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Pythagorean Fuzzy Digital Fine Space Approach for Selecting Menstrual Hygiene Products in Rural Areas

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Abstract. Pythagorean fuzzy digital fine b$^{\approx}$-door space is a novel notion of generalized Pythagorean fuzzy digital fine topological space that is presented in this paper. Additionally, a number of characterizations of Pythagorean fuzzy digital fine $b^{\approx}$-door space are examined. Examples were presented to demonstrate the applicability of these ideas and also some characteristics and connections between other Pythagorean fuzzy digital fine topological spaces and Pythagorean fuzzy digital fine $b^{\approx}$-door space were examined. Furthermore, an application which utilizes Pythagorean fuzzy digital fine topological space to select the best menstrual hygiene product for rural women is investigated.

2020 Mathematics Subject Classifications: 54A40, 03E72

Key Words and Phrases: Pythagorean fuzzy digital fine topological space, Pythagorean fuzzy digital $b_f^{\approx}$-Baire space, Pythagorean fuzzy digital $b_f^{\approx}$D-Baire space Pythagorean fuzzy digital $b_f^{\approx}$-Hausdorff space, Pythagorean fuzzy digital $b_f^{\approx}$-door spaces

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1. Introduction

An extension of classical topology, fuzzy topology makes use of the idea of fuzziness to provide more complex and adaptable interpretations of mathematical ideas. It was created to address circumstances in which the conventional understanding of membership function that is, whether an element is a member of a set or not is too strict to adequately represent actual occurrences. The fundamental concept of fuzzy topology is fuzzy sets, which were initially introduced by Lotfi Zadeh [1] in 1965. Unlike classical sets, where an element is either a member or not (with membership values strictly 1 or 0), fuzzy sets permit varying degrees of membership across a continuum [2]. Developed by Krassimir Atanassov [3] in 1986, intuitionistic fuzzy topology is an extension of fuzzy topology that incorporates the idea of intuitionistic fuzzy sets. This enhancement offers a more comprehensive framework for managing uncertainty by taking into account the degree of membership, the degree of non-membership and a hesitation margin. It is particularly effective in situations where uncertainty exists not only regarding membership value but

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also non-membership value. Pythagorean fuzzy topology is an advanced form of fuzzy and intuitionistic fuzzy topology. It uses Pythagorean fuzzy sets, which provide more flexibility in handling uncertainty. This is especially useful in situations where it is important to balance membership and non-membership values carefully and where there is a high level of indeterminacy. Motivated by this need, Yager R. R. [4] introduced Pythagorean fuzzy sets under a specific constraint $\mu^2 + \nu^2 \leq 1.$, enabling a more refined mathematical framework for addressing uncertainty. As a result, Pythagorean fuzzy topology paves the way for new research directions and practical applications in areas where classical or even intuitionistic methods may be inadequate. Fuzzy Digital topology [5] integrates fuzzy set theory with digital topology [6]. It employs fuzzy sets to represent uncertainty in digital images and utilizes topological principles to analyze and process them. The main objective is to develop techniques capable of effectively managing the inherent fuzziness present in digital imagery. This includes defining fuzzy topological spaces, fuzzy connectedness, and fuzzy boundaries within digital grid structures. To improve the recognition and categorization of objects with partially hidden or ambiguous borders, digital fuzzy topology incorporates fuzziness into the topological modelling of digital images. This method works especially well in real-world applications, including biometric verification, automated visual inspection, and autonomous vehicle navigation systems.

There are numerous uses for the Pythagorean fuzzy topological space [7], which was created with Pythagorean fuzzy sets, in decision-making. Fuzzy door Space was first proposed by Anjalmose S. and Thangaraj G. [8], who also looked into the connections between fuzzy door space and some fuzzy topological space. The notion of b-open sets was presented and examined by Andrijivic [9]. Additionally, Parameswari R.U. and Thangavelu P. [10], presented on the idea of a b#-open set and its fundamental characteristics. Using the intuitionistic fuzzy b-set as a basis, AbdulGawad, A. AL-Qubati, and Mohamed El Sayed [11] developed the concept of Intuitionistic fuzzy b-door space and investigated its properties. Harish Garg [12] explored confidence Pythagorean fuzzy weighted and ordered weighted operators, which are novel averaging and geometric operators. To illustrate their validity and effectiveness, a real-life application has been proposed.

The analysis of door spaces is a well-known topic in fuzzy topology and intuitionistic fuzzy topology. However, it is not yet known whether it can be used to Pythagorean fuzzy digital fine topological spaces. In an intuitionistic fuzzy framework, a great deal of research has been done to produce theoretical findings regarding door spaces. The necessity for a thorough investigation of the function of door spaces with Pythagorean fuzzy digital fine topological space is highlighted by this gap. The following is the motivation behind this study:

- For theoretical developments, it is crucial to comprehend how fine collections behave in a topological space.
- To improve the study of Pythagorean fuzzy topology, the idea of door spaces is extended to Pythagorean fuzzy digital fine door spaces.
- New insights into structural features are obtained by connecting Pythagorean fuzzy

digital fine topological spaces with Pythagorean fuzzy digital fine b$^{\approx}$-door space.

- Filling this research void will advance Pythagorean fuzzy topology and enable its application to more general mathematics and practical issues.

This paper introduces the concept of door space within the framework of Pythagorean fuzzy digital fine topology. Establishing and examining a variety of structures in Pythagorean fuzzy digital fine door spaces is what makes this study new. These structures include:

- Pythagorean fuzzy digital fine topology
- Pythagorean fuzzy digital fine b$^{\approx}$- door space
- Pythagorean fuzzy digital fine b$^{\approx}$- first category space
- Pythagorean fuzzy digital fine b$^{\approx}$- Baire space
- Pythagorean fuzzy digital fine b$^{\approx}$- Hausdorff space

A novel real world application titled as selection of the most suitable menstrual hygiene product for rural women, is presented along with supporting numerical example. This practical case study illustrates how the Pythagorean fuzzy digital fine topology can effectively guide and influence the decision-making process in product selection and the complete structure of this study is given in the Figure 1.

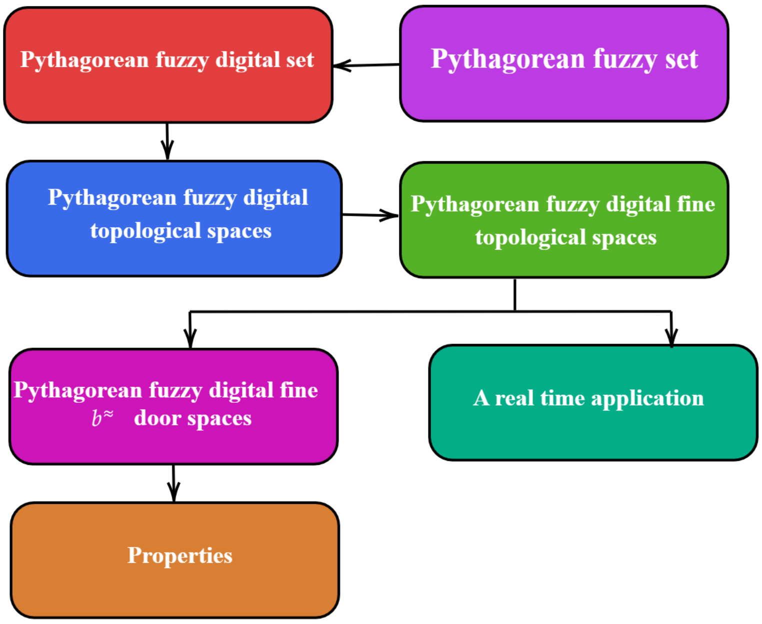

Figure 1: Flow of Pythagorean fuzzy digital space concepts

The structure of the paper is as follows: the fundamental definitions are given in Section 2. The definition of Pythagorean fuzzy digital fine b$^{\approx}$- door space and a discussion of several intriguing characteristics are covered in Section 3. The Pythagorean fuzzy digital $b_f^{\approx}$-quasi compact is suggested and its characteristics are examined in Section 4. The Pythagorean fuzzy digital fine topological space selection algorithm is presented in Section 5. The validation of the present study is provided in Section 6. Finally, Section 7 summarises the findings of the study.

2. Preliminaries

The basic concepts necessary for understanding this study are outlined in this section. Table 1 lists the abbreviations used throughout the work.

Table 1: List of abbreviations and acronyms

Abbreviations	**Acronyms**
Interior	I
Closure	C
Door Space	DS
Membership function	MS function
Non-membership function	NMS function
Fuzzy topological space	FTS
Fuzzy nowhere dense set	$FNDS$
Intuitionistic fuzzy b-set	IF b-set
Intuitionistic fuzzy b-open	IF $b-\mathcal{OS}$
intuitionistic fuzzy b-closed	IF $b-\mathcal{CS}$
Intuitionistic fuzzy b-Door space	IF $b-DS$
Pythagorean fuzzy set	PFS
Pythagorean fuzzy topology	PFT
Pythagorean fuzzy topological space	$PFTS$
Pythagorean fuzzy weighted average	$PFWA$
Pythagorean fuzzy weighted geometric	$PFWG$
Pythagorean fuzzy digital set ($\omega_{\sim}$)	$PFDS(\omega_{\sim})$
Pythagorean fuzzy digital fine	PFD_f
Pythagorean fuzzy digital fine set	PFD_fS
Pythagorean fuzzy digital fine point	PFD_fP
Pythagorean fuzzy digital fine number	PFD_fN
Pythagorean fuzzy digital fine topology	PFD_fT
Pythagorean fuzzy digital fine topological space	PFD_fTS
Pythagorean fuzzy digital fine $b^{\approx}$-topological space	$PFD\mathrm{b}_\mathrm{f}^{\approx}-DS$
Pythagorean fuzzy digital fine $\mathrm{b}^{\approx}$ - open set	$PFD\mathrm{b}_\mathrm{f}^{\approx}-\mathcal{OS}$
Pythagorean fuzzy digital fine $\mathrm{b}^{\approx}$ - closed set	$PFD\mathrm{b}_\mathrm{f}^{\approx}-\mathcal{CS}$
Pythagorean fuzzy digital fine $b^{\approx}$ - closure	$PFDb_f^{\approx}$−C
Pythagorean fuzzy digital fine $b^{\approx}-$ interior	$PFDb_f^{\approx}$−I

Definition 1. *[7] Let ξ be an universal set. Then a PFS ω which is a set of ordered pair over ξ is defined by $\omega = \{< x, \mu_\omega(x), \nu\omega(x) > /x \in \xi\}$ where $\mu_\omega(x) : \xi \to [0,1]$ and $\nu\omega(x) : \xi \to [0,1]$ define the degree of MS and the degree of NMS respectively such that $0 \le \mu^2_\omega(x) + \nu\omega^2(x) \le 1$.*

Definition 2. *[7] Suppose that $\omega = (\mu_\omega, v_\omega)$ and $\varsigma = (\mu_\varsigma, v_\varsigma)$ can be any two PFS of ξ. Then*

- $\omega^{\mathrm{C}} = (v_\omega, \mu_\omega)$ *yields the complement of* ω.
- $\omega \cap \varsigma = \{\min(\mu_\omega, \mu_\varsigma), \max((v_\omega, v_\varsigma)\}$ *yields the intersection of* ω *and* ς.
- $\omega \cup \varsigma = \{\max(\mu_\omega, \mu_\varsigma), \min(v_\omega, v_\varsigma)\ \}$*yields the union of* ω *and* ς.
- *If* $\mu_\omega \le \mu_\varsigma$ *and* $v_\omega \ge v_\varsigma$ *then* ω *is a subset of* $\varsigma(\omega \subseteq \varsigma)$ *or* ς *contains* $\omega(\varsigma \supseteq \omega)$.

Definition 3. *[7] Consider ξ is a set and τ is a family of PFS of ξ.*

- $0_\sim, 1_\sim \in \tau$
- *For any* $\{\omega_{\mathrm{j}}/\mathrm{j} \in \mathrm{I}, \omega_{\mathrm{j}} \in \tau\}, \bigcap_{j=1}^{n} \omega_j \in \tau$
- $\cup_{j=1}\omega_j \in \tau$ *for any* $\{\omega_j/\mathrm{j} \in \mathrm{I}, \omega_{\mathrm{j}} \in \tau\}$, *the notation* τ *is a PFT on* ξ *when I is any arbitrary index set.*

This pair (ξ, τ) is referred to in this context as a PFTS. An open PFS is any member of τ, and the complement of an open PFS is a closed PFS. The topology containing all PFS is called a discrete PFTS.

Definition 4. *[7] In a PFTS (ξ, τ), let ω and ς be any two PFS. If there is an open PFS η in which $\omega \subseteq \eta \subseteq \varsigma$, then ς is regarded as a neighborhood of ω.*

Definition 5. *[6, 13] Let X and Y be points in ξ, the set of integer-coordinate points arranged in a rectangular grid. A path Γ from X to Y is defined as a sequence $X = X_0, X_1, X_2, \ldots, X_n = Y$, such that for each i from 1 to n, the point X_i is adjacent to X_{i-1}, with adjacency being either 4 -adjacent or 8 -adjacent.*

Definition 6. *[6, 13] In the set ξ, points X and Y are said to be connected when a path from X to Y includes all points of a subset P^* of ξ.*

Definition 7. *[14] Consider ξ as a rectangular grid of points with integer coordinates in the Euclidean plane Σ, and let $\omega = (\mu_\omega, v_\omega)$ be a PFS of Σ. The PFDS of ξ, denoted by $\omega\sim$, assigns to each point $\mathrm{X} \in \xi$ a MS degree $\mu_{\omega\sim}(\mathrm{X})$ and a NMS degree $v_{\omega\sim}(\mathrm{X})$ defined as follows: $\mu_{\omega\sim}(X) = \max\{\mu_\omega(S) \mid S \in P^*\}$ and $v_{\omega\sim}(X) = \max\{v_\omega(S) \mid S \in P^*\}$.
Here S is the subset of the plane and P^* is the open unit square has the center X .*

Definition 8. *[8] In a FTS (ξ, τ), a fuzzy set ω is termed as fuzzy dense if $\mathrm{C}(\omega) = 1_\sim$. This implies there is no fuzzy closed set η in (ξ, τ) such that $\omega < \eta < 1$.*

Definition 9. *[8] If there is no non-zero fuzzy open set* ς *in* (ξ, τ) *such that* $\varsigma < \mathrm{C}(\omega)$, *i.e.,* $I(C(\omega)) = 0_{\sim}$, *then a fuzzy set* ω *in a FTS* (ξ, τ) *is said to be a FNDS.*

Definition 10. *[8] Consider the FTS* (ξ, τ). *The fuzzy first category set is a fuzzy set* ω *in* (ξ, τ) *if* $\omega = \bigvee_{n=1}^{\infty} \omega_n$, *where n's are FNDS's in* (ξ, τ) . *In* (ξ, τ), *a fuzzy set that is not a fuzzy first category set is referred to as a fuzzy second category set.*

Definition 11. *[8] The fuzzy first category space is a FTS* (ξ, τ) *where* $1 = \bigvee_{n=1}^{\infty} \omega_n$ *where n's are FNDS's in* (ξ, τ). *A topological space is considered to be of fuzzy second category space if it is not of fuzzy first category space.*

Definition 12. *[8] Assume that the FTS is* (ξ, τ). *If* $I\left(\mathrm{V}_{n=1}^{\infty} \omega_n\right) = 0_{\sim}$, *then* (ξ, τ) *is a fuzzy Baire space. Here* ω_n *'s are FNDS's in* (ξ, τ).

Definition 13. *[8] If every fuzzy first category set in* (ξ, τ) *is a FNDS in* (ξ, τ), *then the FTS* (ξ, τ) *is referred to as a fuzzy D-Baire space.*

Definition 14. *[8] If* $I(\omega)$ *is fuzzy dense in* (ξ, τ) *for any fuzzy dense set* ω *in* (ξ, τ) *with* $I(\omega) \neq 0_{\sim}$ *(the null set), then the FTS* (ξ, τ) *is a fuzzy Quasi-maximal space.*

Definition 15. *[8] Fuzzy Hausdorff spaces are FTS s* (ξ, τ) *if, for every* ω, ς *in* (ξ, τ) *and* $\omega \neq \varsigma$, *we identify the fuzzy open sets* η *and* δ *such that that* $\omega \leq \eta, \varsigma \leq \delta$ *and* $\eta \wedge \delta = 0_{\sim}$.

Definition 16. *[8] Any fuzzy subset of a FTS* (ξ, τ) *that is either fuzzy open or fuzzy closed is referred to as a fuzzy door space.*

Definition 17. *[9, 10] If* $\omega = C(I(\omega)) \cup I(C(\omega))$, *then a subset* ω *of a space* ξ *is b#-open.*

Definition 18. *[11] (i) If* $\omega = (C(I(\omega) \cup (I(C(\omega)))$, *then an IFS* ω *of an IFTS* (ξ, τ) *is IF b-*$\mathcal{OS}$ *and (ii) if* $\omega = (C(I(\omega)) \cap (I(C(\omega)))$, *then an IFS* ω *of an IFTS* (ξ, τ) *is IFb-*$\mathcal{CS}$.

Definition 19. *[15] Let* (ξ, τ) *be a TS and*
$\tau\left(\mathfrak{A}_\alpha\right) = \tau_\alpha = \begin{cases} \mathfrak{G}_\alpha (\neq \xi); \mathfrak{G}_\alpha \cap \mathfrak{A}_\alpha \neq \phi, \text{ for } \mathfrak{A}_\alpha \in \tau \\ \mathfrak{A}_\alpha \neq \phi \text{ for some } \alpha \in J, J \text{ is an index set.} \end{cases}$
And also define $\tau_f = \left\{\phi, \xi, \bigcup_{\alpha \in J} \{\tau_\alpha\}\right\}$.
The fine topological space formed by the topology τ *on* ξ *is denoted by* (ξ, τ, τ_f) *and this collection of* τ_f *subsets of* ξ, *which is known as the fine collections of subsets of* ξ.
Fine open sets in (ξ, τ, τ_f) *are the elements of* τ_f, *while fine closed sets, which are represented by* τ_f^C, *are the complement of fine sets.*

Definition 20. *[12] PFWA and PFWG are two averaging and geometric aggregating operators that have been developed for a set of Pythagorean fuzzy numbers* $\mathfrak{J}_{\mathrm{i}} (1 \leq \mathrm{i} \leq \mathrm{n})$ *as follows*

$$PFWA\left(\mathfrak{J}_1, \mathfrak{J}_2, \mathfrak{J}_3, \ldots\ldots.., \mathfrak{J}_{\mathrm{n}}\right) = \left\langle \sqrt{1 - \prod_{i=1}^{n} \left(1 - \mu_i^2\right)^{\mathfrak{w}_i}}, \prod_{i=1}^{n} \vartheta_i{}^{\mathfrak{w}_i} \right\rangle$$

And

$$PFWG\left(\mathfrak{J}_1, \mathfrak{J}_2, \mathfrak{J}_3, \ldots\ldots\ldots., \mathfrak{J}_{\mathrm{n}}\right) = \left\langle \prod_{i=1}^{n} \mu_i^{\mathfrak{w}_i}, \sqrt{1 - \prod_{i=1}^{n} \left(1 - \vartheta_i^2\right)^{\mathfrak{w}_i}} \right\rangle$$

where $\mathfrak{w} = (\mathfrak{w}_1, \mathfrak{w}_2, \mathfrak{w}_3, \ldots\ldots, \mathfrak{w}_n)^T$ *is the associated normalized weight vector of these Pythagorean fuzzy numbers such that* $\mathfrak{w}_{\mathrm{i}} (\mathrm{i} = 1, 2, 3, \ldots.\mathrm{n}) \in [0, 1]$ *and* $\sum_{i=1}^{n} \mathfrak{w}_i = 1$.

3. Pythagorean Fuzzy Digital Fine $\mathrm{b}^{\approx}$-Door Spaces

This section provides an introduction to PFD_fT and $PFDb_{\mathrm{f}}^{\approx} - DS$, along with a detailed discussion of their properties.

Definition 21. *Let (ξ, τ) be a PFTS.*
Define $\tau(A_\alpha) = \widehat{\tau}_\alpha = \begin{cases} G_\alpha(\neq 1_\sim); G_\alpha \cap A_\alpha \neq 0_\sim, \text{ for } A_\alpha \in \tau \\ A_\alpha \neq 0_\sim \text{ for some } \alpha \in J, J \text{ is an index set.} \end{cases}$
And also define $\widehat{\tau}_f = \left\{0_\sim, 1_\sim, \bigcup_{\alpha \in J} G_\alpha\right\}$. *Here the operator "$\bigcup$" denotes the union of the set of all collections of G_α. The PFD_fS of ξ are therefore denoted by $\hat{\tau}_f$, and the $PFD_{\mathrm{f}}\mathrm{TS}$ produced by the topology τ on ξ is denoted by $(\xi, \tau, \hat{\tau}_f)$.*

All of the elements of ($\xi, \tau, \hat{\tau}_f$) are considered to be PFD fine open sets, and their corresponding complements are considered to be PFD fine closed sets.

Definition 22. *A PFD_fTS (ξ, τ) and its PFD_fS ω are referred to as*

- $PFD\mathrm{b}_{\mathrm{f}}^{\approx} - \mathcal{OS}$ *if* $\omega = ((\mathrm{C}(\mathrm{I}(\omega)) \cup (\mathrm{I}(\mathrm{C}(\omega)))$
- $PFD\mathrm{b}_{\mathrm{f}}^{\approx} - \mathcal{CS}$ *if* $\omega = ((C(I(\omega)) \cap (\mathrm{I}(\mathrm{C}(\omega)))$

Definition 23. *Let ω be any $PFD\mathrm{b}_{\mathrm{f}}^{\approx}$ - set in $PFD_{\mathrm{f}}\mathrm{TS}\,(\xi, \tau, \hat{\tau}_f)$ The $PFDb_f^{\approx}-C$ and $PFDb_f^{\approx}-I$ of ω is given as*
$PFD\ \mathrm{b}_{\mathrm{f}}^{\approx} - \mathrm{C}(\omega) = \cap\{\mathrm{A} : \omega \subseteq \mathrm{A}, \mathrm{A} \text{ is } PFD\mathrm{b}_{\mathrm{f}}^{\approx} - \mathcal{CS} \text{ in } (\xi, \tau, \hat{\tau}_f)\}$
$PFD\mathrm{b}_{\mathrm{f}}^{\approx} - \mathrm{I}(\omega) = \cup\{\mathrm{B} : \mathrm{B} \subseteq \omega, \mathrm{B} \text{ is } PFD\mathrm{b}_{\mathrm{f}}^{\approx} - \mathcal{OS} \text{ in } (\xi, \tau, \hat{\tau}_f)\}$

Definition 24. *A $PFD_{\mathrm{f}}\mathrm{TS}\,(\xi, \tau, \hat{\tau}_f)$ is called as the $PFD\mathrm{b}_{\mathrm{f}}^{\approx} - DS$ if every $PFD\mathrm{b}_{\mathrm{f}}^{\approx}$ set in ($\xi, \tau, \hat{\tau}_f$) is either $PFD\mathrm{b}_{\mathrm{f}}^{\approx}$ open or $PFD\mathrm{b}_{\mathrm{f}}^{\approx}$ closed.*

Example 1. *Consider $\xi = \{l, \mathrm{m}\}$ be a non empty set and $\tau = \{0_\sim, 1_\sim, \xi\}$ is a PFDT on ξ where $0_\sim = \{<0,1>, <0,1>\}$, $1_\sim = \{<1,0>, <1,0>\}$, and $\omega = <x, (l/0.2,\ \mathrm{m}/0.3), (l/0.9,\ \mathrm{m}/0.8)>$. Consider $\mathrm{G} = <x,\ (l/0.2,\ \mathrm{m}/0.1), (l/0.9,\ \mathrm{m}/0.9)>$. Here $\omega \cap \mathrm{G} \neq 0_\sim$. Thus, $(\xi, \tau, \hat{\tau}_f)$ is the $PFD_{\mathrm{f}}\mathrm{TS}$ produced by the topology τ on ξ, and $\widehat{\tau}_f = \{0_\sim, 1_\sim, G\}$ is the PFD_f collections of subsets of ξ. Now $\mathrm{I}(\omega) = \omega; \mathrm{C}(\mathrm{I}(\omega)) = \omega^{\mathrm{c}}$ and $\mathrm{C}(\omega) = \omega^{\mathrm{c}}; \mathrm{I}(\mathrm{C}(\omega)) = \omega$. Then $((\mathrm{C}(\mathrm{I}(\omega) \cup (\mathrm{I}(\mathrm{C}(\omega))) = \omega^{\mathrm{c}}$ and $((\mathrm{C}(\mathrm{I}\,\omega)) \cap (\mathrm{I}(\mathrm{C}(\omega))) = \omega$. So ω is $PFD\mathrm{b}_{\mathrm{f}}^{\approx} - \mathcal{CS}$ and ω is not $PFD\mathrm{b}_{\mathrm{f}}^{\approx} - \mathcal{OS}$. Hence $(\xi, \tau, \hat{\tau}_f)$ is a $PFD\mathrm{b}_{\mathrm{f}}^{\approx} - DS$.*

Definition 25. *Let $(\xi, \tau, \hat{\tau}_f)$ be a $PFD_{\mathrm{f}}\mathrm{TS}$ and let ω be a $PFD\mathrm{b}_{\mathrm{f}}^{\approx}$ set in $(\xi, \tau, \hat{\tau}_f)$. If there is no non-zero $PFD\mathrm{b}_{\mathrm{f}}^{\approx}$-$\mathcal{OS}$ ς in $(\xi, \tau, \hat{\tau}_f)$ such that $\varsigma \subseteq PFD\mathrm{b}_{\mathrm{f}}^{\approx}$-$\mathrm{C}(\omega)$, then ω is a $PFD\mathrm{b}_f^{\approx}-$ nowhere dense set which means that $PFD\mathrm{b}_{\mathrm{f}}^{\approx} - \mathrm{I}\,(\mathrm{C}(\omega)) = 0_\sim$.*

Definition 26. *If ω is a $PFD\mathrm{b}_{\mathrm{f}}^{\approx}$ - nowhere dense set in a $PFD_{\mathrm{f}}\mathrm{TS}\,(\xi, \tau, \hat{\tau}_f)$ then $(\xi, \tau, \hat{\tau}_f)$ is denoted as a $PFD\mathrm{b}_{\mathrm{f}}^{\approx}$-Baire space if $PFD\mathrm{b}_{\mathrm{f}}^{\approx} - \mathrm{I}\,(\cup\omega_\alpha) = 0_\sim$.*

Example 2. *Let $\xi = \{l,\ \mathrm{m}, \mathrm{n}\}$ be a non empty set and $\tau = \{0_\sim, 1_\sim, \mathrm{X}\}$ is a PFDT on ξ where $0_\sim = \{<0,1>, <0,1>, <0,1>\}$, $1_\sim = \{<1,0>, <1,0>, <1,0>\}$, and $\omega = <$*

x, (l/0.6,m/0.6,n/0.6), (l/0.4,m/0.4,n/0.5) >. Consider $\mathrm{G}_1 =<$ x, *(l/0.6,m/0.6,n/0.6), (l/0.5,m/0.5,n/0.5) > and* $\mathrm{G}_2 =<$ x, *(* $l/0.4,\ \mathrm{m}/0.4, \mathrm{n}/0.5$ *), (* $l/0.7,\ \mathrm{m}/0.7, \mathrm{n}/0.6$ *) > Here* $\omega \cap \mathrm{G} \neq 0_{\sim}$. *The* PFD_f *collections of subsets of* ξ *are thus denoted by* $\hat{\tau}_f = \{0_{\sim}, 1_{\sim}, G_1 \cup G_2\}$*, and the* $PFD_{\mathrm{f}}\mathrm{TS}$ *created by the topology* τ *on* ξ *is denoted by* $(\xi, \tau, \hat{\tau}_f)$*. Now* $PFDb_f^{\approx} - I(C(\omega^c)) = 0_{\sim}$ $PFDb_f^{\approx} - I(C(G_2)) = 0_{\sim}$ $PFDb_f^{\approx} - I(C(G_1)) = 1_{\sim}$ *and* $PFDb_f^{\approx} - I(C(\omega)) = 1_{\sim}$*. So here* ω^{c} *and* G_2 *are* $PFD\mathrm{b}_{\mathrm{f}}^{\approx}$*- nowhere dense sets and* ω *and* G_1 *are not* $PFD\mathrm{b}_{\mathrm{f}}^{\approx}$ *- nowhere dense sets. Now* $PFDb_{\mathrm{f}}^{\approx} - \mathrm{I}\,(\omega^{\mathrm{c}} \cup \mathrm{G}_2) = 0_{\sim}$*. Hence* $(\xi, \tau, \hat{\tau}_f)$ *is a PFD* $\mathrm{b}_{\mathrm{f}}^{\approx}$*-Baire space.*

Definition 27. *Let* $(\xi, \tau, \hat{\tau}_f)$ *is a* $PFD_{\mathrm{f}}\mathrm{TS}$*, then* $(\xi, \tau, \hat{\tau}_f)$ *is called a* $PFD\mathrm{b}_f^{\approx}$*-submaximal space if every* $PFD\mathrm{b}_{\mathrm{f}}^{\approx}$ *- dense set is a* $PFD\mathrm{b}_{\mathrm{f}}^{\approx} - \mathcal{OS}$ *in* $(\xi, \tau, \hat{\tau}_f)$

Definition 28. *A* $PFD\mathrm{b}_{\mathrm{f}}^{\approx}$ *set* ω *in a* $PFD_{\mathrm{f}}\mathrm{TS}\,(\xi, \tau, \hat{\tau}_f)$ *is said to be a* $PFD\mathrm{b}_{\mathrm{f}}^{\approx}$*-first category if* $(\cup_{n=1}^{\infty} \omega_n) = \omega$*. Here* ω_{n} *implies that the* $PFD\mathrm{b}_{\mathrm{f}}^{\approx}$ *- nowhere dense set in* $(\xi, \tau, \hat{\tau}_f)$*.*
A $PFD\mathrm{b}_{\mathrm{f}}^{\approx}$ *set* ω *in a* $PFD_{\mathrm{f}}\mathrm{TS}\,(\xi, \tau, \hat{\tau}_f)$ *which is not of the* $PFD\mathrm{b}_{\mathrm{f}}^{\approx}$*-first category, is said to be of the* $PFD\mathrm{b}_{\mathrm{f}}^{\approx}$*-second category.*

Definition 29. A$PFD_{\mathrm{f}}\mathrm{TS}\,(\xi, \tau, \hat{\tau}_f)$ *is said to be a PFD* $\mathrm{b}_{\mathrm{f}}^{\approx}$ *-first category space if* $(\cup_{n=1}^{\infty} \omega_n) = 1_{\sim}$*. Here* ω_{n} *implies that the* $PFD\mathrm{b}_{\mathrm{f}}^{\approx}-$ *nowhere dense set in* $(\xi, \tau, \hat{\tau}_f)$

If $PFD_{\mathrm{f}}\mathrm{TS}\,(\xi, \tau, \hat{\tau}_f)$ *is not the* $PFD\mathrm{b}_{\mathrm{f}}^{\approx}$*-first category, then* $PFD_{\mathrm{f}}\mathrm{TS}\,(\xi, \tau, \hat{\tau}_f)$ *is the* $PFD\mathrm{b}_{\mathrm{f}}^{\approx}-$ *second category space.*

Proposition 1. *Every* $PFD\mathrm{b}_{\mathrm{f}}^{\approx} - DS\,(\xi, \tau, \hat{\tau}_f)$ *is a* $PFD\mathrm{b}_{\mathrm{f}}^{\approx}$*-submaximal space.*

Proof. Consider a $\mathrm{b}_{\mathrm{f}}^{\approx}$-dense set in ($\xi, \tau, \hat{\tau}_f$), denoted by $\omega \subseteq 1_{\sim}$. If ω is not $PFD\mathrm{b}_{\mathrm{f}}^{\approx}$-$\mathcal{OS}$ then ω is a $PFDb_{\mathrm{f}}^{\approx} - \mathcal{CS}$ since $(\xi, \tau, \hat{\tau}_f)$ is a $PFDb_{\mathrm{f}}^{\approx} - DS$. Then $\omega = \bar{\omega} = 1_{\sim}$ and ω is a $PFDb_{\mathrm{f}}^{\approx} - \mathcal{OS}\left(PFDb_{\mathrm{f}}^{\approx} - \mathcal{CS}\right)$. Therefore ($\xi, \tau, \hat{\tau}_f$) is a $PFDb_{\mathrm{f}}^{\approx}$-submaximal space. Since not all $PFDb_{\mathrm{f}}^{\approx} - \mathcal{OS}$ are dense, with the exception of $1_{\sim}$, the theorem's counterpart need not be true.

Proposition 2. $PFDb_{\mathrm{f}}^{\approx} - \mathcal{CS}$ *in (* $\xi, \tau, \hat{\tau}_f$ *) with* $PFDb_{\mathrm{f}}^{\approx} - \mathrm{I}(\omega) = 0_{\sim}$ *implies that* ω *is a* $PFD\mathrm{b}_{\mathrm{f}}^{\approx}$ *- nowhere dense set in (* $\xi, \tau, \hat{\tau}_f$ *).*

Proof. If ω is a $PFDb_{\mathrm{f}}^{\approx} - \mathcal{CS}$ in ($\xi, \tau, \hat{\tau}_f$) with $PFDb_{\mathrm{f}}^{\approx} - \mathrm{I}(\omega) = 0_{\sim}$, in ($\xi, \tau, \hat{\tau}_f$). Then, $PFD\mathrm{b}_{\mathrm{f}}^{\approx} - \mathrm{C}(\omega) = \omega$. So $PFD\mathrm{b}_{\mathrm{f}}^{\approx} - \mathrm{I}\,(\mathrm{C}(\omega)) = PFD\mathrm{b}_{\mathrm{f}}^{\approx} - \mathrm{I}(\omega) = 0_{\sim}$, in $(\xi, \tau, \hat{\tau}_f)$. Hence ω is a $PFD\mathrm{b}_{\mathrm{f}}^{\approx}$ - nowhere dense set in ($\xi, \tau, \hat{\tau}_f$). The converse of the theorem does not necessarily hold.

Proposition 3. *Every PFD subspace of a* $PFD\mathrm{b}_{\mathrm{f}}^{\approx} - DS\,(\xi, \tau, \hat{\tau}_f, \psi)$ *is a* $PFD\mathrm{b}_{\mathrm{f}}^{\approx} - DS$*.*

Proof. If ($G, \tau, \hat{\tau}_f, \psi_G$) is a subspace of a PFD $\mathrm{b}_{\mathrm{f}}^{\approx} - DS\,(\xi, \tau, \hat{\tau}_f, \psi)$, implies $\mathrm{G} \subset \xi$. Let $\omega \subset \mathrm{G}$, since ($\xi, \tau, \hat{\tau}_f, \psi$) is a $PFD\mathrm{b}_{\mathrm{f}}^{\approx} - DS$, then ω is either $PFD\mathrm{b}_{\mathrm{f}}^{\approx} - \mathcal{OS}$ or $PFD\mathrm{b}_{\mathrm{f}}^{\approx} - \mathcal{CS}$ in $(\xi, \tau, \hat{\tau}_f, \psi)$ and hence in ($G, \tau, \hat{\tau}_f, \psi_G$). Therefore, ($G, \tau, \hat{\tau}_f, \psi_G$) is a $PFD\mathrm{b}_{\mathrm{f}}^{\approx} - DS$.

Definition 30. *A PFD* $_{\mathrm{f}}\mathrm{TS}\,(\xi,\tau,\hat{\tau}_f)$ *is considered a PFD* $\mathrm{b}_{\mathrm{f}}^{\approx}$D*-Baire space when every* $PFD\mathrm{b}_{\mathrm{f}}^{\approx}$*-first category set* ω *in (* $\xi,\tau,\hat{\tau}_f$ *) satisfies the condition* $PFD\mathrm{b}_{\mathrm{f}}^{\approx}$ - $\mathrm{I}(\mathrm{C}(\omega)) = 0_{\sim}$*; that is, all such sets are* $PFD\mathrm{b}_{\mathrm{f}}^{\approx}$ *- nowhere dense set in* $(\xi,\tau,\hat{\tau}_f)$.

Proposition 4. *Every* $PFD\mathrm{b}_{\mathrm{f}}^{\approx}$ *- Baire space in (* $\xi,\tau,\hat{\tau}_f$ *) is a* $PFD\mathrm{b}_{\mathrm{f}}^{\approx}$ *D-Baire space.*

Proof. Suppose that ω is a $PFD\mathrm{b}_{\mathrm{f}}^{\approx}$ - first category set in a $PFD\mathrm{b}_{\mathrm{f}}^{\approx}$ D-Baire space ($\xi,\tau,\hat{\tau}_f$). Then $\omega = (\cup_{n=1}^{\infty}\omega_n) = 0_{\sim}$ Here ω and ω_n are $PFD\mathrm{b}_{\mathrm{f}}^{\approx}$-nowhere dense sets in $(\xi,\tau,\hat{\tau}_f)$. So, $PFDb_{\mathrm{f}}^{\approx} - \mathrm{I}(C(\omega)) = 0_{\sim}$, and $PFDb_{\mathrm{f}}^{\approx} - \mathrm{I}(\omega) \leq PFDb_{\mathrm{f}}^{\approx} - \mathrm{I}(\mathrm{C}(\omega))$ gives that PFD $b_{\mathrm{f}}^{\approx} - \mathrm{I}(\omega\ ) = 0_{\sim}$. Hence $PFD\mathrm{b}_{\mathrm{f}}^{\approx} - \mathrm{I}\,(\cup_{n=1}^{\infty}\omega_n) = 0_{\sim}$ where ω_{n} implies $PFD\mathrm{b}_{\mathrm{f}}^{\approx}$ - nowhere dense sets in ($\xi,\tau,\hat{\tau}_f$). Hence ($\xi,\tau,\hat{\tau}_f$) is a $PFD\mathrm{b}_{\mathrm{f}}^{\approx}$ - Baire space.

Proposition 5. *Suppose that (* $\xi,\tau,\hat{\tau}_f$ *) be a* $PFD\mathrm{b}_{\mathrm{f}}^{\approx}$*-first category space. Therefore, (* $\xi,\tau,\hat{\tau}_f$ *) is not a* $PFDb_{\mathrm{f}}^{\approx}$ D*-Baire space.*

Proof. If $(\xi,\tau,\hat{\tau}_f)$ is a $PFD\mathrm{b}_{\mathrm{f}}^{\approx}$-first category space. Hence $(\cup_{n=1}^{\infty}\omega_n) = 1_{\sim}$. Here ω_{n} implies $PFD\mathrm{b}_{\mathrm{f}}^{\approx}$ -nowhere dense sets in $(\xi,\tau,\hat{\tau}_f)$. We obtain $PFD\mathrm{b}_{\mathrm{f}}^{\approx}$-$I\ (C\ (1_{\sim})) = 0_{\sim}$ for the $PFD\mathrm{b}_{\mathrm{f}}^{\approx}$-first category set $1_{\sim}$. So, $(\xi,\tau,\hat{\tau}_f)$ is not a PFD $b_{\mathrm{f}}^{\approx}$D-Baire space.

Definition 31. *If every* PFD_{f} *has two disjoint points that may be separated by* $PFDb_{\mathrm{f}}^{\approx}$ *disjoint open sets, then a* $PFD_{\mathrm{f}}\mathrm{TS}\,(\xi,\tau,\hat{\tau}_f,\psi)$ *is a* $PFD\mathrm{b}_{\mathrm{f}}^{\approx}$*-Hausdorff.*

Lemma 1. *Given* $\mathrm{K} = \mathrm{x}(\mathrm{p},\mathrm{q})$ *is a* PFD_fP *in* $(E,\tau,\hat{\tau}_f,\psi)$*,* $\{\mathrm{K}\}$ *will only be considered a* $PFD\mathrm{b}_{\mathrm{f}}^{\approx} - \mathcal{OS}$ *if it is a* PFD_{f} *open set.*

Lemma 2. *Consider two* $PFD\mathrm{b}_{\mathrm{f}}^{\approx}$ *sets F and G in* $(\xi,\tau,\hat{\tau}_f,\psi)$*. Then* $F\cap\mathrm{G}$ *is a* $PFD\mathrm{b}_{\mathrm{f}}^{\approx}$*-* $\mathcal{OS}$ *in* $(\xi,\tau,\hat{\tau}_f,\psi)$ *if F is a* $PFDb_{\mathrm{f}}^{\approx} - \mathcal{OS}$ *and F is a* PFD_{f} *open set.*

Proposition 6. *Only one limit point exists for a* $PFD\mathrm{b}_{\mathrm{f}}^{\approx}$*-Hausdorff DS (* $\xi,\tau,\hat{\tau}_f$ *).*

Proof. Consider two $PFD_{\mathrm{f}}Ps$ in $(\xi,\tau,\hat{\tau}_f)\,,\mathrm{K} = \mathrm{x}_1(\mathrm{p},\mathrm{q})$ and $\mathrm{L} = \mathrm{x}_2(\mathrm{u},\mathrm{v})$. Since $(\xi,\tau,\hat{\tau}_f)$ is a $PFD\mathrm{b}_{\mathrm{f}}^{\approx}$-Hausdorff, two $PFD\mathrm{b}_{\mathrm{f}}^{\approx}-\mathcal{OS}$ exist, F and G , such that $\mathrm{K}\in\mathrm{F},\mathrm{L}\in\mathrm{G}$ and $\mathrm{F}\cap\mathrm{G} = 0_{\sim}$. Since $(\xi,\tau,\hat{\tau}_f,\psi)$ is a $PFD\mathrm{b}_{\mathrm{f}}^{\approx}-DS$, $\omega = (\mathrm{F}\backslash\mathrm{K})\cup\mathrm{L}$ is either $PFD\mathrm{b}_{\mathrm{f}}^{\approx}-\mathcal{OS}$ or $PFD\mathrm{b}_{\mathrm{f}}^{\approx} - \mathcal{CS}$. Then, according to Lemma 1, $\omega\cap\mathrm{G} = \{\mathrm{L}\}$ is $PFD\mathrm{b}_{\mathrm{f}}^{\approx} - \mathcal{OS}$ in the preceding case, and according to Lemma 2, it is PFD $\mathrm{b}_{\mathrm{f}}^{\approx} - \mathcal{OS}$. In the latter instance, ω^{c} is $PFD\mathrm{b}_{\mathrm{f}}^{\approx} - \mathcal{OS}$; hence, $\omega^{\mathrm{c}}\cap\mathrm{F} = \{\mathrm{K}\}$ is $PFD\mathrm{b}_{\mathrm{f}}^{\approx} - \mathcal{OS}$ and consequently, PFD_{f} open set. In the second case, ω^{c} is PFD $\mathrm{b}_{\mathrm{f}}^{\approx} - \mathcal{OS}$; thus, $\omega^{\mathrm{c}}\cap\mathrm{F} = \{\mathrm{K}\}$ is $PFD\mathrm{b}_{\mathrm{f}}^{\approx} - \mathcal{OS}$ and hence PFD_{f} open set. At least one of the two $PFD_{\mathrm{f}}Ps$ in $(\xi,\tau,\hat{\tau}_f)$ is therefore an isolated $PFD_{\mathrm{f}}P$ in both situations, and this is demonstrated by contradiction.

Definition 32. A$PFD_{\mathrm{f}}\mathrm{TS}\,(\xi,\tau,\hat{\tau}_f,\psi)$ *is a* $PFD\mathrm{b}_{\mathrm{f}}^{\approx} - \mathrm{T}_{1/2}$ *space if every* $PFD_{\mathrm{f}}P$ $\{\mathrm{k}\}$ *is either an* $PFD\mathrm{b}_{\mathrm{f}}^{\approx} - \mathcal{OS}$ *or* $PFD\mathrm{b}_{\mathrm{f}}^{\approx} - \mathcal{CS}$ *in* $(\xi,\tau,\hat{\tau}_f)$.

Proposition 7. *Every* $PFD_{\mathrm{f}}\mathrm{TS}\,(\xi,\tau,\hat{\tau}_f,\psi)$ *is a* $PFD\mathrm{b}_{\mathrm{f}}^{\approx} - \mathrm{T}_{1/2}$ *space.*

Proof. Suppose that $\{K\}$ is a PFD_fP in $(\xi, \tau, \hat{\tau}_f)$. Given that all PFD_f-nowhere dense sets are $PFD\text{b}_f^{\approx} - \mathcal{CS}$, $\{K\}$ is either a $PFD\text{b}_f^{\approx} - \mathcal{OS}$ or a PFD_f-nowhere dense set. Therefore, $\{K\}$ is either $PFD\text{b}_f^{\approx} - \mathcal{OS}$ or $PFD\text{b}_f^{\approx} - \mathcal{CS}$. Hence $(\xi, \tau, \hat{\tau}_f)$ is a $PFD\text{b}_f^{\approx} - T_{1/2}$ space.

Proposition 8. *Let* $(\xi, \tau, \hat{\tau}_f, \psi)$ *be a* PFD_fTS*, The statements listed below are logically equivalent.*
(i) $(\xi, \tau, \hat{\tau}_f, \psi)$ *is a* PFD_f*-discrete space. (ii)* $(\xi, \tau, \hat{\tau}_f, \psi)$ *is a* $PFDb_f^{\approx} - DS$*.*

Proof. (i) ⇒ (ii) Assume that $(\xi, \tau, \hat{\tau}_f, \psi)$ is a PFD_f discrete, then every PFD_fS is a PFD_f open or PFD_f closed, and then $PFD\ b_f^{\approx} - \mathcal{OS}$ or $PFDb_f^{\approx} - \mathcal{CS}$. Hence $(\xi, \tau, \hat{\tau}_f, \psi)$ is a $PFDb_f^{\approx} - DS$.
(ii) ⇒ (i) If $K = x(p, q)$ is a PFD_fP in $(\xi, \tau, \hat{\tau}_f)$. Consequently, K is $PFD\text{b}_f^{\approx} - \mathcal{CS}$ if $\{K\}$ is not $PFDb_f^{\approx} - \mathcal{OS}$ and $\{K\} = (PFDb_f^{\approx} - C(I(K)) \cap (PFDb_f^{\approx} - I(C(K))$. Since $PFDb_f^{\approx} - I(K)$ is not empty and $\{K\} = \left(PFDb_f^{\approx} - C(K)\right)$,
then $\{K\} = \left(PFD\text{b}_f^{\approx} - C(I(K)) \cap \left(PFD\text{b}_f^{\approx} - I(K)\right) = PFD\text{b}_f^{\approx} - I(\{K\})$. In both cases, $\{K\}$ is a PFD_f open set. Therefore, $(\xi, \tau, \hat{\tau}_f, \psi)$ is a PFD_f-discrete space.

Definition 33. *A* $PFD\text{b}_f^{\approx}$ TS $(\xi, \tau, \hat{\tau}_f, \psi)$ *is irreducible if every* $PFD\text{b}_f^{\approx}$ *set in* $(\xi, \tau, \hat{\tau}_f, \psi)$ *is* $PFD\text{b}_f^{\approx}$ *- connected, equivalent to every non-void* $PFD\text{b}_f^{\approx} - \mathcal{OS}$ *in* $(\xi, \tau, \hat{\tau}_f, \psi)$ *being dense.*

Proposition 9. *Every* PFD_f *irreducible submaximal space* $(\xi, \tau, \hat{\tau}_f, \psi)$ *is a* $PFD\text{b}_f^{\approx} - DS$*.*

Proof. Assume that ω be a $PFD\text{b}_f^{\approx}$ set in $(\xi, \tau, \hat{\tau}_f, \psi)$. Suppose ω is $PFD\text{b}_f^{\approx}$-dense, then because $(\xi, \tau, \hat{\tau}_f, \psi)$ is $PFD\text{b}_f^{\approx}$-submaximal, ω is $PFD\text{b}_f^{\approx}$-$\mathcal{OS}$, if ω is not $PFD\text{b}_f^{\approx}$-dense, so a non-void $PFD\text{b}_f^{\approx}$-$\mathcal{OS}$ ς and ω^c can be found. Since $(\xi, \tau, \hat{\tau}_f, \psi)$ is irreducible, ς and ω^c are PFD $b_f^{\approx}$ -dense. In addition, since $(\xi, \tau, \hat{\tau}_f, \psi)$ is PFD $b_f^{\approx}$-submaximal, ω^c is $PFD\text{b}_f^{\approx}$ -$\mathcal{OS}$ or equivalently, ω is $PFD\text{b}_f^{\approx} - \mathcal{CS}$. Thus, in any case, ω is either $PFD\text{b}_f^{\approx} - \mathcal{OS}$ or $PFD\text{b}_f^{\approx} - \mathcal{CS}$. Therefore, $(\xi, \tau, \hat{\tau}_f, \psi)$ is a $PFD\text{b}_f^{\approx} - DS$.

Definition 34. *An* PFD_fTS $(\xi, \tau, \hat{\tau}_f)$ *is called* $PFDb_f^{\approx}$*-extremely disconnected if the* $PFDb_f^{\approx} - C$ *of an* $PFDb_f^{\approx}$*-*$\mathcal{OS}$ *is a* $PFDb_f^{\approx} - \mathcal{OS}$*.*

4. Pythagorean Fuzzy Digital Fine Functions in $b_f^{\approx}$-Door Spaces

This section explores the notion of $PFD\text{b}_f^{\approx}$-quasi compactness and discusses its key properties.

Definition 35. *Let* $\sigma : (\xi, \psi) \to (G, \eta)$ *be a* PFD_f *function. Then,* σ *is* PFD_f *-quasi compact if* $\omega \subseteq 1_\sim$ *is* $PFD\text{b}_f^{\approx} - \mathcal{OS}$ *in* (ξ, ψ) *such that* $\sigma^{-1}(\sigma(\omega)) = \omega$*. Then,* $\sigma(\omega)$ *is* $PFD\text{b}_f^{\approx} - \mathcal{OS}$ *in* (G, η)*.*

Proposition 10. *A* $PFD\text{b}_f^{\approx} - DS$ *has a topological property.*

Proof. Let $\sigma : (\xi, \psi) \to (\mathrm{G}, \eta)$ be an $PFD\mathrm{b}_{\mathrm{f}}^{\approx}$-homeomorphism from a $PFD\mathrm{b}_{\mathrm{f}}^{\approx} - DS$ (ξ, ψ) into another $PFD\mathrm{b}_{\mathrm{f}}^{\approx}$TS (G, η). Let ω be $PFD\mathrm{b}_{\mathrm{f}}^{\approx}S$ in (G, η), that is, $\omega \subseteq \mathrm{G}$. Then, $\sigma^{-1}(\omega\ )$ is $PFD\mathrm{b}_{\mathrm{f}}^{\approx}S$in (ξ, ψ), that is, $\sigma^{-1}(\omega) \subseteq \xi$. Because (ξ, ψ) is an $PFD\mathrm{b}_{\mathrm{f}}^{\approx} - DS$, $\sigma^{-1}(\omega)$ is a $PFD\mathrm{b}_{\mathrm{f}}^{\approx}$ - $\mathcal{OS}$ or a $PFD\mathrm{b}_{\mathrm{f}}^{\approx} - \mathcal{CS}$ in (ξ, ψ), because $\sigma\left(\left(\sigma^{-1}(\omega)\right) = \omega\right.$. Then, ω is either $PFD\mathrm{b}_{\mathrm{f}}^{\approx}- \mathcal{OS}$ or $PFDb_{\mathrm{f}}^{\approx} - CS$ in (G, η). Thus, (G, η) is the $PFDb_{\mathrm{f}}^{\approx} - DS$.

Proposition 11. *A PFD_f-quasi-compact image of an $PFDb_{\mathrm{f}}^{\approx} - DS$ is an $PFDb_{\mathrm{f}}^{\approx} - DS$.*

Proof. Let $\sigma : (\xi, \psi) \to (\mathrm{G}, \eta)$ be a $PFD\mathrm{b}_{\mathrm{f}}^{\approx}$-quasi compact function from a $PFDb_{\mathrm{f}}^{\approx} - DS$ (ξ, ψ) into another PFD $\mathrm{b}_{\mathrm{f}}^{\approx}$TS (G, η). Let ς be $PFD\mathrm{b}_{\mathrm{f}}^{\approx}S$ in (G, η). We have to prove that ς is either $PFD\mathrm{b}_{\mathrm{f}}^{\approx} - \mathcal{OS}$ or $PFDb_{\mathrm{f}}^{\approx} - \mathcal{CS}$ in (G, η). Because (ξ, ψ) is an $PFDb_{\mathrm{f}}^{\approx} - DS$ and $\sigma^{-1}(\xi) = \omega$, then ω is either $PFDb_{\mathrm{f}}^{\approx} - \mathcal{OS}$ or $PFDb_{\mathrm{f}}^{\approx} - \mathcal{CS}$ in (ξ, ψ). Suppose that ω is $PFDb_{\mathrm{f}}^{\approx} - \mathcal{OS}$, and clearly $\sigma(\omega) \subseteq \varsigma$. Thus, $\sigma^{-1}(\sigma(\omega)) \subseteq \sigma^{-1}(\varsigma) = \omega \subseteq \sigma^{-1}(\sigma(\omega))$ or equivalently, $\omega = \sigma^{-1}(\sigma(\omega))$. By assumption, $\sigma(\omega) = \sigma\left(\left(\sigma^{-1}(\varsigma)\right) = \varsigma \cap \sigma(1_{\sim}) = \varsigma \cap 1_{\sim} = \varsigma\right.$ is $PFD\ b_{\mathrm{f}}^{\approx} - \mathcal{OS}$ in (G, η). We now assume that ω is $PFDb_{\mathrm{f}}^{\approx} - \mathcal{CS}$ in (ξ, ψ). Then, $1_{\sim} \backslash \sigma^{-1}(\zeta) = \sigma^{-1}(1_{\sim} \backslash \zeta)$ is $PFDb_{\mathrm{f}}^{\approx} - \mathcal{OS}$ in (ξ, ψ). Hence, $(1_{\sim} \backslash \zeta) \cap \sigma(1_{\sim}) = 1_{\sim} \backslash \zeta$ is $PFD\mathrm{b}_{\mathrm{f}}^{\approx}$ -$\mathcal{OS}$ in (G, η); thus, $1_{\sim} \backslash (1_{\sim} \backslash \zeta) = \zeta$ *is* $PFDb_{\mathrm{f}}^{\approx}$ -$\mathcal{CS}$ (G, η). Therefore, (G, η) is the $PFDb_{\mathrm{f}}^{\approx} - DS$.

Corollary 1. $PFD\mathrm{b}_{\mathrm{f}}^{\approx}$*-open images as well as* $PFD\mathrm{b}_{\mathrm{f}}^{\approx}$*-closed images of a* $PFD\mathrm{b}_{\mathrm{f}}^{\approx} - DS$ *are* $PFD\mathrm{b}_{\mathrm{f}}^{\approx} - DS$.

Proof. Since every $PFD\mathrm{b}_{\mathrm{f}}^{\approx}$-$\mathcal{OS}$ (resp, $PFD\mathrm{b}_{\mathrm{f}}^{\approx}$-$\mathcal{CS}$) surjective function is $PFD\mathrm{b}_{\mathrm{f}}^{\approx}$-quasicompact, it is an $PFD\mathrm{b}_{\mathrm{f}}^{\approx} - DS$.

5. Selection of best menstrual hygiene product for Rural Women

Selecting the best menstrual hygiene product for rural women serves multiple vital purposes that go beyond basic hygiene. First of all, it improves health by lowering the risk of diseases including recurrent and urinary tract infections, which are frequently brought on by the use of unsanitary substitutes like ash, sand, or old fabric. A suitable product also enhances a woman's comfort and confidence, helping her manage menstruation with dignity, free from embarrassment or fear of leaks and odor. Importantly, access to proper menstrual hygiene products enables girls to continue attending school and women to participate fully in work and community life, thereby supporting their educational and economic growth. Moreover, choosing the right product contributes significantly to women's empowerment and gender equality by encouraging open conversations about menstruation and helping women take control of their health. Environmentally, sustainable options such as reusable or biodegradable products help reduce plastic waste, which is especially crucial in rural areas lacking proper waste disposal systems. Economically, reusable products offer long-term cost savings, reducing reliance on continuous external aid. Additionally, promoting locally made menstrual products creates livelihood opportunities for women through community-based production, further boosting rural economies. Lastly, involving communities in menstrual health initiatives raises awareness, dispels myths, and fosters a more supportive environment for women's health and well-being.

In this regard, the current study explores the possibilities and attempts to obtain a meaningful solution using PFD_f collection sets and the PFD_fTS. The structural and topological features of this proposed space are leveraged to establish a clear and coherent framework that aligns with theoretical rigor. This application illustrates how the construction and conditions of the PFD_fTS can facilitate the systematic organization and refinement of the selection process. By anchoring the approach in a robust mathematical foundation, the study highlights the practical relevance and real-world applicability of the theoretical model, particularly in addressing the needs of rural women. Beyond the theoretical framework, the study explores the practical implementation of the PFD_fTS and its associated fine collection sets. It demonstrates the critical role this space plays in identifying the most suitable menstrual hygiene product for rural women. By utilizing defined parameters, theoretical structures, and specific conditions, this approach enhances the effectiveness and precision of the selection process, as detailed in the following subsections.

5.1. Algorithm of Selecting the best menstrual hygiene product using Pythagorean fuzzy digital fine topological space

Step-1: Let $\xi = \{\mathrm{a}\}$ be a non empty set and $\tau = \{0_\sim, 1_\sim, \omega\}$ is a PFD topology on ξ.

Step-2: Define $\hat{\tau}_f = \left\{0_\sim, 1_\sim, \bigcup_{\alpha \in J} G_\alpha\right\}$. Here the operator "$\bigcup$ "denotes the union of the set of all fine collections of G_α. Then $\hat{\tau}_f$ is said to be the PFD_fS of ξ.

Step-3: Construct the PFD_fTS ($\xi, \tau, \hat{\tau}_f$) generated by the topology τ on ξ.

Step-4: Assume the weight vector $\mathfrak{w} = (\mathfrak{w}_1, \mathfrak{w}_2, \mathfrak{w}_3, \ldots\ldots, \mathfrak{w}_\mathrm{n})^\mathrm{T}$ the PFD_fNs such that $\mathfrak{w}_\mathrm{i}(\mathrm{i} = 1, 2, 3, \ldots \mathrm{n}) \in [0, 1]$ and $\sum_{i=1}^{n} \mathfrak{w}_i = 1$ corresponding to the six criteria affordability and accessibility, cultural acceptance and awareness, hygiene and health Impact, product availability and supply consistency, comfort and ease of use, and disposal and waste management.

Step-5: Calculate the score function using $PFWA$ operator formula $PFWA$ $(\mathfrak{J}_1, \mathfrak{J}_2, \mathfrak{J}_3, \ldots \mathfrak{J}_\mathrm{n}) = \left\langle \sqrt{1 - \prod_{i=1}^{n} \left(1 - \mu_i^2\right)^{\mathfrak{w}_i}}, \prod_{i=1}^{n} \vartheta_i^{\mathfrak{w}_i} \right\rangle$, $PFWG$ operator formula $PFWG$ ($\mathfrak{J}_1, \mathfrak{J}_2, \mathfrak{J}_3, \ldots, \mathfrak{J}_\mathrm{n}) = \left\langle \prod_{i=1}^{n} \mu_i^{\mathfrak{w}_i}, \sqrt{1 - \prod_{i=1}^{n} \left(1 - \vartheta_i^2\right)^{\mathfrak{w}_i}} \right\rangle$ and select the best menstrual hygiene product.

5.2. Proposed study to identify the best menstrual hygiene product for Rural Women using Pythagorean fuzzy digital fine topological space

In order to make this process, consider three different menstrual hygiene products as cloth pads, menstrual cups and sanitary pads say $\mathrm{H} = \{\mathrm{H}_1, \mathrm{H}_2, \mathrm{H}_3\}$ and the six factors $\mathfrak{C} = \{\mathfrak{C}_1, \mathfrak{C}_2, \mathfrak{C}_3, \mathfrak{C}_4, \mathfrak{C}_5, \mathfrak{C}_6\}$. The selected evaluation criteria affordability and accessibility, cultural acceptance and awareness, hygiene and health Impact, product availability and supply consistency, comfort and ease of use, and disposal and waste management serve as comprehensive and practical dimensions for assessing menstrual hygiene products, particularly in rural and under-resourced settings. These parameters collectively address essential aspects such as economic feasibility, sociocultural acceptability, health considerations, supply reliability, user experience, and environmental impact. Affordability and accessibility reflect the importance of cost and product reach. Cultural acceptance and awareness consider societal norms and awareness levels that influence product adoption. Hygiene and health impact captures the sanitary and safety implications of use. Product availability and supply consistency ensures the reliability of access over time. Comfort and ease of use examines user-friendliness and physical comfort. Disposal and waste management accounts for environmental concerns and the practical aspects of waste handling. Together, these parameters offer a balanced and realistic framework for decision-making.

In this study, both membership and non-membership values are assigned to each product based on these six criteria. These values are subjectively assigned by the authors based on an analysis of common product attributes, user preferences, and practical challenges encountered in everyday life. For instance, cloth pads are considered economically affordable and culturally acceptable, which leads to higher membership and lower non-membership values in those categories. However, they score lower in terms of hygiene and availability. Menstrual cups exhibit high membership values in hygiene and long-term affordability but face lower cultural acceptance and usability, resulting in higher non-membership in those areas. Sanitary pads generally show moderate performance across most criteria but receive low scores in waste management due to their environmental impact. These values conform to the PFS conditions and serve as input for evaluating the suitability of each product using $PFWA$ and $PFWG$ operators in the proposed model.

The following are the analyses performed to discover the best alternative among the possible ones utilising the $PFWA$ and $PFWG$ operators stated in definition 20.

Step 1: Let $\xi = \{\text{a}\}$ be a non empty set and $\tau = \{0_\sim, 1_\sim, \omega\}$ is a PFD topology on ξ where $0_\sim = \{(0,1)\}, 1_\sim = \{(1,0)\}$ and $\omega = \{(0.21, 0.97)\}$ where ω is a PFD open set on ξ.

Step 2: Define $\tau(\omega) = \widehat{\tau}_\alpha = \begin{cases} \mathfrak{G}_\alpha\,(\neq 1_\sim)\,;\mathfrak{G}_\alpha \cap \omega \neq 0_\sim, \text{ for } \omega \in \tau \\ \omega \neq 0_\sim \text{ for some } \alpha \in J, J \text{ is an index set.} \end{cases}$

- Consider $\mathfrak{G}_1 = \{(0.34, 0.84)\}, \mathfrak{G}_2 = \{(0.38, 0.72)\}, \mathfrak{G}_3 = \{(0.46, 0.77)\}, \mathfrak{G}_4 = \{(0.71, 0.42)\}, \mathfrak{G}_5 = \{(0.91, 0.26)\}, \mathfrak{G}_6 = \{(0.28, 0.83)\}, \mathfrak{G}_7 = \{\ (0.44, 0.85)\}, \mathfrak{G}_8 = \{(0.61, 0.66)\}, \mathfrak{G}_9 = \{(0.75, 0.72)\}, \mathfrak{G}_{10} = \{\ (0.54, 0.77)\}, \mathfrak{G}_{11} = \{(0.49, 0.63)\}, \mathfrak{G}_{12} = \{(0.58, 0.61)\}, G_{13} = \{\ (0.67, 0.51)\}, \mathfrak{G}_{14} = \{(0.62, 0.75)\}, \mathfrak{G}_{15} = \{(0.73, 0.56)\}$ be the $PFDS$'s.
- So the PFD_f collections are $\mathfrak{G}_\alpha = \{(0.34, 0.84)$, (0.38,0.72), (0.46,0.77), (0.71,0.42), (0.91,0.26), (0.28,0.83), (0.44,0.85), (0.61,0.66), (0.75,0.72), (0.54,0.77), (0.49,0.63), $(0.58, 0.61), (0.67, 0.51), (0.62, 0.75), (0.73, 0.56)\}$ which satisfies the condition $\mathfrak{G}_\alpha \cap \omega \neq 0_\sim$.

- Hence $\hat{\tau}_f = \{0_\sim, 1_\sim, \mathfrak{G}_\alpha\}$ is called the PFD_f collections of subsets of ξ and ($\xi, \tau, \hat{\tau}_f$) is said to be the PFD_fTS generated by the topology τ on ξ.

Step 3: All the PFD_f subsets in the PFD_fT on ξ can be represented using the MS and NMS values corresponding to the parameters $\mathfrak{C}_1, \mathfrak{C}_2, \mathfrak{C}_3, \mathfrak{C}_4, \mathfrak{C}_5$, and $\mathfrak{C}_6$. These parameters denote affordability and accessibility, cultural acceptance and awareness, hygiene and health Impact, product availability and supply consistency, comfort and ease of use, and disposal and waste management respectively. The values for $\mathfrak{C}_1$ to $\mathfrak{C}_6$ for the products are assigned randomly, based on the characteristics described by these parameters. For instance, in terms of affordability, menstrual cups are generally less affordable than cloth pads for women in rural areas. Therefore, the MS value for the affordability of menstrual cups is taken as 0.44, which is lower than that of cloth pads. Similarly, for cultural awareness, the MS value of cloth pads is higher than that of sanitary pads, which in turn is higher than that of menstrual cups. Following this rationale, the PFD_f subsets are assigned values as illustrated in the Table 2.

Table 2: Criteria wise membership and non-membership values for menstrual hygiene products

	$\mathfrak{C}_1$	$\mathfrak{C}_2$	$\mathfrak{C}_3$	$\mathfrak{C}_4$	$\mathfrak{C}_5$	$\mathfrak{C}_6$
H1 (Cloth pads)	(0.91,0.26)	(0.71,0.42)	(0.38,0.72)	(0.21,0.97)	(0.46,0.77)	(0.34,0.84)
H_2 (Menstrual cups)	(0.44,0.85)	(0.28,0.83)	$(1,0)$	(0.61,0.66)	(0.75,0.72)	(0.54,0.77)
H_3 (Sanitary pads)	(0.49,0.63)	(0.58,0.61)	(0.67,0.51)	(0.73,0.56)	(0.62,0.75)	$(0,1)$

Step 4: The weight vector corresponding to these six criteria $\mathfrak{C} = \{\mathfrak{C}_1, \mathfrak{C}_2, \mathfrak{C}_3, \mathfrak{C}_4, \mathfrak{C}_5, \mathfrak{C}_6\}$ is $\mathfrak{w} = \{0.17, 0.24, 0.3, 0.04, 0.15, 0.1\}^{\mathrm{T}}$ where $\mathfrak{w}_{\mathrm{i}} \in [0,1]$ and $\sum_{i=1}^{n} \mathfrak{w}_i = 1$.

Step 5: To determine the most suitable product among cloth pads, menstrual cups, and sanitary pads, the scoring function values must be calculated using the formula provided in Equation 2.20.

$PFWA$ calculation:

$$\mathrm{PF}\mathfrak{w}\mathrm{A}\left(\mathfrak{I}_1, \mathfrak{I}_2, \mathfrak{I}_{3,\ldots}, \mathfrak{I}_{\mathrm{n}}\right) = \left\langle \sqrt{1 - \prod_{i=1}^{n}\left(1 - \mu_i^2\right)^{\mathfrak{w}_i}}, \prod_{i=1}^{n} \vartheta_i{}^{\mathfrak{w}_i} \right\rangle$$

The Score values corresponding to them are $\mathrm{SC}(\mathrm{H}_1) = -0.136, \mathrm{SC}(\mathrm{H}_2) = 1.000$ and $\mathrm{SC}(\mathrm{H}_3) = 0.009$. Since $\mathrm{SC}(\mathrm{H}_2) > \mathrm{SC}(\mathrm{H}_3) > \mathrm{SC}(\mathrm{H}_1)$, we have $\mathrm{H}_2 > \mathrm{H}_3 > \mathrm{H}_1$. Hence H_2 (Menstrual cups) is the greatest product for rural women's menstruation hygiene. $PFWG$ calculation:

$$\mathrm{PFWG}\left(\mathfrak{I}_1, \mathfrak{I}_2, \mathfrak{I}_3, \ldots\ldots.., \mathfrak{I}_{\mathrm{n}}\right) = \left\langle \prod_{i=1}^{n} \mu_i^{\mathfrak{w}_i}, \sqrt{1 - \prod_{i=1}^{n}\left(1 - \vartheta_i^2\right)^{\mathfrak{w}_i}} \right\rangle$$

The Score values corresponding to them are $\mathrm{SC}(\mathrm{H}_1) = -0.627, \mathrm{SC}(\mathrm{H}_2) = -0.399$ and $\mathrm{SC}(\mathrm{H}_3) =$ -1.000. Since $\mathrm{SC}(\mathrm{H}_2) > \mathrm{SC}(\mathrm{H}_1) > \mathrm{SC}(\mathrm{H}_3)$, we have $\mathrm{H}_2 > \mathrm{H}_1 > \mathrm{H}_3$. Hence H_2 (Menstrual cups) is the is the greatest product for rural women's menstruation hygiene.

The fuzzy topological framework adopted in this study offers a structured and actionable basis for real world decision making in the domain of menstrual hygiene management.

By utilizing the score outputs derived from $PFWA$ and $PFWG$ operators, policymakers can make informed decisions regarding resource allocation and program design. For instance, if the model identifies menstrual cups as the most suitable product based on the considered criteria, interventions can focus on subsidizing these products and organizing targeted distribution and training programs to ensure cost effectiveness and community acceptance. Furthermore, the inclusion of sociocultural factors such as cultural acceptance and awareness within the model enables local health educators to pinpoint barriers to adoption. This facilitates the development of tailored educational campaigns to promote lesser known but more beneficial products. Additionally, the model's adaptability allows for regular updates using real time feedback from the field, making it a dynamic tool for continuous evaluation and policy refinement. In this way, the proposed fuzzy topological framework serves not only as an evaluative mechanism but also as a practical decision support system for stakeholders working to improve menstrual health outcomes.

6. Validation

In [16], the authors indicate that menstruation cups surpass sanitary pads in terms of comfort, leakage protection, capacity, and discretion. Menstrual cups and sanitary pads are comparable in terms of accessibility and usability. Higher scores were observed for menstrual cups regarding skin rash prevention and reduced frequency of change compared to sanitary pads. Overall, menstrual cups demonstrate significant improvement with each menstrual cycle when compared to sanitary pads. However, sanitary pads are still suggested during the first few cycles of menstrual cup use. Notably, all performance scores show improvement as users become more accustomed to menstrual cups. Further research is warranted to evaluate cost-effectiveness and effectiveness in specific subgroups, such as individuals with menorrhagia. Promoting awareness and educating the population on the environmental benefits and initial learning curve associated with menstrual cup use is essential. Accordingly, the current study conveys the concept through a topological framework in a clear and simplified way.

7. Conclusion

Utilizing the $PFD\mathrm{b}_{\mathrm{f}}^{\approx}S$ as a basis, we proposed the notion of $PFD\mathrm{b}_{\mathrm{f}}^{\approx}-DS$ in this study and studied its features. Furthermore, we provided the ideas of $PFDb_{\mathrm{f}}^{\approx}$-Baire space and $PFDb_{\mathrm{f}}^{\approx}$-submaximal space. The characteristics of the $PFDb_{\mathrm{f}}^{\approx}-DS$ have been defined. This relate with various $PFDb_{\mathrm{f}}^{\approx}$-spaces was shown, and the results were interpreted appropriately. In addition, it is demonstrated that a quasi compact image of a $PFDb_{\mathrm{f}}^{\approx}-DS$ is a $PFD\mathrm{b}_{\mathrm{f}}^{\approx}-DS$ and that a $PFD\mathrm{b}_{\mathrm{f}}^{\approx}-DS$ has a topological property. Moreover, the framework and conditions of the PFD_fTS, together with its associated fine collection sets, are systematically utilized to determine the most appropriate menstrual hygiene product for women in rural areas.

In rural areas, cultural awareness regarding the use of menstrual cups remains relatively low. Many women are unfamiliar with menstrual cups due to limited access to menstrual health education and prevailing cultural taboos surrounding menstruation. Traditional practices, such as the use of cloth pads, are deeply rooted and often preferred, while myths and misconceptions-particularly concerning virginity, insertion, and safety create additional barriers to acceptance. The resistance to change is often reinforced by generational norms and a lack of open dialogue about menstrual health. Consequently, menstrual cups are perceived as unfamiliar and intimidating. To improve awareness and acceptance, targeted education initiatives and awareness programs are essential. These efforts should include hands-on demonstrations, involvement of local health workers, and the support of community leaders to dispel myths and encourage cultural acceptance of menstrual cups as a safe, sustainable, and effective menstrual hygiene option.

The present work explains the selection of the most suitable menstrual hygiene product using PFD_fT. This topological approach provides a structured framework, and in the future, it can be applied to real-time data collected from specific rural areas to identify the most appropriate product based on local conditions and preferences.

Disclosure Statement

The author(s) disclosed no potential conflicts of interest.

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