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Some Spectral Radius Inequalities for Certain Matrices

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Abstract. In this paper we present upper bounds for spectral radius inequalities of 2×2 block accretive-dissipative matrices.

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1. Introduction

The study of matrix theory has become more and more popular in the last few decades. Researchers are attracted to this subject because of its connections with other pure and applied areas.

In particular, the eigenvalues are crucial in solving systems of differential equations, analyzing population growth models and calculating powers of matrices. It is not always easy to calculate the eigenvalues. However, in many scientific problems it is enough to know that the eigenvalues lie in specific region. Such information is provided by comparing between spectral radius and unitarily invariant norms.

A large number of inequalities involving spectral radius in addition to matrix norm were studied in many books that is concerning with inequalities, like Bhatia, 2007 [1]. Some investigations on norm and spectral radius inequalities were obtained by Kittaneh in 2005 [2], Elhaddad and Kittaneh in 2007[3]. In 2015, Abu-Omar and Kittaneh studied similar topics; they applied spectral radius and norm inequalities for any two by two block matrices [4].

Moreover, in 2025, Sakkijha and Hasan studied sum, difference and commutators for spectral radius inequalities involving accretive-dissipative matrices [5].

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The spectral radius $r(X)$ of a matrix $X \in M_n(\mathbb{C})$ is defined as

$$r(X) = \max\{|\lambda| : \lambda \in \sigma(X)\}. \tag{1}$$

It's well known that

$$r(X) \leqslant \|X\| \quad \text{for every} \quad X \in M_n(\mathbb{C}), \tag{2}$$

If X is positive semidefinite, then

$$r(X) = \|X\|, \tag{3}$$

where $\|X\|$ is the spectral norm of X which is defined as $max_{\|\nu\|=1}\|X\nu\|$ and satisfies

$$\|X\| = \|X^*\| = \|X^*X\|^{1/2} = \|XX^*\|^{1/2}. \tag{4}$$

Moreover, for any $X, \Upsilon \in M_n(\mathbb{C}), \delta \in \mathbb{C}$ and a positive integer n,

$$r(\delta X) = |\delta| r(X). \tag{5}$$

A special case of the spectral mapping theorem, which asserts that

$$r\left(X^n\right) = r^n(X). \tag{6}$$

In addition, two useful facts that are related to the spectral radius are as follows:

$$r\left(X^*\right) = r(X) \tag{7}$$

and

$$r\left(UXU^*\right) = r(X) \tag{8}$$

for every unitary matrix U, i.e. $U * U = I$,
A commutative property which asserts that

$$r(X\Upsilon) = r(\Upsilon X). \tag{9}$$

The last property is an immediate consequence of the fact that the spectra of the operators $X\Upsilon$ and ΥX have the same nonzero elements.
A matrix $\Psi \in M_n(\mathbb{C})$ is referred to as positive semidefinite (p.s.d.) matrix if $(\Psi\nu, \nu) \geq 0 \quad \forall \nu \in \mathbb{C}^n$. It is called accretive-dissipative (Acc-Dis) if in its Cartesian decomposition (CD) $\Psi = \psi_1 + i\psi_2$, the matrices $\psi_1 = \mathrm{Re}(\Psi) = \frac{\Psi+\Psi^*}{2}$ and $\psi_2 = \mathrm{Im}(\Psi) = \frac{\Psi-\Psi^*}{2i}$ are p.s.d. Accretive-dissipative matrices found many applications. For example, Gunzburger and Plemmons used the results in their study of energy conserving norms for the solution of hyperbolic systems of partial differential equations, see [6]. Many researchers are interested with this kind of Matrices like, George and Ikramor (2005)[7], also Sakkijha and Hasan (2024)[8].

2. Basic Lemmas

In order to prove our main results, we need the following lemmas.

Lemma 1. *[9] If $\psi_1, \psi_2 \in M_n(\mathbb{C})$ are p.s.d, then*

$$\|\psi_1 + i\psi_2\| \leqslant \|\psi_1 + \psi_2\|.$$

Lemma 2. *[10] If $\psi_1, \psi_2 \in M_n(\mathbb{C})$ are p.s.d, then*

$$\|\psi_1 + \psi_2\| \leqslant \max(\|\psi_1\|, \|\psi_2\|) + \left\|\psi_1^{1/2}\psi_2^{1/2}\right\|.$$

Lemma 3. *[11] If $\psi_1, \psi_2 \in M_n(\mathbb{C})$ are p.s.d, then*

$$\|\psi_1\psi_2 - \psi_2\psi_1\| \leqslant \frac{1}{2}\|\psi_1\|\|\psi_2\|.$$

Lemma 4. *[12] If $\psi_1, \psi_2, \psi_3, \psi_4 \in M_n(\mathbb{C})$, then*

$$r\left(\begin{bmatrix} \psi_1 & \psi_2 \\ \psi_3 & \psi_4 \end{bmatrix}\right) \leqslant r\left(\begin{bmatrix} \|\psi_1\| & \|\psi_2\| \\ \|\psi_3\| & \|\psi_4\| \end{bmatrix}\right).$$

Lemma 5. *[13] If $\psi_1, \psi_2 \in M_n(\mathbb{C})$, then*

$$\left\|\begin{bmatrix} \psi_1 & 0 \\ 0 & \psi_2 \end{bmatrix}\right\| = \max(\|\psi_1\|, \|\psi_2\|).$$

Lemma 6. *[14] If $\psi_1, \psi_2 \in M_n(\mathbb{C})$ are p.s.d, then*

$$\left\|\psi_1^{1/2}\psi_2^{1/2}\right\| \leqslant \|\psi_1\psi_2\|^{1/2}.$$

Lemma 7. *[10] If $\psi_1, \psi_2 \in M_n(\mathbb{C})$ are p.s.d, then*

$$\|\psi_1 - \psi_2\| \leq \max(\|\psi_1\|, \|\psi_2\|).$$

3. Main Results

In this section, we will present some spectral radius inequalities for 2×2 block accretive-dissipative matrices.

Theorem 1. *Let $\Psi, \Phi \in M_n(\mathbb{C})$ be Acc-Dis matrices with CD $\Psi = \psi_1 + i\psi_2$ and $\Phi = \phi_1 + i\phi_2$. Then*

$$r\left(\begin{bmatrix} \Psi & 0 \\ \Phi & 0 \end{bmatrix}\right) \leqslant \sqrt{\max\left(r^2(\psi_1), r^2(\psi_2)\right) + 2r(\psi_1)r(\psi_2)} = \alpha.$$

Proof.

$$\begin{aligned}
\text{Consider} \quad r^2\left(\begin{bmatrix}\Psi & 0\\ \Phi & 0\end{bmatrix}\right) &= r\left(\begin{bmatrix}\Psi & 0\\ \Phi & 0\end{bmatrix}\right)^2 \qquad (by (6))\\
&= r\left(\begin{bmatrix}\Psi^2 & 0\\ \Phi\Psi & 0\end{bmatrix}\right)\\
&= r\left(\begin{bmatrix}\Psi & 0\\ \Phi & 0\end{bmatrix}\begin{bmatrix}\Psi & 0\\ 0 & 0\end{bmatrix}\right)\\
&= r\left(\begin{bmatrix}\Psi & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\Psi & 0\\ \Phi & 0\end{bmatrix}\right) \qquad (by (9))\\
&= r\left(\begin{bmatrix}\Psi^2 & 0\\ 0 & 0\end{bmatrix}\right)\\
&\leqslant r\left(\begin{bmatrix}\|\Psi^2\| & 0\\ 0 & 0\end{bmatrix}\right) \quad \text{(by Lemma 4)}\\
&= r\left(\begin{bmatrix}\left\|\left(\psi_1^2-\psi_2^2\right)+i(\psi_1\psi_2+\psi_2\psi_1)\right\| & 0\\ 0 & 0\end{bmatrix}\right)\\
&= \left\|\left(\psi_1^2-\psi_2^2\right)+i(\psi_1\psi_2+\psi_2\psi_1)\right\|\\
&\leqslant \left\|\psi_1^2-\psi_2^2\right\| + \|\psi_1\psi_2+\psi_2\psi_1\|\\
&\leqslant \left\|\psi_1^2-\psi_2^2\right\| + \|\psi_1\psi_2\| + \|\psi_2\psi_1\|\\
&\leqslant \max\left(\left\|\psi_1^2\right\|, \left\|\psi_2^2\right\|\right) + \|\psi_1\psi_2\| + \|\psi_2\psi_1\| \qquad \text{(by Lemma 7)}\\
&\leq \max\left(r^2(\psi_1), r^2(\psi_2)\right) + 2r(\psi_1)r(\psi_2) \quad \text{(by (6) and (9))}.
\end{aligned}$$

The proof is obvious by taking the square root.

Theorem 2. *Let $\Psi, \Phi \in M_n(\mathbb{C})$ be Acc-Dis matrices with CD $\Psi = \psi_1 + i\psi_2, \Phi = \phi_1 + i\phi_2$. Then*

$$r\left(\begin{bmatrix}\Psi & 0\\ \Phi & 0\end{bmatrix}\right) \leqslant \sqrt{\max\left(r^2(\psi_1), r^2(\psi_2)\right) + \max\left(r^2(\phi_1), r^2(\phi_2)\right) + \frac{3}{2}r(\psi_1)r(\psi_2) + \frac{3}{2}r(\phi_1)r(\phi_2)} = \beta$$

Proof. Consider $r\left(\begin{bmatrix}\Psi & 0\\ \Phi & 0\end{bmatrix}\right) \leqslant \left\|\begin{bmatrix}\Psi & 0\\ \Phi & 0\end{bmatrix}\right\| \qquad \text{(by (3))}$

$$
\begin{aligned}
&= \left\| \begin{bmatrix} \Psi^* & \Phi^* \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Psi & 0 \\ \Phi & 0 \end{bmatrix} \right\|^{1/2} \qquad (\text{ by } (4)) \\
&= \left\| \begin{bmatrix} \Psi^*\Psi + \Phi^*\Phi & 0 \\ 0 & 0 \end{bmatrix} \right\|^{1/2} = \|\Psi^*\Psi + \Phi^*\Phi\|^{1/2} \\
&= \|(\psi_1 - i\psi_2)(\psi_1 + i\psi_2) + (\phi_1 - i\phi_2)(\phi_1 + i\phi_2)\|^{1/2} \\
&= \left\| \left(\psi_1^2 + \psi_2^2\right) + \left(\phi_1^2 + \phi_2^2\right) + i(\psi_1\psi_2 - \psi_2\psi_1) + i(\phi_1\phi_2 - \phi_2\phi_1) \right\|^{1/2} \\
&\leqslant \left(\left\|\psi_1^2 + \psi_2^2\right\| + \left\|\phi_1^2 + \phi_2^2\right\| + \|\psi_1\psi_2 - \psi_2\psi_1\| + \|\phi_1\phi_2 - \phi_2\phi_1\| \right)^{1/2} \\
&\leqslant (\max(\|\psi_1^2\|, \|\psi_2^2\|) + \|\psi_1\psi_2\| + \max(\|\phi_1^2\|, \|\phi_2^2\|) + \|\phi_1\phi_2\| \\
&+ \frac{1}{2}\|\psi_1\|\|\psi_2\| + \frac{1}{2}\|\phi_1\|\|\phi_2\|)^{\frac{1}{2}} \qquad (\text{by Lemma 2 and Lemma 3}) \\
&\leqslant (\max(\|\psi_1^2\|, \|\psi_2^2\|) + \|\psi_1\|\|\psi_2\| + \max(\|\phi_1^2\|, \|\phi_2^2\|) + \|\phi_1\|\|\phi_2\| \\
&+ \frac{1}{2}\|\psi_1\|\|\psi_2\| + \frac{1}{2}\|\phi_1\|\|\phi_2\|)^{\frac{1}{2}} \\
&= \sqrt{\max\left(r^2(\psi_1), r^2(\psi_2)\right) + \max\left(r^2(\phi_1), r^2(\phi_2)\right) + \frac{3}{2}r(\psi_1)r(\psi_2) + \frac{3}{2}r(\phi_1)r(\phi_2)}
\end{aligned}
$$

By Theorem 1 and 2, we conclude that

$$
r\left(\begin{bmatrix} \Psi & 0 \\ \Phi & 0 \end{bmatrix}\right) \leq \min(\alpha, \beta).
$$

Corollary 1. *Let $\Psi, \Phi \in M_n(\mathbb{C}\ )$ be Acc-Dis matrices with CD $\Psi = \psi_1 + i\psi_2$, $\Phi = \phi_1 + i\phi_2$. Then*

$$
r\left(\begin{bmatrix} 0 & \Phi \\ 0 & \Psi \end{bmatrix}\right) \leqslant \min(\alpha, \beta).
$$

Proof. The proof is observed by using (8) as follows:

$$
r\left(\begin{bmatrix} 0 & \Phi \\ 0 & \Psi \end{bmatrix}\right) = r\left(U^* \begin{bmatrix} \Psi & 0 \\ \Phi & 0 \end{bmatrix} U\right) = r\left(\begin{bmatrix} \Psi & 0 \\ \Phi & 0 \end{bmatrix}\right),
$$

where $U = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$.

Corollary 2. *Let $\Psi, \Phi \in M_n(\mathbb{C})$ be Acc-Dis matrices with CD $\Psi = \psi_1 + i\psi_2$, $\Phi = \phi_1 + i\phi_2$. Then*

$$
r\left(\begin{bmatrix} \Psi & 0 \\ -\Phi & 0 \end{bmatrix}\right) \leq \min(\alpha, \beta).
$$

Proof. The proof is due to using (8), since

$$r\left(\begin{bmatrix} \Psi & 0 \\ -\Phi & 0 \end{bmatrix}\right) = r\left(U^*\begin{bmatrix} \Psi & 0 \\ \Phi & 0 \end{bmatrix}U\right) = r\left(\begin{bmatrix} \Psi & 0 \\ \Phi & 0 \end{bmatrix}\right),$$

where $U = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$.

Corollary 3. *Let* $\Phi, \Psi \in M_n(\mathbb{C})$ *be Acc-Dis matrices with CD* $\Psi = \psi_1 + i\psi_2, \Phi = \phi_1 + i\phi_2$. *Then*

$$r\left(\begin{bmatrix} \Psi & \Phi \\ 0 & 0 \end{bmatrix}\right) \leq \min(\alpha, \beta).$$

Proof. Consider $r\left(\begin{bmatrix} \Psi & \Phi \\ 0 & 0 \end{bmatrix}\right) = r\left(\begin{bmatrix} \Psi^* & 0 \\ \Phi^* & 0 \end{bmatrix}\right)$ (by (7))

Now, using the same procedure in Theorem 1, we get

$$r\left(\begin{bmatrix} \Psi & \Phi \\ 0 & 0 \end{bmatrix}\right) \leqslant \left(\left\|(\Psi^*)^2\right\|\right)^{1/2} = \left(\left\|\left(\Psi^2\right)^*\right\|\right)^{1/2} = \left(\left\|\Psi^2\right\|\right)^{1/2} \leqslant \alpha.$$

Also, $r\left(\begin{bmatrix} \Psi & \Phi \\ 0 & 0 \end{bmatrix}\right) \leqslant \left\|\begin{bmatrix} \Psi & \Phi \\ 0 & 0 \end{bmatrix}\right\| = \left\|\begin{bmatrix} \Psi & \Phi \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \Psi^* & 0 \\ \Phi^* & 0 \end{bmatrix}\right\|^{1/2} \leqslant \beta$, which completes the proof.

Theorem 3. *Let* $\Psi, \Phi \in M_n(\mathbb{C})$ *be Acc-Dis matrices with CD* $\Psi = \psi_1 + i\psi_2, \Phi = \phi_1 + i\phi_2$. *Then*

$$r\left(\begin{bmatrix} \Psi & \Phi \\ \Phi & \Psi \end{bmatrix}\right) \leq r(\psi_1 + \phi_1) + r(\psi_2 + \phi_2).$$

Proof. Consider $r\left(U^*\begin{bmatrix} \Psi & \Phi \\ \Phi & \Psi \end{bmatrix}U\right) = r\left(\begin{bmatrix} \Psi & \Phi \\ \Phi & \Psi \end{bmatrix}\right)$, (by (8))

where $U = \frac{1}{\sqrt{2}}\begin{bmatrix} I & I \\ -I & I \end{bmatrix}$.

Now,

$$
\begin{aligned}
r\left(\begin{bmatrix} \Psi & \Phi \\ \Phi & \Psi \end{bmatrix}\right) &= \frac{1}{2} r\left(\begin{bmatrix} I & I \\ -I & I \end{bmatrix}\begin{bmatrix} \Psi & \Phi \\ \Phi & \Psi \end{bmatrix}\begin{bmatrix} I & -I \\ I & I \end{bmatrix}\right) \\
&= \frac{1}{2} r\left(\begin{bmatrix} 2(\Psi+\Phi) & 0 \\ 0 & 2(\Psi-\Phi) \end{bmatrix}\right) \\
&= r\left(\begin{bmatrix} \Psi+\Phi & 0 \\ 0 & \Psi-\Phi \end{bmatrix}\right) \quad \text{(by (5))} \\
&\leqslant \left\|\begin{bmatrix} \Psi+\Phi & 0 \\ 0 & \Psi-\Phi \end{bmatrix}\right\| \quad \text{(by (3))} \\
&= \left\|\begin{bmatrix} (\psi_1+\phi_1)+i(\psi_2+\phi_2) & 0 \\ 0 & (\psi_1-\phi_1)+i(\psi_2-\phi_2) \end{bmatrix}\right\| \\
&= \left\|\begin{bmatrix} (\psi_1+\phi_1) & 0 \\ 0 & (\psi_1-\phi_1) \end{bmatrix} + i\begin{bmatrix} (\psi_2+\phi_2) & 0 \\ 0 & (\psi_2-\phi_2) \end{bmatrix}\right\| \\
&\leqslant \left\|\begin{bmatrix} (\psi_1+\phi_1) & 0 \\ 0 & (\psi_1-\phi_1) \end{bmatrix}\right\| + \left\|\begin{bmatrix} (\psi_2+\phi_2) & 0 \\ 0 & (\psi_2-\phi_2) \end{bmatrix}\right\| \\
&= \max\left(\|\psi_1+\phi_1\|, \|\psi_1-\phi_1\|\right) + \max\left(\|\psi_2+\phi_2\|, \|\psi_2-\phi_2\|\right) \quad \text{(by Lemma 5)} \\
&= \|\psi_1+\phi_1\| + \|\psi_2+\phi_2\|, \text{(Since } \psi_1, \psi_2, \phi_1, \phi_2 \text{ are p.s.d)} \\
&= r(\psi_1+\phi_1) + r(\psi_2+\phi_2) \qquad \text{(by(3))}.
\end{aligned}
$$

Theorem 4. *Let $\Psi \in M_n(\mathbb{C})$ be Acc-Dis matrix with CD $\Psi = \psi_1 + i\psi_2$.Then*

$$r\left(\begin{bmatrix} \Psi & \Psi \\ -\Psi & -\Psi \end{bmatrix}\right) \leq 2r(\psi_2 + \psi_2).$$

Proof. Consider $r\left(\begin{bmatrix} \Psi & \Psi \\ -\Psi & -\Psi \end{bmatrix}\right) = r\left(U\begin{bmatrix} \Psi & \Psi \\ -\Psi & -\Psi \end{bmatrix}U^*\right)$, (by (8)),
where $U = \frac{1}{\sqrt{2}}\begin{bmatrix} I & I \\ -I & I \end{bmatrix}$

$$
\begin{aligned}
r\left(\begin{bmatrix} \Psi & \Psi \\ -\Psi & -\Psi \end{bmatrix}\right) &= \frac{1}{2} r\left(\begin{bmatrix} I & I \\ -I & I \end{bmatrix}\begin{bmatrix} \Psi & \Psi \\ -\Psi & -\Psi \end{bmatrix}\begin{bmatrix} I & -I \\ I & I \end{bmatrix}\right) \\
&= \frac{1}{2} r\left(\begin{bmatrix} 0 & 0 \\ -4\Psi & 0 \end{bmatrix}\right) \\
&= 2r\left(\begin{bmatrix} 0 & 0 \\ \Psi & 0 \end{bmatrix}\right) \\
&\leqslant 2\left\|\begin{bmatrix} 0 & 0 \\ \Psi & 0 \end{bmatrix}\right\| \\
&= 2\|\Psi\| \\
&= 2\|\psi_1 + i\psi_2\| \\
&\leqslant 2\|\psi_1 + \psi_2\| \qquad \text{(by Lemma 1)} \\
&= 2r(\psi_1 + \psi_2) \qquad \text{(by (3))}
\end{aligned}
$$

The following is an example of Theorem 4.

Example 1. *Let* $\Psi = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + i \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1+i & 1 \\ 1 & 1+i \end{bmatrix}$.
Note that $r\left(\begin{bmatrix} \Psi & \Psi \\ -\Psi & -\Psi \end{bmatrix}\right) = r\left(U \begin{bmatrix} \Psi & \Psi \\ -\Psi & -\Psi \end{bmatrix} U^*\right) = \frac{1}{2} r\left(\begin{bmatrix} 0 & 0 \\ -4\Psi & 0 \end{bmatrix}\right) = 2\|\Psi\| = 2r(\Psi)$, *where* $U = \frac{1}{\sqrt{2}} \begin{bmatrix} I & I \\ -I & I \end{bmatrix}$.
To find $r(\Psi)$, *we compute the eigenvalues of* Ψ *as follows:*

$$
\begin{aligned}
det(\lambda I - \Psi) &= \det \begin{bmatrix} \lambda - 1 - i & -1 \\ -1 & \lambda - 1 - i \end{bmatrix} \\
&= \det \begin{bmatrix} \lambda - (1+i) & -1 \\ -1 & \lambda - (1+i) \end{bmatrix} \\
&= (\lambda - (1+i))^2 - 1 = 0.
\end{aligned}
$$

$\Rightarrow (\lambda - (1+i))^2 = 1 \Rightarrow \lambda \in \{2+i, i\}$.
So, $r(\Psi) = \max|\lambda| = \sqrt{4+1} = \sqrt{5}$.
Now it is very easy to find $r(\psi_1 + \psi_2) = r\left(\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\right)$ *by calculating their eigenvalues which are* $1, 3$. *Consequently,* $r(\psi_1 + \psi_2) = 3$.
Hence, $2\sqrt{5} = 2r(\Psi) \leq 2r(\psi_1 + \psi_2) = 6$ *as it was concluded by Theorem 4.*

4. Conclusion

New results for spectral radius of 2×2 block matrices involving accretive-dissipative matrices were discussed and proved, also an example is given.

In future, our plan is going to make extension of our inequalities for accretive-dissipative matrices, which are matrices whose only real part is p.s.d.

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