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Generalized $(\alpha, *)$-Derivations on Rings and C^*-Algebras

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Abstract. In the presence of an automorphism denoted as α of a ring specifically referred to as $\mathcal{R}$, and with $*$ symbolizing involution on $\mathcal{R}$, this study endeavors to establish the conceptual framework for $(\alpha, *)$-derivations on $\mathcal{R}$. By leveraging the functions of α and $*$, we extract certain commutativity theorems applicable to prime rings. Furthermore, the demonstrations of these theorems, particularly in the context of non-commutative prime rings, alongside the conditions in which a generalized $(\alpha, *)$-derivation functions as a α-centralizer, will be scrutinized. Pertinent examples are presented to elucidate the proposed concepts.

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1. Introduction

The symbol $\mathcal{Z}(\mathcal{R})$ represents the center of an associative ring $\mathcal{R}$ throughout the paper. The mathematical expression $ax - xa$ represents the commutator of $a, x \in \mathcal{R}$, which is specified by the symbol $[a, x]$. A ring $\mathcal{R}$ is a t-torsion free ring if $tr = 0$ implies $r = 0$ for every $r \in \mathcal{R}$ and $t > 1$ is a fixed integer. If $a\mathcal{R}x = \{0\}$ indicates that either $a = 0$ or $x = 0$, then a ring $\mathcal{R}$ is a prime. If it meets the condition that $a\mathcal{R}a = \{0\}$ results in $a = 0$, it is referred to as semiprime.

The next attempt is to define certain key terms and concepts before going on to the literature review of this section. Involution $*$ from $\mathcal{R}$ to $\mathcal{R}$ is an additive mapping that satisfies the condition for every $d, j \in \mathcal{R}$., $(dj)^* = j^*d^*$ and $(d^*)^* = d$ for each $d, j \in \mathcal{R}$.

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The most common types of involution are identity maps, negation, and complex conjugation. For example, consider the negation of the function $(x)^* = -x$ on the set of real numbers. Then $(x+y)^* = -(x+y) = (-x)+(-y) = (x)^* + (y)^*$ is clearly additive. Also, $(xy)^* = -(xy) = -(-y)(-x) = (y)^*(x)^*$ and $((x)^*)^* = -(-x) = x$, hence $*$ is an involution. An involution ring, or ring having an involution $*$, is commonly used for a $*$-ring. The most influential research to start reading about generalized derivations, involution, centralizers, etc., are [1–6].

In short, if $\zeta(c)c + c\zeta(c) = 0$ for each of $c \in \mathcal{R}$, then the mapping ζ is (skew)-commuting on $\mathcal{R}$. A map ζ from $\mathcal{R}$ to $\mathcal{R}$ is considered to be (skew)-centralizing on $\mathcal{R}$ if $\zeta(c)c + c\zeta(c) \in \mathcal{Z}(\mathcal{R})$ for each $c \in \mathcal{R}$. The mapping η from $\mathcal{R}$ to $\mathcal{R}$ is considered a derivation on $\mathcal{R}$ if it satisfies the equation $\eta(ce) = \eta(c)e + c\eta(e)$, for each of $c, e \in \mathcal{R}$. An additive mapping $\eta : R \longrightarrow \mathcal{R}$ is said to be $*$-derivation if it fulfills the condition for every $x, y \in \mathcal{R}$, $\eta(xy) = \eta(x)y^* + x\eta(y)$. An additive mapping $\zeta : \mathcal{R} \longrightarrow \mathcal{R}$ is said to be α-derivation if it fulfills the condition for every $x, y \in \mathcal{R}$,

$$\zeta(xy) = \zeta(x)\alpha(y) + x\zeta(y).$$

Example 1. *Let a ring* $\mathcal{R} = \left\{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \,\middle|\, a, b, c \in 2\mathbb{Z}_4 \right\}$. *Define the map* D *and* α *in such a way* $D : \mathcal{R} \to \mathcal{R}$, $D\left(\begin{pmatrix} a & 0 \\ b & c \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix}$, *and*
$\alpha\left(\begin{pmatrix} a & 0 \\ b & c \end{pmatrix}\right) = \begin{pmatrix} c & 0 \\ b & a \end{pmatrix}$. *Then* D *is a* α*-derivation on* $\mathcal{R}$.

Consider the additive mappings $F, d : \mathcal{R} \longrightarrow \mathcal{R}$, F is said to be a generalized derivation on $\mathcal{R}$ associated with d if it satisfies the condition $F(xy) = F(x)y + xd(y)$, for every $x, y \in \mathcal{R}$. For example define $F : R \to \mathcal{R}$ defined by $F(x) = 2x$ and the associated derivation $d : \mathcal{R} \to \mathcal{R}$ as $d(x) = 0$. Then for any $x, y \in \mathcal{R}$, we have $F(xy) = 2xy$, and $F(x)y + xd(y) = (2x)y + x(0) = 2xy$. So $F(x) = 2x$ is a generalized derivation with $d(x) = 0$.

As stated by [7], if $T(xy) = T(x)y$ $(T(xy) = xT(y))$ holds for all $x, y \in \mathcal{R}$ and T is additive, then a mapping $T : \mathcal{R} \to \mathcal{R}$ is referred to as a left (right) centralizer respectively. The equation for $*$-centralizer in the same line of inquiry is as follows: The left $*$-centralizer and the right $*$-centralizer on $\mathcal{R}$ shall be identifiers for a mapping T on $\mathcal{R}$ that is additive and satisfies $T(xy) = T(x)y^*$ and $T(xy) = x^*T(y)$ for all $x, y \in \mathcal{R}$. In [8–13], an outstanding analysis of the theory of centralizers and $*$-centralizers was offered. The extended idea of $*$-derivation on standard operator algebra has been developed by the authors in [3].

Definition 1. *An additive map* $d : \mathcal{R} \to \mathcal{R}$ *is said to be* $(\alpha, *)$*-derivation if is satisfies*

$$d(xy) = d(x)\alpha(y) + x^*d(y)$$

$$or \quad d(xy) = d(x)y^* + \alpha(x)d(y)$$

for all $x, y \in \mathcal{R}$.

Definition 2. *An additive map* $F : \mathcal{R} \to \mathcal{R}$ *is said to be generalized* $(\alpha, *)$*-derivation with associative derivation* d*, if*

$$F(xy) = F(x)\alpha(y) + x^*d(y)$$

$$or \quad F(xy) = F(x)y^* + \alpha(x)d(y)$$

for all $x, y \in \mathcal{R}$.

To understand the concept well, we pose the following example:

Example 2. *Consider a ring as follows;*

$$\mathcal{R} = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \middle| a, b, c \in 2\mathbb{Z}_4 \right\}.$$

Define the mappings

$$F, d, \alpha, * : \mathcal{R} \to \mathcal{R}$$

such that

$$F\left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \right\} = \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$d\left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \right\} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix},$$

$$\alpha\left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \right\} = \begin{pmatrix} 0 & c & 0 \\ 0 & 0 & a \\ 0 & 0 & 0 \end{pmatrix},$$

and

$$\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}^* = \begin{pmatrix} 0 & a & -b \\ 0 & 0 & -c \\ 0 & 0 & 0 \end{pmatrix}.$$

We can verify that F *is a generalized* $(\alpha, *)$*-derivation associated with* $(\alpha, *)$*-derivation on* $\mathcal{R}$.

A lot of research has been done in the context of involution involved with derivation, generalized derivation, Jordan derivation, left derivation, etc. Our present research is motivated by all the above theories and the role of automorphism and involution on $\mathcal{R}$. We will prove some commutativity theorems in the setting of prime and semiprime rings. We will observe that α and $*$ play a crucial role in our proofs. The commutativity theorem on prime rings possessing automorphisms/ endomorphisms, n-commuting mappings, was proved in [4, 14–16]. Further, we refer the reader to the extensive bibliography contained in it.

Motivated by the above literature review and concepts, we put out the extension of generalized derivation to generalized $(\alpha, *)$-derivations on rings. In the present paper, our objective is to investigate some differential identities related to generalized $(\alpha, *)$-derivation on rings and algebras.

2. Main results

Throughout, α and $*$ will represent the automorphism and involution of $\mathcal{R}$. To prove our theorems, we use only one of the given conditions in Definition 2 (or Definition 1) randomly. We begin with the following theorems:

Lemma 1. *[17] The center of $\mathcal{R}$ includes the center of a nonzero ideal (one-sided) for a semiprime ring $\mathcal{R}$. Any commutative ideal (one-sided) is immediately enclosed in $Z(\mathcal{R})$.*

Theorem 1. *Let $\mathcal{R}$ be a semiprime $*$-ring and F be a $(\alpha, *)$-generalized derivation on $\mathcal{R}$ with associative $(\alpha, *)$-derivation. Then F maps $\mathcal{R}$ into $Z(\mathcal{R})$.*

Proof. We are given that

$$F(xy) = F(x)y^* + \alpha(x)d(y) \quad \text{for each} \quad x, y \in \mathcal{R}. \tag{1}$$

Put yz for y in (1) to observe

$$F(x(yz)) = F(x)(yz)^* + \alpha(x)d(yz) \quad \text{for each} \quad x, y, z \in \mathcal{R}. \tag{2}$$

This implies that

$$F(xyz) = F(x)z^*y^* + \alpha(x)d(y)z^* + \alpha(x)\alpha(y)d(z) \quad \text{for each} \quad x, y, z \in \mathcal{R}. \tag{3}$$

Also, expand the left hand side of (2) as

$$F((xy)z) = F(xy)z^* + \alpha(xy)d(z) \quad \text{for each} \quad x, y, z \in \mathcal{R}. \tag{4}$$

Which yields that

$$F(xyz) = F(x)y^*z^* + \alpha(x)d(y)z^* + \alpha(x)\alpha(y)d(z) \quad \text{for each} \quad x, y, z \in \mathcal{R}. \tag{5}$$

Comparing (6) and (7) together, we obtain

$$F(x)[y^*, z^*] = 0 \quad \text{for each} \quad x, y, z \in \mathcal{R}. \tag{6}$$

Substitute y and z for y^* and z^* respectively in (6) to get

$$F(x)[y,z] = 0 \text{ for each } x,y,z \in \mathcal{R}. \tag{7}$$

Putting $yF(x)$ in place of y in (7), we have

$$F(x)y[F(x),z] = 0 \text{ for each } x,y,z \in \mathcal{R}. \tag{8}$$

Reword the above expression in terms

$$F(x)zy[F(x),z] = 0 \text{ for each } x,y,z \in \mathcal{R}. \tag{9}$$

Multiply (8) by z from left to get

$$zF(x)y[F(x),z] = 0 \text{ for each } x,y,z \in \mathcal{R}. \tag{10}$$

On subtracting (9) and (10), we obtain

$$[F(x),z]y[F(x),z] = 0 \text{ for each } x,y,z \in \mathcal{R}. \tag{11}$$

Since $\mathcal{R}$ is semiprime, thus $[F(x),z] = 0$ for all $x,z \in \mathcal{R}$. Hence F maps $\mathcal{R}$ into $Z(\mathcal{R})$.

The preceding findings wield significant influence, forging new pathways and opening unprecedented avenues for exploration.

Corollary 1. *Let $\mathcal{R}$ be a semiprime $*$-ring and d be a $(\alpha,*)$-derivation on $\mathcal{R}$. Then d maps $\mathcal{R}$ into $Z(\mathcal{R})$.*

Corollary 2. *If d is a $(\alpha,*)$-derivation on a prime $*$-ring $\mathcal{R}$, then d is commuting on $\mathcal{R}$.*

Theorem 2. *Let $\mathcal{R}$ be a prime $*$-ring. If F is a $(\alpha,*)$-generalized derivation on $\mathcal{R}$, then $F = 0$ or $\mathcal{R}$ is commutative.*

Proof. Following the foot steps of Theorem 1 and from (7), we have

$$F(x)[y,z] = 0 \text{ for each } x,y,z \in \mathcal{R}. \tag{12}$$

Replace y by ry in (12), we get

$$F(x)r[y,z] = 0 \text{ for each } x,y,z,r \in \mathcal{R}. \tag{13}$$

Primeness yields that either $F = 0$ or $[y,z] = 0$ for all $y,z \in \mathcal{R}$. Commutativity of $\mathcal{R}$ guaranteed from the last case.

Theorem 3. *Let $\mathcal{R}$ be a prime $*$-ring with involution $*$. If F is a nonzero $(\alpha,*)$-generalized derivation with associated $(\alpha,*)$-derivation D such that*

$$F(\varsigma)[y,z]=0 \quad \textit{for each} \quad \varsigma,y,z\in\mathcal{R},$$

then one of the following holds:

(i) $\mathcal{R}$ is commutative,

(ii) $D=0$, moreover, F acts as an α-centralizer.

Proof. We are given that by hypothesis

$$F(\varsigma)[y,z]=0 \text{ for every } \varsigma,y,z\in\mathcal{R}. \tag{14}$$

Put ςw in place of ς in (14) to obtain

$$F(\varsigma w)[y,z]=0 \text{ for every } \varsigma,y,z,w\in\mathcal{R}. \tag{15}$$

By the definition of F, we have

$$F(\varsigma)\alpha(w)[y,z]+\varsigma^*D(w)[y,z]=0 \quad \text{for every } \varsigma,y,z,w\in\mathcal{R}.$$

Now replace w by $\alpha^{-1}(w)$ in above equation to find

$$F(\varsigma)w[y,z]+\varsigma^*D(\alpha^{-1}(w))[y,z]=0 \text{ for every } \varsigma,y,z,w\in\mathcal{R}. \tag{16}$$

Substitute wy for y in equation (14) to find

$$F(\varsigma)[wy,z]=0 \quad \text{for every } \varsigma,y,z,w\in\mathcal{R} \tag{17}$$

This implies that

$$F(\varsigma)w[y,z]+F(\varsigma)[w,z]y=0 \text{ for every } \varsigma,y,z,w\in\mathcal{R}. \tag{18}$$

In view of (14),(18) gives that

$$F(\varsigma)w[y,z]=0 \text{ for every } \varsigma,w,y,z\in\mathcal{R}. \tag{19}$$

Encounter (16) and (19) together to find

$$\varsigma^*D(\alpha^{-1}(w))[y,z]=0 \quad \text{for every}\varsigma,y,z\in\mathcal{R}.$$

Again replacing $w=\alpha(w)$ and $\varsigma^*=\varsigma$ in above expression, we arrive at

$$\varsigma D(w)[y,z]=0 \text{ for every } \varsigma,w,y,z\in\mathcal{R}. \tag{20}$$

Reword the last equation after substituting ry for y to observe

$$\varsigma D(w)r[y,z] = 0 \text{ for every } \varsigma, w, y, z, r \in \mathcal{R}. \tag{21}$$

Thus, either $\varsigma D(w) = 0$ or $[y,z] = 0$. From (21), we say that $\mathcal{R}$ can be viewed as joint $K_1^+ \cup K_2^+$, where

$$K_1^+ = \{[y,z] = 0 \mid z, y \in \mathcal{R}\}$$

and

$$K_2^+ = \{\varsigma, w \in \mathcal{R} \mid \varsigma D(w) = 0\}$$

are two additive subgroup of $\mathcal{R}$. Which is a contradiction to the fact that $\mathcal{R}$ cannot be determined by the union of two additive subgroups, namely K_1^+ and K_2^+. Hence, primeness implies that either $K_1^+ = \mathcal{R}$ or $K_2^+ = \mathcal{R}$.

If $K_1^+ = \mathcal{R}$, then $\mathcal{R}$ is commutative by Lemma 1. In case $K_2^+ = \mathcal{R}$, we say that after simple manipulation if $\varsigma D(w) = 0$, then $D(w)\varsigma D(w) = 0$, which implies $D = 0$. In case $D = 0$, F possessing the form $F(\varsigma y) = F(\varsigma)\alpha(y)$ acting as $\alpha-$centralizer on $\mathcal{R}$.

Theorem 4. *Let $\mathcal{R}$ be a prime $*$-ring. If F is a nonzero generalized $(\alpha, *)$-derivation with associated derivation D such that*

$$[F(\varsigma), y] = 0 \quad \textit{for every } \varsigma, y \in \mathcal{R},$$

then either $\mathcal{R}$ is commutative or F acts as an α-centralizer.

Proof. We are given that

$$[F(\varsigma), y] = 0 \quad \text{for every } \varsigma, y \in \mathcal{R}, \tag{22}$$

Put ςz in place of ς in equation (22) to obtain the following

$$[F(\varsigma z), y] = 0 \quad \text{for every } \varsigma, z, y \in \mathcal{R}$$

Expanding the above expression, we get

$$[F(\varsigma)\alpha(z) + \varsigma^* D(z), y] = 0 \quad \text{for every } \varsigma, z, y \in \mathcal{R}$$

This implies that

$$[F(\varsigma)\alpha(z), y] + [\varsigma^* D(z), y] = 0 \quad \text{for every } \varsigma, z, y \in \mathcal{R}.$$

Using the properties of commutators, we find

$$[F(\varsigma), y]\alpha(z) + F(\varsigma)[\alpha(z), y] + \varsigma^*[D(z), y] + [\varsigma^*, y]D(z) = 0 \quad \text{for every } \varsigma, z, y \in \mathcal{R}$$

Put ς^* for y, and use the fact that $[F(\varsigma), y] = 0$ to get

$$F(\varsigma)[\alpha(z), y] + y[D(z), y] = 0 \quad \text{for every } \varsigma, z, y \in \mathcal{R}.$$

Put $\alpha(z)$ for w in above expression to obtain

$$F(\varsigma)[w, y] + y[D(z), y] = 0 \quad \text{for every } \varsigma, w, z, y \in \mathcal{R}.$$

Considering w for y, we get

$$y[D(z), y] = 0 \quad \text{for every } z, y \in \mathcal{R}.$$

Since $y \neq 0$, then it follows that

$$[D(z), y] = 0 \quad \text{for every } z, y \in \mathcal{R}. \tag{23}$$

Further, putting zw for z in the above equation, we obtain the following

$$[D(zw), y] = 0 \quad \text{for every } z, w, y \in \mathcal{R}.$$

Expanding the derivation

$$[D(z)\alpha(w), y] + [z^* D(w), y] = 0 \quad \text{for every } z, w, y \in \mathcal{R}.$$

Using the properties of commutators, we obtain

$$[D(z), y]\alpha(w) + D(z)[\alpha(w), y] + z^*[D(w), y] + [z^*, y]D(w) = 0 \quad \text{for every } z, w, y \in \mathcal{R}.$$

Putting y for z^*, we get

$$D(z)[\alpha(w), y] = 0$$

Now, substitute $\alpha^{-1}(w)$ for w, we find that

$$D(z)[w, y] = 0 \quad \text{for every } z, w, y \in \mathcal{R}. \tag{24}$$

Next, replacing rw for w, we have

$$D(z)[rw, y] = 0 \quad \text{for every } z, w, y, r \in \mathcal{R}.$$

Simplify the above equation, we get

$$D(z)r[w, y] + D(z)[r, y]w = 0 \quad \text{for every } z, w, y, r \in \mathcal{R}. \tag{25}$$

In view of (24), (25) yields that

$$D(z)r[w, y] = 0 \quad \text{for every } z, w, y, r \in \mathcal{R}. \tag{26}$$

Making use of primeness of $\mathcal{R}$, we obtain either $D(z) = 0$ or $[w, y] = 0$ for each $w, y, z \in \mathcal{R}$. This implies $\mathcal{R}$ is commutative in the last condition. In case $D = 0$, F acting as α-centralizer.

Theorem 5. *Let $\mathcal{R}$ be a prime $*$-ring. If F is a nonzero generalized $(\alpha,*)$-derivation with associated derivation D such that*

$$F(\varsigma)F(y)[z,u]=0 \quad \textit{for every} \;\; \varsigma,y,z,u\in\mathcal{R},$$

then either $\mathcal{R}$ is commutative or F acts as an α-centralizer.

Proof. We are given that

$$F(\varsigma)F(y)[z,u]=0 \text{ for every } \varsigma,y,y',z,u\in\mathcal{R}. \tag{27}$$

Put $y'z$ in place of z in (27) and making use of commutator identities to obtain

$$F(\varsigma)F(y)y'[z,u]+F(\varsigma)F(y)[y',u]z=0 \text{ for every } \varsigma,y,y',z,u\in\mathcal{R}. \tag{28}$$

Using (27) and (28), we observe

$$F(\varsigma)F(y)y'[z,u]=0 \text{ for every } \varsigma,y,y',z,u\in\mathcal{R}. \tag{29}$$

Substitute yy' for y in (29) to find

$$F(\varsigma)\big(F(y)\alpha(y')+y^*D(y')\big)[z,u]=0 \text{ for every } \varsigma,y,y',z,u\in\mathcal{R}. \tag{30}$$

Simplifying (30) after putting $y'=\alpha^{-1}(y')$, we get

$$F(\varsigma)F(y)y'[z,u]+F(\varsigma)y^*D(\alpha^{-1}(y'))[z,u]=0 \text{ for every } \varsigma,y,y',z,u\in\mathcal{R}. \tag{31}$$

In view of (29), (31) reduces to the form

$$F(\varsigma)y^*D(\alpha^{-1}(y'))[z,u]=0 \text{ for every } \varsigma,y,y',z,u\in\mathcal{R}. \tag{32}$$

Again replace y' by $\alpha(y')$ and y by y^* to obtain

$$F(\varsigma)yD(y')[z,u]=0 \text{ for every } \varsigma,y,y',z,u\in\mathcal{R}. \tag{33}$$

Which yields that

$$F(\varsigma)yrD(y')[z,u]=0 \text{ for every } \varsigma,y,y',z,u,r\in\mathcal{R}. \tag{34}$$

Using primeness of $\mathcal{R}$, we conclude that either $F(\varsigma)y=0$ or $D(y')[z,u]=0$ for every $\varsigma,y,y',z,u\in\mathcal{R}$. Consider the first case $F(\varsigma)y=0$ for every $\varsigma,y\in\mathcal{R}$. Primeness argument gives that $F(\varsigma)=0$ for each $\varsigma\in\mathcal{R}$, hence $F=0$, which leads to a contradiction.

Next, investigate the second case if $D(y')[z,u]=0$ for every $u,y',z\in\mathcal{R}$. Last equation gives that $D(y')\mathcal{R}[z,u]=0$ for every $u,y',z\in\mathcal{R}$. Again, applying primeness of $\mathcal{R}$, we get either $D(y')=0$ or $[z,u]=0$ for each $u,y',z\in\mathcal{R}$. Second case guaranteed the commutativity of $\mathcal{R}$. In case $D(y')=0$ for each $y'\in\mathcal{R}$, F will act as left α-centralizer.

Theorem 6. *Let $\mathcal{R}$ be a prime $*$-ring. If F is a generalized $(\alpha,*)$-derivation with associated derivation D such that*

$$F(\varsigma)(y \circ z) = 0 \quad \textit{for every } \varsigma, y, z \in \mathcal{R},$$

then either $\mathcal{R}$ is commutative or F acts as an α-centralizer.

Proof. By the given hypothesis, we have

$$F(\varsigma)(y \circ z) = 0 \text{ for every } \varsigma, y, z \in \mathcal{R}. \tag{35}$$

Recall that $(y \circ z)$ represents the Jordan product, which in this context is defined as $yz + zy$. Substituting zu for z in (35) and using the identity $(y \circ zu) = (y \circ z)u + z[u, y]$, we obtain

$$F(\varsigma)(y \circ z)u + F(\varsigma)z[u, y] = 0 \text{ for every } \varsigma, y, u, z \in \mathcal{R}. \tag{36}$$

Encounter (35) and (36) together to find

$$F(\varsigma)z[u, y] = 0 \text{ for every } \varsigma, y, u, z \in \mathcal{R}. \tag{37}$$

Multiplying (37) by r from left to get

$$rF(\varsigma)z[u, y] = 0 \text{ for every } \varsigma, y, u, r, z \in \mathcal{R}. \tag{38}$$

From (37), we also write

$$F(\varsigma)rz[u, y] = 0 \text{ for every } \varsigma, y, u, r, z \in \mathcal{R}. \tag{39}$$

Evaluate (38) and (39) to obtain

$$[F(\varsigma), r]z[u, y] = 0 \text{ for every } \varsigma, y, u, r, z \in \mathcal{R}. \tag{40}$$

This implies that either $[F(\varsigma), r] = 0$ or $[u, y] = 0$ for each $\varsigma, y, u, r \in \mathcal{R}$. Take first if $[F(\varsigma), r] = 0$ for each $\varsigma, r \in \mathcal{R}$, then conclusion follows from Theorem 4. The commutativity is straightforward from the second condition. This completes the proof.

Theorem 7. *Let $\mathcal{R}$ be a prime $*$-ring. If F is a generalized $(\alpha,*)$-derivation with associated derivation D such that*

$$F(\mu)F(\varsigma)(y \circ z) = 0 \quad \textit{for every } \mu, \varsigma, y, z \in \mathcal{R},$$

then either $\mathcal{R}$ is commutative or F acts as an α-centralizer.

Proof. By the given hypothesis, we have

$$F(\mu)F(\varsigma)(y \circ z) = 0 \text{ for every } \mu, \varsigma, y, z \in \mathcal{R}. \tag{41}$$

Substituting zu for z in (41) and simplifying, we obtain

$$F(\mu)F(\varsigma)(y \circ z)u + F(\mu)F(\varsigma)z[u, y] = 0 \text{ for every } \mu, \varsigma, y, u, z \in \mathcal{R}. \tag{42}$$

Encounter the lats two equations in combination to determine

$$F(\mu)F(\varsigma)z[u, y] = 0 \text{ for every } \mu, \varsigma, y, u, z \in \mathcal{R}. \tag{43}$$

Following the footsteps of Theorem 5 from equation (29), we conclude the desired result.

3. Application on C^*-algebra

An algebra equipped with an involution is called a $*$-algebra. A C^*-algebra $\mathfrak{C}$ is a Banach $*$-algebra with the additional norm condition $||\rho^*\rho|| = ||\rho||^2$ for all $\rho \in \mathfrak{A}$. Since a C^*-algebra $\mathfrak{C}$ is primitive if its zero ideal is primitive. Hence $\mathfrak{C}$ has a faithful nonzero irreducible structure. we consider $\mathfrak{C}$ is non-unital unless indicate otherwise throughout.

Theorem 8. *Let $\mathfrak{C}$ be a C^*-algebra. If $F : \mathfrak{C} \to \mathfrak{C}$ is a linear generalized $(\alpha, *)$-derivation with associated $(\alpha, *)$-derivation $d : \mathfrak{C} \to \mathfrak{C}$, then F is commuting on $\mathfrak{C}$.*

Proof. Since every C^*-algebra is a semiprime ring [18]. Hence $\mathfrak{C}$ is semiprime. We divide the proof into the following two cases:

Case(1) : If $d \neq 0$, then by Theorem 1, we have F maps $\mathfrak{C}$ into $Z(\mathfrak{C})$. This implies that $[F(r), r] = 0$ for all $r \in \mathfrak{C}$. Hence F is commuting.

Case(2) : If $d = 0$, then by definition $F(x)$ has one possible form, that is

$$F(xy) = F(x)y^*.$$

Similarly,

$$F(x(yz)) = F(x)(yz)^* = F(x)z^*y^*. \tag{44}$$

Also, we may have

$$F((xy)z) = F(xy)z^* = F(x)y^*z^*. \tag{45}$$

Subtract (45) from (44) to get

$$F(x)[y^*, z^*] = 0 \text{ for all } x, y, z \in \mathfrak{C}. \tag{46}$$

This implies that

$$F(x)[y, z] = 0 \text{ for all } x, y, z \in \mathfrak{C}. \tag{47}$$

Put $yF(x)$ in place of y in (47) and use (47) to find

$$F(x)y[F(x), z] = 0 \text{ for all } x, y, z \in \mathfrak{C}. \tag{48}$$

Reword (48) in the form

$$F(x)zy[F(x), z] = 0 \text{ for all } x, y, z \in \mathfrak{C}. \tag{49}$$

Left multiplication to (48) by z yields that

$$zF(x)y[F(x), z] = 0 \text{ for all } x, y, z \in \mathfrak{C}. \tag{50}$$

Simplifies the equations (49) and (50) to obtain

$$[F(x), z]y[F(x), z] = 0 \text{ for all } x, y, z \in \mathfrak{C}. \tag{51}$$

Making use of primness of $\mathfrak{C}$, we find $[F(x), x] = 0$ for all $x \in \mathfrak{C}$. This is the desired conclusion.

Moreover, in case $d = 0$, we have another form of F by following Definition 2 as $F(xy) = F(x)\alpha(y)$, $x, y \in \mathfrak{C}$. Putting $\alpha^{-1}(y)$ for y in the last expression to get $F(xy) = F(x)y$ for each $x, y \in \mathfrak{C}$. Which implies that after some manipulation $[F(x), x] = 0$ for all $x \in \mathfrak{C}$. Hence, in both definitions, we have seen that F is commuting on $\mathfrak{C}$. This completes the proof of the theorem.

Theorem 9. *Let $\mathcal{R}$ be a semisimple $*$-ring. If $\mathcal{R}$ admits a generalized $(\alpha, *)$-derivation F, associate with $(\alpha, *)$-derivation, then F maps $\mathcal{R}$ into $Z(\mathcal{R})$.*

Proof. Since every semisimple $*$-ring is a semiprime $*$-ring. By application of Theorem 8, we obtain $[F(x), x] = 0$ for all $x \in \mathcal{R}$. Hence F maps $\mathcal{R}$ into $Z(\mathcal{R})$.

Conclusions

Our goal is to investigate the commutative structure within the specified framework. We determine that in certain scenarios, generalized $(\alpha, *)$-derivations are found within $Z(R)$. Moreover, we elucidate the structure of all generalized $(\alpha, *)$-derivations on $*$-prime (semiprime) rings. Furthermore, we have established that certain differential identities involving generalized $(\alpha, *)$-derivation F lead to the conclusion that either F operates as a α-centralizer or $\mathcal{R}$ exhibits commutativity. Ultimately, we applied our theorems to semisimple $*$-rings, demonstrating that if $\mathcal{R}$ permits a generalized $(\alpha, *)$-derivation F associated with a $(\alpha, *)$-derivation, then F maps $\mathcal{R}$ into $Z(\mathcal{R})$. The roles of α, $*$, the Jordan product, and the Lie product (Lie identities) are intricate and intriguing within our proofs.

The exploration of generalized $(\alpha, *)$-derivations, α-centralizer, and $*$-centralizer processes within rings (potentially incorporating an involution of the second kind), in conjunction with the CSL subalgebra of the von Neumann algebra, constitutes a fertile domain for future scholarly inquiry. This study aspires to unveil continuity theorems throughout diverse algebraic frameworks, spanning Banach and semi-simple Banach algebras to Lie and C^* algebras, underscoring its auspicious promise. Scholars are encouraged to investigate functional identities pertinent to specific derivation types, such as α-centralizer and $*$-centralizer in the CSL subalgebra of the von Neumann algebra.

This investigation into additive mappings about rings and their subsets reveals a profound level of mathematical sophistication and theoretical insight. Moreover, such structural analyses are indispensable to numerous applications within the realms of computer science, engineering, and various other disciplines. These structures garner significant interest among researchers, facilitating future endeavors to efficiently organize, compare, and optimize data, thus providing essential tools for addressing complex challenges across a confined spectrum of fields.

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