q\rangle^2 \varepsilon^2}{(\langle 2; q\rangle} + \frac{2\left|F[\varkappa]_q\right|\varepsilon}{\langle 3; q\rangle},$$

and

$$\left|a_3 - \eta a_2^2\right| \leq \begin{cases} \frac{2\left|F\langle\varkappa;q\rangle\right|\varepsilon}{\langle 3;q\rangle\,\Gamma_q(4)}, & |1-\eta| \leq \left|1 - \frac{\langle 2;q\rangle^3\,\langle 3-\delta;q\rangle\left(2\left(\langle\varkappa;q^2\rangle+\langle\varkappa;q\rangle^2\right)\varepsilon^2-\langle\varkappa;q^2\rangle\right)}{8\langle\varkappa;q\rangle^2\langle 3;q\rangle^2\,\langle 2-\delta;q\rangle\,F\,\varepsilon}\right|, \\ 4\left|\langle\varkappa;q\rangle\right|\varepsilon\left|\mathcal{H}(\eta)\right|, & |1-\eta| \geq \left|1 - \frac{\langle 2;q\rangle^3\,\langle 3-\delta;q\rangle\left(2\left(\langle\varkappa;q^2\rangle+\langle\varkappa;q\rangle^2\right)\varepsilon^2-\langle\varkappa;q^2\rangle\right)}{8\langle\varkappa;q\rangle^2\langle 3;q\rangle^2\,\langle 2-\delta;q\rangle\,F\,\varepsilon}\right|. \end{cases}$$

where

$$\mathcal{H}(\eta) = \frac{(1-\eta)\,[2-\delta]_q^2\langle 3-\delta;q\rangle\,F^2\left[C_1^{(\varkappa)}(\varepsilon;q)\right]^2}{2\Gamma_q(3)\left(\langle 3;q\rangle^2\langle 2-\delta;q\rangle\,F\,\left[C_1^{(\varkappa)}(\varepsilon;q)\right]^2 - \langle 2;q\rangle^3\,\langle 3-\delta;q\rangle\,C_2^{(\varkappa)}(\varepsilon;q)\right)}.$$

5. Conclusion

The present study investigates the coefficient of three novel subclasses of bi-univalent functions defined in the open unit disk $\mathbb{U}$:

$$\mathcal{B}_\Sigma(F,\lambda,\delta,\mathscr{B}^{(\varkappa)}(\varepsilon,z;q)), \mathcal{B}_\Sigma(F,0,\delta,\mathscr{B}^{(\varkappa)}(\varepsilon,z;q)), \text{ and } \mathcal{B}_\Sigma(F,1,\delta,\mathscr{B}^{(\varkappa)}(\varepsilon,z;q))$$

as defined in Definition 2. We obtained the Taylor-Maclaurin coefficients $|a_2|, |a_3|$ for all three subclasses and provided estimates for the Fekete-Szegö functional problems. We also explored additional results arising from specific parameter configurations within our primary framework. Future studies could examine the Hankel determinants of these subclasses to gain deeper insights into their structure. We expect practical applications of the q-differintegral operator in various scientific domains, including mathematics and technology.

In future studies, one could investigate the maximum bounds of the Zalcman conjecture and analyze Hankel determinants for the classes of bi-convex and bi-close-to-convex functions. These directions present promising opportunities for novel discoveries and deeper exploration in the field.

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