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On the Diophantine Equation $p^x + (p+5k)^y = z^2$

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Abstract. With the use of modular arithmetic and other fundamental number theoretic methods, as well as the concepts of the floor function and the principle of mathematical induction, this study searches for possible nonnegative integer solutions of exponential Diophantine equations of the form $p^x + (p+5k)^y = z^2$, where $k \in \mathbb{N}$. Results are obtained for the following cases:

a) when $p = 2$; or

b) when p and $p + 5k$ are prime pairs.

In addition, the study is limited only to solutions where x and y are not both greater than 1.

2020 Mathematics Subject Classifications: 11D61, 11D72, 11A41

Key Words and Phrases: Exponential Diophantine equation, Prime pairs, Mihăilescu's theorem

1. Introduction

A *prime number* is a natural number greater than 1 whose divisors are 1 and itself. The difference between two prime numbers is called *prime gap*, and the two prime numbers are called *prime pair*. The prime pairs that have two gaps are called *twin primes*, those with four and six gaps are called *cousin primes* and *sexy primes*, respectively. In this present study, we incorporate prime bases and prime pairs with gaps of multiples of five. They are part of the study of exponential Diophantine equations.

Diophantine equations are any equations that seek rational solutions, but usually integers. These equations are believed to have been introduced by Diophantus of Alexandria. There are two types of Diophantine equation, namely, linear and nonlinear. Each type could have no solution, unique solution, finitely many solutions, or infinitely many solutions.

The *Linear Diophantine Equation* (LDE) is the famous one. In n unknowns, it is of the form

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = c, \tag{1}$$

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where $x_1, x_2, \ldots, x_n$ are the unknowns, and $a_1, a_2, \ldots, a_n, c$ are fixed integers, and not all coefficients are zero. The simplest LDE is the one with two unknowns, say x and y.

An equation that is not of the form (1) is considered *Nonlinear Diophantine Equation* (NDE). Examples of this type include the Pythagorean equation (in 3 unknowns) $x^2 + y^2 = z^2$, Pell equation (in 2 unknowns) $x^2 - Dy^2 = 1$, Nagell-Ljunggren equation (in 4 unknowns) $\dfrac{x^n - 1}{x - 1} = y^q$, and the Catalan equation (in 4 unknowns) $a^x - b^y = 1$. The last equation is also known as an *Exponential Diophantine Equation* (EDE). It is an equation that involves variable exponents.

In the past few years, many researchers have studied EDEs of the form

$$p^x + q^y = z^2 \tag{2}$$

for various fixed values of p and q. In 2011, Singta et al. [1] proved that there are no solutions in the set $\mathbb{N}_0$ of non-negative integers when $p = 4$ and $q = 7$ or 11. In 2012, Sroysang [2] showed that $(1, 0, 2)$ is the only solution in $\mathbb{N}_0$ for the case $p = 3$ and $q = 5$. In the same year, he also established in [3] that $(1, 0, 3)$ is the only solution in $\mathbb{N}_0$ when $p = 8$ and $q = 19$, and posed an open problem for the case $q = 17$. This problem was later addressed by Rabago [4], who demonstrated that (2) has only finitely many non-negative integer solutions when $p = 8$ and $q = 17$, namely, $(1, 0, 3), (1, 1, 5), (2, 1, 9)$, and $(3, 1, 23)$. A year later, Sroysang [5] examined the same class of EDEs with $p = 5$ and $q = 7$, but found no solutions in $\mathbb{N}_0$. In 2013, Chotchaisthit [6] studied the case where p and q are two consecutive integers, but p must be a Mersenne prime. Other Diophantine equations that involve bases of primes and Mersenne primes are found in the works of Gayo, Mina and Bacani (cf. [7–11]). Most of these studies utilize Mihăilescu's theorem (known originally as Catalan's conjecture). Basically, this theorem states that the only solution to the Diophantine equation $p^x - q^y = 1$ is $(p, q, x, y) = (3, 2, 2, 3)$, with the assumption that all variables have values greater than 1 [12].

It is also observed that the EDE (2) has been studied for specific prime pairs (p, q). Some examples are found in Gupta's and Sroysang's papers [2, 5, 13]. In 2015, Bacani and Rabago [14] investigated the case where p and q are twin primes. In 2018, Burshtein ([15], [16]) published two articles, namely, when p and q are cousin primes and when they are sexy primes with the condition that $x + y = 2, 3, 4$. In the same year, Neres [17] proved the solvability of (2) when p and q have a prime gap of 8, and $p > 3$. Recently, Tadee [18] studied the case where the prime gap is 14. A more interesting case was what Mina and Bacani [19] published in 2021, where they studied the case where p and q have prime gaps of multiples of four. In 2022, Orosram, et al. [20] also studied the same prime pairs with an additional constraint that $p \equiv 7 \pmod{12}$.

Motivated by the papers mentioned above, the present study searches for nonnegative integer solutions of the Diophantine equation of the form

$$p^x + (p + 5k)^y = z^2, \tag{3}$$

where $k \in \mathbb{N}$, and p satisfies any of the following conditions:

a) $p = 2$; or

b) p and $p + 5k$ are prime pairs.

The principle of mathematical induction is also used in this study, as well as the concept of floor function, which is usually not seen in related literature. Furthermore, the results presented here extend those of Dokchan and Panngam [21], who investigated the same equation under the condition $p \equiv 1 \pmod 5$ and sought solutions in $\mathbb{N}$.

2. Main results

The results are divided into two parts. First, we present interesting results for the case $p = 2$. Next, we discuss the solutions of the equation when p and $p + 5k$ are prime pairs.

2.1. On the Diophantine Equation $2^x + (2 + 5k)^y = z^2$

This subsection talks about the Diophantine equation (3), where $p = 2$, that is,

$$2^x + (2 + 5k)^y = z^2. \tag{4}$$

It is further divided into two parts. The first part discusses the case when $y = 1$, and the second part contains all other claims when $y \neq 1$.

2.1.1. Part I: The case where $y = 1$.

Consider the equation

$$2^x + (2 + 5k) = z^2. \tag{5}$$

For this case, the following variables will be used:

Variable	Meaning
x_n	the value of x at a specific value of $n \in \mathbb{N}_0$
$z_{(0,x_n)}$	the first value of z at a specific value of x_n
$z_{(m,x_n)}$	the $(m+1)$st value of z at a specific value of x_n
$k_{(0,x_n)}$	the first value of k at a specific value of x_n
$k_{(1,x_n)}$	the second value of k at a specific value of x_n
α_{x_n}	the difference between the first and second value of k at a specific value of x_n
$k_{(m,x_n)}$	the $(m+1)$st value of k at a specific value of x_n

Table 1: Variables Considered in Solving $2^x + (2 + 5k) = z^2$

We derive $k_{(0,x_n)}$ and $k_{(m,x_n)}$ from (5), and define α_{x_n} as the difference of $k_{(0,x_n)}$ and $k_{(1,x_n)}$:

$$k_{(0,x_n)} = \frac{z^2_{(0,x_n)} - 2^{x_n} - 2}{5}, \tag{6a}$$

$$k_{(m,x_n)} = \frac{z_{(m,x_n)}^2 - 2^{x_n} - 2}{5}, \tag{6b}$$

$$\alpha_{x_n} = k_{(1,x_n)} - k_{(0,x_n)}. \tag{6c}$$

Here, $m \in \mathbb{N}$. We begin the discussion with the following lemmas.

Lemma 1. *Suppose the exponential Diophantine equation (5) has a solution. Then,* $x \equiv 1 \pmod 4$ *if and only if* $z \equiv 2 \pmod 5$ *or* $z \equiv 3 \pmod 5$.

Proof. Let $x \equiv 1 \pmod 4$ in the exponential Diophantine equation (5). This means that $2^x \equiv 2 \pmod 5$ and $2 + 5k \equiv 2 \pmod 5$ for all $k \in \mathbb{N}$. Thus, $z^2 \equiv 2^x + (2 + 5k) \equiv 4 \pmod 5$. We claim that $z \equiv 2 \pmod 5$ or $z \equiv 3 \pmod 5$.

Suppose on the contrary that $z \equiv 0 \pmod 5$, $z \equiv 1 \pmod 5$, or $z \equiv 4 \pmod 5$. Then, $z^2 \equiv 0 \pmod 5$, $z^2 \equiv 1 \pmod 5$, or $z^2 \equiv 1 \pmod 5$, respectively. However, any of these is impossible since we already established that $z^2 \equiv 4 \pmod 5$. This implies that $z \equiv 2 \pmod 5$ or $z \equiv 3 \pmod 5$.

Now, let $z \equiv 2 \pmod 5$ or $z \equiv 3 \pmod 5$ in equation (5). Then, $z^2 \equiv 4 \pmod 5$. Assume the contrary that $x \not\equiv 1 \pmod 4$. We first consider $x \equiv 2 \pmod 4$. Then, $2^x \equiv 4 \pmod 5$ and the equation $2^x + (2 + 5k) \equiv 4 + 2 \equiv 1 \pmod 5$. Next, we let $x \equiv 3 \pmod 4$. Consequently, $2^x \equiv 3 \pmod 5$ and $2^x + (2 + 5k) \equiv 3 + 2 \equiv 0 \pmod 5$. Lastly, we take $x \equiv 0 \pmod 4$. Then, $2^x \equiv 1 \pmod 5$ and $2^x + (2 + 5k) \equiv 1 + 2 \equiv 3 \pmod 5$. For all these cases, we have $2^x + (2 + 5k) \not\equiv z^2 \pmod 5$. Hence, x must be congruent to $1 \pmod 4$.

Therefore, the equation $2^x + (2 + 5k) = z^2$ has a solution when $x \equiv 1 \pmod 4$ if and only if $z \equiv 2 \pmod 5$ or $z \equiv 3 \pmod 5$. □

From equation (5), we can first look at the smallest possible value of z and then add a multiple of five to get the other values of z. To get the smallest number of possible conditions for $z_{(0,x_n)}$, we use $\lfloor \sqrt{2^x} \rfloor$ because $\sqrt{2^x}$ will never be an integer since $x \equiv 1 \pmod 4$. As a result, we obtain the following formulas:

$$x := x_n = 1 + 4n \tag{7a}$$

$$z_{(0,x_n)} = \begin{cases} \lfloor \sqrt{2^{x_n}} \rfloor + 6 & \text{if } x = 1, \\ \lfloor \sqrt{2^{x_n}} \rfloor + 1 & \text{if } \lfloor \sqrt{2^{x_n}} \rfloor \equiv 1 \pmod 5, \\ \lfloor \sqrt{2^{x_n}} \rfloor + 5 & \text{if } \lfloor \sqrt{2^{x_n}} \rfloor \equiv 2 \pmod 5, \\ \lfloor \sqrt{2^{x_n}} \rfloor + 4 & \text{if } \lfloor \sqrt{2^{x_n}} \rfloor \equiv 3 \pmod 5, \\ \lfloor \sqrt{2^{x_n}} \rfloor + 3 & \text{if } \lfloor \sqrt{2^{x_n}} \rfloor \equiv 4 \pmod 5, \\ \lfloor \sqrt{2^{x_n}} \rfloor + 2 & \text{if } \lfloor \sqrt{2^{x_n}} \rfloor \equiv 0 \pmod 5, \end{cases} \tag{7b}$$

or

$$z_{(0,x_n)} = \begin{cases} \lfloor\sqrt{2^{x_n}}\rfloor + 2 & \text{if } \lfloor\sqrt{2^{x_n}}\rfloor \equiv 1 \pmod 5, \\ \lfloor\sqrt{2^{x_n}}\rfloor + 1 & \text{if } \lfloor\sqrt{2^{x_n}}\rfloor \equiv 2 \pmod 5, \\ \lfloor\sqrt{2^{x_n}}\rfloor + 5 & \text{if } \lfloor\sqrt{2^{x_n}}\rfloor \equiv 3 \pmod 5, \\ \lfloor\sqrt{2^{x_n}}\rfloor + 4 & \text{if } \lfloor\sqrt{2^{x_n}}\rfloor \equiv 4 \pmod 5, \\ \lfloor\sqrt{2^{x_n}}\rfloor + 3 & \text{if } \lfloor\sqrt{2^{x_n}}\rfloor \equiv 0 \pmod 5, \end{cases} \tag{7c}$$

$$z_{(m,x_n)} = z_{(0,x_n)} + 5m, \tag{7d}$$

where $n \in \mathbb{N}_0$ and $m \in \mathbb{N}$. The five-tuple $(p,k,x,y,z) = (2, k_{(m,x_n)}, x_n, 1, z_{(m,x_n)})$ is a solution of (3).

As an example, we use our equations when $x = x_n = 1 + 4n$ and $z \equiv 2 \pmod 5$. If $n = 0$, we have $x = 1$, $z_{(0,1)} = 7$, and $z_{(1,1)} = 12$ from (7a), (7b), and (7d), respectively. Note that when $x = x_0 = 1$ and z is congruent to 2 (mod 5), the smallest value of z is $\lfloor\sqrt{2^{x_0}}\rfloor + 6$ because we must have $k > 0$. From (6a) and (6b), we have that $k_{(0,1)} = 9$ and $k_{(1,1)} = 28$. Thus, the first two solutions when $x = 1$ and $z \equiv 2 \pmod 5$ are $(p,k,x,y,z) = (2,9,1,1,7)$ and $(2,28,1,1,12)$.

When $z \equiv 3 \pmod 5$, we have $x = 1$, $z_{(0,1)} = 3$, and $z_{(1,1)} = 8$ from (7a), (7c), and (7d), respectively, for $n = 0$. We also have $k_{(0,1)} = 1$ and $k_{(1,1)} = 12$ from (6a) and (6b). Thus, the first two solutions when $x = 1$ and $z \equiv 3 \pmod 5$ are $(p,k,x,y,z) = (2,1,1,1,3)$ and $(2,12,1,1,8)$.

Lemma 2. *Suppose the exponential Diophantine equation (5) has a solution. Then,* $x \equiv 2 \pmod 4$ *if and only if* $z \equiv 1 \pmod 5$ *or* $z \equiv 4 \pmod 5$.

Proof. Consider the Diophantine equation (5). Let $x \equiv 2 \pmod 4$. This means that $2^x \equiv 4 \pmod 5$ and $2 + 5k \equiv 2 \pmod 5$ for all $k \in \mathbb{N}$. Thus, $2^x + (2 + 5k) \equiv 1 \pmod 5 \equiv z^2$. We claim that $z \equiv 1 \pmod 5$ or $z \equiv 4 \pmod 5$.

Suppose otherwise that $z \equiv 0 \pmod 5$, $z \equiv 2 \pmod 5$, or $z \equiv 3 \pmod 5$. Then, we have $z^2 \equiv 0 \pmod 5$, $z^2 \equiv 4 \pmod 5$, or $z^2 \equiv 4 \pmod 5$, respectively. Any of these is a contradiction since we already established that $z^2 \equiv 1 \pmod 5$. Now, let $z \equiv 1 \pmod 5$ or $z \equiv 4 \pmod 5$. Then, both will give $z^2 \equiv 1 \pmod 5$. Thus, when $x \equiv 2 \pmod 4$, (5) only has a solution when $z \equiv 1 \pmod 5$ or $z \equiv 4 \pmod 5$.

Now, let $z \equiv 1 \pmod 5$ or $z \equiv 4 \pmod 5$ in (5). Then, $z^2 \equiv 1 \pmod 5$. Assume the contrary that $x \not\equiv 2 \pmod 4$. If $x \equiv 1 \pmod 4$, then $2^x \equiv 2 \pmod 5$ and $2^x + (2+5k) \equiv 4 \pmod 5$. If $x \equiv 3 \pmod 4$, then $2^x \equiv 3 \pmod 5$ and $2^x + (2 + 5k) \equiv 0 \pmod 5$. Lastly, if $x \equiv 0 \pmod 4$, then $2^x \equiv 1 \pmod 5$ and $2^x + (2 + 5k) \equiv 3 \pmod 5$. In any case, we have $2^x + (2 + 5k) \not\equiv z^2$. Hence, when $z \equiv 1 \pmod 5$ or $z \equiv 4 \pmod 5$, (5) only has a solution when $x \equiv 2 \pmod 4$.

Therefore, the equation $2^x + (2 + 5k) = z^2$ has a solution when $x \equiv 2 \pmod 4$ if and only if $z \equiv 1 \pmod 5$ or $z \equiv 4 \pmod 5$. □

We note that for this case, $\sqrt{2^x}$ is always an integer. To reduce the number of possible conditions to be considered for the smallest possible value of z, we use $\sqrt{2^x}$. With this, we have the following formulas.

$$x := x_n = 2 + 4n \tag{8a}$$

$$z_{(0,x_n)} = \begin{cases} \sqrt{2^{x_n}} + 4 & \text{if } n \text{ is even,} \\ \sqrt{2^{x_n}} + 3 & \text{if } n \text{ is odd,} \end{cases} \tag{8b}$$

or

$$z_{(0,x_n)} = \begin{cases} \sqrt{2^{x_n}} + 2 & \text{if } n \text{ is even,} \\ \sqrt{2^{x_n}} + 1 & \text{if } n \text{ is odd,} \end{cases} \tag{8c}$$

$$z_{(m,x_n)} = z_{(0,x_n)} + 5m, \tag{8d}$$

where $n \in \mathbb{N}_0$ and $m \in \mathbb{N}$. The five-tuple $(p, k, x, y, z) = (2, k_{(m,x_n)}, x_n, 1, z_{(m,x_n)})$ is a solution of (3).

As an example, we use our equations when $x = x_n = 2 + 4n$ and $z \equiv 1 \pmod 4$. If $n = 0$, we have $x = 2$, $z_{(0,2)} = 6$, and $z_{(1,2)} = 11$ from (8a), (8b), and (8d), respectively. From (6a) and (6b), we have that $k_{(0,2)} = 6$ and $k_{(1,2)} = 23$. Thus, the first two solutions when $x = 2$ and $z \equiv 1 \pmod 5$ are $(p, k, x, y, z) = (2, 6, 2, 1, 6)$ and $(2, 23, 2, 1, 11)$.

When $z \equiv 4 \pmod 5$, we have $x = 2$, $z_{(0,2)} = 4$, and $z_{(1,2)} = 9$ from (8a), (8c), and (8d), respectively, for $n = 0$. We also have $k_{(0,2)} = 2$ and $k_{(1,2)} = 15$ from (6a) and (6b). Thus, the first two solutions when $x = 2$ and $z \equiv 4 \pmod 5$ are $(p, k, x, y, z) = (2, 2, 2, 1, 4)$ and $(2, 15, 2, 1, 9)$.

Lemma 3. *Suppose the exponential Diophantine equation (5) has a solution. Then,* $x \equiv 3 \pmod 4$ *if and only if* $z \equiv 0 \pmod 5$.

The proof is omitted as it follows the same reasoning as in the proof of Lemma 1.

Similar to the case when $x \equiv 1 \pmod 4$, we use $\lfloor \sqrt{2^x} \rfloor$ because $\sqrt{2^x}$ will never be an integer since $x \equiv 3 \pmod 4$. Thus, we will have the smallest number of possible condition for $z_{(0,x_n)}$. Hence, we have these formulas:

$$x = x_n = 3 + 4n, \tag{9a}$$

$$z_{(0,x_n)} = \begin{cases} \lfloor \sqrt{2^{x_n}} \rfloor + 4 & \text{if } \lfloor \sqrt{2^{x_n}} \rfloor \equiv 1 \pmod 5, \\ \lfloor \sqrt{2^{x_n}} \rfloor + 3 & \text{if } \lfloor \sqrt{2^{x_n}} \rfloor \equiv 2 \pmod 5, \\ \lfloor \sqrt{2^{x_n}} \rfloor + 2 & \text{if } \lfloor \sqrt{2^{x_n}} \rfloor \equiv 3 \pmod 5, \\ \lfloor \sqrt{2^{x_n}} \rfloor + 1 & \text{if } \lfloor \sqrt{2^{x_n}} \rfloor \equiv 4 \pmod 5, \\ \lfloor \sqrt{2^{x_n}} \rfloor + 5 & \text{if } \lfloor \sqrt{2^{x_n}} \rfloor \equiv 0 \pmod 5, \end{cases} \tag{9b}$$

$$z_{(m,x_n)} = z_{(0,x_n)} + 5m, \tag{9c}$$

where $n \in \mathbb{N}_0$ and $m \in \mathbb{N}$. The five-tuple $(p, k, x, y, z) = (2, k_{(m,x_n)}, x_n, 1, z_{(m,x_n)})$ is a solution of (5).

As an example, if $n = 0$ in $x = x_n = 3 + 4n$, we have $x = 3$, $z_{(0,3)} = 5$, and $z_{(1,3)} = 10$ from (9a), (9b), and (9c), respectively. From (6a) and (6b), we have that

$k_{(0,3)} = 3$ and $k_{(1,3)} = 18$. Thus, the first two solutions when $x = 3$ and $z \equiv 0 \pmod 5$ are $(p, k, x, y, z) = (2, 3, 3, 1, 5)$ and $(2, 18, 3, 1, 10)$.

For these values of x_n, we provide an easier way to solve for $k_{(m,x_n)}$ by using the next theorem.

Theorem 1. *Suppose the exponential Diophantine equation (5), where $x = x_n = 1 + 4n$, $x = x_n = 2 + 4n$, or $x = x_n = 3 + 4n$, for $n \in \mathbb{N}_0$, has a solution. Then,*

$$k_{(m,x_n)} = k_{(0,x_n)} + \sum_{i=0}^{m-1} (\alpha_{x_n} + 10i), \tag{10}$$

where $m \in \mathbb{N}$.

Proof. Consider the Diophantine equation $2^x + (2 + 5k) = z^2$, where $x = x_n = 1 + 4n$, $x = x_n = 2 + 4n$, or $x = x_n = 3 + 4n$, for $n \in \mathbb{N}_0$. We prove using the principle of mathematical induction on m that

$$k_{(m,x_n)} = k_{(0,x_n)} + \sum_{i=0}^{m-1} (\alpha_{x_n} + 10i),$$

where $m \in \mathbb{N}$.

Firstly, for $m = 1$, and by using the definition of α_{x_n} in (6c), we have:

$$k_{(1,x_n)} = k_{(0,x_n)} + \alpha_{x_n}.$$

Thus, equation (10) is satisfied when $m = 1$.

We now assume that equation (10) is true for $m = a$, that is,

$$k_{(a,x_n)} = k_{(0,x_n)} + \sum_{i=0}^{a-1} (\alpha_{x_n} + 10i).$$

We need to show that it is also true for $m = a + 1$. By using (6b), we have the following simplification:

$$k_{(a,x_n)} + (\alpha_{x_n} + 10a) = \left(k_{(0,x_n)} + \sum_{i=0}^{a-1} (\alpha_{x_n} + 10i) \right) + (\alpha_{x_n} + 10a),$$

$$\frac{z^2_{(a,x_n)} - 2^{x_n} - 2}{5} + \alpha_{x_n} + 10a = k_{(0,x_n)} + \sum_{i=0}^{a} (\alpha_{x_n} + 10i).$$

Thus, we obtain

$$\frac{z^2_{(a,x_n)} - 2^{x_n} - 2 + 5\alpha_{x_n} + 50a}{5} = k_{(0,x_n)} + \sum_{i=0}^{a} (\alpha_{x_n} + 10i). \tag{11}$$

By definition, $\alpha_{x_n} = k_{(1,x_n)} - k_{(0,x_n)}$. Using (6b) and (6a), we get

$$\alpha_{x_n} = \frac{z^2_{(1,x_n)} - 2^{x_n} - 2}{5} - \frac{z^2_{(0,x_n)} - 2^{x_n} - 2}{5} = \frac{z^2_{(1,x_n)} - z^2_{(0,x_n)}}{5}.$$

Also, recalling that $z_{(m,x_n)} = z_{(0,x_n)} + 5m$ from (7d), (8d), and (9c), we have $z_{(1,x_n)} = z_{(0,x_n)} + 5$ and $z_{(a,x_n)} = z_{(0,x_n)} + 5a$. Hence, we can simplify the expression as follows:

$$\begin{aligned}
&\frac{z^2_{(a,x_n)} - 2^{x_n} - 2 + 5\alpha_{x_n} + 50a}{5} \\
&= \frac{z^2_{(a,x_n)} - 2^{x_n} - 2 + z^2_{(1,x_n)} - z^2_{(0,x_n)} + 50a}{5} \\
&= \frac{z^2_{(a,x_n)} + z^2_{(1,x_n)} - z^2_{(0,x_n)} + 50a - 2^{x_n} - 2}{5} \\
&= \frac{z^2_{(a,x_n)} + (z_{(0,x_n)} + 5)^2 - z^2_{(0,x_n)} + 50a - 2^{x_n} - 2}{5} \\
&= \frac{z^2_{(a,x_n)} + z^2_{(0,x_n)} + 10z_{(0,x_n)} + 25 - z^2_{(0,x_n)} + 50a - 2^{x_n} - 2}{5} \\
&= \frac{z^2_{(a,x_n)} + 10z_{(0,x_n)} + 50a + 25 - 2^{x_n} - 2}{5} \\
&= \frac{z^2_{(a,x_n)} + 10(z_{(0,x_n)} + 5a) + 25 - 2^{x_n} - 2}{5} \\
&= \frac{z^2_{(a,x_n)} + 10z_{(a,x_n)} + 25 - 2^{x_n} - 2}{5} \\
&= \frac{(z_{(a,x_n)} + 5)^2 - 2^{x_n} - 2}{5} \\
&= \frac{z^2_{(a+1,x_n)} - 2^{x_n} - 2}{5} \\
&= k_{(a+1,x_n)}.
\end{aligned}$$

In the manipulations, we have used the fact that $z_{(a+1,x_n)} = z_{(0,x_n)} + 5(a+1) = z_{(0,x_n)} + 5a + 5 = z_{(a,x_n)} + 5$, and the definition of $k_{(m,x_n)}$. Thus, (11) can be written as

$$k_{(a+1,x_n)} = k_{(0,x_n)} + \sum_{i=0}^{a}(\alpha_{x_n} + 10i),$$

showing that equation (10) is also true for $m = a + 1$. Therefore, (10) is true for any $x = x_n = 1 + 4n$, $x = x_n = 2 + 4n$, or $x = x_n = 3 + 4n$, where $n \in \mathbb{N}_0$. □

The next remark follows directly from Theorem 1.

Remark 1. *Consider the exponential Diophantine equation* $2^x + (2 + 5k) = z^2$, *where* $x = x_n = 1 + 4n$, $x = x_n = 2 + 4n$, *or* $x = x_n = 3 + 4n$, *for* $n \in \mathbb{N}_0$. *Then,* $k_{(m,x_n)} = k_{(m-1,x_n)} + \alpha_{x_n} + 10(m-1)$.

Given below are tables containing the first five solutions of equation (5), where $x \equiv 1, 2$ or 3 (mod 4). The expression for $k_{(m,x_n)}$ is obtained by applying either (6b), Theorem 1, or Remark 1.

If $n = 0$ in $x = x_n = 1 + 4n$, then we have $x = 1$, and Tables 2 and 3 present some solutions where $z \equiv 2 \pmod 5$ or $z \equiv 3 \pmod 5$. For $z \equiv 2 \pmod 5$, we have the following:

$$z_{(0,1)} = 7, \quad z_{(1,1)} = 12, \quad k_{(0,1)} = 9, \quad k_{(1,1)} = 28, \quad \alpha_1 = 19.$$

For $z \equiv 3 \pmod 5$, we have $z_{(0,1)} = 3, \quad z_{(1,1)} = 8, \quad k_{(0,1)} = 1, \quad k_{(1,1)} = 12, \quad \alpha_1 = 11.$

m	k	z	Equation	Solution (p, k, x, y, z)
	9	7	$2 + (2 + 5(9)) = 7^2$	(2, 9, 1, 1, 7)
1	28	12	$2 + (2 + 5(28)) = 12^2$	(2, 28, 1, 1, 12)
2	57	17	$2 + (2 + 5(57)) = 17^2$	(2, 57, 1, 1, 17)
3	96	22	$2 + (2 + 5(96)) = 22^2$	(2, 96, 1, 1, 22)
4	145	27	$2 + (2 + 5(145)) = 27^2$	(2, 145, 1, 1, 27)

Table 2: Some Solutions of (5) with $x = 1, y = 1$ and $z \equiv 2 \pmod 5$

m	k	z	Equation	Solution (p, k, x, y, z)
	1	3	$2 + (2 + 5(1)) = 3^2$	(2, 1, 1, 1, 3)
1	12	8	$2 + (2 + 5(12)) = 8^2$	(2, 12, 1, 1, 8)
2	33	13	$2 + (2 + 5(33)) = 13^2$	(2, 33, 1, 1, 13)
3	64	18	$2 + (2 + 5(64)) = 18^2$	(2, 64, 1, 1, 18)
4	105	23	$2 + (2 + 5(105)) = 23^2$	(2, 105, 1, 1, 23)

Table 3: Some Solutions of (5) with $x = 1, y = 1$ and $z \equiv 3 \pmod 5$

If $n = 1$ in $x = x_n = 1 + 4n$, we get $x = 5$, and Tables 4 and 5 present some solutions where $z \equiv 2 \pmod 5$ or $z \equiv 3 \pmod 5$. For $z \equiv 2 \pmod 5$, we have $z_{(0,5)} = 7,\ z_{(1,5)} = 12,\ k_{(0,5)} = 3,\ k_{(1,5)} = 22,$ and $\alpha_5 = 19$. For $z \equiv 3 \pmod 5$, we have $z_{(0,5)} = 8,\ z_{(1,5)} = 13,\ k_{(0,5)} = 6,\ k_{(1,5)} = 27,$ and $\alpha_5 = 21$.

m	k	z	Equation	Solution (p, k, x, y, z)
	3	7	$2^5 + (2 + 5(3)) = 7^2$	(2, 3, 5, 1, 7)
1	22	12	$2^5 + (2 + 5(22)) = 12^2$	(2, 22, 5, 1, 12)
2	51	17	$2^5 + (2 + 5(51)) = 17^2$	(2, 51, 5, 1, 17)
3	90	22	$2^5 + (2 + 5(90)) = 22^2$	(2, 90, 5, 1, 22)
4	139	27	$2^5 + (2 + 5(139)) = 27^2$	(2, 139, 5, 1, 27)

Table 4: Some Solutions of (5) with $x = 5, y = 1$ and $z \equiv 2 \pmod 5$

m	k	z	Equation	Solution (p, k, x, y, z)
	6	8	$2^5 + (2 + 5(6)) = 8^2$	(2, 6, 5, 1, 8)
1	27	13	$2^5 + (2 + 5(27)) = 13^2$	(2, 27, 5, 1, 13)
2	58	18	$2^5 + (2 + 5(58)) = 18^2$	(2, 58, 5, 1, 18)
3	99	23	$2^5 + (2 + 5(99)) = 23^2$	(2, 99, 5, 1, 23)
4	150	28	$2^5 + (2 + 5(150)) = 28^2$	(2, 150, 5, 1, 28)

Table 5: Some Solutions of (5) with $x = 5, y = 1$ and $z \equiv 3 \pmod 5$

If $n = 0$ in $x = x_n = 2 + 4n$, we get $x = 2$, and we have Tables 6 - 7 below that present some solutions where $z \equiv 1 \pmod 5$ or $z \equiv 4 \pmod 5$.

m	k	z	Equation	Solution (p, k, x, y, z)
	6	6	$2^2 + (2 + 5(6)) = 6^2$	(2, 6, 2, 1, 6)
1	23	11	$2^2 + (2 + 5(23)) = 11^2$	(2, 23, 2, 1, 11)
2	50	16	$2^2 + (2 + 5(50)) = 16^2$	(2, 50, 2, 1, 16)
3	87	21	$2^2 + (2 + 5(87)) = 21^2$	(2, 87, 2, 1, 21)
4	134	26	$2^2 + (2 + 5(134)) = 26^2$	(2, 134, 2, 1, 26)

Table 6: Some Solutions of (5) with $x = 2, y = 1$ and $z \equiv 1 \pmod 5$

m	k	z	Equation	Solution (p, k, x, y, z)
	2	4	$2^2 + (2 + 5(2)) = 4^2$	(2, 2, 2, 1, 4)
1	15	9	$2^2 + (2 + 5(15)) = 9^2$	(2, 15, 2, 1, 9)
2	38	14	$2^2 + (2 + 5(38)) = 14^2$	(2, 38, 2, 1, 14)
3	71	19	$2^2 + (2 + 5(71)) = 19^2$	(2, 71, 2, 1, 19)
4	114	24	$2^2 + (2 + 5(114)) = 24^2$	(2, 114, 2, 1, 24)

Table 7: Some Solutions of (5) with $x = 2, y = 1$ and $z \equiv 4 \pmod 5$

If $n = 1$ in $x = x_n = 2 + 4n$, we get $x = 6$ and Tables 8 - 9 show some solutions where $z \equiv 1 \pmod 5$ or $z \equiv 4 \pmod 5$.

m	k	z	Equation	Solution (p, k, x, y, z)
	3	9	$2^6 + (2 + 5(3)) = 9^2$	(2, 3, 6, 1, 9)
1	26	14	$2^6 + (2 + 5(26)) = 14^2$	(2, 26, 6, 1, 14)
2	59	19	$2^6 + (2 + 5(59)) = 19^2$	(2, 59, 6, 1, 19)
3	102	24	$2^6 + (2 + 5(102)) = 24^2$	(2, 102, 6, 1, 24)
4	155	29	$2^6 + (2 + 5(155)) = 29^2$	(2, 155, 6, 1, 29)

Table 9: Some Solutions of (5) with $x = 6, y = 1$ and $z \equiv 4 \pmod 5$

m	k	z	Equation	Solution (p, k, x, y, z)
	11	11	$2^6 + (2 + 5(11)) = 11^2$	(2, 11, 6, 1, 11)
1	38	16	$2^6 + (2 + 5(38)) = 16^2$	(2, 38, 6, 1, 16)
2	75	21	$2^6 + (2 + 5(75)) = 21^2$	(2, 75, 6, 1, 21)
3	122	26	$2^6 + (2 + 5(122)) = 26^2$	(2, 122, 6, 1, 26)
4	179	31	$2^6 + (2 + 5(179)) = 31^2$	(2, 179, 6, 1, 31)

Table 8: Some Solutions of (5) with $x = 6, y = 1$ and $z \equiv 1 \pmod 5$

If $n = 0, 1$ in $x = x_n = 3 + 4n$, we get $x = 3, 7$, respectively, and Tables 10 - 11 show some solutions wherein $z \equiv 0 \pmod 5$.

m	k	z	Equation	Solution (p, k, x, y, z)
	3	5	$2^3 + (2 + 5(3)) = 5^2$	(2, 3, 3, 1, 5)
1	18	10	$2^3 + (2 + 5(18)) = 10^2$	(2, 18, 3, 1, 10)
2	43	15	$2^3 + (2 + 5(43)) = 15^2$	(2, 43, 3, 1, 15)
3	78	20	$2^3 + (2 + 5(78)) = 20^2$	(2, 78, 3, 1, 20)
4	123	25	$2^3 + (2 + 5(123)) = 25^2$	(2, 123, 3, 1, 25)

Table 10: Some Solutions of (5) with $x = 3, y = 1$ and $z \equiv 0 \pmod 5$

m	k	z	Equation	Solution (p, k, x, y, z)
	19	15	$2^7 + (2 + 5(19)) = 15^2$	(2, 19, 7, 1, 15)
1	54	20	$2^7 + (2 + 5(54)) = 20^2$	(2, 54, 7, 1, 20)
2	99	25	$2^7 + (2 + 5(99)) = 25^2$	(2, 99, 7, 1, 25)
3	154	30	$2^7 + (2 + 5(154)) = 30^2$	(2, 154, 7, 1, 30)
4	219	35	$2^7 + (2 + 5(219)) = 35^2$	(2, 219, 7, 1, 35)

Table 11: Some Solutions of (5) with $x = 7, y = 1$ and $z \equiv 0 \pmod 5$

The last result for this subsection discusses the case when $x \equiv 0 \pmod 4$.

Theorem 2. *The exponential Diophantine equation $2^x + (2 + 5k) = z^2$ has no solution if $x \equiv 0 \pmod 4$.*

Proof. Consider the Diophantine equation $2^x + (2 + 5k) = z^2$, where $x \equiv 0 \pmod 4$. Then, $2^x \equiv 1 \pmod 5$. Hence, $2^x + (2 + 5k) \equiv 1 + 2 \equiv 3 \pmod 5$. On the other hand, $z^2 \equiv 0, 1, \text{ or } 4 \pmod 5$. Thus, the equation can never be true whenever $x \equiv 0 \pmod 4$. □

2.1.2. Part II: $2^x + (2 + 5k)^y = z^2$, where $y \neq 1$.

For this part, we study

$$2^x + (2 + 5k)^y = z^2, \tag{12}$$

where $y \neq 1$. The discussion begins with the case where $x = 0$ or $y = 0$, followed by a claim when $\min(x, y) > 1$.

Theorem 3. *Consider the Diophantine equation (12). Let* $y = 0$*. Then, (12) has a solution only when* $x = 3$*. Consequently,* $(p, k, x, y, z) = (2, k, 3, 0, 3)$ *is a solution for all* $k \in \mathbb{N}$*. On the other hand, (12) has no solution when* $x = 0$*.*

Proof. **Case 1.** Let $y = 0$ in (12). Then, for all $k \in \mathbb{N}$, $(2 + 5k)^y = 1$. We can express (12) as $2^x + 1 = z^2$, or equivalently, $z^2 - 2^x = 1$. If $z = 0$ or $z = 1$, it is obvious that it has no solution. However, if $z > 1$, then this makes it a Catalan equation, so it has a unique solution. Hence, $(p, k, x, y, z) = (2, k, 3, 0, 3)$ is a solution of (12). Therefore, if $y = 0$, (12) has a solution only if $x = z = 3$ for all $k \in \mathbb{N}$.

Case 2. Let $x = 0$ in (12). Then, equation (12) will be equivalent to $z^2 - (2+5k)^y = 1$. By Mihailescu's theorem, this will only have a solution when $z = 3$, $y = 3$ and $k = 0$. However, we choose $k \in \mathbb{N}$ and so it is impossible for $k = 0$ to happen. Therefore, (12) has no solution if $x = 0$. □

Solutions wherein $\min(x, y) > 1$ remains an open problem. Below is a claim that has been verified numerically but has not been proven rigorously.

Conjecture 1. *Let* $k \leq 5000$ *and* $x \leq 50$*. If* $y = 2$*, then equation (12) has 21 solutions, and if* $y = 3$ *then (12) has 9 solutions.*

Tables 12 and 13 are given below to illustrate the list of solutions when $y = 2$ and $y = 3$, respectively.

k	z	Equation	Solution (p, k, x, y, z)
1	9	$2^5 + (2 + 5(1))^2 = 9^2$	(2, 1, 5, 2, 9)
2	20	$2^8 + (2 + 5(2))^2 = 20^2$	(2, 2, 8, 2, 20)
12	66	$2^9 + (2 + 5(12))^2 = 66^2$	(2, 12, 9, 2, 66)
25	129	$2^9 + (2 + 5(25))^2 = 129^2$	(2, 25, 9, 2, 129)
50	260	$2^{12} + (2 + 5(50))^2 = 260^2$	(2, 50, 12, 2, 260)
6	96	$2^{13} + (2 + 5(6))^2 = 96^2$	(2, 6, 13, 2, 96)
22	144	$2^{13} + (2 + 5(22))^2 = 144^2$	(2, 22, 13, 2, 144)
204	1026	$2^{13} + (2 + 5(204))^2 = 1026^2$	(2, 204, 13, 2, 1026)
409	2049	$2^{13} + (2 + 5(409))^2 = 2049^2$	(2, 409, 13, 2, 2049)
38	320	$2^{16} + (2 + 5(38))^2 = 320^2$	(2, 38, 16, 2, 320)
818	4100	$2^{16} + (2 + 5(818))^2 = 4100^2$	(2, 818, 16, 2, 4100)
198	1056	$2^{17} + (2 + 5(198))^2 = 1056^2$	(2, 198, 17, 2, 1056)
406	2064	$2^{17} + (2 + 5(406))^2 = 2064^2$	(2, 406, 17, 2, 2064)
3276	16386	$2^{17} + (2 + 5(3276))^2 = 16386^2$	(2, 3276, 17, 2, 16386)
806	4160	$2^{20} + (2 + 5(806))^2 = 4160^2$	(2, 806, 20, 2, 4160)
102	1536	$2^{21} + (2 + 5(102))^2 = 1536^2$	(2, 102, 21, 2, 1536)
358	2304	$2^{21} + (2 + 5(358))^2 = 2304^2$	(2, 358, 21, 2, 2304)
3270	16416	$2^{21} + (2 + 5(3270))^2 = 16416^2$	(2, 3270, 21, 2, 16416)
614	5120	$2^{24} + (2 + 5(614))^2 = 5120^2$	(2, 614, 24, 2, 5120)
3174	16896	$2^{25} + (2 + 5(3174))^2 = 16896^2$	(2, 3174, 25, 2, 16896)
1638	24576	$2^{29} + (2 + 5(1638))^2 = 24576^2$	(2, 1638, 29, 2, 24576)

Table 12: Solutions of $2^x + (2 + 5k)^y = z^2$ when $y = 2$

k	z	Equation	Solution (p, k, x, y, z)
3	71	$2^7 + (2 + 5(3))^3 = 71^2$	(2, 3, 7, 3, 71)
6	192	$2^{12} + (2 + 5(6))^3 = 192^2$	(2, 6, 12, 3, 192)
6	256	$2^{15} + (2 + 5(6))^3 = 256^2$	(2, 6, 15, 3, 256)
54	4544	$2^{19} + (2 + 5(54))^3 = 4544^2$	(2, 54, 19, 3, 4544)
102	12288	$2^{24} + (2 + 5(102))^3 = 12288^2$	(2, 102, 24, 3, 12288)
102	16384	$2^{27} + (2 + 5(102))^3 = 16384^2$	(2, 102, 27, 3, 16384)
870	290816	$2^{31} + (2 + 5(870))^3 = 290816^2$	(2, 870, 31, 3, 290816)
1638	786432	$2^{36} + (2 + 5(1638))^3 = 786432^2$	(2, 1638, 36, 3, 786432)
1638	1048576	$2^{39} + (2 + 5(1638))^3 = 1048576^2$	(2, 1638, 39, 3, 1048576)

Table 13: Solutions of $2^x + (2 + 5k)^y = z^2$ when $y = 3$

2.2. On the Diophantine Equation $p^x + (p + 5k)^y = z^2$ for Prime Pairs p and $p + 5k$

This subsection discusses some findings regarding solutions of (3), where p and $p + 5k$ are prime pairs. This is further divided into two sub-cases based on the parity of k; that is, when k is odd, and when k is even.

2.2.1. Sub-case I: k is odd.

The case where $p = 2$ falls in this sub-case. Some results have already been discussed in Section 2.1. Additionally, using similar arguments as in proving claims in that section, we obtain the results below.

We note that in the Diophantine equation $p^x + (p + 5k)^y = z^2$, where p and $p + 5k$ are prime pairs and k is odd, the prime gap $5k$ is also odd. It is clear that $p + 5k$ must be odd to be prime and for it to have a prime gap of an odd number, p must be even, that is, $p = 2$. Hence, some of the solutions were already included in Section 2.1.

Let us start with the discussion of solutions when x is odd. That is, $x \equiv 1 \pmod 4$ and $x \equiv 3 \pmod 4$. Note that unlike the previous section that uses modulo 5 for z, this subsection uses modulo 10. We start the discussion with the following lemmas.

Lemma 4. *Let k be an odd integer. Suppose the Diophantine equation (5) has a solution. Then, $x \equiv 1 \pmod 4$ if and only if $z \equiv 3 \pmod{10}$ or $z \equiv 7 \pmod{10}$.*

Proof. Consider equation (5) and let $x \equiv 1 \pmod 4$ and k be odd. This means that $2^x + (2 + 5k) \equiv 2 + 7 \equiv 9 \pmod{10} \equiv z^2$. It follows that z^2 must be odd and that z is odd too. We claim that $z \equiv 3 \pmod{10}$ or $z \equiv 7 \pmod{10}$.

Suppose on the contrary that $z \equiv 1, 5$ or $9 \pmod{10}$. If $z \equiv 1, 9 \pmod{10}$, then $z^2 \equiv 1 \pmod{10}$. If $z \equiv 5 \pmod{10}$, then $z^2 \equiv 5 \pmod{10}$. Any of these is a contradiction since we established that $z^2 \equiv 9 \pmod{10}$. Now, if $z \equiv 3, 7 \pmod{10}$, then $z^2 \equiv 9 \pmod{10}$. Hence, when $x \equiv 1 \pmod 4$, (5) only has a solution when $z \equiv 3 \pmod{10}$ or $z \equiv 7 \pmod{10}$.

Now, let $z \equiv 3 \pmod{10}$ or $z \equiv 7 \pmod{10}$ in (5), where k is odd. Then, $z^2 \equiv 9 \pmod{10}$. Suppose on the contrary that $x \equiv 2, 3, 0 \pmod 4$, then $2^x \equiv 4, 8, 6 \pmod{10}$, respectively. Hence, we will have $2^x + (2 + 5k) \equiv 1, 5, 3 \pmod{10}$, respectively. This will be a contradiction from the assumption that $z^2 \equiv 9 \pmod{10}$. Thus, if $z \equiv 3 \pmod{10}$ or $z \equiv 7 \pmod{10}$, (5) only has a solution when $x \equiv 1 \pmod 4$.

Therefore, the equation $2^x + (2 + 5k) = z^2$ where 2 and $2 + 5k$ are prime pairs and k is odd, has a solution when $x \equiv 1 \pmod 4$ if and only if $z \equiv 3 \pmod{10}$ or $z \equiv 7 \pmod{10}$. □

Using this result, the first two solutions when $x = 1$ and $z \equiv 3 \pmod{10}$ are $(p, k, x, y, z) = (2, 1, 1, 1, 3)$ and $(2, 33, 1, 1, 13)$. On the other hand, the first solution when $x = 1$ and $z \equiv 7 \pmod{10}$ is $(p, k, x, y, z) = (2, 9, 1, 1, 7)$.

Lemma 5. *Let k be an odd integer. Suppose the Diophantine equation (5) has a solution. Then, $x \equiv 3 \pmod 4$ if and only if $z \equiv 5 \pmod{10}$.*

Proof. Let $x \equiv 3 \pmod 4$ in (5), where $p = 2$ and $p + 5k$ are prime pairs and k is odd. This means that $2^x + (2 + 5k) \equiv 8 + 7 \equiv 5 \pmod{10} \equiv z^2$. Then, z^2 is odd and z must be odd too. We claim that $z \equiv 5 \pmod{10}$.

Suppose on the contrary that $z \not\equiv 5 \pmod{10}$. If $z \equiv 1, 9 \pmod{10}$, then $z^2 \equiv 1 \pmod{10}$. If $z \equiv 3, 7 \pmod{10}$, then $z^2 \equiv 1 \pmod{10}$. Any of these is a contradiction since we established that $z^2 \equiv 5 \pmod{10}$. Now, if $z \equiv 5 \pmod{10}$, then $z^2 \equiv 5 \pmod{10}$. Hence, when $x \equiv 3 \pmod 4$, (5) only has a solution when $z \equiv 5 \pmod{10}$.

Now, let $z \equiv 5 \pmod{10}$. Then, $z^2 \equiv 5 \pmod{10}$. Suppose on the contrary that $x \equiv 1, 2, 0 \pmod{4}$, then $2^x \equiv 2, 4, 6 \pmod{10}$, respectively. Thus, we will have equation (5) as $2^x + (2+5k)^y \equiv 9, 1, 3 \pmod{10} \equiv z^2$, respectively. This will be a contradiction from the assumption that $z^2 \equiv 5 \pmod{10}$. Hence, if $z \equiv 5 \pmod{10}$, (5) only has a solution when $x \equiv 3 \pmod{4}$.

Therefore, the equation $2^x + (2+5k) = z^2$, where $p = 2$ and $p + 5k$ are prime pairs and k is odd, only has a solution when $x \equiv 3 \pmod{4}$ if and only if $z \equiv 5 \pmod{10}$. □

One can verify that the first solution when $x = 3$ and $z \equiv 5 \pmod{10}$ is $(p, k, x, y, z) = (2, 3, 3, 1, 5)$.

For values of x_n that are either in the form $1 + 4n$ or $3 + 4n$, an easier way to solve for $k_{(m,x_n)}$ is given by the next theorem, which can be proven by induction.

Theorem 4. *Suppose the exponential Diophantine equation (5), where $x = x_n = 1 + 4n$ or $x = x_n = 3 + 4n$, $n \in \mathbb{N}_0$, has a solution. Then,*

$$k_{(m,x_n)} = k_{(0,x_n)} + \sum_{i=0}^{m-1} (\alpha_{x_n} + 40i), \tag{13}$$

where $m \in \mathbb{N}$.

Proof. Consider the Diophantine equation $2^x + (2+5k) = z^2$, where $x := x_n = 1 + 4n$ or $x := x_n = 3 + 4n$, $n \in \mathbb{N}_0$. We prove by induction on m that

$$k_{(m,x_n)} = k_{(0,x_n)} + \sum_{i=0}^{m-1} (\alpha_{x_n} + 40i),$$

where $m \in \mathbb{N}$.

We first note that by using the definition of α_{x_n}, equation (13) is satisfied when $m = 1$, as seen below:

$$k_{(1,x_n)} = k_{(0,x_n)} + \alpha_{x_n}.$$

We now assume that equation (13) is true for $m = a$, that is,

$$k_{(a,x_n)} = k_{(0,x_n)} + \sum_{i=0}^{a-1} (\alpha_{x_n} + 40i).$$

By recalling that $k_{(a,x_n)} = \frac{z^2_{(a,x_n)} - 2^{x_n} - 2}{5}$, we get

$$k_{(a,x_n)} + (\alpha_{x_n} + 40a) = \left(k_{(0,x_n)} + \sum_{i=0}^{a-1} (\alpha_{x_n} + 40i) \right) + (\alpha_{x_n} + 40a),$$

$$\frac{z^2_{(a,x_n)} - 2^{x_n} - 2}{5} + \alpha_{x_n} + 40a = k_{(0,x_n)} + \sum_{i=0}^{a} (\alpha_{x_n} + 40i).$$

Thus, we have

$$\frac{z_{(a,x_n)}^2 - 2^{x_n} - 2 + 5\alpha_{x_n} + 200a}{5} = k_{(0,x_n)} + \sum_{i=0}^{a}(\alpha_{x_n} + 40i). \tag{14}$$

Note that α_{x_n} can also be written as follows:

$$\alpha_{x_n} = \frac{z_{(1,x_n)}^2 - 2^{x_n} - 2}{5} - \frac{z_{(0,x_n)}^2 - 2^{x_n} - 2}{5} = \frac{z_{(1,x_n)}^2 - z_{(0,x_n)}^2}{5}.$$

This, together with the equations $z_{(1,x_n)} = z_{(0,x_n)} + 10$ and $z_{(a,x_n)} = z_{(0,x_n)} + 10a$, will lead us to the following simplification:

$$\begin{aligned}
&\frac{z_{(a,x_n)}^2 - 2^{x_n} - 2 + 5\alpha_{x_n} + 200a}{5} \\
&\quad = \frac{z_{(a,x_n)}^2 - 2^{x_n} - 2 + z_{(1,x_n)}^2 - z_{(0,x_n)}^2 + 200a}{5} \\
&\quad = \frac{z_{(a,x_n)}^2 + z_{(1,x_n)}^2 - z_{(0,x_n)}^2 + 200a - 2^{x_n} - 2}{5} \\
&\quad = \frac{z_{(a,x_n)}^2 + (z_{(0,x_n)} + 10)^2 - z_{(0,x_n)}^2 + 200a - 2^{x_n} - 2}{5} \\
&\quad = \frac{z_{(a,x_n)}^2 + z_{(0,x_n)}^2 + 20z_{(0,x_n)} + 100 - z_{(0,x_n)}^2 + 200a - 2^{x_n} - 2}{5} \\
&\quad = \frac{z_{(a,x_n)}^2 + 20z_{(0,x_n)} + 200a + 100 - 2^{x_n} - 2}{5} \\
&\quad = \frac{z_{(a,x_n)}^2 + 20(z_{(0,x_n)} + 10a) + 100 - 2^{x_n} - 2}{5} \\
&\quad = \frac{z_{(a,x_n)}^2 + 20z_{(a,x_n)} + 100 - 2^{x_n} - 2}{5} \\
&\quad = \frac{(z_{(a,x_n)} + 10)^2 - 2^{x_n} - 2}{5}.
\end{aligned}$$

Since $z_{(a+1,x_n)} = z_{(0,x_n)} + 10(a+1) = z_{(0,x_n)} + 10a + 10 = z_{(a,x_n)} + 10$, and by definition of $k_{(m,x_n)}$, we have

$$\begin{aligned}
\frac{z_{(a,x_n)}^2 - 2^{x_n} - 2 + 5\alpha_{x_n} + 200a}{5} &= \frac{z_{(a+1,x_n)}^2 - 2^{x_n} - 2}{5} \\
&= k_{(a+1,x_n)}.
\end{aligned}$$

Hence,

$$k_{(a+1,x_n)} = k_{(0,x_n)} + \sum_{i=0}^{a}(\alpha_{x_n} + 40i),$$

showing that (13) is also true for $m = a+1$. Therefore, (13) is true for any $m \in \mathbb{N}$, where $x_n = 1+4n$ or $x_n = 3+4n$, $n \in \mathbb{N}_0$. □

The next remark follows directly from the theorem above.

Remark 2. *Consider the exponential Diophantine equation* $2^x + (2+5k) = z^2$, *where* $x := x_n = 1+4n$ *or* $x := x_n = 3+4n$, $n \in \mathbb{N}_0$. *Then,*

$$k_{(m,x_n)} = k_{(m-1,x_n)} + \alpha_{x_n} + 40(m-1).$$

Tables 14 - 17 each contain the first five solutions of equation (5) where $x \equiv 1$ or 3 (mod 4), and $k_{(m,x_n)}$ is obtained by applying equation (6b), Theorem 4, or Remark 2.

m	k	z	Equation	Solution (p,k,x,y,z)
	1	3	$2+(2+5(1)) = 3^2$	(2, 1, 1, 1, 3)
1	33	13	$2+(2+5(33)) = 13^2$	(2, 33, 1, 1, 13)
3	217	33	$2+(2+5(217)) = 33^2$	(2, 217, 1, 1, 33)
4	369	43	$2+(2+5(369)) = 43^2$	(2, 369, 1, 1, 43)
6	793	63	$2+(2+5(793)) = 63^2$	(2, 793, 1, 1, 63)

Table 14: Some Solutions of (5) with $x = y = 1$ and $z \equiv 3 \pmod{10}$

m	k	z	Equation	Solution (p,k,x,y,z)
	9	7	$2+(2+5(9)) = 7^2$	(2, 9, 1, 1, 7)
2	145	27	$2+(2+5(145)) = 27^2$	(2, 145, 1, 1, 27)
3	273	37	$2+(2+5(273)) = 37^2$	(2, 273, 1, 1, 37)
4	441	47	$2+(2+5(441)) = 47^2$	(2, 441, 1, 1, 47)
7	1185	77	$2+(2+5(1185)) = 77^2$	(2, 1185, 1, 1, 77)

Table 15: Some Solutions of (5) with $x = y = 1$ and $z \equiv 7 \pmod{10}$

m	k	z	Equation	Solution (p,k,x,y,z)
	27	13	$2^5+(2+5(27)) = 13^2$	(2, 27, 5, 1, 13)
4	555	53	$2^5+(2+5(555)) = 53^2$	(2, 555, 5, 1, 53)
6	1059	73	$2^5+(2+5(1059)) = 73^2$	(2, 1059, 5, 1, 73)
7	1371	83	$2^5+(2+5(1371)) = 83^2$	(2, 1371, 5, 1, 83)
12	3531	133	$2^5+(2+5(3531)) = 133^2$	(2, 3531, 5, 1, 133)

Table 16: Some Solutions of (5) with $x = 5, y = 1$ and $z \equiv 3 \pmod{10}$

m	k	z	Equation	Solution (p,k,x,y,z)
	3	7	$2^5+(2+5(3)) = 7^2$	(2, 3, 5, 1, 7)
1	51	17	$2^5+(2+5(51)) = 17^2$	(2, 51, 5, 1, 17)
5	643	57	$2^5+(2+5(643)) = 57^2$	(2, 643, 5, 1, 57)
6	891	67	$2^5+(2+5(891)) = 67^2$	(2, 891, 5, 1, 67)
7	1179	77	$2^5+(2+5(1179)) = 77^2$	(2, 1179, 5, 1, 77)

Table 17: Some Solutions of (5) with $x = 5, y = 1$ and $z \equiv 7 \pmod{10}$

Tables 18 - 19 present some solutions of (5), where $z \equiv 5 \pmod{10}$.

Now, we discuss the case when x is even, i.e., $x \equiv 0 \pmod 4$ and $x \equiv 2 \pmod 4$. We begin with a remark, followed by a proven claim.

m	k	z	Equation	Solution (p, k, x, y, z)
	3	5	$2^3 + (2 + 5(3)) = 5^2$	(2, 3, 3, 1, 5)
2	123	25	$2^3 + (2 + 5(123)) = 25^2$	(2, 123, 3, 1, 25)
3	243	35	$2^3 + (2 + 5(243)) = 35^2$	(2, 243, 3, 1, 35)
4	405	45	$2^3 + (2 + 5(405)) = 45^2$	(2, 405, 3, 1, 45)
6	843	65	$2^3 + (2 + 5(843)) = 65^2$	(2, 843, 3, 1, 65)

Table 18: Some Solutions of (5) with $x = 3, y = 1$ and $z \equiv 5 \pmod{10}$

m	k	z	Equation	Solution (p, k, x, y, z)
	19	15	$2^7 + (2 + 5(19)) = 15^2$	(2, 19, 7, 1, 15)
2	219	35	$2^7 + (2 + 5(219)) = 35^2$	(2, 219, 7, 1, 35)
4	579	55	$2^7 + (2 + 5(579)) = 55^2$	(2, 579, 7, 1, 55)
11	3099	125	$2^7 + (2 + 5(3099)) = 125^2$	(2, 3099, 7, 1, 125)
12	3619	135	$2^7 + (2 + 5(3619)) = 135^2$	(2, 3619, 7, 1, 135)

Table 19: Some Solutions of (5) with $x = 7, y = 1$ and $z \equiv 5 \pmod{10}$

Remark 3. *From Theorem 2, the equation* $2^x + (2 + 5k) = z^2$*, where* $2 + 5k$ *is prime and* k *is odd, has no solution when* $x \equiv 0 \pmod 4$.

Theorem 5. *Let* $2+5k$ *be prime, and* $x = 2+4n$ *with odd number* n*. Then the exponential Diophantine equation* (5)*, has solutions where* $k = \dfrac{2^{2+2n} - 1}{5}$ *and* $z = \dfrac{5k + 3}{2}$.

Proof. Let $x = 2 + 4n$, n be odd, and $q = 2 + 5k$ be prime. Then, we can express $2^x + (2 + 5k) = z^2$ as $2^{2+4n} + q = z^2$. Finding for q we have,

$$q = z^2 - 2^{2+4n} = (z - 2^{1+2n})(z + 2^{1+2n}).$$

Since q is prime, we know that $\gcd(z - 2^{1+2n}, z + 2^{1+2n}) = 1$. It follows that $z - 2^{1+2n} = 1$ and $z + 2^{1+2n} = q$. Having this system of equations, we solve for z. Thus, we get

$$2z = q + 1 \implies z = \frac{q+1}{2} \implies z = \frac{5k+3}{2}.$$

Substituting to find k, we obtain

$$\begin{aligned} q &= z + 2^{1+2n} \\ 2 + 5k &= \frac{5k+3}{2} + 2^{1+2n} \\ 2(2 + 5k) &= 5k + 3 + 2 \cdot 2^{1+2n} \\ 4 + 10k &= 5k + 3 + 2^{2+2n} \\ 10k - 5k &= 2^{2+2n} - 1 \\ 5k &= 2^{2+2n} - 1 \\ k &= \frac{2^{2+2n} - 1}{5}. \end{aligned}$$

We are sure that k is an odd integer because n is odd. Thus, $2^x + (2 + 5k) = z^2$ has a solution if $y = 1$, $x = 2 + 4n$, $k = \dfrac{2^{2+2n} - 1}{5}$, and $z = \dfrac{5k + 3}{2}$, where n is an odd number.

Using Theorem 5, one can verify that $(p, k, x, y, z) = (2, 3, 6, 1, 9), (2, 51, 14, 1, 129)$, and $(2, 13107, 30, 1, 32769)$ are indeed solutions of (3).

Theorem 6. *Consider the exponential Diophantine equation (3), where $2+5k$ is prime. If $y = 0$, the equation has only solution when $x = 3$. Consequently, $(p, k, x, y, z) = (2, k, 3, 0, 3)$ is a solution for some odd $k \in \mathbb{N}$. On the other hand, the equation has no solution when $x = 0$.*

The proof of Theorem 6 follows from Theorem 3. The next conjecture also follows from Conjecture 1.

Conjecture 2. *Let $2 + 5k$ be prime, $k \leq 5000$, and $x \leq 50$. If $y = 2$, then equation $2^x + (2+5k)^y = z^2$ has 2 solutions, namely, $(p, k, x, y, z) = (2, 1, 5, 2, 9)$ and $(2, 25, 9, 2, 129)$. If $y = 3$, then the equation has only the solution $(p, k, x, y, z) = (2, 3, 7, 3, 71)$.*

2.2.2. Sub-case II: k is even.

This sub-case will discuss the solutions of (3) when k is even, that is, $k \equiv 0 \pmod 4$ or $k \equiv 2 \pmod 4$, and p and $p + 5k$ are prime pairs.

We note that since k is even, then the prime gap, $5k$, is also even. Thus, for p and $p + 5k$ to be prime pairs, both of them must be odd integers. We begin the discussion with $k \equiv 2 \pmod 4$.

Lemma 6. *Let $p \geq 3$ and $p + 5k$ be prime pairs and $k \equiv 2 \pmod 4$. Then, the equation (3) has no solution if x and y are both even integers.*

Proof. Consider the Diophantine equation $p^x + (p + 5k)^y = z^2$, where p and $q = p + 5k$ are odd prime pairs and $k \equiv 2 \pmod 4$. If $p \equiv 1, 3 \pmod 4$, then $q \equiv 3, 1 \pmod 4$, respectively.

Let x and y be both even integers. In either case, we have $p^x \equiv 1 \pmod 4$ and $q^y \equiv 1 \pmod 4$ so that $p^x + q^y \equiv 1 + 1 \equiv 2 \pmod 4$. Also, since p^x and q^y are both odd, then their sum is even. Thus, $z^2 \equiv 0 \pmod 4$ and we'll have a contradiction since $p^x + q^y \equiv 2 \not\equiv 0 \equiv z^2 \pmod 4$. Conclusion follows. □

Notice that when $x = y = 1$, we have the equation

$$p + (p + 5k) = z^2. \tag{15}$$

We use the following variables to study (15).

Variable	Meaning
k_n	the value of k at a specific value of n
$z_{(0,k_n)}$	the first value of z at a specific value of k_n
$z_{(m,k_n)}$	the $(m+1)$st value of z at a specific value of k_n
$p_{(0,k_n)}$	the first value of p at a specific value of k_n
$p_{(1,k_n)}$	the second value of p at a specific value of k_n
ω_{k_n}	the difference between $p_{(0,k_n)}$ and $p_{(1,k_n)}$ at a specific value of k_n
$p_{(m,k_n)}$	the $(m+1)$st value of p at a specific value of k_n

Table 20: Variables Considered for $p + (p + 5k) = z^2$, where $p \geq 3$ and $p + 5k$ are Prime Pairs

We derive the following formulas:

$$k = k_n = 2 + 4n, \tag{16a}$$

$$z_{(0,k_n)} = \begin{cases} \sqrt{6+5k} & \text{if } \sqrt{6+5k} \text{ is an even integer,} \\ \left\lfloor \sqrt{6+5k} \right\rfloor + 1 & \text{if } \left\lfloor \sqrt{6+5k} \right\rfloor \text{ is odd,} \\ \left\lfloor \sqrt{6+5k} \right\rfloor + 2 & \text{if } \left\lfloor \sqrt{6+5k} \right\rfloor \text{ is even,} \end{cases} \tag{16b}$$

$$z_{(m,k_n)} = z_{(0,k_n)} + 2m, \tag{16c}$$

$$p_{(0,k_n)} = \frac{z^2_{(0,k_n)} - 5k_n}{2}, \tag{16d}$$

$$p_{(m,k_n)} = \frac{z^2_{(m,k_n)} - 5k_n}{2}, \tag{16e}$$

$$\omega_{k_n} = p_{(1,k_n)} - p_{(0,k_n)}, \tag{16f}$$

where $n \in \mathbb{N}_0$ and for some $m \in \mathbb{N}$. We use the floor function of $\sqrt{6+5k}$ for $z_{(0,k_n)}$ since we know that the smallest possible value of p is 3. Also, from equation (15), we have $3 + 3 + 5k = z^2$ will mean that $z = \sqrt{6+5k}$. Since k is even, $\sqrt{6+5k}$ is even if it is an integer. Otherwise, we have another two conditions as seen below. The five-tuple $(p, k, x, y, z) = (p_{(m,k_n)}, k_n, 1, 1, z_{(m,k_n)})$ will become a solution of equation (3).

Below is another result. The proof is similar to the theorems above that use mathematical induction, hence we omit it.

Theorem 7. *Suppose that the exponential Diophantine equation (15), where $k := k_n = 2 + 4n$, $n \in \mathbb{N}_0$, has a solution. Then,*

$$p_{(m,k_n)} = p_{(0,k_n)} + \sum_{i=0}^{m-1} (\omega_{k_n} + 4i).$$

This remark follows directly from the Theorem 7.

Corollary 1. *Consider the Diophantine equation (15), where $k = k_n = 2 + 4n$, $n \in \mathbb{N}_0$. Then,*

$$p_{(m,k_n)} = p_{(m-1,k_n)} + \omega_{k_n} + 4(m-1).$$

Given below are tables containing the first five solutions of equation (15), where $k := k_n = 2 + 4n$, and $p_{(m,k_n)}$ is obtained by applying (16e), or Theorem 7, or Remark 1.

Tables 21 and 22 present some solutions of (15) when $n = 0$ and $n = 1$ in $k = k_n = 2 + 4n$, respectively. If $n = 0$, we have $k = 2$, $z_{(0,2)} = 4$, $z_{(1,2)} = 6$, $p_{(0,2)} = 3$, $p_{(1,1)} = 13$, and $\omega_2 = 10$. For $n = 1$, we have $k = 6$, $z_{(0,6)} = 6$, $z_{(1,6)} = 8$, $p_{(0,6)} = 3$, $p_{(1,6)} = 17$, and $\omega_6 = 14$.

m	p	z	Equation	Solution (p, k, x, y, z)
	3	4	$3 + 13 = 4^2$	(3, 2, 1, 1, 4)
1	13	6	$13 + 23 = 6^2$	(13, 2, 1, 1, 6)
7	157	18	$157 + 167 = 18^2$	(157, 2, 1, 1, 18)
10	283	24	$283 + 293 = 24^2$	(283, 2, 1, 1, 24)
16	643	36	$643 + 653 = 36^2$	(643, 2, 1, 1, 36)

Table 21: Some Solutions of (15) with $k = 2$, $x = 1, y = 1$ and z is even

m	p	z	Equation	Solution (p, k, x, y, z)
1	17	8	$17 + (17 + 5(6)) = 8^2$	(17, 6, 1, 1, 8)
4	83	14	$83 + (83 + 5(6)) = 14^2$	(83, 6, 1, 1, 14)
8	227	22	$227 + (227 + 5(6)) = 22^2$	(227, 6, 1, 1, 22)
14	563	34	$563 + (563 + 5(6)) = 34^2$	(563, 6, 1, 1, 34)
19	953	44	$953 + (953 + 5(6)) = 44^2$	(953, 6, 1, 1, 44)

Table 22: Some Solutions of (15) with $k = 6$, $x = 1, y = 1$ and z is even

For the next theorem, we discuss the case when k is even and exactly either x or y is zero. The case when both x and y are zero are impossible since 2 is not a perfect square.

Theorem 8. *Consider the Diophantine equation (3) where $p \geq 3$ and $p + 5k$ are prime pairs, and k is a positive even number. Let $x = 0$ or $y = 0$. Then, $(p, k, x, y, z) = (3, k, 1, 0, 2)$ for some even $k \in \mathbb{N}$ is the only solution of the equation.*

Proof. Consider the equation (3). Let $p \geq 3$ and $q = p + 5k$ be prime pairs, k is even, and $x = 0$ or $y = 0$.

Case 1. Let $x = 0$. Then, we have $1 + (p + 5k)^y = z^2$. If $y > 1$, then by Mihailescu's theorem, this equation will only have a solution when $z = 3$ and $p + 5k = 2$, which is absurd since $p \geq 3$.

If $y = 1$, then the equation reduces to $1 + p + 5k = z^2$. We first claim that the only solution to the equation $1 + q = z^2$, where q is a prime number, is $(q, z) = (3, 2)$. Indeed, rewriting the equation gives $q = z^2 - 1 = (z - 1)(z + 1)$. Since $z - 1$ and $z + 1$ are consecutive even integers, their product is greater than 1 for all $z > 2$, making q composite in those cases. Hence, the only valid solution occurs when $z = 2$, which gives $q = 3$.

Returning to the original equation, this implies that $p + 5k = 3$. However, since k is assumed to be a positive integer, the left-hand side $p + 5k > 3$, and thus no prime value of p satisfies the equation. Therefore, there is no solution with p prime and $y = 1$.

Case 2. Let $y = 0$. Then, $(p + 5k)^y = 1$ for all even k. From that, we have $p^x = z^2 - 1 = (z + 1)(z - 1)$. It follows that $\gcd(z + 1, z - 1) = 1$, which further implies that $z-1=1$ and $z+1=p^x$. From $z-1=1$, we obtain $z=2$. Therefore, $z+1=3=p^x$, which implies that $p=3$ and $x=1$.

Therefore, $(p,k,x,y,z)=(3,k,1,0,2)$ is the only solution of $p^x+(p+5k)^y=z^2$ for even k, where $p \geq 3$ and $p+5k$ are prime pairs, and $x=0$ or $y=0$. □

The following result is obtained when $k \equiv 0 \pmod 4$.

Theorem 9. *The Diophantine equation (3), where $p \geq 3$ and $p+5k$ are prime pairs such that $k \equiv 0 \pmod 4$ and x and y have the same parity, has no solution.*

Proof. Suppose $x \geq 1$ and $y \geq 1$ are of the same parity. Let x and y be odd integers, then $p^x \equiv (p+5k)^y \equiv 1 \pmod 4$ or $p^x \equiv (p+5k)^y \equiv 3 \pmod 4$. If x and y are even integers, then $p^x \equiv 1 \pmod 4$ and $(p+5k)^y \equiv 1 \pmod 4$. In any case, $p^x+(p+5k)^y \equiv 2 \pmod 4$. On the other hand, since p^x and $(p+5k)^y$ are both odd, their sum is even. Hence, $z^2 \equiv 0 \pmod 4$ and thus, $p^x+(p+5k)^y \not\equiv z^2$. □

3. Conclusions

In this paper, the authors tried to provide solutions of an exponential Diophantine equation of the form

$$p^x+(p+5k)^y=z^2,$$

where $k \in \mathbb{N}$. The authors considered the cases (i) when $p=2$; or (ii) when p and $p+5k$ are prime pairs are given. In addition, the study is limited only to integer solutions (x,y,z), where x and y are not simultaneously greater than 1.

For the first case, it is concluded that if $y=1$, then the Diophantine equation has infinitely many solutions that can be further divided based on the value of x taking modulo 4. If $x \equiv 1 \pmod 4$, z can be either $z \equiv 2,3 \pmod 5$. If $x \equiv 2 \pmod 4$, then $z \equiv 1,4 \pmod 5$. If $x \equiv 3 \pmod 4$, then $z \equiv 0 \pmod 5$. In the case where z have two possible equivalence modulo 4, the union of all these solutions form all the solutions at a specific value of x.

Moreover, when $y=0$, $(p,k,x,y,z)=(2,k,3,0,3)$ is the only solution of the equation for all $k \in \mathbb{N}$. When $y=2,3$ such that $k \leq 5000$ and $x \leq 50$, there are finitely many solutions mentioned. In particular, 21 solutions were mentioned when $y=2$ and 9 solutions were mentioned when $y=3$. The equation has no solution when $y=1$ and x is a multiple of 4, i.e., $x \equiv 0 \pmod 4$, and if $x=0$.

The discussion for the second case was divided into two, based on the parity of k. If k is odd, some solutions are already mentioned in the first case since $p=2$. In this case, modulo 10 was used for the value of z since we are dealing with prime numbers and z should be odd for $p+5k$ to be an odd integer and prime number. Similar to that of the first case, when $y=1$, the Diophantine equation also have infinitely many solutions for some value of x with the note that there are infinitely many prime numbers of the form $p+5k$. If $x \equiv 1 \pmod 4$, then the equation has a solution for some $z \equiv 3,7 \pmod{10}$ and if $x \equiv 3 \pmod 4$, then the equation has a solution for some $z \equiv 5 \pmod{10}$.

There are finitely many solutions when $y = 1$ and $x = 2 + 4n, n \in \mathbb{N}_0$, where the value of k and z are given by $k = (2^{2+2n} - 1)/5$ and $z = (5k + 3)/2$. Three of them were mentioned and they are the following: $(p, k, x, y, z) = (2, 3, 6, 1, 9), (2, 51, 14, 1, 129)$, and $(2, 13107, 30, 1, 129)$. When $y = 2$ and $y = 3$ with the same values of k and x in the first case, there are three solutions that were mentioned. In other words, there are three $(2 + 5k)$ out of the 30 solutions mentioned in the first case that were prime, namely, $(p, k, x, y, z) = (2, 1, 5, 2, 9), (2, 25, 9, 2, 129)$, and $(2, 3, 7, 3, 71)$. Similar to the first case, the equation has no solution when $y = 1$ and $x \equiv 0 \pmod 4$, and when $x = 0$.

For even integers k, the discussion was also divided into two: when $k \equiv 2 \pmod 4$; and when $k \equiv 0 \pmod 4$. It has infinitely many solutions when $k \equiv 2 \pmod 4$ and $x = y = 1$. Moreover, $(p, k, x, y, z) = (3, k, 1, 0, 2)$ is a solution for some even integers k, that is, only when $3 + 5k$ is a prime. It has no solution for the most part.

For the last case of this study, it is concluded that the equation has infinitely many solution when $x = y = 1$, and that z and k have the same parity.

For future study, the authors recommend the examination of the same cases, but to explore solutions where $\min(x, y) > 1$.

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