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Collectionwise Pre-Normality in Topological Spaces

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Abstract. This paper introduces and studies a new topological property called collectionwise pre-normality. A space X is said to be *collectionwise pre-normal* if and only if X is T_1 and for every discrete family $\mathcal{F} = \{F_s\}_{s\in S}$ of closed subsets of X, there exists a discrete family $\mathcal{U} = \{U_s\}_{s\in S}$ of pre-open subsets of X such that $F_s \subseteq U_s$ for each $s \in S$. We investigate this property and present examples that illustrate its relationship with other known topological properties.

2020 Mathematics Subject Classifications: 54C10, 54D10, 54D20, 54D15, 54D70

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1. Introduction

In this paper, we introduce and study a weak version of collectionwise normality called collectionwise pre-normality, which is a generalization of collectionwise normality. The space X means a topological space in whole paper. We need to recall that: a subset A of a space X is said to be a *closed domain* subset if it is the closure of its own interior [1]. The complement of a closed domain subset is called open domain. A subset A of a space X is called π-*closed* if it is a finite intersection of closed domain subsets [2]. The complement of a π-closed subset is called π-open. A subset A of X is said to be *pre-open* [3], if $A \subseteq \operatorname{int}(\overline{A})$. The complement of a pre-open set is called pre-closed. The intersection of all pre-closed sets containing A is called a pre-closure of A [4, 5], and denoted by $p\,\mathrm{cl}(A)$. The pre-interior of A, denoted by $p\,\mathrm{int}(A)$, is defined to be the union of all pre-open sets contained in A. A subset A is said to be a *pre-neighborhood* of x, [5], if there exists a

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pre-open set U such that $x \in U \subseteq A$. The family of all pre-open subsets of X is denoted by $PO(X)$ and the family of all pre-closed subsets is denoted by $PC(X)$.Observe that:

closed domain $\Longrightarrow$ π-closed $\Longrightarrow$ closed $\Longrightarrow$ pre-closed

open domain $\Longrightarrow$ π-open $\Longrightarrow$ open $\Longrightarrow$ pre-open

A space X is called *pre-normal* if for every pair of disjoint closed subsets A and B, there exist disjoint pre-open subsets U and V such that $A \subseteq U$ and $B \subseteq V$ [6]. A space X is said to be a *sub-maximal* if every dense subset of X is an open [6]. A space X is called an *pre-regular* if for each closed set F and each $x \notin F$, there exist disjoint pre-open sets U and V such that $x \in U$ and $F \subseteq V$ [3, 7]. A space X is called a *pre-*T_2, if for any distinct two points $x \neq y$, there exist two disjoint pre-open sets U and V in X such that $x \in U$ and $y \in V$. A space X is called a *pre-*T_1*-space* if for each $x, y \in X$ with $x \neq y$, there exist pre-open sets U and V such that $x \in U$, $y \in V$ and $x \notin V$, $y \notin U$. A space X is called a p_1*-paracompact* if every pre-open cover of X has a locally finite open refinement [3]. A space X is called a *pre-compact* space if every pre-open cover of X has a finite subcover. A space X is called a *pre-Lindelöf* space if every pre-open cover of X has a countable subcover. A family $\mathcal{U} = \{A_s\}_{s\in S}$ of subsets of a space X is called a *discrete family* if every point x of X has a neighborhood that intersects at most one element of $\mathcal{U}$ [8]. A space X is *paracompact* if every open cover of X has a locally finite open refinement [8–10]. A space X is called *countably paracompact* if every countable open cover for X has a locally finite open-refinement, [8, 9]. A space X is called a *collectionwise normal* space if and only if X is a T_1-space and for every discrete family $\mathcal{F} = \{F_s\}_{s\in S}$ of closed subsets of X, there exits a discrete family $\mathcal{U} = \{U_s\}_{s\in S}$ of open subsets of X such that $F_s \subseteq U_s$ for each $s \in S$ [9]. Observe that: every normal space is pre-normal. .

2. Preliminaries

First, we present the main definitions of this work.

Definition 1. A space X is called a *collectionwise pre-normal* space if and only if X is T_1 and for every discrete family $\mathcal{F} = \{F_s\}_{s\in S}$ of closed subsets of X, there exits a discrete family $\mathcal{U} = \{U_s\}_{s\in S}$ of pre-open subsets of X such that $F_s \subseteq U_s$ for each $s \in S$.

From Definition 1, clearly that: every collectionwise pre-normal space is T_1 and any non T_1-space cannot be collectionwise pre-normal. First, we give the following basic results:

Theorem 1. *Every collectionwise normal space is collectionwise pre-normal.*

Proof. Let X be a collectionwise normal space. We show that X is collectionwise pre-normal. For that, let $\{F_s\}_{s\in S}$ be a discrete family of closed subsets of X. Since X is collectionwise normal, there exists a discrete family $\{U_s\}_{s\in S}$ of open subsets of X such that $F_s \subseteq U_s$ for each $s \in S$. Since every open set is pre-open, $\{U_s\}_{s\in S}$ is a discrete family of pre-open subsets of X such that $F_s \subseteq U_s$ for each $s \in S$. Therefore, X is collectionwise pre-normal.

The converse of Theorem 1 is not true in general. Here is an example of a collectionwise pre-normal space which is not collectionwise normal:

Example 1. *The finite complement topology:* [10, Example 19], $(\mathbb{R}, \mathcal{CF})$ is a T_1, compact, countably compact, Lindelöf, separable and paracompact space which is neither regular, normal, first countable nor second countable [10]. The finite complement topology is a pre-normal space which is not normal [11]. Hence, the finite complement topology is not collectionwise normal. Since X is T_1 countably compact pre-normal space, by Theorem 11 the finite complement topology is collectionwise pre-normal.

Theorem 2. *Every collectionwise pre-normal space is pre-normal.*

Proof. Let F_r and F_t be any two disjoint closed subsets of X. Consider $\mathcal{F} = \{F_s : s \in S\}$ be a discrete family of all pairwise disjoint closed subsets of a collectionwise pre-normal space X. By collectionwise pre-normality of X, there exists a discrete family $\mathcal{V} = \{V_s : s \in S\}$ of pre-open subsets of X such that $F_s \subseteq V_s$ for each $s \in S$. Thus, there exist $V_r, V_t \in \mathcal{V}$ such that $F_r \subseteq V_r$, $F_t \subseteq V_t$ and $V_r \cap V_t = \emptyset$. Hence, X is pre-normal.

The converse of Theorem 2 is not true in general. Here is an example of a pre-normal space which is not collectionwise pre-normal:

Example 2. The left ray topology $(\mathbb{R}, \mathcal{L})$ and the right ray topology $(\mathbb{R}, \mathcal{R})$ are normal and almost completely regular spaces. Since the two spaces are normal, we conclude $(\mathbb{R}, \mathcal{L})$ and $(\mathbb{R}, \mathcal{R})$ are pre-normal. Since the two spaces are not T_1, we get $(\mathbb{R}, \mathcal{L})$ and $(\mathbb{R}, \mathcal{R})$ are not collectionwise pre-normal. Therefore, $(\mathbb{R}, \mathcal{L})$ and $(\mathbb{R}, \mathcal{R})$ are examples of pre-normal spaces which are not collectionwise pre-normal.

Since every Hausdorff paracompact space is collectionwise normal [9], and every collectionwise normal is collectionwise pre-normal, we conclude the next corollary:

Corollary 1. Every Hausdorff paracompact space is collectionwise pre-normal.

Observe that: every p_1-paracompact space is paracompact [11, 12], we get:

Corollary 2. Every regular p_1-paracompact T_1-space is collectionwise pre-normal.

Theorem 3. *Every T_1 pre-regular p_1-paracompact space is pre-normal.*

Proof. Let X be a pre-regular paracompact space. We show that X is pre-normal. Let A and B be any disjoint closed sets in X, i.e. $A \cap B = \emptyset$. Then for each $x \in A$, we have $x \notin B$. Therefore, $X \setminus B$ is an open containing x and hence $X \setminus B$ is pre-open. By pre-regularity of X, there exists a pre-open set U_x such that $x \in U_x$ and $\text{cl}(U_x) \cap B = \emptyset$. So, the family $\{U_x : x \in A\} \cup \{X \setminus B\}$ is pre-open cover of X. Since X is p_1-paracompact, there exists a locally finite pre-open refinement of it. Let $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ denotes to the members of the family which have a non-empty intersection with A. Let $V_1 = \cup_{\alpha \in \Lambda} U_\alpha$. Then, V_1 is pre-open such that $A \subseteq V_1$. Let $V_2 = X \setminus \cup_{\alpha \in \Lambda} \text{cl}(U_\alpha)$. Then, V_2 is pre-open because $\{U_\alpha : \alpha \in \Lambda\}$ is locally finite and $\text{cl}(\cup_{\alpha \in \Lambda} U_\alpha) = \cup_{\alpha \in \Lambda} \text{cl}(U_\alpha)$. Thus, $V_1 \cap V_2 = \emptyset$. Since $\mathcal{U}$ is refinement and each member of it intersects A, for each $U_\alpha \in \mathcal{U}$ there exists $x \in A$ such that $U_\alpha \subseteq \text{cl}(U_x)$. Now, $\text{cl}(U_\alpha) \subseteq X \setminus B$. Thus, $B \subseteq X \setminus \text{cl}(U_\alpha)$ for each $U_\alpha \in \mathcal{U}$. So, $B \subseteq \cap_{\alpha \in \Lambda}(X \setminus \text{cl}(U_\alpha)) = X \setminus \cup_{\alpha \in \Lambda} \text{cl}(U_\alpha) = V_2$. Thus, $B \subseteq V_2$. Therefore, V_1 and V_2 are disjoint pre-open subsets of X such that $A \subseteq V_1$ and $B \subseteq V_2$. Hence, X is pre-normal.

Theorem 4. *Every T_1 pre-regular space is pre-T_2.*

Proof. Let X be a T_1 pre-regular space. Let $x, y \in X$ such that $x \neq y$. Since X is T_1, $\{x\}$ and $\{y\}$ are closed sets in X such that $x \notin \{y\}$. By pre-regularity of X, there exist two pre-open sets U and V in X such that $x \in U$, $\{y\} \subseteq V$ and $U \cap V = \emptyset$. Thus, there exist two pre-open sets U and V in X such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$. Therefore, X is pre-T_2.

Theorem 5. *Every pre-T_2 p_1-paracompact space is collectionwise pre-normal.*

Proof. Let $\mathcal{F} = \{B_s : s \in S\}$ be a discrete family of closed subsets of a p_1-paracompact space X. Then, for each $x \in X$, choose a pre-open neighborhood H_x of a point x whose closure meets at most one set B_s. Thus, $\{H_x : x \in X\}$ is a pre-open cover for X. By p_1-paracompactness of X, there exists a locally finite pre-open refinement $\mathcal{W}$ of $\{H_x : x \in X\}$. Now, for each $s \in S$, let $V_s = X \setminus \bigcup\{\mathrm{cl}(W) : W \in \mathcal{W} \text{ and } \mathrm{cl}(W) \cap B_s = \emptyset\}$, which is pre-open in X for each $s \in S$ such that $B_s \subseteq V_s$. Since for each $W \in \mathcal{W}$, $\mathrm{cl}(W)$ meets at most one set B_s. Then, W meets at most one set B_s. So, $\{V_s : s \in S\}$ is a discrete family of pre-open subsets of X such that $B_s \subseteq V_s$ for each $s \in S$. Since X is T_1, we get X is collectionwise pre-normal.

Since every T_2-space is pre-T_2-space, we conclude:

Corollary 3. Every T_2 p_1-paracompact space is collectionwise pre-normal.

Since every pre-compact space is p_1-paracompact, we get:

Corollary 4. Every pre-T_2 pre-compact space is collectionwise pre-normal.

Corollary 5. Every pre-regular pre-compact T_1-space is collectionwise pre-normal.

The proofs of the next results is similar to that of the corresponding results for normality.

Theorem 6. *Every T_1-pre-normal space is pre-regular.*

Proof. Let X be a T_1 pre-normal space. Let $x \in X$ and F be any closed set in X such that $x \notin F$. Since X is T_1, we have $\{x\}$ is closed set in X and $\{x\} \cap F = \emptyset$. By pre-normality of X, there exist two disjoint pre-open sets U and V in X such that $\{x\} \subseteq U$ and $F \subseteq V$. Hence, $x \in U$, $F \subseteq V$ and $U \cap V = \emptyset$. Therefore, X is pre-regular.

Since every collectionwise pre-normal space is T_1, we get:

Corollary 6. Every collectionwise pre-normal space is pre-regular.

Theorem 7. *Every pre-regular pre-Lindelöf space is pre-normal.*

Proof. Let X be a pre-regular pre-Lindelöf space. Let A and B be any disjoint closed subsets of X, i.e. $A \cap B = \emptyset$. Then for each $x \in A$, we have $x \notin B$. Therefore, $X \setminus B$ is an open containing x and hence B is pre-open. By pre-regularity of X, there exists a pre-open set U_x such that $x \in U_x$, $U_x \cap B = \emptyset$ and $\mathrm{cl}(U_x) \cap B = \emptyset$. So, the family $\{U_x : x \in A\} \cup \{X \setminus B\}$ is pre-open cover of X. Since X is pre-Lindelöf, X has a countable subcover say $\{U_{x_i} : i \in \mathbb{N}\}$. Observe that $A \subseteq \bigcup\limits_{i=1}^{\infty} U_{x_i}$ and $B \subseteq X \setminus \mathrm{cl}(\bigcup\limits_{i=1}^{\infty} U_{x_i})$. Let $U = \bigcup\limits_{i=1}^{\infty} U_{x_i}$ and $V = X \setminus \mathrm{cl}(\bigcup\limits_{i=1}^{\infty} U_{x_i})$. Then, U and V are disjoint pre-open sets in X such that $A \subseteq U$ and $B \subseteq V$. Therefore, X is pre-normal.

Theorem 8. *Every T_1 pre-regular pre-Lindelöf space is collectionwise pre-normal.*

Proof. Let X be a pre-regular pre-Lindelöf space. By Theorem 7 X is pre-normal. Let $\mathcal{F} = \{F_s : s \in S\}$ be a discrete family of pairwise disjoint closed subsets of X. Then, $F_s \cap F_t = \emptyset$ for each $s \neq t$. By pre-normality of X, there exist two disjoint pre-open sets U_s and U_t in X such that $F_s \subseteq U_s$, $F_t \subseteq U_t$ and $\mathrm{cl}(U_s) \cap \mathrm{cl}(U_t) = \emptyset$ where $s \neq t$. Then, the family $\{U_s\}_{s\in S}$ is a family of pre-open sets in X. Now, we show that $\{U_s\}_{s\in S}$ is discrete. If not, there exists $x \in X$ such that for any pre-open neighborhood W_x of x we have $W_x \cap U_s \neq \emptyset \neq W_x \cap U_t$ with $s \neq t$. Thus, $x \in \mathrm{cl}(U_s)$ and $x \in \mathrm{cl}(U_t)$. Hence, $x \in \mathrm{cl}(U_s) \cap \mathrm{cl}(U_t)$, which is a contradiction. Hence, the family $\{U_s\}_{s\in S}$ must be a discrete family of pre-open sets in X such that $F_s \subseteq U_s$ for each $s \in S$. Since X is T_1, we obtain X is collectionwise pre-normal.

Recall that: a space X is called *collectionwise Hausdorff* if X is T_1 and for every discrete collection $\{x_s\}_{s\in S}$ of points of X, there exists a disjoint collection $\{V_s\}_{s\in S}$ of open subsets of X such that $x_s \in V_s$ for each $s \in S$ [13]. Since every collectionwise Hausdorff p_1-paracompact space is Hausdorff paracompact, and every pre-compact space is p_1-paracompact, we conclude:

Corollary 7. Every collectionwise Hausdorff p_1-paracompact space is collectionwise pre-normal.

Observe that: every p_1-paracompact space is paracompact, every pre-Lindelöf space is Lindelöf, every pre-compact space is compact, every p_1-paracompact space is sub-maximal [12], every sub-maximal pre-regular space is regular [12], $\mathrm{int}(A) \subseteq p\,\mathrm{int}(A) \subseteq A \subseteq p\,\mathrm{cl}(A) \subseteq \overline{A}$ for each $A \subseteq X$ [11], and if X is sub-maximal space, then $p\,\mathrm{cl}(A) = \overline{A}$ for each $A \subseteq X$.

Lemma 1. [11], Let X be a space. Then:

(1) Any dense subset of X is pre-open. If D is dense subset of X and A is closed subset of X, then $D \cup A$ and $D \setminus A$ are pre-open.

(2) Let D be a dense subset of X. For any two disjoint closed subsets A and B, the sets $U = (D \setminus A) \bigcup B$ and $V = (D \setminus B) \bigcup A$ are pre-open subsets.

(3) If X has two disjoint dense subsets, then X is pre-normal.

Theorem 9. [11], *Let X be a sub-maximal space. Fix a point $p \in X$ and let $M = X \setminus \{p\}$. Then, M is a sub-maximal subspace of X.*

Observe that: the product space $\omega_1 \times \omega_1 + 1$ is not pre-normal, the product space $X = (\omega_0 + 1) \times (\omega_1 + 1)$ is pre-normal sub-maximal space and hence X is collectionwise pre-normal, the Tychonoff plank $M = (\omega_0+1) \times (\omega_1+1) \setminus \{(\omega_0, \omega_1)\}$ is dense sub-maximal subspace of X, which is not collectionwise pre-normal, every pre-normal sub-maximal space is normal, the product of two sub-maximal spaces is sub-maximal [11] and every p_1-paracompact space is sub-maximal and paracompact [12].

Theorem 10. *Every collectionwise pre-normal sub-maximal space is collectionwise normal.*

Proof. Let $\{F_s\}_{s \in S}$ be a discrete family of closed subsets of X. Since X is collectionwise pre-normal, there exists a discrete family $\{U_s\}_{s \in S}$ of pre-open subsets of X such that $F_s \subseteq U_s$ for each $s \in S$. Since X is sub-maximal, every pre-open set in X is open. Therefore, $\{U_s\}_{s \in S}$ is a discrete family of open subsets of X such that $F_s \subseteq U_s$ for each $s \in S$. Hence, X is collectionwise normal.

Since every Hausdorff p_1-paracompact space is Hausdorff paracompact, we get:

Corollary 8. Every Hausdorff p_1-paracompact space is collectionwise normal and hence collectionwise pre-normal.

Note that: every Hausdorff countably compact normal space is collectionwise normal [9], thus we get the following results:

Theorem 11. *Every T_1 countably compact pre-normal space is collectionwise pre-normal.*

Proof. Let $\{F_s\}_{s \in S}$ be any discrete family of closed subsets of X. Since every discrete family is locally finite family, the family $\{F_s\}_{s \in S}$ is locally finite family of closed subsets of X. Since X is countably compact, the family $\{F_s\}_{s \in S}$ is finite family of pairwise disjoint closed subsets of X. Then, the family can be rewritten as $\{F_{s_i}\}_{i=1}^{n}$, for some $n \in \mathbb{N}$. By pre-normality of X, for any disjoint closed sets F_{s_i} and F_{s_j}, there exist two disjoint pre-open sets U_{s_i} and U_{s_j} in X such that $F_{s_i} \subseteq U_{s_i}$ and $F_{s_j} \subseteq U_{s_j}$, $\overline{U_{s_i}} \cap \overline{U_{s_j}} = \emptyset$ and thus $\mathrm{cl}(U_{s_i}) \cap \mathrm{cl}(U_{s_j}) = \emptyset$ for each $i \neq j$. Then, the family $\{U_{s_i}\}_{i=1}^{n}$ is a family of pre-open sets in X such that $F_{s_i} \subseteq U_{s_i}$ for each $i = 1, 2, 3, \ldots, n$. It can be observed that the family $\{U_{s_i}\}_{i=1}^{n}$ is discrete. Therefore, the family $\{U_{s_i}\}_{i=1}^{n}$ is discrete family of pre-open sets in X such that $F_{s_i} \subseteq U_{s_i}$ for each $i = 1, 2, 3, \ldots, n$. Hence, X is collectionwise pre-normal.

Since every countable countably-compact space is separable compact [9], we obtain:

Corollary 9. Every Countable Hausdorff countably compact space is collectionwise pre-normal.

Since every collectionwise pre-normal space is T_1 and every finite T_1-space is discrete, we get the following corollary:

Corollary 10. Every finite collectionwise pre-normal space is discrete and hence it is collectionwise normal.

Theorem 12. *Collectionwise pre-normality is a topological property.*

Proof. Let $X \cong Y$ and X be a collectionwise pre-normal space. Then, there exists a function $f : X \to Y$ such that f is 1-1, onto, continuous and f^{-1} is continuous. We show that Y is collectionwise pre-normal. Let $\mathcal{F} = \{F_s : s \in S\}$ be any discrete family of closed subsets of Y. Then, F_s is closed in Y for each $s \in S$. Since f is continuous, $f^{-1}(F_s)$ is a closed subset of X for each $s \in S$. Note that: $\{f^{-1}(F_s) : s \in S\}$ is a discrete family of closed subsets of X. Since X is collectionwise pre-normal, there is a discrete family $\{V_s : s \in S\}$ of pre-open subsets of X such that $f^{-1}(F_s) \subseteq V_s$ for each $s \in S$. So, $F_s \subseteq f(V_s)$ for each $s \in S$. Since f is homeomorphism, we have $f(V_s)$ is a pre-open subset of Y for each $s \in S$. Thus, we have $\{f(V_s)\}_{s \in S}$ is a discrete family of pre-open subsets of Y such that $F_s \subseteq f(V_s)$ for each $s \in S$. Therefore, Y is collectionwise pre-normal.

Theorem 13. *The sum $X = \oplus_{s \in S} X_s$, $X_s \neq \emptyset$ for each $s \in S$, is collectionwise pre-normal if and only if each X_s is collectionwise pre-normal.*

Proof. Let $X = \underset{s \in S}{\oplus} X_s$ be a collectionwise pre-normal space. Since $X_s \subseteq X$ is a clopen subspace of a collectionwise pre-normal space X and a clopen subspace of a collectionwise pre-normal space is collectionwise pre-normal (Corollary 12), we have X_s is collectionwise pre-normal for each $s \in S$. Now, let X_s be a collectionwise pre-normal space for each $s \in S$. We show that $X = \underset{s \in S}{\oplus} X_s$ is collectionwise pre-normal. Let $\{F_i : i \in I\}$ be a discrete family of closed subsets of X. Then, $\{F_i \cap X_s : i \in I\}$ is a discrete family of closed subsets of X_s for each $s \in S$. By collectionwise pre-normality of X_s, there exists a discrete family $\{U_{is} : i \in I\}$ of pre-open subsets of X_s such that $F_i \cap X_s \subseteq U_{is}$ for each $s \in S$. Thus, $\underset{s \in S}{\cup} (F_i \cap X_s) \subseteq \underset{s \in S}{\cup} U_{is}$. Put $U_i = \underset{s \in S}{\cup} U_{is}$, which is a pre-open set in X for each $i \in I$. So, we have $F_i \subseteq U_i$ for each $i \in I$. Hence, $\{U_i : i \in I\}$ is a discrete family of pre-open subsets of X such that $F_i \subseteq U_i$ for each $i \in I$. Therefore, $X = \underset{s \in S}{\oplus} X_s$ is collectionwise pre-normal.

Corollary 11. Collectionwise pre-normality is an additive property.

3. Characterizations of collectionwise pre-normality

Now, we give some characterizations of collectionwise pre-normal spaces. First, we need to recall the next definitions:

Definition 2. A subset A of X is called:

- *generalized closed* (briefly; *g-closed*) if $\overline{A} \subseteq U$ whenever $A \subseteq U$ and U is open [14].
- *generalized pre-open* (briefly; *g-pre-open*) if $F \subseteq p\,\mathrm{int}(A)$ whenever $F \subseteq A$ and F is closed[15].

- *strongly generalized pre-open* (briefly; *g^*-pre-open*) if $F \subseteq p\,\text{int}(A)$ whenever $F \subseteq A$ and F is g-closed [16].
- *π-generalized pre-open*, (briefly; *πg-pre-open*) if $F \subseteq p\,\text{int}(A)$ whenever $F \subseteq A$ and F is π-closed.[17]

Observe that: every open set is pre-open and every closed set is pre-closed. From the Definition 2, we have:

$$\text{pre-open} \Longrightarrow g^*\text{-pre-open} \Longrightarrow g\text{-pre-open} \Longrightarrow \pi g\text{-pre-open}$$

$$g^*\text{-closed } (g\text{-closed}, \pi g\text{-closed}) \Longrightarrow g^*\text{-pre-closed } (g\text{-pre-closed}, \pi g\text{-pre-closed})$$

Now, we give the following theorem, which is useful for giving some characterizations of collectionwise pre-normal spaces.

Theorem 14. *Let X be a space. The following statements are equivalent:*

(1) X is collectionwise pre-normal.

(2) for any discrete family $\{F_s\}_{s\in S}$ of closed sets in X, there exists a discrete family $\{U_s\}_{s\in S}$ of $g^\star$-pre-open sets in X such that $F_s \subseteq p\,\text{int}(U_s)$ for each $s \in S$.

(3) for any discrete family $\{F_s\}_{s\in S}$ of closed sets in X, there exists a discrete family $\{U_s\}_{s\in S}$ of g-pre-open sets in X such that $F_s \subseteq p\,\text{int}(U_s)$ for each $s \in S$.

(4) for any discrete family $\{F_s\}_{s\in S}$ of closed sets in X, there exists a discrete family $\{U_s\}_{s\in S}$ of πg-pre-open sets in X such that $F_s \subseteq p\,\text{int}(U_s)$ for each $s \in S$.

Proof. (1) $\Longrightarrow$ (2): Let X be collectionwise pre-normal. Let $\{F_s\}_{s\in S}$ be a discrete family of closed subsets of X. By collectionwise pre-normality of X, there exists a discrete family $\{U_s\}_{s\in S}$ of pre-open sets in X such that $F_s \subseteq U_s$ for each $s \in S$. Since every pre-open set is $g^\star$-pre-open, we have $\{U_s\}_{s\in S}$ is a discrete family of $g^\star$-pre-open sets in X such that $F_s \subseteq U_s$ for each $s \in S$. Since U_s is g-pre-open as every pre-open set is g-pre-open, and $F_s \subseteq U_s$ we have $F_s \subseteq p\,\text{int}(U_s)$ for each $s \in S$.
(2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) are obvious.
(4) $\Longrightarrow$ (1): Suppose (4) holds. We show that X is collectionwise pre-normal. Let $\{F_s\}_{s\in S}$ be a discrete family of closed subsets of X. By (4), there exists a discrete family $\{U_s\}_{s\in S}$ of πg-pre-open sets in X such that $F_s \subseteq p\,\text{int}(U_s)$ for each $s \in S$. Put $V_s = p\,\text{int}(U_s)$ for each $s \in S$. Then, V_s is pre-open subset of X for each $s \in S$. Since $\{U_s\}_{s\in S}$ is discrete family and $V_s \subseteq U_s$ for each $s \in S$, we obtain $\{V_s\}_{s\in S}$ is a discrete family of pre-open subsets of X such that $F_s \subseteq V_s$ for each $s \in S$. Therefore, X is collectionwise pre-normal.

Theorem 15. *A space X is collectionwise pre-normal if one of the next equivalent statements holds:*

(1) for any discrete family $\{F_s\}_{s\in S}$ of g-closed sets in X, there exists a discrete family $\{U_s\}_{s\in S}$ of π-pre-open sets in X such that $p\,\text{cl}(F_s) \subseteq U_s$ for each $s \in S$.

(2) for any discrete family $\{F_s\}_{s\in S}$ of g-closed sets in X, there exists a discrete family $\{U_s\}_{s\in S}$ of pre-open sets in X such that $p\,\mathrm{cl}(F_s) \subseteq U_s$ for each $s \in S$.

(3) for any discrete family $\{F_s\}_{s\in S}$ of g-closed sets in X, there exists a discrete family $\{U_s\}_{s\in S}$ of $g^\star$-pre-open sets in X such that $p\,\mathrm{cl}(F_s) \subseteq p\,\mathrm{int}(U_s)$ for each $s \in S$.

(4) for any discrete family $\{F_s\}_{s\in S}$ of g-closed sets in X, there exists a discrete family $\{U_s\}_{s\in S}$ of g-pre-open sets in X such that $p\,\mathrm{cl}(F_s) \subseteq p\,\mathrm{int}(U_s)$ for each $s \in S$.

(5) for any discrete family $\{F_s\}_{s\in S}$ of g-closed sets in X, there exists a discrete family $\{U_s\}_{s\in S}$ of πg-pre-open sets in X such that $p\,\mathrm{cl}(F_s) \subseteq p\,\mathrm{int}(U_s)$ for each $s \in S$.

Proof. (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) $\Longrightarrow$ (5) are obvious. Now, we show that:
(5) $\Longrightarrow$ *collectionwise pre − normality*: Suppose (5) holds. Let $\{F_s\}_{s\in S}$ be a discrete family of closed subsets of X. Since every closed set is g-closed, the family $\{F_s\}_{s\in S}$ is a discrete family of g-closed subsets of X. By (5), there exists a discrete family $\{U_s\}_{s\in S}$ of πg-pre-open sets in X such that $p\,\mathrm{cl}(F_s) \subseteq p\,\mathrm{int}(U_s)$ for each $s \in S$. Since F_s is pre-closed for each $s \in S$, we get $F_s \subseteq p\,\mathrm{int}(U_s)$ for each $s \in S$. Let $V_s = p\,\mathrm{int}(U_s)$ for each $s \in S$. Then, V_s is pre-open set in X for each $s \in S$. Since $V_s \subseteq U_s$ for each $s \in S$ and $\{U_s\}_{s\in S}$ is discrete, we conclude that $\{V_s\}_{s\in S}$ is a discrete family of pre-open sets in X such that $F_s \subseteq V_s$ for each $s \in S$. Therefore, X is collectionwise pre-normal.

4. Collectionwise pre-normality in subspaces

Now, we study collectionwise pre-normality in subspaces. The next example shows that collectionwise pre-normality is not a hereditary property in general.

Example 3. Consider the product space $X = (\omega_0+1)\times(\omega_1+1)$, which is pre-normal, but the subspace $M = X \setminus \{\langle\omega_0, \omega_1\rangle\}$ is not pre-normal [11]. Since $X = (\omega_0 + 1) \times (\omega_1 + 1)$ is normal, it is pre-normal. The Tychonoff plank $M = X\setminus\{\langle\omega_0, \omega_1\rangle\}$ is dense subspace of X. Since both ω_0+1 and ω_1+1 are sub-maximal spaces and the product of two sub-maximal spaces is sub-maximal, we obtain the space $X = (\omega_0 + 1) \times (\omega_1 + 1)$ is sub-maximal. By Theorem 9, $M = X \setminus \{\langle\omega_0, \omega_1\rangle\}$ is sub-maximal subspace of X. Since the subspace M is not normal, we obtain M is not pre-normal. Since $M = X\setminus\{\langle\omega_0, \omega_1\rangle\}$ is not collectionwise normal, we get $M = X \setminus \{\langle\omega_0, \omega_1\rangle\}$ is not collectionwise pre-normal.

Lemma 2. [11], Let M be a closed domain subspace of X and $A \subseteq M$. A is pre-closed (pre-open) in M, if and only if A is pre-closed (pre-open) in X.

Theorem 16. *Let M be a closed domain subspace of X. Then:*

(1) A family $\{F_s\}_{s\in S}$ is discrete family of closed sets in M if and only if $\{F_s\}_{s\in S}$ is discrete family of closed sets in X, where $F_s \subseteq M$ for each $s \in S$.

(2) A family $\{F_s\}_{s\in S}$ is discrete family of pre-open sets in M if and only if $\{F_s\}_{s\in S}$ is discrete family of pre-open sets in X, where $F_s \subseteq M$ for each $s \in S$.

Proof. Let M be a closed domain subspace of X. Then:
(1): Let $\{F_s\}_{s\in S}$ be a discrete family of closed sets in M. Then, F_s is closed subset of M for each $s \in S$. Since M is closed subset of X, we have F_s is closed set in X for each $s \in S$. Hence, $\{F_s\}_{s\in S}$ is discrete family of closed sets in X, where $F_s \subseteq M$ for each $s \in S$. Conversely, let $\{F_s\}_{s\in S}$ be a discrete family of closed sets in X, where $F_s \subseteq M$ for each $s \in S$. Then, F_s is closed in X for each $s \in S$. Since M is closed in X, we have $F_s \cap M = F_s$ is closed set in M for each $s \in S$. Then, $\{F_s\}_{s\in S}$ is a discrete family of closed sets in M.
(2): Let $\{F_s\}_{s\in S}$ be a discrete family of pre-open sets in M. Then, F_s is pre-open set in M for each $s \in S$. Since M is closed domain set in X, by Lemma 2 F_s is pre-open set in X for each $s \in S$. Hence, $\{F_s\}_{s\in S}$ is a discrete family of pre-open sets in X, where $F_s \subseteq M$ for each $s \in S$. Conversely, let $\{F_s\}_{s\in S}$ be a discrete family of pre-open sets in X, where $F_s \subseteq M$ for each $s \in S$. Then, F_s is pre-open set in X for each $s \in S$. Since M is closed domain in X, by Lemma 2 we have F_s is pre-open set in M for each $s \in S$. Then, $\{F_s\}_{s\in S}$ is a discrete family of pre-open sets in M.

Lemma 3. [11], Let M be a closed domain subspace of X and $A \subseteq M$. Then:

(1) A is pre-closed (pre-open) in M if and only if A is pre-closed (pre-open) in X.

(2) If $A \subseteq X$ and A is pre-closed (pre-open) in X, then $A \cap M$ is pre-closed (pre-open) in M.

Theorem 17. *A closed domain subspace of a collectionwise pre-normal space is collectionwise pre-normal.*

Proof. Let $\{F_s : s \in S\}$ be a discrete family of closed subsets of M. By Theorem 16, $\{F_s : s \in S\}$ is a discrete family of closed subsets of X. Since X is collectionwise pre-normal, there exists a family $\{U_s : s \in S\}$ of pre-open subsets of X such that $F_s \subseteq U_s$ for each $s \in S$. Thus, $F_s \cap M \subseteq U_s \cap M$ and so $F_s \subseteq U_s \cap M$ for each $s \in S$. By Lemma 3, we have $U_s \cap M$ is pre-open set in M for each $s \in S$. Hence, $\{U_s \cap M : s \in S\}$ is a discrete family of pre-open subsets of M such that $F_s \subseteq U_s \cap M$ for each $s \in S$. Therefore, M is collectionwise pre-normal.

Since every clopen subset of a space X is closed domain, we conclude the next corollary:

Corollary 12. A clopen subspace of a collectionwise pre-normal space is collectionwise pre-normal.

5. The product of collectionwise pre-normality

In this section, we study the product of collectionwise pre-normality as follows:

Theorem 18. *Let $(X_i, \mathcal{T}_i)$ be a topological space for each $i \in \{1, 2, 3, ..., n\}$, $n \in \mathbb{N}$. Let $\mathcal{T}$ be the product topology on $X = \prod_{i=1}^{n} X_i$. If $(X, \mathcal{T})$ is collectionwise pre-normal, then $(X_i, \mathcal{T}_i)$ is collectionwise pre-normal for each $i \in \{1, 2, 3, ..., n\}$.*

Proof. Let $X = \prod\limits_{i=1}^{n} X_i$ be a collectionwise pre-normal space. Let $m \in \{1, 2, 3, ..., n\}$ be arbitrary. Let $\{F_{s_m}\}_{s\in S}$ be any discrete family of closed subsets of X_m. Let $\pi_m : \prod\limits_{i=1}^{n} X_i \longrightarrow X_m$ be the natural projection map from X onto X_m. Now, $\pi_m^{-1}(F_{s_m}) = \prod\limits_{i=1}^{n} W_i$, (where $W_i = X_i$ for each $i \neq m$) is closed in X. Then, $\{\pi_m^{-1}(F_{s_m})\}_{s\in S}$ is a discrete family of closed sets in X. Since X is collectionwise pre-normal, there exists a discrete family $\{U_s\}_{s\in S}$ of pre-open sets in X such that $\pi_m^{-1}(F_{s_m}) \subseteq U_s$ for each $s \in S$. Then, we have $F_{s_m} \subseteq \pi_m(U_s)$ for each $s \in S$. Since π_m is a clopen onto continuous function, then $\pi_m(U_s)$ is pre-open set in X_m for each $s \in S$. Thus, $\{\pi_m(U_s)\}_{s\in S}$ is a discrete family of pre-open sets in X_m such that $F_{s_m} \subseteq \pi_m(U_s)$ for each $s \in S$. Hence, X_m is collectionwise pre-normal. Since m was arbitrary, then $(X_i, \mathcal{T}_i)$ is collectionwise pre-normal for each $i \in \{1, 2, 3, ..., n\}$.

Corollary 13.

- If the product space $X \times Y$ is collectionwise pre-normal, then both X and Y are collectionwise pre-normal.
- If $X \times I$ is collectionwise pre-normal, then X is collectionwise pre-normal.
- A space X is collectionwise pre-normal if and only if $X \times \{0\}$ is collectionwise pre-normal.

Note that: collectionwise pre-normality is not productive in general. Here is an example:

Example 4. *The space* $\omega_1 \times (\omega_1 + 1)$, [10], is Tychonoff, mildly normal, locally compact and countably compact space which is neither almost normal, normal, compact nor Lindelöf. Since X is not normal, the space $\omega_1 \times (\omega_1 + 1)$ is not collectionwise normal. Since ω_1 and $\omega_1 + 1$ are sub-maximal spaces [11], we get $\omega_1 \times (\omega_1 + 1)$ is sub-maximal. Since $\omega_1 \times (\omega_1 + 1)$ is not normal, we conclude that $\omega_1 \times (\omega_1 + 1)$ is not pre-normal. Therefore, $\omega_1 \times (\omega_1 + 1)$ is not collectionwise pre-normal. This example shows that the product of two collectionwise pre-normal spaces cannot be collectionwise pre-normal.

Observe that: any Tychonoff space Y has a one-point compactification $X = Y \cup \{p\}$, $p \notin Y$ and X is a Hausdorff compact space [18], we get:

Corollary 14. Any compactification X of a Tychonoff space Y is collectionwise pre-normal. In particular, any Tychonoff space Y has a one-point compactification $X = Y \cup \{p\}$, $p \notin Y$ and X is collectionwise pre-normal.

6. The closed extension and the discrete extension spaces of collectionwise pre-normality

Now, we study the closed extension and the discrete extension spaces of collectionwise pre-normality. In fact, collectionwise pre-normality is not preserved by the discrete extension space X_M in general. Here is a counterexample:

Example 5. [18, Example 8], *The rational sequence topology* [10, Example 65], is a first countable, zero-dimensional, Tychonoff, locally compact, separable space which is neither paracompact, normal nor Lindelöf [10]. By Corollary 14, $\mathbb{R}$ with the rational sequence topology has a one-point compactification. Let $X = \mathbb{R} \cup \{p\}$, $p \notin \mathbb{R}$, be a one-point compactification of $\mathbb{R}$. By Corollary 14, X is Hausdorff compact. Hence, X is collectionwise pre-normal. Now, let $X_{\mathbb{R}} = \mathbb{R} \cup \{p\}$. Then, $X_{\mathbb{R}}$ is first countable, separable and Tychonoff space which is not normal and $\{p\}$ is clopen subset [18]. Since $\mathbb{R}$ is clopen subspace in $X_{\mathbb{R}}$ and $\mathbb{R}$ is not pre-normal, we conclude that $X_{\mathbb{R}}$ is not pre-normal. Therefore, $X_{\mathbb{R}}$ is neither collectionwise normal nor collectionwise pre-normal because $\mathbb{R}$ with the rational sequence topology is sub-maximal [11]. Hence, $X_{\mathbb{R}}$ is a discrete extension space of a collectionwise pre-normal space $X = \mathbb{R} \cup \{p\}$ which is not collectionwise pre-normal.

Now, we give the following results:

Lemma 4. [11], Let M be a closed subspace of X and $A \subseteq M$. Then: if A is pre-closed (pre-open) in M, then A is pre-closed (pre-open) in X.

Lemma 5. Let M be a closed subspace of X. Then:

(1) If A is a pre-closed (pre-open) set in X, then A is pre-closed (pre-open) set in X_M.

(2) If A is a closed set in X_M, then $A \cap M$ is closed subset of a subspace M in X.

(3) If A is a closed set in X_M, then $A_1 = A \cap M$ is a closed set in X.

Proof. Let M be a closed subspace of X.

(1) Let A be a pre-closed (pre-open) set in X. Since $X \subset X_M$ is a closed subspace of X_M [18], and A is pre-closed (pre-open) in X, by Lemma 4 we conclude that A is pre-closed (pre-open) in X_M.

(2) Let A be a closed set in X_M. Since $M \subset X_M$ is closed in both X and X_M and its topology coincides with the topology on M by the topology on X, i.e. $\mathcal{T}_M = \mathcal{T}_{(M)_M}$ [18], we get $A \cap M$ is a closed subset of a subspace M in X_M. Since $\mathcal{T}_M = \mathcal{T}_{(M)_M}$, $A \cap M$ is a closed subset of a subspace M in X.

(3) Let A be a closed set in X_M. By part (2), $A_1 = A \cap M$ is a closed subset of a subspace M in X. Since M is closed subspace of X and $A \cap M$ is closed subset of M in X, we have $A_1 = A \cap M$ is a closed set in X.

Lemma 6. Let M be a closed subspace of a space X. Then: If $\{F_s\}_{s\in S}$ is a discrete family of closed sets in X_M, then $\{F_s \cap M\}_{s\in S}$ is a discrete family of closed sets in X.

Proof. Let M be a closed subspace of X and $\{F_s\}_{s\in S}$ be a discrete family of closed sets in X_M. Then, F_s is closed set in X_M for each $s \in S$. By Lemma 5, we get $F_s \cap M$ is closed subset of a subspace M in X for each $s \in S$. Put $G_s = F_s \cap M$ for each $s \in S$. Then, $\{G_s\}_{s\in S}$ is a family of closed subsets of X. Since $\{F_s\}_{s\in S}$ is a discrete family, $X = X_M$, $\mathcal{T} \subseteq \mathcal{T}_{(M)}$ and $G_s \subseteq F_s$ for each $s \in S$, we obtain $\{G_s\}_{s\in S}$ is a discrete family of closed subsets of X. Hence, $\{F_s \cap M\}_{s\in S}$ is a discrete family of closed sets in X.

Theorem 19. *If X is collectionwise pre-normal and M is a closed subspace of X, then X_M is collectionwise pre-normal.*

Proof. Let $\{F_s\}_{s\in S}$ be a discrete family of closed sets in X_M. Since M is closed in X, by Lemma 6 we get $\{F_s \cap M\}_{s\in S}$ is a discrete family of closed sets in X. By collectionwise pre-normality of X, there exists a discrete family $\{U_s\}_{s\in S}$ of pre-open sets in X such that $F_s \cap M \subseteq U_s$ for each $s \in S$. Let $V_s = U_s \cup F_s \setminus M$ for each $s \in S$. Then, V_s is pre-open set in X_M and $F_s \subseteq V_s$ for each $s \in S$. Observe that $\{V_s : s \in S\}$ is discrete. Hence, $\{V_s\}_{s\in S}$ is a discrete family of pre-open sets in X_M such that $F_s \subseteq V_s$ for each $s \in S$. Therefore, X_M is collectionwise pre-normal.

Since the closed extension space $(X^p, \mathcal{T}^*)$ of a space $(X, \mathcal{T})$ is separable, first countable, second countable and T_0-space which is neither T_1, Hausdorff, regular nor normal [19], we get the next corollary:

Corollary 15. Any closed extension space $(X^p, \mathcal{T}^*)$ of a collectionwise pre-normal space $(X, \mathcal{T})$ cannot be collectionwise pre-normal. That is: collectionwise pre-normality is not preserved by the closed extension spaces.

Proof. Since the closed extension space $(X^p, \mathcal{T}^*)$ of a space $(X, \mathcal{T})$ is not T_1-space, and every collectionwise pre-normal space is T_1, we conclude that any closed extension space $(X^p, \mathcal{T}^*)$ of a collectionwise pre-normal space $(X, \mathcal{T})$ is not collectionwise pre-normal.

Now, we present the next examples. Here is a Tychonoff space which is not collectionwise pre-normal:

Example 6. *The rational sequence topology* [10, Example 65], $(\mathbb{R}, \mathcal{RS})$ is a Tychonoff first countable, zero-dimensional, locally compact, separable and almost normal space which is neither paracompact, normal, extremally disconnected, π-normal nor Lindelöf [10, 11]. Observe that: the rational sequence topology is an example of a Tychonoff space which is neither pre-normal nor normal being sub-maximal space [11]. Therefore, the rational sequence topology is neither collectionwise pre-normal nor collectionwise normal

The following problems are still open in this research: is there an example of a T_1 pre-normal space which is not collectionwise pre-normal?, is there a Tychonoff collectionwise pre-normal space which is not collectionwise normal?, is a closed subspace of a collectionwise pre-normal space, collectionwise pre-normal?, are the Niemytzki plane topology and the countable complement topology $(\mathbb{R}, \mathcal{CC})$, collectionwise pre-normal?, and is a quotient space of a collectionwise pre-normal space, collectionwise pre-normal?.

7. Conclusion

New topological property, called collectionwise pre-normality has been studied in this work. Some results, properties, relationships, characterizations and counterexamples were given and discussed. The importance of this study is to open a window for future studies and to help us for obtaining some new results of several weak versions of collectionwise normality in the future researches.

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