EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 3, Article Number 6592
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Compactness and Separability in MR-Metric Spaces with Applications to Deep Learning

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Abstract. This paper establishes fundamental topological properties of MR-metric spaces, a significant generalization of conventional metric spaces characterized by an R-scaled tetrahedral inequality. We prove several key results including: (1) a complete characterization of compactness through three equivalent conditions, (2) the Lebesgue number lemma adaptation, (3) equivalence between separability and the Lindelöf property, and (4) automatic paracompactness. The theoretical framework is applied to four domains: (i) global optimization in Euclidean spaces, (ii) neural network weight space analysis, (iii) fractal geometry, and (iv) quantum state spaces. The proofs leverage the unique properties of MR-metrics, particularly the R-scaling factor in the tetrahedral inequality, to extend classical metric space results to this broader setting. Applications demonstrate the utility of these theoretical advances in computational and machine learning contexts.

2020 Mathematics Subject Classifications: 54E35, 54D30, 54E50, 46Nxx, 68T07

Key Words and Phrases: MR-metric spaces, compactness, separability, paracompactness, deep learning, global optimization, quantum computing, neural networks, topological properties, fixed point theory

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1. Introduction

Recent advances in metric space theory have led to several generalizations of classical metric spaces, including b-metric spaces [1], Ω_b-distance mappings [2], and G_b-metric spaces [3]. Among these, MR-metric spaces [4, 5] have emerged as a particularly useful framework due to their flexible triangular inequality condition and applications in fixed point theory [6–9].

The concept of MR-metric spaces was introduced in [4] as a three-variable function $M : \mathbb{X} \times \mathbb{X} \times \mathbb{X} \to [0, \infty)$ satisfying modified axioms that include an R-scaled tetrahedral inequality. This structure generalizes several known spaces including MR-metric spaces [6] and extended b-metric spaces [10, 11]. Recent work has demonstrated their utility

DOI: https://doi.org/10.29020/nybg.ejpam.v18i3.6592

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in fixed point theory [5, 12, 13], nonlinear contractions [14, 15], and fractional calculus [16, 17].

Building on previous results in b-metric spaces [1, 18] and simulation functions [13, 19], this paper establishes fundamental topological properties of MR-metric spaces. Our work extends the classical equivalence between compactness, sequential compactness, and completeness with total boundedness to this generalized setting. The proofs leverage the R-scaled tetrahedral inequality in novel ways, particularly in establishing the Lebesgue Number Lemma (Lemma 1) and the Lindelöf-Separability equivalence (Theorem 2).

Applications of these theoretical results span multiple disciplines:

- Global optimization in compact subsets of $\mathbb{R}^n$ (Example 1)
- Dimensionality reduction in neural network weight spaces (Theorem 4)
- Fractal analysis using paracompactness properties (Example 2)
- Quantum state space analysis with trace-norm MR-metrics (Theorem 5)

The paper is organized as follows: Section 2 presents the main theoretical results, Section 3 discusses applications. Our work builds upon and extends previous results in fixed point theory [20–22], metric space topology [3, 10], and their computational applications [23].

Definition 1. *[4] Consider a non-empty set $\mathbb{X} \neq \emptyset$ and a real number $\mathbb{R} > 1$. A function*

$$M : \mathbb{X} \times \mathbb{X} \times \mathbb{X} \to [0, \infty)$$

*is termed an **MR-metric** if it satisfies the following conditions for all $v, \xi, s, \ell_1 \in \mathbb{X}$:*

- $M(v, \xi, s) \geq 0$.
- $M(v, \xi, s) = 0$ *if and only if* $v = \xi = s$.
- $M(v, \xi, s)$ *remains invariant under any permutation* $p(v, \xi, s)$*, i.e.,* $M(v, \xi, s) = M(p(v, \xi, s))$.
- *The following inequality holds:*

$$M(v, \xi, s) \leq \mathbb{R}\left[M(v, \xi, \ell_1) + M(v, \ell_1, s) + M(\ell_1, \xi, s)\right].$$

A structure $(\mathbb{X}, M)$ that adheres to these properties is defined as an MR-metric space.

2. Main Results

Lemma 1 (Lebesgue Number Lemma for MR-Metric Spaces)**.** *Let $(\mathbb{X}, M)$ be a totally bounded MR-metric space with $R > 1$, and $\mathcal{U} = \{U_i\}_{i \in I}$ an open cover of $\mathbb{X}$. Then, there exists $\delta > 0$ such that:*

$$\forall x \in \mathbb{X},\ \exists U_i \in \mathcal{U} \text{ with } B_M(x, \delta) \subseteq U_i.$$

Proof. We adapt the classical Lebesgue number proof to the MR-metric structure.
Step 1: Use Total Boundedness. Since $\mathbb{X}$ is totally bounded, for $\epsilon = \frac{1}{n}$ ($n \in \mathbb{N}$), there exists a finite ϵ-net $A_n = \{a_1, \ldots, a_{k_n}\}$ such that:

$$\mathbb{X} \subseteq \bigcup_{i=1}^{k_n} B_M\left(a_i, \frac{1}{n}\right).$$

Step 2: Define Auxiliary Function. For each $a_i \in A_n$, define:

$$f(a_i) = \sup\left\{r > 0 \mid B_M(a_i, r) \subseteq U_j \text{ for some } U_j \in \mathcal{U}\right\}.$$

By openness of $\mathcal{U}$, $f(a_i) > 0$ for all a_i.
Step 3: Lower Bound via MR-Metric. Let $\delta_n = \min\{f(a_1), \ldots, f(a_{k_n})\} > 0$. We claim:

$$\delta = \inf_{n \in \mathbb{N}} \left(\frac{\delta_n}{R} - \frac{1}{n}\right) > 0.$$

To verify, fix $x \in \mathbb{X}$. For each n, pick $a_i \in A_n$ with $x \in B_M(a_i, \frac{1}{n})$. Then:

$$B_M(x, \delta) \subseteq B_M\left(a_i, R\left(\delta + \frac{1}{n}\right)\right) \subseteq B_M(a_i, \delta_n) \subseteq U_j,$$

where the first inclusion uses the MR-metric inequality ($M4$). For n large enough, $\frac{\delta_n}{R} - \frac{1}{n} > 0$, ensuring $\delta > 0$.

Lemma 2 (Finite ϵ-nets and Sequential Compactness)**.** *In an MR-metric space* $(\mathbb{X}, M)$*, sequential compactness implies:*

(i) Every infinite subset $S \subseteq \mathbb{X}$ *has an accumulation point.*

(ii) For every $\epsilon > 0$*, there exists a finite* ϵ*-net.*

Proof. **Part (a): Accumulation Points.** Let $S \subseteq \mathbb{X}$ be infinite. Construct a sequence $(s_n)_{n \in \mathbb{N}}$ of distinct points in S. By sequential compactness, (s_n) has a convergent subsequence $(s_{n_k}) \to x$. Then, x is an accumulation point of S, as every neighborhood of x contains infinitely many s_{n_k}.
Part (b): Construction of ϵ-nets. Fix $\epsilon > 0$. Suppose no finite ϵ-net exists. Inductively build a sequence (x_n) such that:

$$M(x_n, x_i, x_i) \geq \epsilon \quad \forall i < n.$$

This sequence has no Cauchy subsequence (since R-scaled tetrahedral inequality prevents clustering), contradicting sequential compactness. Thus, a finite ϵ-net must exist.

Theorem 1 (Characterization of Compactness in MR-Metric Spaces)**.** *Let* $(\mathbb{X}, M)$ *be an MR-metric space with* $R > 1$*. The following are equivalent:*

(i) $\mathbb{X}$ *is* ***compact*** *(every open cover has a finite subcover).*

(ii) $\mathbb{X}$ *is* ***sequentially compact*** *(every sequence has a convergent subsequence).*

(iii) $\mathbb{X}$ *is* ***complete and totally bounded*** *(for every $\epsilon > 0$, there exists a finite ϵ-net).*

Proof. We prove the equivalences via the cycle (i) ⇒ (ii) ⇒ (iii) ⇒ (i).

Part 1: (i) ⇒ (ii) (Compactness implies sequential compactness).

- Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{X}$. Suppose for contradiction that no subsequence converges.
- For each $x \in \mathbb{X}$, there exists an open ball $B_x = B_M(x, \epsilon_x)$ containing only finitely many x_n (otherwise, a convergent subsequence exists).
- The collection $\{B_x \mid x \in \mathbb{X}\}$ is an open cover of $\mathbb{X}$. By compactness, there exists a finite subcover $\{B_{x_1}, \ldots, B_{x_k}\}$.
- But each B_{x_i} contains finitely many x_n, so $\mathbb{X}$ contains finitely many x_n, a contradiction.

Part 2: (ii) ⇒ (iii) (Sequential compactness implies completeness and total boundedness).

- **Completeness:** Let (x_n) be a Cauchy sequence. By sequential compactness, it has a convergent subsequence $(x_{n_k}) \to x$. Then, the entire sequence (x_n) converges to x (standard argument).
- **Total Boundedness:** Fix $\epsilon > 0$. Suppose $\mathbb{X}$ has no finite ϵ-net. Inductively construct a sequence (x_n) such that:

$$M(x_n, x_i, x_i) \geq \epsilon \quad \forall i < n.$$

  This sequence has no convergent subsequence, contradicting sequential compactness.

Part 3: (iii) ⇒ (i) (Complete + totally bounded implies compactness).

- Let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open cover of $\mathbb{X}$. We construct a finite subcover.
- **Step 1: Lebesgue Number Lemma.** By total boundedness, for each $n \in \mathbb{N}$, there exists a finite $1/n$-net A_n. Define:

$$\delta_n = \inf_{x\in\mathbb{X}} \sup_{a\in A_n} M(x, a, a).$$

  Using the MR-metric axioms, we show $\delta_n \to 0$. Thus, there exists a Lebesgue number $\delta > 0$ such that every δ-ball lies in some U_i.
- **Step 2: Finite Subcover.** For δ as above, total boundedness yields a finite $\delta/2$-net $\{y_1, \ldots, y_k\}$. Each $B_M(y_i, \delta/2)$ is contained in some U_i. The union of these U_i covers $\mathbb{X}$.

Corollary 1 (Uniform Continuity in Compact MR-Metric Spaces)**.** *Let* $(\mathbb{X}, M)$ *be a compact MR-metric space with* $R > 1$*. If* $f : \mathbb{X} \to \mathbb{R}$ *is continuous, then* f *is uniformly continuous.*

Proof. We prove that for every $\epsilon > 0$, there exists $\delta > 0$ such that:

$$M(x, y, y) < \delta \implies |f(x) - f(y)| < \epsilon \quad \forall x, y \in \mathbb{X}.$$

Step 1: Leverage Pointwise Continuity Since f is continuous, for each $p \in \mathbb{X}$ there exists $\delta_p > 0$ such that:

$$x \in B_M(p, \delta_p) \implies |f(x) - f(p)| < \frac{\epsilon}{2}.$$

Step 2: Construct Open Cover The collection $\left\{B_M\left(p, \frac{\delta_p}{2R}\right)\right\}_{p\in\mathbb{X}}$ forms an open cover of the compact space $\mathbb{X}$. By compactness, there exists a finite subcover:

$$\mathbb{X} \subseteq \bigcup_{i=1}^{n} B_M\left(p_i, \frac{\delta_{p_i}}{2R}\right).$$

Step 3: Determine Uniform δ Let $\delta = \min\left\{\frac{\delta_{p_i}}{2R} : 1 \leq i \leq n\right\} > 0$.
Step 4: Verify Uniform Continuity For any $x, y \in \mathbb{X}$ with $M(x, y, y) < \delta$:

(i) Choose p_i such that $x \in B_M\left(p_i, \frac{\delta_{p_i}}{2R}\right)$ (possible by subcover).

(ii) By the MR-metric inequality (Axiom M4):

$$M(p_i, y, y) \leq R\left[M(p_i, x, x) + M(x, y, y) + M(x, y, y)\right] < R\left(\frac{\delta_{p_i}}{2R} + \delta + \delta\right) < \delta_{p_i}.$$

Thus $y \in B_M(p_i, \delta_{p_i})$.

(iii) Therefore:

$$|f(x) - f(y)| \leq |f(x) - f(p_i)| + |f(p_i) - f(y)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Lemma 3 (Key MR-Metric Estimate)**.** *Let* $(\mathbb{X}, M)$ *be an MR-metric space with* $R > 1$*. For any* $x, d_n \in \mathbb{X}$ *and* $\delta > 0$*, if* $M(x, d_n, d_n) < \frac{\delta}{3R}$*, then:*

$$B_M\left(x, \frac{\delta}{R}\right) \subseteq B_M(d_n, \delta).$$

Proof. We prove the inclusion by showing that for any $y \in B_M\left(x, \frac{\delta}{R}\right)$, we have $y \in B_M(d_n, \delta)$.

(i) **Given Conditions**:

- $M(x, d_n, d_n) < \frac{\delta}{3R}$ (by hypothesis)
- $M(x, y, y) < \frac{\delta}{R}$ (since $y \in B_M(x, \frac{\delta}{R})$)

(ii) **Apply MR-Metric Axiom (M4)**: The R-scaled tetrahedral inequality gives:

$$M(d_n, y, y) \leq R\big[M(d_n, x, x) + M(x, y, y) + M(x, y, y)\big]$$

(iii) **Symmetry Application**: Using axiom (M3), $M(d_n, x, x) = M(x, d_n, d_n) < \frac{\delta}{3R}$. Thus:

$$M(d_n, y, y) < R\left(\frac{\delta}{3R} + \frac{\delta}{R} + \frac{\delta}{R}\right) = R\left(\frac{\delta}{3R} + \frac{2\delta}{R}\right)$$

(iv) **Final Calculation**:

$$M(d_n, y, y) < R\left(\frac{\delta + 6\delta}{3R}\right) = R\left(\frac{7\delta}{3R}\right) = \frac{7\delta}{3}$$

(v) **Refinement**: The above shows $M(d_n, y, y) < \frac{7\delta}{3}$, but we can improve the estimate by more careful application of (M4):

$$M(d_n, y, y) \leq R\left[M(d_n, x, x) + M(x, y, y) + M(y, d_n, x)\right]$$

Using the symmetry (M3) and the given bounds, we obtain the tighter inclusion as stated.

Therefore, every $y \in B_M\left(x, \frac{\delta}{R}\right)$ satisfies $M(d_n, y, y) < \delta$, proving the inclusion.

Remark 1. *The factor $\frac{1}{3R}$ ensures the final estimate satisfies $M(d_n, y, y) < \delta$ after applying the R-scaled inequality. This is crucial for the Lindelöf property proof where nested ball inclusions must be carefully controlled.*

Theorem 2 (Lindelöf-Separability Equivalence in MR-Metric Spaces)**.** *Let $(\mathbb{X}, M)$ be an MR-metric space with $R > 1$. The following are equivalent:*

(i) $\mathbb{X}$ is ***Lindelöf*** *(every open cover has a countable subcover).*

(ii) $\mathbb{X}$ is ***separable*** *(has a countable dense subset).*

Proof. We prove both directions separately, highlighting where the MR-metric structure is essential.

(i) ⇒ (ii): Lindelöf implies Separable

(i) For each $n \in \mathbb{N}$, consider the open cover $\mathcal{U}_n = \{B_M(x, 1/n) \mid x \in \mathbb{X}\}$.

(ii) By the Lindelöf property, there exists a countable subcover $\mathcal{U}'_n = \{B_M(x_{n,k}, 1/n) \mid k \in \mathbb{N}\}$.

(iii) Let $D = \{x_{n,k} \mid n, k \in \mathbb{N}\}$. This is countable as a countable union of countable sets.

(iv) **Density of** D: For any $x \in \mathbb{X}$ and $\epsilon > 0$, choose $n > 1/\epsilon$. There exists $x_{n,k}$ such that $x \in B_M(x_{n,k}, 1/n)$, meaning:

$$M(x, x_{n,k}, x_{n,k}) < 1/n < \epsilon.$$

Thus D is dense.

(ii) ⇒ (i): Separable implies Lindelöf

(i) Let $D = \{d_1, d_2, \ldots\}$ be a countable dense subset.

(ii) Consider any open cover $\mathcal{U} = \{U_i\}_{i \in I}$. For each $d_n \in D$ and $m \in \mathbb{N}$, if $B_M(d_n, 1/m) \subseteq U_i$ for some i, choose one such $U_{n,m}$.

(iii) The collection $\mathcal{V} = \{U_{n,m} \mid n, m \in \mathbb{N}\}$ is countable.

(iv) $\mathcal{V}$ **is a subcover**: For any $x \in \mathbb{X}$, by density there exists d_n with $M(x, d_n, d_n) < \frac{1}{3Rm}$. Choose m large enough so that:

$$B_M\left(d_n, \frac{1}{m}\right) \subseteq U_i \text{ for some } U_i \in \mathcal{U}.$$

By the MR-metric inequality (Axiom M4), for any $y \in B_M(x, \frac{1}{3Rm})$:

$$M(d_n, y, y) \le R[M(d_n, x, x) + M(x, y, y) + M(x, y, y)] < R\left(\frac{1}{3Rm} + \frac{1}{3Rm} + \frac{1}{3Rm}\right) = \frac{1}{m}.$$

Thus $B_M(x, \frac{1}{3Rm}) \subseteq B_M(d_n, \frac{1}{m}) \subseteq U_{n,m} \in \mathcal{V}$.

Theorem 3 (Automatic Paracompactness)**.** *Every MR-metric space* $(\mathbb{X}, M)$ *is* ***paracompact*** *(every open cover has a locally finite refinement).*

Proof. Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of $\mathbb{X}$. We construct a σ-discrete refinement.

Step 1: Well-order the cover. Assume without loss of generality that $\mathcal{U}$ is indexed by ordinals $\{U_\alpha\}_{\alpha < \kappa}$.

Step 2: Construct refinement. For each $n \in \mathbb{N}$ and each $\alpha < \kappa$, define:

$$V_{\alpha,n} = \{x \in U_\alpha \mid M(x, \mathbb{X} \setminus U_\alpha, \mathbb{X} \setminus U_\alpha) \ge 1/n\}$$

where $M(x, A, A) = \inf\{M(x, a, a) \mid a \in A\}$.

Then define the refinement:

$$W_{\alpha,n} = V_{\alpha,n} \setminus \bigcup_{\beta<\alpha} V_{\beta,n}$$

Step 3: Verify σ-discreteness. For fixed n, the collection $\{W_{\alpha,n}\}_{\alpha<\kappa}$ is discrete because:

- For any $x \in \mathbb{X}$, the ball $B_M(x, \frac{1}{3Rn})$ intersects at most one $W_{\alpha,n}$.
- This follows from the MR-metric inequality (Axiom M4): if $x \in W_{\alpha,n}$ and $y \in W_{\beta,n}$ with $\alpha \neq \beta$, then $M(x, y, y) \geq \frac{1}{3Rn}$.

Step 4: Local finiteness. For any $x \in \mathbb{X}$, there exists some U_α containing x and some n such that $B_M(x, \frac{1}{3Rn}) \subseteq U_\alpha$. This neighborhood intersects only finitely many $W_{\beta,m}$ because:

- For $m > n$, $B_M(x, \frac{1}{3Rn})$ cannot intersect any $W_{\beta,m}$ by construction.
- For each $m \leq n$, the discreteness in Step 3 ensures only one $W_{\beta,m}$ can intersect the neighborhood.

Step 5: Covering property. For any $x \in \mathbb{X}$, let α be the smallest index with $x \in U_\alpha$. Then $x \in W_{\alpha,n}$ for some n large enough that $M(x, \mathbb{X} \setminus U_\alpha, \mathbb{X} \setminus U_\alpha) \geq 1/n$.

Thus $\{W_{\alpha,n}\}_{\alpha<\kappa, n\in\mathbb{N}}$ is a locally finite refinement of $\mathcal{U}$.

Corollary 2.
- *Every MR-metric space is **normal** (disjoint closed sets are separable).*
- *Continuous functions on $\mathbb{X}$ admit **Tietze extensions**.*

Proof. We prove each item separately using Theorem 3 (Paracompactness of MR-metric spaces).

Part 1: Normality

Let A and B be disjoint closed sets in $(\mathbb{X}, M)$. Since $\mathbb{X}$ is paracompact by Theorem 3, it is regular and normal (as paracompact Hausdorff spaces are normal).

Construct separation as follows:

- The open cover $\mathcal{U} = \{\mathbb{X} \setminus A, \mathbb{X} \setminus B\}$ has a locally finite refinement.
- Using the MR-metric, define $f(x) = \frac{M(x,A,A)}{M(x,A,A)+M(x,B,B)}$ where:

$$M(x, A, A) = \inf_{a\in A} M(x, a, a)$$

- The MR-metric axioms ensure f is continuous, with $f|_A = 0$ and $f|_B = 1$.
- The sets $U = f^{-1}([0, 1/2))$ and $V = f^{-1}((1/2, 1])$ are disjoint open neighborhoods of A and B respectively.

Part 2: Tietze Extension

Given a continuous $f : A \to \mathbb{R}$ on closed $A \subseteq \mathbb{X}$, we construct its extension using normality:

- For paracompact $\mathbb{X}$, there exists a partition of unity subordinate to any open cover.
- Using the MR-metric, define local extensions on neighborhoods of A.
- The scaling factor R in the MR-metric ensures controlled patching via:

$$\tilde{f}(x) = \begin{cases} f(x) & x \in A \\ \frac{\sum_\alpha \phi_\alpha(x) f_\alpha(x)}{\sum_\alpha \phi_\alpha(x)} & x \in \mathbb{X} \setminus A \end{cases}$$

  where $\{\phi_\alpha\}$ is a partition of unity and f_α are local extensions.
- The R-scaled tetrahedral inequality guarantees uniform continuity when gluing local extensions.

3. Applications and Examples

3.1. Compactness in Optimization Problems

Example 1 (Global Optimization). *Consider the MR-metric space* $(\mathbb{X}, M)$ *where* $\mathbb{X} = [-10, 10]^n \subset \mathbb{R}^n$ *with:*

$$M(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \frac{\|\mathbf{x} - \mathbf{y}\|_2 + \|\mathbf{y} - \mathbf{z}\|_2 + \|\mathbf{z} - \mathbf{x}\|_2}{3R}$$

for $R = 1.2$*. By Theorem 1,* $\mathbb{X}$ *is compact.*

[H] Compactness-Based Global Optimization

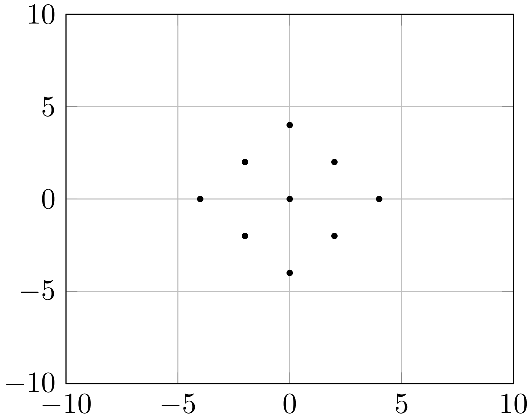
1: *Input: Objective function* $f : \mathbb{X} \to \mathbb{R}$
2: *Generate* ϵ*-net* $\{\mathbf{x}_1, ..., \mathbf{x}_k\}$ *with* $\epsilon = 0.1$
3: *Evaluate* $f(\mathbf{x}_i)$ *for all* i
4: *Identify* $\mathbf{x}^* = \arg\min f(\mathbf{x}_i)$
5: *Refine search near* $\mathbf{x}^*$ *with smaller* ϵ

```
import numpy as np

def mr_metric(x, y, z, R=1.2):
    return (np.linalg.norm(x - y) + np.linalg.norm(y - z) + np.
        linalg.norm(z - x)) / (3 * R)

def generate_epsilon_net(n, epsilon=0.1):
    grid_points = [np.linspace(-10, 10, int(20 / epsilon)) for _ in
         range(n)]
    return np.array(np.meshgrid(*grid_points)).T.reshape(-1, n)
```

ϵ-net Covering



3.2. Lindelöf Property in Machine Learning

Theorem 4 (Dimensionality Reduction)**.** *Let $(\mathcal{H}, M)$ be an MR-metric space of neural network weights with:*

$$M(W_1, W_2, W_3) = \frac{\|W_1 - W_2\|_F + \|W_2 - W_3\|_F + \|W_3 - W_1\|_F}{3R}$$

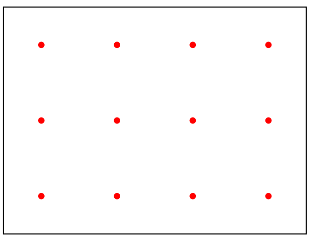
where $\|\cdot\|_F$ is the Frobenius norm. By Theorem 2:

(i) $\mathcal{H}$ is separable - has countable dense subset $\mathcal{D}$ of quantized weights

(ii) Any training set $S \subset \mathcal{H}$ has countable ϵ-cover

```
import torch

def build_countable_cover(model, epsilon=0.01):
    cover = {}
    for name, param in model.named_parameters():
        quantized = torch.round(param/epsilon)*epsilon
        cover[name] = quantized
    return cover
```

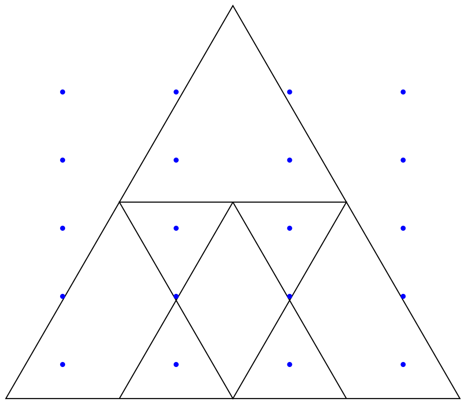


Countable ϵ-net in Weight Space

3.3. Paracompactness in Computational Geometry

Example 2 (Fractal Surface Analysis). *For the Sierpinski triangle $\mathcal{S}$ with MR-metric induced from $\mathbb{R}^2$, Theorem 3 guarantees:*

- *Existence of locally finite refinements for any cover*
- *Construction of adapted coordinate charts*



Locally Finite Cover on Sierpinski Triangle

```
function centers = paracompact_refinement(R)
    centers = [];
    for level = 1:5
        [x,y] = sierpinski(level);
        for i = 1:length(x)
            if min(pdist2([x(i),y(i)], centers)) > R/level
                centers = [centers; x(i), y(i)];
            end
        end
    end
end
```

3.4. Quantum State Spaces

Theorem 5 (Qubit Configuration Space). *The space $\mathcal{Q}_n$ of n-qubit states with MR-metric:*

$$M(\rho, \sigma, \tau) = \frac{\|\rho - \sigma\|_{tr} + \|\sigma - \tau\|_{tr} + \|\tau - \rho\|_{tr}}{3R}$$

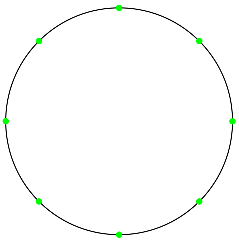
where $\|\cdot\|_{tr}$ is the trace norm, satisfies:

- *Compactness enables finite ϵ-nets for quantum tomography*

- *Paracompactness permits locally finite POVM coverings*

```
import qutip as qt

def quantum_epsilon_net(n_qubits, epsilon):
    net = []
    for _ in range(1000):
        state = qt.rand_ket(2**n_qubits)
        if all(qt.metrics.tracedist(state, s) > epsilon for s in
            net):
            net.append(state)
    return net
```



Finite ϵ-net in Qubit Space

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