EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 4, Article Number 6587
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Random Exact Solutions for the Stochastic Korteweg-de Vries Equation

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Abstract. This paper considers the stochastic Korteweg-de–de Vries (SKdV) equation perturbed by multiplicative Brownian motion, which is an important model reflecting the nonlinear science. After a systematic change and a rescaling, the SKdV equation is exactly recast into a deterministic KdV equation with random variable coefficients (KdV-RVCs). By using the Jacobi elliptic equation method and the generalized Riccati equation mapping approach, we obtain new exact solutions (rational, hyperbolic, trigonometric, and elliptic) for the KdV-RVCs. After that, we use the obtained solutions to obtain the stochastic solutions for the SKdV equation. Of practical interest, these results are related to specific physical systems: magnetized plasmas in astrophysics and in 1D/2D fusion, soliton propagation in fiber optical communication, and surface-wave dynamics in fluid mechanics. For example, the resulting solutions explain how noise-induced perturbations change soliton propagation in optical fibers and stabilize wave patterns in Turbulence. To give some (visual) impression of how multiplicative noise influences the solution behavior, we use pictures of the probability density distributions and ensemble-averaged trajectories as examples. These findings show that multiplicative Brownian motion has a stabilizing effect on the SKdV solutions by keeping their variations more or less near zero. This connection between stochastic modeling and experimental observations in plasma turbulence, nonlinear optical signal processing, and fluid wave dynamics has implications for the development of predictive theories for noise-driven nonlinear systems.

2020 Mathematics Subject Classifications: 35A20, 60H10, 60H15, 35Q51, 83C15

Key Words and Phrases: Exact stochastic solutions, analytical different methods, Brownian motion, random variable coefficients

1. Introduction

The KDV equation, also known as the Korteweg-de Vries equation, is a mathematical model that describes the behavior of waves in shallow water. The KDV equation derives its name from the Dutch mathematicians Diederik Korteweg and Gustav de Vries, who first

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DOI: https://doi.org/10.29020/nybg.ejpam.v18i4.6587

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introduced it in 1895 [1]. Over the years, the KDV equation has gained plenty of attention for its ability to represent complicated wave dynamics and its relevance in understanding various natural phenomena. It has been widely utilized in different areas, such as nonlinear optics, plasma physics, and fluid dynamics [2].

On the other side, it is crucial to consider the effect of random fluctuations in the KDV equation in order to accurately describe and predict the behavior of waves. By considering random fluctuations, the KDV equation can better capture the complex interactions and variations in wave dynamics, leading to more accurate predictions and simulations. Waves in nature rarely propagate in isolation, but rather interact with other waves, obstacles, and boundaries. These interactions can lead to wave-breaking, formation of solitons or rogue waves, and the creation of complex interference patterns. Random fluctuations in the KDV equation enable the modeling of such interactions and phenomena, offering insights into the behavior and characteristics of waves in different situations. This understanding is crucial in various fields, including marine engineering, coastal management, and the study of natural disasters such as tsunamis.

Here, we consider the following stochastic KDV equation perturbed by multiplicative noise as follows:

$$\mathcal{P}_t + a\mathcal{P}\mathcal{P}_x + b\mathcal{P}_{xxx} = \sigma\mathcal{P}\mathcal{B}_t, \tag{1}$$

where $\mathcal{P}(x,t)$ the profile of waves at position x and time t, $a\mathcal{P}\mathcal{P}_x$ represents the advection of the wave with a velocity of $a\mathcal{P}$, and $b\mathcal{P}_{xxx}$ represents the dispersion of the wave; σ is the amplitude of noise, and $\mathcal{B}(t)$ is the Brownian motion, $\mathcal{B}_t = \frac{\partial\mathcal{B}}{\partial t}$.

One of the most distinctive properties of the KDV equation is its soliton solutions. Solitons are solitary waves that retain their speed and shape as they propagate without dispersion or attenuation. The KDV equation generates a class of solitons known as KDV solitons. These solitons have a bell-shaped profile and show intriguing features such as stability, particle-like behavior, and non-interacting properties. They arise due to the delicate balance between the wave advection and dispersion terms within the equation. Therefore, many authors, for example [3–10], have extensively studied multiple versions of the KDV equation utilizing diverse methodology and methods from distinct viewpoints, whereas the analytical solutions of SKDV Eq. (1) is acquired by Mohammed et al. [11]. Furthermore, there are other authors, such as [12–18], have obtained the solutions of Eq. (1) in deterministic case.

Our aim of this study is to obtain the analytical stochastic solutions of the SKDV Eq. (1). This is accomplished by transforming the SKDV equation into a different KDV equation with random variable coefficients (KDVE-RVCs) by using the appropriate transformation. Moreover, the generalized Riccati equation mapping method (GREM-method) and the Jacobi elliptic method (JEF-method) are used to derive exact solutions for KDVE-RVCs. Finally, utilizing the employed transformation, we may obtain the stochastic solutions for the SKDV equation. Since all prior studies supposed that the solutions of wave equation of the SKdV equation was deterministic, the novelty of this work is that we assumed it to be stochastic. The achieved solutions are vital in comprehending different complex physical phenomena due to the relevance of the KDV equation (1) in plasma physics, nonlinear optics, and fluid dynamics. Also, we extend some of the previous findings, such the solutions given in [13]. Finally, we use MATLAB software to provide various graphs that illustrate the influence of the stochastic term on the acquired solutions.

The outline of this paper as follows: In Section 2, we conclude KDVE-RVCs from SKDV Eq. (1), then we use the JEF-method and GREM-method to discover the solutions of KDVE-RVCs. Section 3 provides the solutions for SKDV Eq. (1). Section 4, we see the impact of stochastic term on the achieved solutions. Lastly, we present this article's conclusions.

2. The Derivation and Solutions of KDVE-RVCs

To derive the KDVE-RVCs, we take the following transformation

$$\mathcal{P}(x,t) = \mathcal{V}(x,t)e^{\sigma\mathcal{B}(t)}. \tag{2}$$

Differentiating Eq. (2) with regards to x and t and utilizing the Itô derivatives rule, we get

$$\mathcal{P}_x = \mathcal{V}_x e^{\sigma\mathcal{B}(t)},\ \mathcal{P}_{xxx} = \mathcal{V}_{xxx} e^{\sigma\mathcal{B}(t)}. \tag{3}$$

and

$$\mathcal{P}_t = \mathcal{V}_t e^{\sigma\mathcal{B}(t)} + (\sigma\mathcal{V}\mathcal{B}_t + \frac{1}{2}\sigma^2\mathcal{V})e^{\sigma\mathcal{B}(t)}, \tag{4}$$

where the term $\frac{1}{2}\sigma^2\mathcal{V}$ is called the Itô correction term. Now, substituting from Eqs (4) and (3) into Eq. (1), we have the following KDVE-RVCs:

$$\mathcal{V}_t + b\mathcal{V}_{xxx} + A(t)\mathcal{V}\mathcal{V}_x + \frac{1}{2}\sigma^2\mathcal{V} = 0, \tag{5}$$

where $A(t) = ae^{\sigma\mathcal{B}(t)}$ and $\mathcal{V}$ is a real stochastic function.

2.1. JEF-method

We apply the JEF-method, as indicated in [19]. Assuming the solutions of KdVE-RVCs (5) have the form

$$\mathcal{V}(x,t) = \sum_{k=0}^{n} a_k(t)\mathcal{J}^k(\eta), \quad \eta = kx + \int_0^t \lambda(s)ds, \tag{6}$$

where $\mathcal{J}(\eta)$ represents one of these elliptic functions: $sn(\varpi\eta, m)$, $cn(\varpi\eta, m)$ or $dn(\varpi\eta, m)$. First let us balance $\mathcal{V}'''$ with $\mathcal{V}\mathcal{V}'$ to find n in Eq. (6) as:

$$n + 3 = n + n + 1 \quad \Longrightarrow n = 2.$$

Rewriting Eq. (6) as

$$\mathcal{V}(x,t) = a_0(t) + a_1(t)\mathcal{J}(\eta) + a_2(t)\mathcal{J}^2(\eta). \tag{7}$$

By differentiating Eq. (7) with regards to t and x, we obtain

$$\begin{aligned}
\mathcal{V}_t &= \dot{a}_0 + \dot{a}_1\mathcal{J} + \varpi\lambda a_1\mathcal{J}' + \dot{a}_2\mathcal{J}^2 + 2\varpi\lambda a_2\mathcal{J}'\mathcal{J}, \\
\mathcal{V}_x &= (\varpi k a_1 + 2\varpi k a_2\mathcal{J})\mathcal{J}', \\
\mathcal{V}_{xx} &= k^2(a_1 + 2a_2)(B_1\mathcal{J} + B_2\mathcal{J}^3) + 2\varpi^2k^2a_2\mathcal{J}'^2, \\
\mathcal{V}_{xxx} &= \varpi k^3 a_1(B_1 + 3B_2\mathcal{J}^2)\mathcal{J}' + 4\varpi k^3 a_2(2B_1\mathcal{J} + 3B_2\mathcal{J}^3)\mathcal{J}', \\
\mathcal{V}_x\mathcal{V} &= \varpi k(a_0a_1 + 2a_0a_2\mathcal{J} + a_1^2\mathcal{J} + 3a_1a_2\mathcal{J}^2 + 2a_2^2\mathcal{J}^3)\mathcal{J}',
\end{aligned} \tag{8}$$

where B_1 and B_2 are constants that based on ϖ, and m, and will be explained later. Putting Eqs. (7) and (8) into KdVE-RVCs (5) and equating all the coefficients of $\mathcal{J}'\mathcal{J}^n$ to zero, we attain

$$\begin{aligned}
\mathcal{J}^0 &: \quad \dot{a}_0 + \frac{1}{2}\sigma^2 a_0 = 0, \\
\mathcal{J} &: \quad \dot{a}_1 + \frac{1}{2}\sigma^2 a_1 = 0, \\
\mathcal{J}^2 &: \quad \dot{a}_2 + \frac{1}{2}\sigma^2 a_2 = 0, \\
\mathcal{J}^0\mathcal{J}' &: \quad \varpi a_1[\lambda + bk^3B_1 + ka_0^2A(t)] = 0, \\
\mathcal{J}\mathcal{J}' &: \quad 2\varpi\lambda a_2 + 8b\varpi k^3a_2B_1 + A\varpi k a_1^2 + 2\varpi kA(t)a_0a_2 = 0, \\
\mathcal{J}^2\mathcal{J}' &: \quad 3b\varpi k^3a_1B_2 + 3\varpi k a_1a_2A(t) = 0,
\end{aligned}$$

and

$$\mathcal{J}^3\mathcal{J}' : \quad 12b\varpi k^3B_2a_2 + 2\varpi kA(t)a_2^2 = 0.$$

Solving these equations yields

$$a_0(t) = \ell_0 e^{-\frac{1}{2}\sigma^2 t}, \quad a_1 = 0, \; a_2(t) = \ell_2 e^{-\frac{1}{2}\sigma^2 t},$$

$$b = \frac{-\ell_2 A(t)}{6k^2B_2}e^{-\frac{1}{2}\sigma^2 t}, \text{ and } \; \lambda(t) = -k(\frac{2\ell_2B_1}{3B_2} + \ell_0)A(t)e^{-\frac{1}{2}\sigma^2 t},$$

where ℓ_0 and ℓ_2 are constants. Hence, the solution of the KdVE-RVCs (5) is

$$\mathcal{V}(x,t) = [\ell_0 + \ell_2\mathcal{J}^2(\eta)]e^{-\frac{1}{2}\sigma^2 t}, \; \eta = kx - k(\frac{2\ell_2B_1}{3B_2} + \ell_0)\int_0^t e^{\sigma\mathcal{B}(\tau) - \frac{1}{2}\sigma^2\tau}d\tau. \tag{9}$$

In the following, $\mathcal{J}(\eta)$ is defined as:

Set 1: When $\mathcal{J}(\eta) = sn(\varpi\eta, m)$, then Eq. (9) takes the form

$$\mathcal{V}(x,t) = \Big(\ell_0 + \ell_2\Big(sn(k\varpi x - k\varpi(\frac{2\ell_2B_1}{3B_2} + \ell_0)\int_0^t e^{\sigma\mathcal{B}(\tau) - \frac{1}{2}\sigma^2\tau}d\tau, m)\Big)^2\Big)e^{-\frac{1}{2}\sigma^2 t}, \tag{10}$$

where the integral $\int_0^t e^{\sigma\mathcal{B}(\tau) - \frac{1}{2}\sigma^2\tau}d\tau$ converges almost surely,

$$B_1 = -\varpi^2(1 + m^2) \text{ and } B_2 = 2\varpi^2m^2.$$

Set 2: When $\mathcal{J}(\eta) = cn(\varpi\eta, m)$, then Eq. (9) becomes

$$\mathcal{V}(x,t) = \Big(\ell_0 + \ell_2\Big(cn(k\varpi x - k\varpi(\frac{2\ell_2 B_1}{3B_2} + \ell_0)\int_0^t e^{\sigma\mathcal{B}(\tau)-\frac{1}{2}\sigma^2\tau} d\tau, m)\Big)^2\Big)e^{-\frac{1}{2}\sigma^2 t}, \quad (11)$$

where

$$B_1 = \varpi^2(1-2m^2) \text{ and } B_2 = -2\varpi^2 m^2.$$

Set 3: When $\mathcal{J}(\eta) = dn(\varpi\eta, m)$, then Eq. (9) becomes

$$\mathcal{V}(x,t) = \Big(\ell_0 + \ell_2\Big(dn(k\varpi x - k\varpi(\frac{2\ell_2 B_1}{3B_2} + \ell_0)\int_0^t e^{\sigma\mathcal{B}(\tau)-\frac{1}{2}\sigma^2\tau} d\tau, m)\Big)^2\Big)e^{-\frac{1}{2}\sigma^2 t}, \quad (12)$$

where

$$B_1 = \varpi^2(2-m^2) \text{ and } B_2 = 2\varpi^2.$$

2.2. GREM-method

In this subsection, we employ the GREM-method reported in [20] to obtain the KDVE-RVC solutions. Let the solutions of Eq. (5), with $n = 2$, take the form

$$\mathcal{V}(x,t) = \alpha_0(t) + \alpha_1(t)\mathcal{X}(\eta) + \alpha_2(t)\mathcal{X}^2(\eta), \quad (13)$$

where

$$\mathcal{X}' = s\mathcal{X}^2 + r\mathcal{X} + p. \quad (14)$$

Differentiating Eq. (13) with regards to x and t, we attain

$$\begin{aligned}
\mathcal{V}_t &= (\dot{\alpha_0} + p\alpha_1\lambda) + (\dot{\alpha_1} + \alpha_1 r\lambda + 2p\lambda\alpha_2)\mathcal{X} \\
&\quad + (s\lambda\alpha_1 + \dot{\alpha_2} + 2\lambda r\alpha_2)\mathcal{X}^2 + 2s\lambda\alpha_2\mathcal{X}^3, \\
\mathcal{V}_x &= k[2s\alpha_2\mathcal{X}^3 + (s\alpha_1 + 2r\alpha_2)\mathcal{X}^2 + (r\alpha_1 + 2p\alpha_2)\mathcal{X} + p\alpha_1], \\
\mathcal{V}_{xxx} &= k^3[24\alpha_2 s^3\mathcal{X}^5 + (6s^3\alpha_1 + 54rs^2\alpha_2)\mathcal{X}^4 \\
&\quad + (12rs^2\alpha_1 + 48ps^2\alpha_2 + 32sr^2\alpha_2)\mathcal{X}^3 \\
&\quad + (7r^2 s\alpha_1 + 8ps^2\alpha_1 + 50rps\alpha_2 + 8r^3\alpha_2)\mathcal{X}^2 \\
&\quad + (r^3\alpha_1 + 8rps\alpha_1 + 14pr^2\alpha_2)\mathcal{X} \\
&\quad + (pr^2\alpha_1 + 2p^2 s\alpha_1 + 6p^2 r\alpha_2)], \\
\mathcal{V}\mathcal{V}_x &= k[2s\alpha_2^2\mathcal{X}^5 + (s\alpha_1\alpha_2 + 2r\alpha_2^2)\mathcal{X}^4 + \\
&\quad (2s\alpha_0\alpha_2 + s\alpha_1^2 + 3r\alpha_1\alpha_2 + 2p\alpha_2^2)\mathcal{X}^3 \\
&\quad + (s\alpha_0\alpha_1 + 2r\alpha_0\alpha_2 + r\alpha_1^2 + 3p\alpha_2\alpha_1)\mathcal{X}^2 \\
&\quad + (r\alpha_0\alpha_1 + 2p\alpha_0\alpha_2 + p\alpha_0^2)\mathcal{X} + p\alpha_0\alpha_1].
\end{aligned} \quad (15)$$

Plugging Eqs. (13) and (15) into Eq. (5), we have the following polynomial of degree 5 in $\mathcal{X}$:

$$[24bk^3 s^3\alpha_2 + 2Aks\alpha_2^2]\mathcal{X}^5$$

$$+[skA\alpha_1\alpha_2 + 2kAr\alpha_2^2 + 6bk^3s^3\alpha_1 + 54bk^3rs^2\alpha_2]\mathcal{X}^4$$

$$+[2\lambda s\alpha_2 + 12rbk^3s^2\alpha_1 + 48pbk^3s^2\alpha_2 + 32sbk^3r^2\alpha_2 + 2skA\alpha_0\alpha_2$$
$$+skA\alpha_1^2 + 3rkA\alpha_1\alpha_2 + 2kAp\alpha_2^2]\mathcal{X}^3$$

$$+[s\lambda\alpha_1 + \dot{\alpha}_2 + 2r\alpha_2 + 7bk^3r^2s\alpha_1 + 8pbk^3s^2\alpha_1 + 50rpsbk^3\alpha_2 + 8bk^3r^3\alpha_2$$
$$+skA\alpha_0\alpha_1 + 2rkA\alpha_0\alpha_2 + rkA\alpha_1^2 + 3kAp\alpha_2\alpha_1 + \frac{1}{2}\sigma^2\alpha_2]\mathcal{X}^2$$

$$+[\dot{\alpha}_1 + \alpha_1 r\lambda + 2p\lambda\alpha_2 + bk^3r^3\alpha_1 + 8rpsbk^3\alpha_1 + 14pbk^3r^2\alpha_2$$
$$+rkA\alpha_0\alpha_1 + 2pkA\alpha_0\alpha_2 + pkA\alpha_0^2 + \frac{1}{2}\sigma^2\alpha_1]\mathcal{X}$$

$$+[\dot{\alpha}_0 + p\alpha_1\lambda + pbk^3r^2\alpha_1 + 2bsk^3p^2\alpha_1 + 6rbk^3p^2\alpha_2 + pkA\alpha_0\alpha_1 + \frac{1}{2}\sigma^2\alpha_0] = 0.$$

After putting each coefficient of $\mathcal{X}^k$ to zero, we attain

$$24bk^3s^3\alpha_2 + 2Aks\alpha_2^2 = 0,$$

$$skA\alpha_1\alpha_2 + 2kAr\alpha_2^2 + 6bk^3s^3\alpha_1 + 54bk^3rs^2\alpha_2 = 0,$$

$$2s\alpha_2 + 12rbk^3s^2\alpha_1 + 48pbk^3s^2\alpha_2 + 32sbk^3r^2\alpha_2$$
$$+2skA\alpha_0\alpha_2 + skA\alpha_1^2 + 3rkA\alpha_1\alpha_2 + 2kAp\alpha_2^2 = 0,$$

$$s\lambda\alpha_1 + \dot{\alpha}_2 + 2r\alpha_2 + 7bk^3r^2s\alpha_1 + 8pbk^3s^2\alpha_1 + 50rpsbk^3\alpha_2$$
$$+8bk^3r^3\alpha_2 + skA\alpha_0\alpha_1 + 2rkA\alpha_0\alpha_2 + rkA\alpha_1^2 + 3kAp\alpha_2\alpha_1 + \tfrac{1}{2}\sigma^2\alpha_2 = 0,$$

$$\dot{\alpha}_1 + \alpha_1 r\lambda + 2p\alpha_2 + bk^3r^3\alpha_1 + 8rpsbk^3\alpha_1 + 14pbk^3r^2\alpha_2$$
$$+rkA\alpha_0\alpha_1 + 2pkA\alpha_0\alpha_2 + pkA\alpha_1^2 + \tfrac{1}{2}\sigma^2\alpha_1 = 0,$$

and

$$\dot{\alpha}_0 + p\alpha_1\lambda + pbk^3r^2\alpha_1 + 2bsk^3p^2\alpha_1$$
$$+6rbk^3p^2\alpha_1 + pkA\alpha_0\alpha_1 + \tfrac{1}{2}\sigma^2\alpha_0 = 0.$$

Solving these equations yields

$$\alpha_0(t) = \ell_0 e^{-\frac{1}{2}\sigma^2 t}, \quad \alpha_1 = r = 0, \;\; \alpha_2 = \ell_2 e^{-\frac{1}{2}\sigma^2 t},$$

$$b = \frac{-A\ell_2}{12k^2s^2}e^{-\frac{1}{2}\sigma^2 t}, \quad \text{and} \quad \lambda(t) = -ak\ell_0 e^{\sigma\mathcal{B}(t)-\frac{1}{2}\sigma^2 t},$$

where ℓ_0 and ℓ_2 are constants. Therefore, by using Eq. (13), the solution of KDVE-RVCs (5) takes the form

$$\mathcal{V}(x,t) = (\ell_0 + \ell_2\mathcal{X}^2(\eta))e^{-\frac{1}{2}\sigma^2 t}, \; \eta = kx - ak\ell_0\int_0^t e^{\sigma\mathcal{B}(\tau)-\frac{1}{2}\sigma^2\tau}d\tau. \tag{16}$$

To find $\mathcal{X}$, there are many families for the solutions of Eq. (14) relaying on s and p:
Family I: If $sp > 0$, then the solutions of Eq. (14) are

$$\mathcal{X}_1(\eta) = \sqrt{\frac{p}{s}} \tan\left(\sqrt{ps}\eta\right),$$

$$\mathcal{X}_2(\eta) = -\sqrt{\frac{p}{s}} \cot\left(\sqrt{ps}\eta\right),$$

$$\mathcal{X}_3(\eta) = \sqrt{\frac{p}{s}}\Big(\tan(\sqrt{4ps}\eta) \pm \sec(\sqrt{4ps}\eta)\Big),$$

$$\mathcal{X}_4(\eta) = -\sqrt{\frac{p}{s}}\Big(\cot(\sqrt{4ps}\eta) \pm \csc(\sqrt{4ps}\eta)\Big),$$

$$\mathcal{X}_5(\eta) = \frac{1}{2}\sqrt{\frac{p}{s}}\Big(\tan(\frac{1}{2}\sqrt{ps}\eta) - \cot(\frac{1}{2}\sqrt{ps}\eta)\Big),$$

Then, KDVE-RVCs (5) possess the trigonometric functions solution:

$$\mathcal{V}_1(x,t) = \Big(\ell_0 + \frac{\ell_2 p}{s}\tan^2(\sqrt{ps}\eta)\Big)e^{-\frac{1}{2}\sigma^2 t}, \tag{17}$$

$$\mathcal{V}_2(x,t) = \Big(\ell_0 - \frac{\ell_2 p}{s}\cot^2(\sqrt{ps}\eta)\Big)e^{-\frac{1}{2}\sigma^2 t}, \tag{18}$$

$$\mathcal{V}_3(x,t) = \Big(\ell_0 + \frac{\ell_2 p}{s}\Big(\tan(\sqrt{4ps}\eta) \pm \sec(\sqrt{4ps}\eta)\Big)^2\Big)e^{-\frac{1}{2}\sigma^2 t}, \tag{19}$$

$$\mathcal{V}_4(x,t) = \Big(\ell_0 - \frac{\ell_2 p}{s}\Big(\cot(\sqrt{4ps}\eta) \pm \csc(\sqrt{4ps}\eta)\Big)^2\Big)e^{-\frac{1}{2}\sigma^2 t}, \tag{20}$$

$$\mathcal{V}_5(x,t) = \Big(\ell_0 + \frac{\ell_2 p}{2s}\Big(\tan(\frac{1}{2}\sqrt{ps}\eta) - \cot(\frac{1}{2}\sqrt{ps}\eta)\Big)^2\Big)e^{-\frac{1}{2}\sigma^2 t}, \tag{21}$$

where $\eta = kx - ak\ell_0 \int_0^t e^{\sigma\mathcal{B}(\tau)-\frac{1}{2}\sigma^2\tau} d\tau$.
Family II: If $sp < 0$, then the solutions of Eq. (14) are

$$\mathcal{X}_6(\eta) = -\sqrt{\frac{-p}{s}} \tanh\left(\sqrt{-ps}\eta\right),$$

$$\mathcal{X}_7(\eta) = -\sqrt{\frac{-p}{s}} \coth\left(\sqrt{-ps}\eta\right),$$

$$\mathcal{X}_8(\eta) = -\sqrt{\frac{-p}{s}}\Big(\coth(\sqrt{-4ps}\eta) \pm \operatorname{csch}(\sqrt{-4ps}\eta)\Big),$$

$$\mathcal{X}_9(\eta) = \frac{-1}{2}\sqrt{\frac{-p}{s}}\Big(\tanh(\frac{1}{2}\sqrt{-ps}\eta) + \coth(\frac{1}{2}\sqrt{-ps}\eta)\Big).$$

Hence, KDVE-RVCs (5) possess the following hyperbolic functions solution:

$$\mathcal{V}_6(x,t) = \Big(\ell_0 + \frac{\ell_2 p}{s}\tanh^2(\sqrt{-ps}\eta)\Big)e^{-\frac{1}{2}\sigma^2 t}, \tag{22}$$

$$\mathcal{V}_7(x,t) = \Big(\ell_0 + \frac{\ell_2 p}{s}\coth^2(\sqrt{-ps}\eta)\Big)e^{-\frac{1}{2}\sigma^2 t}, \tag{23}$$

$$\mathcal{V}_8(x,t) = \Big(\ell_0 + \frac{\ell_2 p}{s}\Big(\coth(\sqrt{-4ps}\eta) \pm \operatorname{csch}(\sqrt{-4ps}\eta)\Big)^2\Big)e^{-\frac{1}{2}\sigma^2 t}, \tag{24}$$

$$\mathcal{V}_9(x,t) = \Big(\ell_0 + \frac{\ell_2 p}{2s}\Big(\tanh(\frac{1}{2}\sqrt{-ps}\eta) + \coth(\frac{1}{2}\sqrt{-ps}\eta)\Big)\Big)e^{-\frac{1}{2}\sigma^2 t}, \tag{25}$$

where $\eta = kx - ak\ell_0 \int_0^t e^{\sigma\mathcal{B}(\tau)-\frac{1}{2}\sigma^2\tau}d\tau$.

Family III: If $s \neq 0$ and $p = 0$, then Eq. (14) has the solution:

$$\mathcal{X}_{10}(\eta) = \frac{-1}{s\eta}.$$

Therefore, the KDVE-RVCs (5) has the following rational function solution

$$\mathcal{V}_{10}(x,t) = \Big(\ell_0 + \frac{\ell_2}{s^2\eta^2}\Big)e^{-\frac{1}{2}\sigma^2 t}, \tag{26}$$

where $\eta = kx - ak\ell_0 \int_0^t e^{\sigma\mathcal{B}(\tau)-\frac{1}{2}\sigma^2\tau}d\tau$.

3. Exact Solutions of SKDV Equation

Here, we apply the results from the preceding section to derive solutions to the SKDV Eq. (1).

3.1. JEF-method

Substituting Eqs (10)-(12) into Eq. (2), we have the solutions of SKDV Eq. (1):

$$\begin{aligned}\mathcal{P}(x,t) &= e^{\sigma\mathcal{B}(t)-\frac{1}{2}\sigma^2 t}\Big(\ell_0 \\ &\quad + \ell_2 sn^2\Big(k\varpi x - k\varpi(\frac{-\ell_2(1+m^2)}{3m^2} + \ell_0)\int_0^t e^{\sigma\mathcal{B}(\tau)-\frac{1}{2}\sigma^2\tau}d\tau, m\Big)\Big),\end{aligned} \tag{27}$$

$$\begin{aligned}\mathcal{P}(x,t) &= e^{\sigma\mathcal{B}(t)-\frac{1}{2}\sigma^2 t}\Big(\ell_0 \\ &\quad + \ell_2 cn^2\Big(k\varpi x - k\varpi(\frac{\ell_2(2m^2-1)}{3m^2} + \ell_0)\int_0^t e^{\sigma\mathcal{B}(\tau)-\frac{1}{2}\sigma^2\tau}d\tau, m\Big)\Big),\end{aligned} \tag{28}$$

and

$$\mathcal{P}(x,t) = e^{\sigma\mathcal{B}(t)-\frac{1}{2}\sigma^2 t}\Big(\ell_0$$

$$+\ell_2 dn^2\Big(k\varpi x - k\varpi(\frac{\ell_2(2-m^2)}{3B_2}+\ell_0)\int_0^t e^{\sigma\mathcal{B}(\tau)-\frac{1}{2}\sigma^2\tau}d\tau, m\Big)\Big). \quad (29)$$

If $m \to 1$, then the Eqs (27)-(29) turn into

$$\mathcal{P}(x,t) = e^{\sigma\mathcal{B}(t)-\frac{1}{2}\sigma^2 t}\Big(\ell_0 + \ell_2\tanh^2\Big(k\varpi x - k\varpi(\frac{-2\ell_2}{3}+\ell_0)\int_0^t e^{\sigma\mathcal{B}(\tau)-\frac{1}{2}\sigma^2\tau}d\tau\Big)\Big), \quad (30)$$

and

$$\mathcal{P}(x,t) = e^{\sigma\mathcal{B}(t)-\frac{1}{2}\sigma^2 t}\Big(\ell_0 + \ell_2\text{sech}^2\Big(k\varpi x - k\varpi(\frac{\ell_2}{3}+\ell_0)\int_0^t e^{\sigma\mathcal{B}(\tau)-\frac{1}{2}\sigma^2\tau}d\tau)\Big)\Big). \quad (31)$$

3.2. GREM-method

Plugging Eq. (16) into Eqs (2), we acquire the solutions of SKDV Eq. (1) as

$$\mathcal{P}(x,t) = \mathcal{V}(\eta)e^{[\sigma\mathcal{B}(t)-\frac{1}{2}\sigma^2 t]},\ \eta = kx - ak\ell_0\int_0^t e^{\sigma\mathcal{B}(\tau)-\frac{1}{2}\sigma^2\tau}d\tau. \quad (32)$$

If $ps > 0$, then the SKDV Eq. (1), utilizing (17)-(21), has the solutions:

$$\mathcal{P}_1(x,t) = \Big(\ell_0 + \frac{\ell_2 p}{s}\tan^2(\sqrt{ps}\eta)\Big)e^{[\sigma\mathcal{B}(t)-\frac{1}{2}\sigma^2 t]}, \quad (33)$$

$$\mathcal{P}_2(x,t) = \Big(\ell_0 - \frac{\ell_2 p}{s}\cot^2(\sqrt{ps}\eta)\Big)e^{[\sigma\mathcal{B}(t)-\frac{1}{2}\sigma^2 t]}, \quad (34)$$

$$\mathcal{P}_3(x,t) = \Big(\ell_0 + \frac{\ell_2 p}{s}\Big(\tan(\sqrt{4ps}\eta) \pm \sec(\sqrt{4ps}\eta)\Big)^2\Big)e^{[\sigma\mathcal{B}(t)-\frac{1}{2}\sigma^2 t]}, \quad (35)$$

$$\mathcal{P}_4(x,t) = \Big(\ell_0 - \frac{\ell_2 p}{s}\Big(\cot(\sqrt{4ps}\eta) \pm \csc(\sqrt{4ps}\eta)\Big)^2\Big)e^{[\sigma\mathcal{B}(t)-\frac{1}{2}\sigma^2 t]}, \quad (36)$$

$$\mathcal{P}_5(x,t) = \Big(\ell_0 + \frac{\ell_2 p}{2s}\Big(\tan(\frac{1}{2}\sqrt{ps}\eta) - \cot(\frac{1}{2}\sqrt{ps}\eta)\Big)^2\Big)e^{[\sigma\mathcal{B}(t)-\frac{1}{2}\sigma^2 t]}. \quad (37)$$

While, if $ps < 0$, then the SKDV Eq. (1), utilizing (22)-(25), has the solutions:

$$\mathcal{P}_6(x,t) = \Big(\ell_0 + \frac{\ell_2 p}{s}\tanh^2(\sqrt{-ps}\eta)\Big)e^{[\sigma\mathcal{B}(t)-\frac{1}{2}\sigma^2 t]}, \quad (38)$$

$$\mathcal{P}_7(x,t) = \Big(\ell_0 + \frac{\ell_2 p}{s}\coth^2(\sqrt{-ps}\eta)\Big)e^{[\sigma\mathcal{B}(t)-\frac{1}{2}\sigma^2 t]}, \quad (39)$$

$$\mathcal{P}_8(x,t) = \Big(\ell_0 + \frac{\ell_2 p}{s}\Big(\coth(\sqrt{-4ps}\eta) \pm \text{csch}(\sqrt{-4ps}\eta)\Big)^2\Big)e^{[\sigma\mathcal{B}(t)-\frac{1}{2}\sigma^2 t]}, \quad (40)$$

$$\mathcal{P}_9(x,t)=\Big(\ell_0+\frac{\ell_2 p}{2s}\Big(\tanh(\frac{1}{2}\sqrt{-ps}\eta)+\coth(\frac{1}{2}\sqrt{-ps}\eta)\Big)^2\Big)e^{[\sigma\mathcal{B}(t)-\frac{1}{2}\sigma^2 t]}. \quad (41)$$

If $s\neq 0$ and $p=0$, then SKDV Eq. (1), utilizing (26), has the solution:

$$\mathcal{P}_{10}(x,t)=\Big(\frac{-1}{s\eta}\Big)e^{[\sigma\mathcal{B}(t)-\frac{1}{2}\sigma^2 t]}, \quad (42)$$

where $\eta=kx-ak\ell_0\int_0^t e^{\sigma\mathcal{B}(\tau)-\frac{1}{2}\sigma^2\tau}d\tau$.

Remark 1. *Putting $p=s=\frac{3}{2}$, $\ell_0=\frac{-c}{6}$, $\ell_2=\frac{-c}{2}$, $k=\frac{\sqrt{c}}{3}$ and $\sigma=0$ (i.e. no noise) in Eqs (33), (34), (38) and (39), we acquired the results that reported in [13] as follows:*

$$\mathcal{P}(x,t)=\frac{-c}{6}\Big(1+3\tan^2(\frac{\sqrt{-c}}{2}(x-ct))\Big),$$

$$\mathcal{P}(x,t)=\frac{-c}{6}\Big(1+3\cot^2(\frac{\sqrt{-c}}{2}(x-ct))\Big),$$

$$\mathcal{P}(x,t)=\frac{-c}{6}\Big(1-3\tanh^2(\frac{\sqrt{-c}}{2}(x-ct))\Big),$$

and

$$\mathcal{P}(x,t)=\frac{-c}{6}\Big(1-3\coth^2(\frac{\sqrt{-c}}{2}(x-ct))\Big).$$

Remark 2. *If we choose $\ell_0=\frac{-\lambda}{6}$ and $\ell_2=\frac{-\lambda}{2}$ in Eqs (33), (34), (38) and (39), then we get*

$$\mathcal{P}(x,t)=\Big(\frac{-\lambda}{6}+\frac{\lambda}{2}\tan^2(\frac{\sqrt{\lambda}}{2}(x-\lambda\int_0^t e^{\sigma\mathcal{B}(\tau)-\frac{1}{2}\sigma^2\tau}d\tau))\Big)e^{[\sigma\mathcal{B}(t)-\frac{1}{2}\sigma^2 t]}$$

$$\mathcal{P}(x,t)=\Big(\frac{-\lambda}{6}+\frac{\lambda}{2}\cot^2(\frac{\sqrt{\lambda}}{2}(x-\lambda\int_0^t e^{\sigma\mathcal{B}(\tau)-\frac{1}{2}\sigma^2\tau}d\tau))\Big)e^{[\sigma\mathcal{B}(t)-\frac{1}{2}\sigma^2 t]},$$

$$\mathcal{P}(x,t)=\Big(\frac{-\lambda}{6}+\frac{\lambda}{2}\tanh^2(\frac{\sqrt{-\lambda}}{2}(x+\lambda\int_0^t e^{\sigma\mathcal{B}(\tau)-\frac{1}{2}\sigma^2\tau}d\tau))\Big)e^{[\sigma\mathcal{B}(t)-\frac{1}{2}\sigma^2 t]},$$

and

$$\mathcal{P}(x,t)=\Big(\frac{-\lambda}{6}+\frac{\lambda}{2}\coth^2(k\sqrt{-ps}(x-6\ell_0\int_0^t e^{\sigma\mathcal{B}(\tau)-\frac{1}{2}\sigma^2\tau}d\tau))\Big)e^{[\sigma\mathcal{B}(t)-\frac{1}{2}\sigma^2 t]}.$$

These results extend the results that obtained in [11] as follows:

$$\mathcal{P}(x,t)=\Big(\frac{-\lambda}{6}+\frac{\lambda}{2}\tan^2(\frac{\sqrt{\lambda}}{2}(x-\lambda t))\Big)e^{[\sigma\mathcal{B}(t)-\frac{1}{2}\sigma^2 t]}$$

$$\mathcal{P}(x,t)=\Big(\frac{-\lambda}{6}+\frac{\lambda}{2}\cot^2(\frac{\sqrt{\lambda}}{2}(x-\lambda t))\Big)e^{[\sigma\mathcal{B}(t)-\frac{1}{2}\sigma^2 t]},$$

$$\mathcal{P}(x,t)=\Big(\frac{-\lambda}{6}+\frac{\lambda}{2}\tanh^2(\frac{\sqrt{-\lambda}}{2}(x-\lambda t))\Big)e^{[\sigma\mathcal{B}(t)-\frac{1}{2}\sigma^2 t]},$$

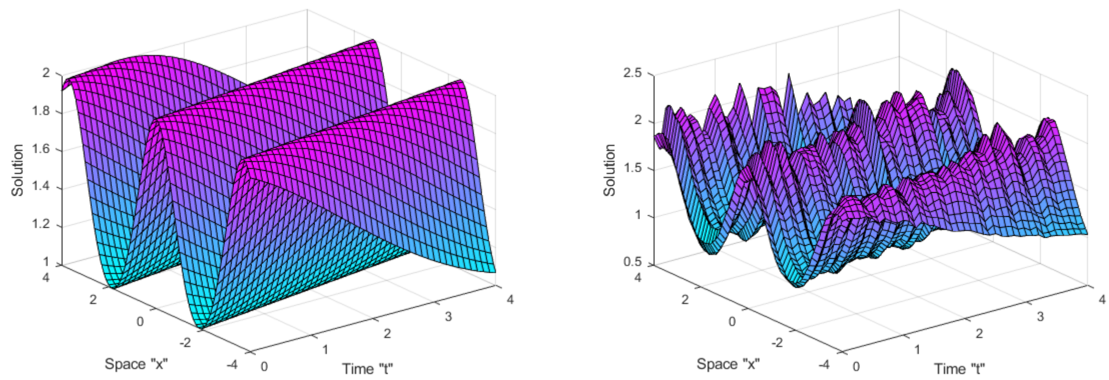
and

$$\mathcal{P}(x,t)=\Big(\frac{-\lambda}{6}+\frac{\lambda}{2}\coth^2(k\sqrt{-ps}(x-\lambda t))\Big)e^{[\sigma\mathcal{B}(t)-\frac{1}{2}\sigma^2 t]}.$$

4. Discussion and effect of noise

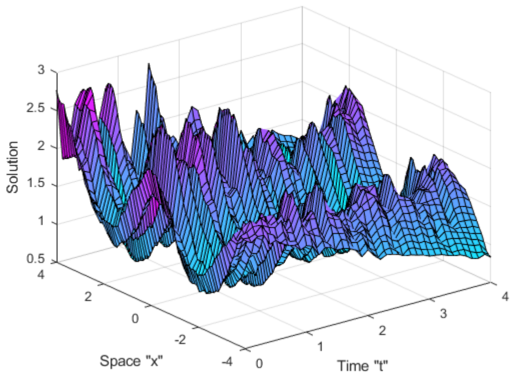
Discussion: In this study, we acquired the stochastic solutions of the SKDV Eq. (1). We employed two methods, namely the JEF-method and the GREM-method. The JEF-method is suggested to create the exact periodic solutions of nonlinear wave equations. Moreover, the JEF-method is more general than the hyperbolic tangent function expansion method. While the GREM-method is a powerful and effective approach that provides solutions in several forms such as the hyperbolic function, the trigonometric function, and the rational functional form. By using these methods, we acquired a variety of solutions, including the elliptic solutions (27)-(29), kink solution (38), singular solution (39), singular periodic (33) and (34) and etc. One of the key aspects of singular solitons in the KdV equation is their stability properties. Unlike traditional wave solutions that disperse and dissipate over time, singular solitons have the remarkable feature of maintaining their form and amplitude as they propagate through a medium. This stability allows solitons to travel long distances without distortion, making them valuable in modeling the behavior of waves in oceans, rivers, and other fluid systems.

Effect of noise: The impact of multiplicative noise on the exact solutions of the SKDV Eq. (1) is examined here. A number of graphs representing different solutions with distinct value of noise intensity are shown. The main distinction between the solutions supplied here and those obtained in [11] is the amplitude functions $\mathcal{V}(x,t)$. Here $\mathcal{V}(x,t)$ is a stochastic functions, while $\mathcal{V}(x,t)$ is supposed deterministic function in [11]. Figures 1, 2 and 3 illustrate the solutions $\mathcal{P}(x,t)$ given in Eqs (28), (31) and (38) for distinct value of the noise strength σ as follows:

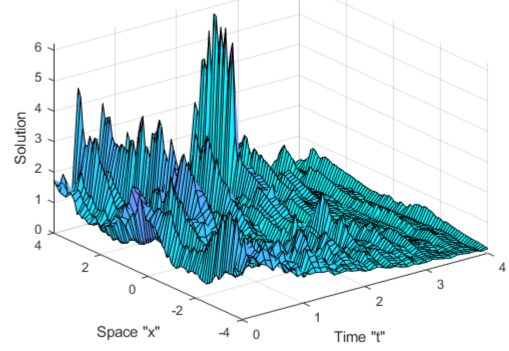


(i) $\sigma = 0$

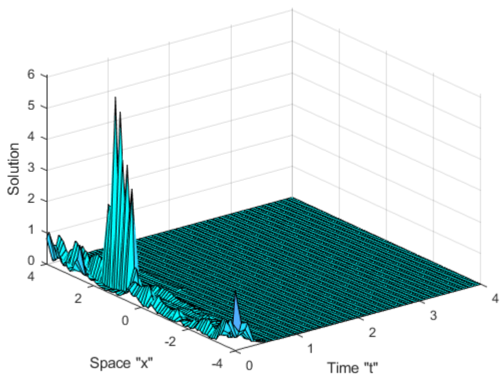
(ii) $\sigma = 0.1$



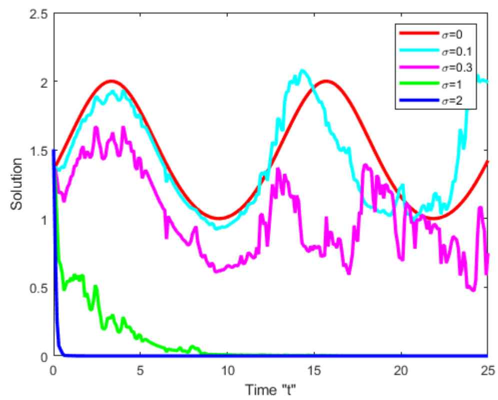
(iii) $\sigma = 0.3$



(iv) $\sigma = 1$

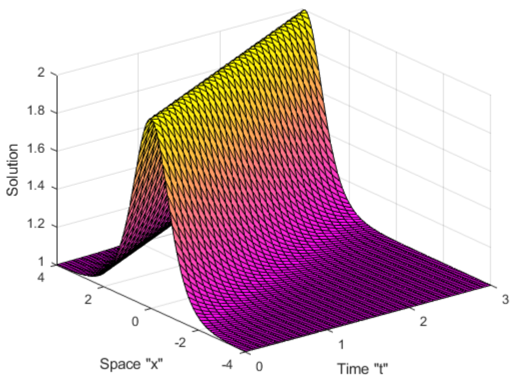


(v) $\sigma = 2$

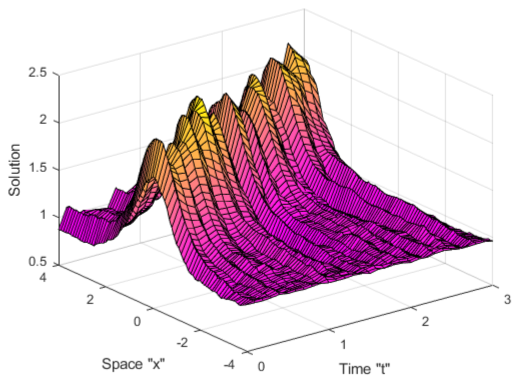


(vi) $\sigma = 0,\ 0.1,\ 0.3,\ 1,\ 2$

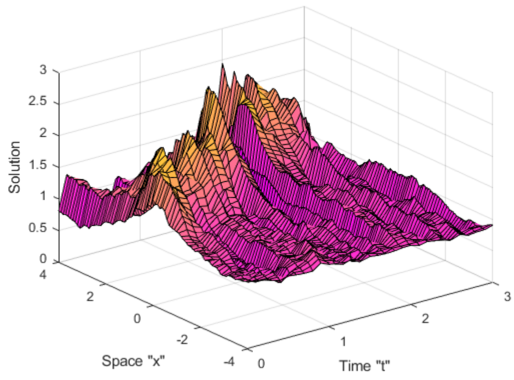
Figure 1. (i-v) depict 3D periodic solution of $\mathcal{P}(x,t)$ stated in Eq (28) with $\varpi = k = \ell_0 = \ell_2 = a = 1$, $t \in [0,4]$ and $x \in [-4,4]$ (vi) shows 2D-shape of Eq. (28) with distinct value of σ and $x = 0.8$.



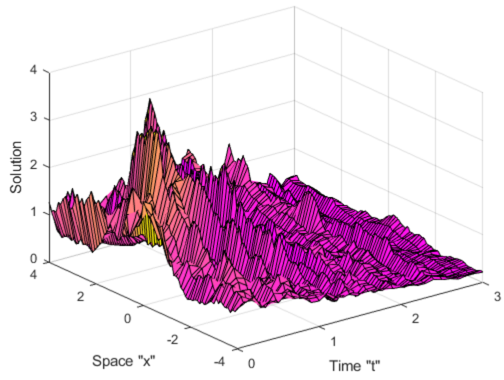
(i) $\sigma = 0$



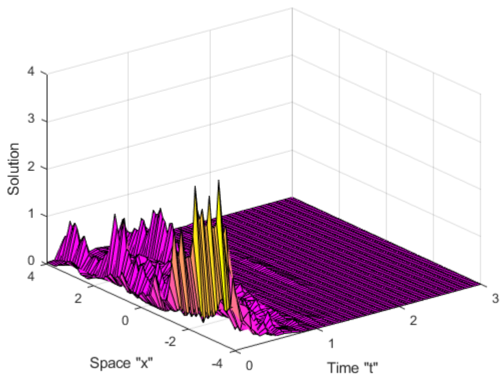
(ii) $\sigma = 0.1$



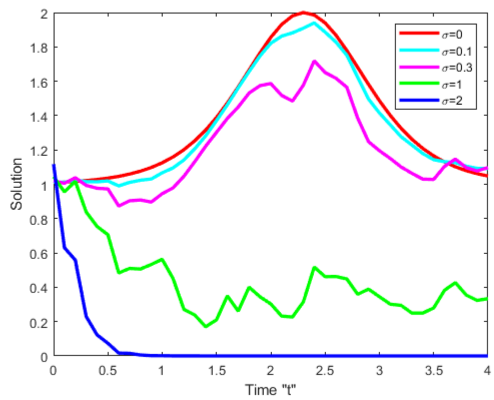
(iii) $\sigma = 0.3$



(iv) $\sigma = 1$

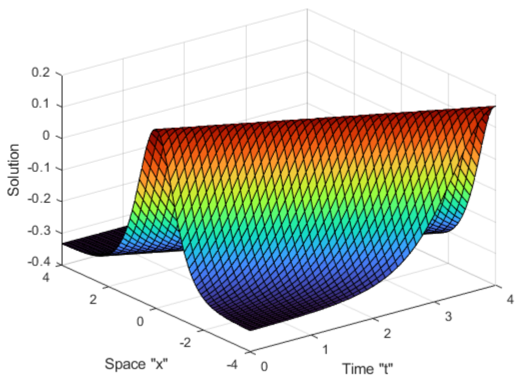


(v) $\sigma = 2$

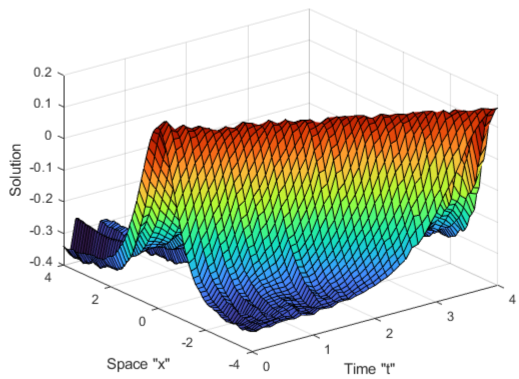


(vi) $\sigma = 0,\ 0.1,\ 0.3,\ 1,\ 2$

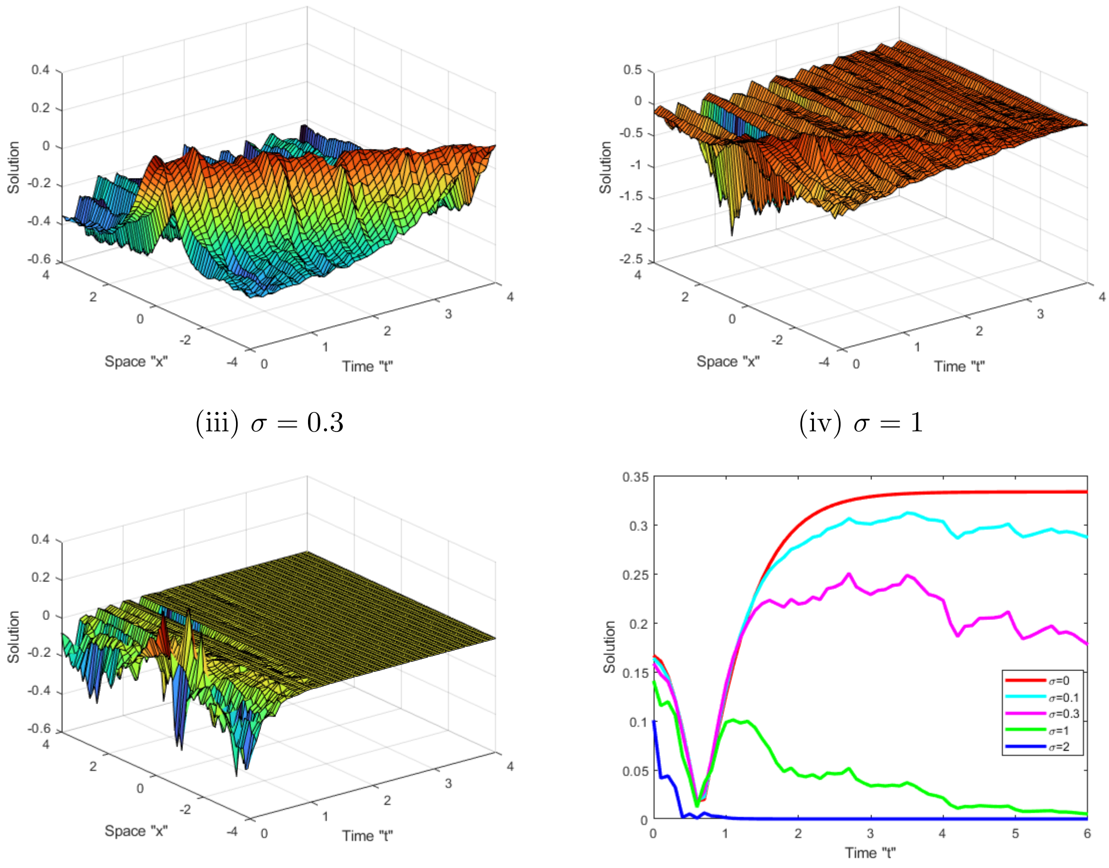
Figure 2. (i-v) depict bell-shape bright solution of $\mathcal{P}(x,t)$ stated in Eq. (31) with $p = \ell_0 = a = 1, s = \ell_2 = -1$, $t \in [0,4]$ and $x \in [-4,4]$ (vi) shows 2D-shape of Eq. (31) with distinct value of σ and $x = 1.2$.



(i) $\sigma = 0$



(ii) $\sigma = 1$



(iii) $\sigma = 0.3$ (iv) $\sigma = 1$

(v) $\sigma = 2$ (vi) $\sigma = 0,\ 0.1,\ 0.3,\ 1,\ 2$

Figure 3. (i-v) depict bell-shape dark solution of $\mathcal{P}(x,t)$ stated in Eq (38) with $m = 0.5,\ \varpi = k = \ell_0 = \ell_2 = a = 1,\ t \in [0,4]$ and $x \in [-4,4]$ (vi) shows 2D-shape of Eq. (38) with distinct value of σ and $x = 0.01$.

As mentioned before, in the absence of noise (i.e., $\sigma = 0$), a variety of solutions appear including periodic solutions, bell-shape bright solutions, and bell-shape dark solutions, as illustrated in Figures 1(i)-3(i). When noise is introduced as illustrated in Figures 1(ii)-1(v), 2(ii)-2(v) and 3(ii)-3(v), the surface flattens after a few transit patterns. This work shows that the SKDV solutions of Eq. (1) may be stabilized around zero when we added the multiplicative noise term into Eq. (1).

5. Conclusions

The stochastic KDV Eq. (1) forced by multiplicative noise in the Itô sense was studied in this study. We converted the SKDV equation into a KDV-RVCs (5) by using a suitable transformation. Abundant exact stochastic solutions for KDV-RVCs in the type of elliptic, rational, trigonometric, and hyperbolic functions was discovered by employing the JEF-method and GREM-method. Next, we obtained the solutions of SKDV Eq. (1). Additionally, we expanded on a few earlier solutions, such the solutions stated in [13]. Due to the relevance of the KDV equation in nonlinear optics, plasma physics, and fluid dynamics, the obtained solutions are essential in comprehending various complex physical phenomena. Finally, several illustrations were provided to show how the multiplicative noise term affected the exact stochastic solutions of the SKDV equation.

Acknowledgements

This research has been funded by Scientific Research Deanship at the University of Ha'il-Saudi Arabia through project number RG-25005.

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