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Edge Irregular Reflexive Labeling of Ladder Graph Corona Null Graph Families

Diari Indriati[1], Muhammad Yogi Sentosa[1], Zhafirah Miranti Verdiana[1], Titin Sri Martini[1], Putranto Hadi Utomo[1], Nughthoh Arfawi Kurdi[1], Isnaini Rosyida[2,*]

[1] *Department of Mathematics, Faculty of Mathematics and Natural Sciences, Universitas Sebelas Maret, Surakarta, Indonesia*
[2] *Department of Mathematics, Faculty of Mathematics and Natural Sciences, Universitas Negeri Semarang, Indonesia*

Abstract. Given a graph $G(V, E)$ or simply written as G. The graph labeling was first introduced in 1960, it was a function that mapping integers or labels to graph elements (vertices, edges, or both of them) which must satisfy some certain criteria. This concepts was applied in some real problems. In 2007, there was a new concept on labeling, i.e., "vertex irregular total labeling" and "edge irregular total labeling". In 2017, there was a new concept, i.e., "vertex irregular reflexive labeling" and "edge irregular reflexive labeling". The "edge irregular reflexive $k-$ labeling" of G is a mapping that puts an even number label from 0 to $2k_v$ to every vertex and a positive integer label from 1 to k_e to every edge with different weights for each edge. The minimum k of the biggest label among all possible "edge irregular reflexive $k-$ labeling" of G is called the "reflexive edge strength" of G, denoted as $res(G)$. There are only few result on "vertex irregular reflexive labeling" or "edge irregular reflexive labeling" of corona product graphs. Therefore, the goal of this study is to examine the res of corona product of some ladder graphs and null graphs. We get the results as follows: $res(SL_n \odot N_m)$ for "$n \geq 2$ and $m \geq 1$" are "$\lceil \frac{2nm+3n-3}{3} \rceil$" for "$2nm + 3n - 3 \not\equiv 2, 3 \pmod 6$" and "$\lceil \frac{2nm+3n-3}{3} \rceil + 1$" for "$2nm + 3n - 3 \equiv 2, 3 \pmod 6$". Moreover, $res(DL_n \odot N_m)$ for $n \geq 2$ and $m \geq 1$ are "$\lceil \frac{2nm+5n-4}{3} \rceil$" for "$2nm + 5n - 4 \not\equiv 2, 3 \pmod 6$" and "$\lceil \frac{2nm+5n-4}{3} \rceil + 1$" for "$2nm + 5n - 4 \equiv 2, 3 \pmod 6$". These results contribute to developing the theory of reflexive labeling.

2020 Mathematics Subject Classifications: 05C78

Key Words and Phrases: Corona, slanting ladder graph, diagonal ladder graph, null graph, edge irregular reflexive k-labeling

*Corresponding author.
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Email addresses: diari_indri@staff.uns.ac.id (D. Indriati), isnaini@mail.unnes.ac.id (I. Rosyida)

1. Introduction

Graph theory is a branch of mathematics that studies relationship between objects through visual representation in the form of graphs. Graphs are used to represent relationships in various fields such as biology, social networks, and transportation systems. According to Diestel [1], A graph consists of a non-empty set of vertices $\{v_1, v_2, v_3, \cdots, v_n\}$ and an edge set $\{e_1, e_2, e_3, \cdots, e_m\}$. Every edge has two end vertices to indicate it and is often depicted as a line segment joining these two vertices.

One of the topics discussed in graph theory is graph labeling. This labeling was first introduced in 1960 and according to Wallis [2], a mapping called "graph labeling" gives each vertex and edge in the graph a label or an integer, which must satisfy some certain criteria or rules.

The "vertex irregular total $k-$labeling" and "edge irregular total $k-$ labeling" are two novel ideas in graph theory that were introduced by Bača *et al* in 2007 [3]. However, in 2017, this concept was developed again by Ryan *et al* [4] by introducing "vertex irregular reflexive k-labeling" and "edge irregular reflexive k labeling". In G, a reflexive edge irregular k-labeling is defined as a function that can put an even number label from 0 to $2k_v$ to each vertex and a positive integer label from 1 to k_e to each edge where k=max$\{k_e, 2k_v\}$ with a different weight for each edge. The reflexive edge strength, represented by $res(G)$, is the lowest value k of the largest label [4]. The following lower bound of $res(G)$ was given by Ryan *et al* [4].

Lemma 1. *Considering each graph G,*

$$res(G) \geq \begin{cases} \left\lceil \frac{|E(G)|}{3} \right\rceil, & \text{if } |E(G)| \not\equiv 2, 3 \pmod 6, \\ \left\lceil \frac{|E(G)|}{3} \right\rceil + 1, & \text{if } |E(G)| \equiv 2, 3 \pmod 6. \end{cases}$$

The "edge irregular reflexive k-labeling" of several graphs have been investigated, includes Tanna *et al.* [5] proved of prism graph (D_n), wheel graph (W_n), fan graph (F_n) and basket graph (B_n). Budi *et al.* [6] investigated cycles graphs (C_n). Bača *et al.* [3] examined tadpole graphs $T_{m,1}$ and $T_{m,2}$. Indriati *et al.* found *res* of "corona of path" and other graphs [7]. Agustin *et al.* provided the *res* of some trees and some almost regular graphs [8, 9]. Santoso *et al.* constructed an algorithm for the non-inclusive vertex irregular labeling [10]. For more results on the *res* of various graphs, the readers could see [11].

In this research, we focuss on the families related to ladder graphs, i.e., slanting ladder and diagonal ladder [12],[13]. The reflexive edge strength of slanting ladder corona null graph ($SL_n \odot N_m$), $n \geq 2$ and $m \geq 1$ and diagonal ladder corona null graph ($DL_n \odot N_m$), $n \geq 2$ and $m \geq 1$ will be determined.

2. Main Results

2.1. Corona of Slanting Ladder and Null Graph

The corona of "slanting ladder" and "null graph", symbolized by $SL_n \odot N_m$, is a connected graph with $V(SL_n \odot N_m) = \{u_i, v_i : 1 \leq i \leq n\} \cup \{u_{i,j}, v_{i,j} : 1 \leq i \leq n, 1 \leq j \leq$

$m\}$ and $E(SL_n \odot N_m) = \{u_iu_{i+1}, v_iv_{i+1}, u_iv_{i+1} : 1 \leq i \leq n-1\} \cup \{v_iv_{i,j}, u_iu_{i,j} : 1 \leq i \leq n, 1 \leq j \leq m\}$. Therefore, this graph has order $2nm + 2n$ and size $2nm + 3n - 3$. This graph is shown in Figure 1.

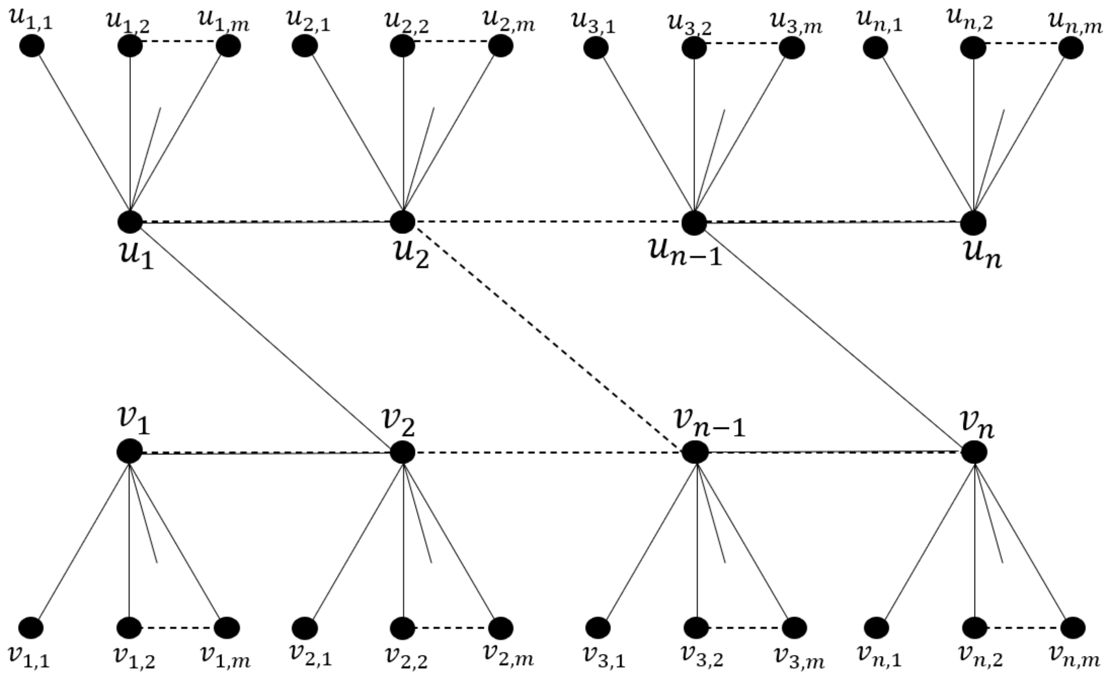

Figure 1: The corona graph $SL_n \odot N_m$

The *res* of $SL_n \odot N_m$ is proved in Theorem 1.

Theorem 1. *Given $SL_n \odot N_m$ for $n \geq 2$ and $m \geq 1$,*

$$res(SL_n \odot N_m) = \begin{cases} \left\lceil \frac{2nm+3n-3}{3} \right\rceil, \textit{if } 2nm + 3n - 3 \not\equiv 2, 3 \pmod 6, \\ \left\lceil \frac{2nm+3n-3}{3} \right\rceil + 1, \textit{if } 2nm + 3n - 3 \equiv 2, 3 \pmod 6. \end{cases} \quad (1)$$

Proof. Since the size is $2nm + 3n - 3$, then by using *Lemma* 1 we get:

$$res(SL_n \odot N_m) \geq \begin{cases} \left\lceil \frac{2nm+3n-3}{3} \right\rceil, \text{if } 2nm + 3n - 3 \not\equiv 2, 3 \pmod 6, \\ \left\lceil \frac{2nm+3n-3}{3} \right\rceil + 1, \text{if } 2nm + 3n - 3 \equiv 2, 3 \pmod 6. \end{cases} \quad (2)$$

Statement (2) is a lower bound of $res(SL_n \odot N_m)$. Further, we verify the upper bound of the *res*. Construct θ as k-labeling of corona of slanting ladder and null graph with $k = \lceil \frac{2nm+3n-3}{3} \rceil$ for $2nm + 3n - 3 \not\equiv 2, 3 \pmod 6$ and $k = \lceil \frac{2nm+3n-3}{3} \rceil + 1$ for $2nm + 3n - 3 \equiv 2, 3 \pmod 6$ below.

Label of vertex v_i for $1 \leq i \leq n$,

$$\theta(v_i) = \begin{cases} 0, & i = 1, \\ \frac{2m+6-2a}{3}, & i = 2,\ m \equiv a \pmod 3,\ a = 0,1,2, \\ \frac{8m+6}{3}, & i = 4,\ m \equiv 0 \pmod 3, \\ \frac{10m+10}{3}, & i = 5,\ m \equiv 2 \pmod 3, \\ \frac{2im+3i}{3}, & i \equiv 0 \pmod 6, \\ & i \equiv 2 \pmod 6,\ m \equiv 0 \pmod 3,\ i \neq 2, \\ & i \equiv 4 \pmod 3,\ m \equiv 0 \pmod 3,\ i \neq 4, \\ \frac{2im+3i-3}{3}, & i \equiv 1 \pmod 6,\ m \equiv 0 \pmod 3, \\ & i \equiv 3 \pmod 6, \\ & i \equiv 5 \pmod 3,\ m \equiv 0 \pmod 3, \\ \frac{2im+3i-4}{3}, & i \equiv 2 \pmod 6,\ m \equiv 1 \pmod 3,\ i \neq 2 \\ & i \equiv 4 \pmod 6,\ m \equiv 2 \pmod 3, \\ \frac{2im+3i-1}{3}, & i \equiv 1 \pmod 6,\ m \equiv 2 \pmod 3,\ i \neq 1 \\ & i \equiv 5 \pmod 6,\ m \equiv 1 \pmod 3, \\ \frac{2im+3i-2}{3}, & i \equiv 2 \pmod 6,\ m \equiv 2 \pmod 3,\ i \neq 2 \\ & i \equiv 4 \pmod 6,\ m \equiv 1 \pmod 3, \\ \frac{2im+3i+1}{3}, & i \equiv 1 \pmod 6,\ m \equiv 1 \pmod 3,\ i \neq 1, \\ & i \equiv 5 \pmod 6,\ m \equiv 2 \pmod 3,\ i \neq 5. \end{cases}$$

The label of u_i for $i = 1, 2, \cdots, n$:

$$\theta(u_i) = \begin{cases} 0, & i = 1, \\ \frac{4m}{3}, & i = 2,\ m \equiv 0 \pmod 3, \\ \frac{8m+6}{3}, & i = 4,\ m \equiv 0 \pmod 3, \\ \frac{10m+10}{3}, & i = 5,\ m \equiv 2 \pmod 3, \\ \frac{2im+3i}{3}, & i \equiv 0 \pmod 6, \\ & i \equiv 2 \pmod 6,\ m \equiv 0 \pmod 3,\ i \neq 2, \\ & i \equiv 4 \pmod 6,\ m \equiv 0 \pmod 3,\text{`}i \neq 4, \\ \frac{2im+3i-3}{3}, & i \equiv 1 \pmod 6,\ m \equiv 0 \pmod 3,\ i \neq 1, \\ & i \equiv 3 \pmod 6, \\ & i \equiv 5 \pmod 6,\ m \equiv 0 \pmod 3, \\ \frac{2im+3i-4}{3}, & i \equiv 2 \pmod 6,\ m \equiv 1 \pmod 3, \\ & i \equiv 4 \pmod 6,\ m \equiv 2 \pmod 3, \end{cases}$$

Continue of label of u_i for $i = 1, 2, \cdots, n$:

$$\theta(u_i) = \begin{cases} \frac{2im+3i-1}{3}, & i \equiv 1 \pmod 6,\ m \equiv 2 \pmod 3,\ i \neq 1 \\ & i \equiv 2 \pmod 6,\ m \equiv 1 \pmod 3, \\ \frac{2im+3i-2}{3}, & i \equiv 2 \pmod 6,\ m \equiv 2 \pmod 3, \\ & i \equiv 4 \pmod 3,\ m \equiv 1 \pmod 3, \\ \frac{2im+3i+1}{3}, & i \equiv 1 \pmod 6,\ m \equiv 1 \pmod 3,\ i \neq 1 \\ & i \equiv 5 \pmod 6,\ m \equiv 2 \pmod 3,\ i \neq 5. \end{cases}$$

Label of $v_{i,j}$ for $1 \leq i \leq n$ and $1 \leq j \leq m$:

$$\theta(v_{i,j}) = \begin{cases} 0, & i = 1, \\ \frac{4m}{3}, & i = 2,\ m \equiv 0 \pmod 3, \\ \frac{2im+3i}{3}, & i \equiv 0 \pmod 6, \\ & i \equiv 2 \pmod 3,\ m \equiv 0 \pmod 3, \\ & i \equiv 4 \pmod 6,\ m \equiv 0 \pmod 3,\ i \neq 2, \\ \frac{2im+3i-3}{3}, & i \equiv 1 \pmod 6,\ m \equiv 0 \pmod 3, \\ & i \equiv 3 \pmod 6, \\ & i \equiv 5 \pmod 6,\ m \equiv 0 \pmod 3, \\ \frac{2im+3i-4}{3}, & i \equiv 2 \pmod 6,\ m \equiv 1 \pmod 3, \\ & i \equiv 4 \pmod 6,\ m \equiv 2 \pmod 3, \\ \frac{2im+3i-1}{3}, & i \equiv 1 \pmod 6,\ m \equiv 2 \pmod 3,\ i \neq 1 \\ & i \equiv 5 \pmod 6,\ m \equiv 1 \pmod 3, \\ \frac{2im+3i-2}{3}, & i \equiv 2 \pmod 6,\ m \equiv 2 \pmod 3, \\ & i \equiv 4 \pmod 6,\ m \equiv 1 \pmod 3, \\ \frac{2im+3i+1}{3}, & i \equiv 1 \pmod 6,\ m \equiv 1 \pmod 3,\ i \neq 1, \\ & i \equiv 5 \pmod 6,\ m \equiv 2 \pmod 3). \end{cases}$$

Label of $u_{i,j}$ for $1 \leq i \leq n$ and $j = 1, 2, \cdots, m$:

$$\theta(u_{i,j}) = \begin{cases} \frac{2m-2}{3}, & i = 1,\ m \equiv 1 \pmod 3, \\ \frac{2m-4}{3}, & i = 1,\ m \equiv 2 \pmod 3, \\ \frac{2im+3i}{3}, & i \equiv 0 \pmod 6, \\ & i \equiv 2 \pmod 6,\ m \equiv 0 \pmod 3, \\ & i \equiv 4 \pmod 6,\ m \equiv 0 \pmod 3, \end{cases}$$

$$\theta(u_{i,j}) = \begin{cases} \frac{2im+3i-3}{3}, & i \equiv 1 \pmod 6,\ m \equiv 0 \pmod 3, \\ & i \equiv 3 \pmod 6, \\ & i \equiv 5 \pmod 6,\ m \equiv 0 \pmod 3, \\ \frac{2im+3i-4}{3}, & i \equiv 2 \pmod 6,\ m \equiv 1 \pmod 3, \\ & i \equiv 4 \pmod 6,\ m \equiv 2 \pmod 3, \\ \frac{2im+3i-1}{3}, & i \equiv 1 \pmod 6,\ m \equiv 2 \pmod 3,\ i \neq 1 \\ & i \equiv 5 \pmod 6,\ m \equiv 1 \pmod 3, \\ \frac{2im+3i-2}{3}, & i \equiv 2 \pmod 6,\ m \equiv 2 \pmod 3, \\ & i \equiv 4 \pmod 6,\ m \equiv 1 \pmod 3, \\ \frac{2im+3i+1}{3}, & i \equiv 1 \pmod 6,\ m \equiv 1 \pmod 3,\ i \neq 1, \\ & i \equiv 5 \pmod 6,\ m \equiv 2 \pmod 3. \end{cases}$$

Label of edge v_iv_{i+1} for $i = 1, 2, \cdots, n-1$,

$$\theta(v_iv_{i+1}) = \begin{cases} \frac{4m+2a-3}{3}, & i = 1,\ m \equiv a \pmod 3,\ a = 0,1,2, \\ \frac{4m+2a}{3}, & i = 2,\ m \equiv a \pmod 3,\ a = 0,1,2, \\ \frac{4m+9}{3}, & i = 3,\ m \equiv 0 \pmod 3, \\ \frac{8m+11}{3}, & i = 5,\ m \equiv 2 \pmod 3, \\ \frac{(2i-2)m+3i-6}{3}, & i \equiv 0,3 \pmod 6,\ m \equiv 0 \pmod 3,\ i \neq 3, \\ & i \equiv 1,4 \pmod 6,\ i \neq 1, \\ & i \equiv 2,5 \pmod 6,\ m \equiv 0 \pmod 3,\ i \neq 2, \\ \frac{(2i-2)m+3i-8}{3}, & i \equiv 0 \pmod 6,\ m \equiv 2 \pmod 3, \\ & i \equiv 5 \pmod 6,\ m \equiv 1 \pmod 3, \\ \frac{(2i-2)m+3i-2}{3}, & i \equiv 2 \pmod 6,\ m \equiv 1 \pmod 3,\ i \neq 2 \\ & i \equiv 3 \pmod 6,\ m \equiv 2 \pmod 3, \\ \frac{(2i-2)m+3i-4}{3}, & i \equiv 2 \pmod 6,\ m \equiv 2 \pmod 3,\ i \neq 2 \\ & i \equiv 3 \pmod 6,\ m \equiv 1 \pmod 3, \\ \frac{(2i-2)m+3i-10}{3}, & i \equiv 0 \pmod 6,\ m \equiv 1 \pmod 3, \\ & i \equiv 5 \pmod 6,\ m \equiv 2 \pmod 3,\ i \neq 5. \end{cases}$$

Label of edge u_iu_{i+1} where $i = 1, 2, \cdots, n-1$,

$$\theta(u_iu_{i+1}) = \begin{cases} \frac{2m+9-2a}{3}, & i = 1,\ m \equiv a \pmod 3,\ a = 0,1,2, \\ \frac{2m+12}{3}, & i = 2,\ m \equiv 0 \pmod 3, \\ \frac{4m+15}{3}, & i = 3,\ m \equiv 0 \pmod 3, \\ \frac{6m+18}{3}, & i = 4,\ m \equiv 0,2 \pmod 3, \end{cases}$$

Continue to label of u_iu_{i+1}:

$$\theta(u_iu_{i+1}) = \begin{cases} \frac{2m+9-2a}{3}, & i = 1,\, m \equiv a \pmod 3,\, a = 0,1,2, \\ \frac{2m+12}{3}, & i = 2,\, m \equiv 0 \pmod 3, \\ \frac{4m+15}{3}, & i = 3,\, m \equiv 0 \pmod 3, \\ \frac{6m+18}{3}, & i = 4,\, m \equiv 0,2 \pmod 3, \\ \frac{8m+17}{3}, & i = 5,\, m \equiv 2 \pmod 3, \\ \frac{(2i-2)m+3i}{3}, & i \equiv 0,3 \pmod 6,\, i \neq 3, \\ & i \equiv 1,4 \pmod 6,\, i \neq 1,4 \\ & i \equiv 2,5 \pmod 6,\, m \equiv 0 \pmod 3,\, i \neq 2, \\ \frac{(2i-2)m+3i-2}{3}, & i \equiv 0 \pmod 6,\, m \equiv 2 \pmod 3, \\ & i \equiv 5 \pmod 6,\, m \equiv 1 \pmod 3, \\ \frac{(2i-2)m+3i+4}{3}, & i \equiv 2 \pmod 6,\, m \equiv 1 \pmod 3, \\ & i \equiv 3 \pmod 6,\, m \equiv 2 \pmod 3, \\ \frac{(2i-2)m+3i+2}{3}, & i \equiv 2 \pmod 6,\, m \equiv 2 \pmod 3, \\ & i \equiv 3 \pmod 6,\, m \equiv 1 \pmod 3, \\ \frac{(2i-2)m+3i-4}{3}, & i \equiv 0 \pmod 6,\, m \equiv 1 \pmod 3, \\ & i \equiv 5 \pmod 6,\, m \equiv 2 \pmod 3,\, i \neq 5. \end{cases}$$

Label of edge u_iv_{i+1} for $i = 1, 2, \cdots, n-1$:

$$\theta(u_iv_{i+1}) = \begin{cases} \frac{4m+2a}{3}, & i = 1,\, m \equiv a \pmod 3,\, a = 0,1,2, \\ \frac{2m+9}{3}, & i = 2,\, m \equiv 0 \pmod 3, \\ \frac{4m+12}{3}, & i = 3,\, m \equiv 0 \pmod 3, \\ \frac{8m+14}{3}, & i = 5,\, m \equiv 2 \pmod 3, \\ \frac{(2i-2)m+3i-3}{3}, & i \equiv 0,3 \pmod 6,\, m \equiv 0 \pmod 3,\, i \neq 3, \\ & i \equiv 1,4 \pmod 6,\, i \neq 1 \\ & i \equiv 2,5 \pmod 6,\, m \equiv 0 \pmod 3,\, i \neq 2, \\ \frac{(2i-2)m+3i+1}{3}, & i \equiv 2 \pmod 6,\, m \equiv 1 \pmod 3, \\ & i \equiv 3 \pmod 6,\, m \equiv 2 \pmod 3, \\ \frac{(2i-2)m+3i-5}{3}, & i \equiv 0 \pmod 6,\, m \equiv 2 \pmod 3, \\ & i \equiv 5 \pmod 6,\, m \equiv 1 \pmod 3, \\ \frac{(2i-2)m+3i-1}{3}, & i \equiv 2 \pmod 6,\, m \equiv 2 \pmod 3, \\ & i \equiv 3 \pmod 6,\, m \equiv 1 \pmod 3, \\ \frac{(2i-2)m+3i-7}{3}, & i \equiv 0 \pmod 6,\, m \equiv 1 \pmod 3, \\ & i \equiv 5 \pmod 6,\, m \equiv 2 \pmod 3,\, i \neq 5. \end{cases}$$

Label of edge $v_iv_{i,j}$ for $i = 1, 2, \cdots, n$ and $j = 1, 2, \cdots, m$:

$$\theta(v_iv_{i,j}) = \begin{cases} i+j-1, & i = 1,2,3, \\ (\frac{2m+12}{3}) + j - 1, & i = 4,\ m \equiv 0 \pmod 3, \\ (\frac{4m+13}{3}) + j - 1, & i = 5,\ m \equiv 2 \pmod 3, \\ (\frac{(2i-6)m+3i}{3}) + j - 1, & i \equiv 1 \pmod 6,\ m \equiv 0 \pmod 3,\ i \neq 1, \\ & i \equiv 0 \pmod 6, \\ & i \equiv 5 \pmod 6,\ m \equiv 0 \pmod 3, \\ (\frac{(2i-6)m+3i-6}{3}) + j - 1, & i \equiv 0 \pmod 6, \\ & i \equiv 2 \pmod 6,\ m \equiv 0 \pmod 3,\ i \neq 2, \\ & i \equiv 4 \pmod 6,\ m \equiv 0 \pmod 3,\ i \neq 4, \\ (\frac{(2i-6)m+3i-2}{3}) + j - 1, & i \equiv 2 \pmod 6,\ m \equiv 2 \pmod 3,\ i \neq 2, \\ & i \equiv 4 \pmod 6,\ m \equiv 1 \pmod 3, \\ (\frac{(2i-6)m+3i-8}{3}) + j - 1, & i \equiv 1 \pmod 6,\ m \equiv 1 \pmod 3,\ i \neq 1 \\ & i \equiv 5 \pmod 6,\ m \equiv 2 \pmod 3,\ i \neq 5, \\ (\frac{(2i-6)m+3i+2}{3}) + j - 1, & i \equiv 2 \pmod 6,\ m \equiv 1 \pmod 3,\ i \neq 2, \\ & i \equiv 4 \pmod 6,\ m \equiv 2 \pmod 3, \\ (\frac{(2i-6)m+3i-4}{3}) + j - 1, & i \equiv 1 \pmod 6,\ m \equiv 2 \pmod 3,\ i \neq 1 \\ & i \equiv 5 \pmod 6,\ m \equiv 1 \pmod 3. \end{cases}$$

Label of $u_iu_{i,j}$ where $i = 1, 2, \cdots, n$ and $j = 1, 2, \cdots, m$:

$$\theta(u_iu_{i,j}) = \begin{cases} \frac{(m+2a+3}{3}) + j - 1, & i = 1,\ m \equiv a \pmod 3,\ a = 1,2,3, \\ \frac{(m+6}{3}) + j - 1, & i = 2,\ m \equiv 0 \pmod 3, \\ \frac{(5m+12}{3}) + j - 1, & i = 4,\ m \equiv 0 \pmod 3, \\ \frac{(7m+13}{3}) + j - 1, & i = 5,\ m \equiv 2 \pmod 3, \\ (\frac{(2i-3)m+3i-2}{3}) + j - 1, & i \equiv 4 \pmod 6,\ m \equiv 1 \pmod 3, \\ & i \equiv 5 \pmod 6,\ m \equiv 2 \pmod 3, \\ (\frac{(2i-3)m+3i-8}{3}) + j - 1, & i \equiv 1 \pmod 3,\ m \equiv 1 \pmod 3, \\ & i \equiv 2 \pmod 6,\ m \equiv 2 \pmod 3,\ i \neq 2, \\ (\frac{(2i-3)m+3i+2}{3}) + j - 1, & i \equiv 2 \pmod 6,\ m \equiv 1 \pmod 3, \\ & i \equiv 4 \pmod 6,\ m \equiv 2 \pmod 3, \\ (\frac{(2i-3)m+3i-4}{3}) + j - 1, & i \equiv 1 \pmod 6,\ m \equiv 2 \pmod 3, \\ & i \equiv 5 \pmod 6,\ m \equiv 1 \pmod 3,\ i \neq 2, \\ (\frac{(2i-3)m+3i-6}{3}) + j - 1, & i \equiv 0 \pmod 6,\ m \geq 1, \\ & i \equiv 2 \pmod 6,\ m \equiv 0 \pmod 3,\ i \neq 2, \\ & i \equiv 4 \pmod 6,\ m \equiv 0 \pmod 3,\ i \neq 4, \end{cases}$$

Continue of label of edge $u_iu_{i,j}$ where $1 \le i \le n$ and $1 \le j \le m$:

$$\theta(u_iu_{i,j}) = \left(\frac{(2i-3)m+3i}{3}\right) + j - 1,$$

for $i \equiv 3 \pmod 6, m \ge 1; i \equiv 1 \pmod 6, m \equiv 0 \pmod 3$; and $i \equiv 2 \pmod 6, m \equiv 0 \pmod 3$.

Based on the above labeling, the maximum value of the labels is $\lceil \frac{2nm+3n-3}{3} \rceil$ when $2nm+3n-3 \not\equiv 2,3 \pmod 6$ and $\lceil \frac{2nm+3n-3}{3} \rceil + 1$ when $2nm+3n-3 \equiv 2,3 \pmod 6$. The weight of an edge uv is $wt_\theta(uv) = \theta(u) + \theta(v) + \theta(uv)$. We get the edge weights as follows:

$$\begin{aligned}
wt_\theta(v_iv_{i+1}) &= 2im + 3i - 2, \ 1 \le i \le n-1.\\
wt_\theta(u_iu_{i+1}) &= 2im + 3i, \ 1 \le i \le n-1.\\
wt_\theta(u_iv_{i+1}) &= 2im + 3i - 1, \ 1 \le i \le n-1.\\
wt_\theta(v_1v_{1,j}) &= j,\\
wt_\theta(v_iv_{i,j}) &= ((2i-2)m + 3i - 2) + j - 1, \ 2 \le i \le n, 1 \le j \le m.\\
wt_\theta(u_iu_{i,j}) &= ((2i-2)m + 3i - 2) + j - 1, \ 1 \le i \le n, 1 \le j \le m.
\end{aligned}$$

The weights of all edges on the corona $SL_n \odot N_m$ are distinct. We get the lower bound of $res(SL_n \odot N_m)$ same as this upper bound. Then, $res(SL_n \odot N_m)$ is obtained. Therefore θ satisfies the edge irregular reflexive k-labeling and the res is shown in Theorem 1. It completes the proof.

Figure 2 gives the illustration of $SL_2 \odot N_2$-labeling.

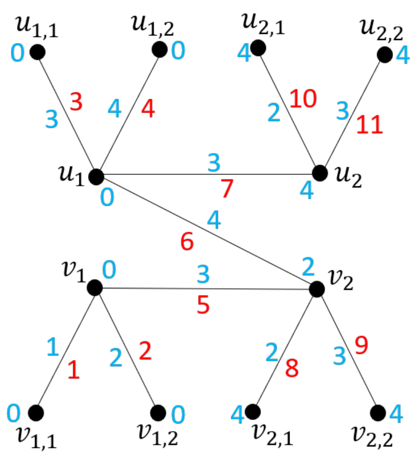

Figure 2: The edge irregular reflexive 4-labeling of $SL_2 \odot N_2$

On the labeling, the label of each edge and each vertex is shown with blue numbers. A number in red indicates the weight of each edge.

2.2. Corona of Diagonal Ladder and Null Graph

Corona of "diagonal ladder" and "null graph", symbolized by $DL_n \odot N_m$, is a connected graph with $V(DL_n \odot N_m) = \{u_i, v_i : 1 \leq i \leq n\} \cup \{u_{i,j}, v_{i,j} : 1 \leq i \leq n, 1 \leq j \leq m\}$ and $E(DL_n \odot N_m) = \{u_iu_{i+1}, v_iv_{i+1}, u_iv_{i+1}, v_iu_{i+1} : 1 \leq i \leq n-1\} \cup \{u_iv_i : 1 \leq i \leq n\} \cup \{u_iu_{i,j}, v_iv_{i,j} : 1 \leq i \leq n, 1 \leq j \leq m\}$. The *order* of this graph is $= 2nm + 2n$ and the *size* is $2nm + 5n - 4$. Figure 3 gives an illustration of $DL_n \odot N_m$.

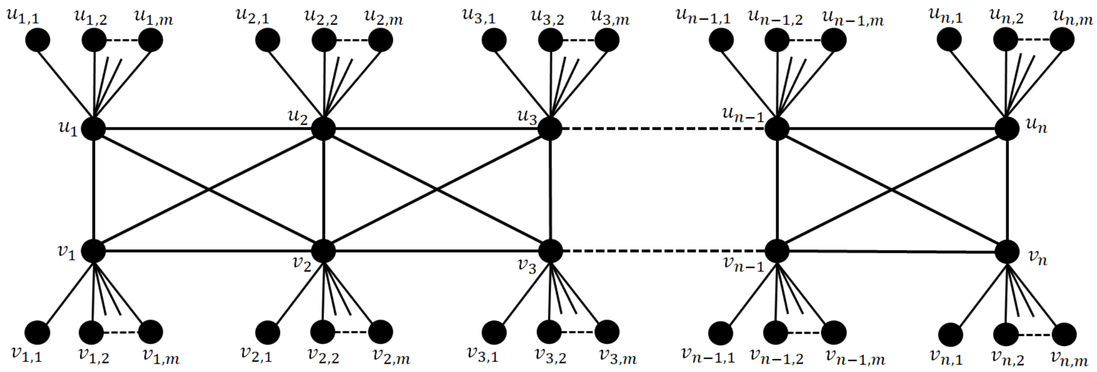

Figure 3: Corona of "diagonal ladder" and "null graph" $DL_n \odot N_m$

The reflexive edge strength of $DL_n \odot N_m$ is presented in Theorem 2.

Theorem 2. *For $DL_n \odot N_m$ with $n \geq 2$ and $m \geq 1$,*

$$res(DL_n \odot N_m) = \begin{cases} \left\lceil \frac{2nm+5n-4}{3} \right\rceil, & \text{for } 2nm + 5n - 4 \not\equiv 2, 3 \pmod 6, \\ \left\lceil \frac{2nm+5n-4}{3} \right\rceil + 1, & \text{for } 2nm + 5n - 4 \equiv 2, 3 \pmod 6. \end{cases} \quad (3)$$

Proof. Since $|E|$ of $DL_n \odot N_m$ is $2nm + 5n - 4$, then using *Lemma* 1, obtained the lower bound of $res(DL_n \odot N_m)$

$$res(DL_n \odot N_m) \geq \begin{cases} \left\lceil \frac{2nm+5n-4}{3} \right\rceil, & 2nm + 5n - 4 \not\equiv 2, 3 \pmod 6, \\ \left\lceil \frac{2nm+5n-4}{3} \right\rceil + 1, & 2nm + 5n - 4 \equiv 2, 3 \pmod 6. \end{cases} \quad (4)$$

We will prove the upper bound of $res(DL_n \odot N_m)$ for $n \geq 2$ and $m \geq 1$ by constructing a k-label θ of corona of diagonal ladder and null graph with $k = \left\lceil \frac{2nm+5n-4}{3} \right\rceil$ for $2nm + 5n - 4 \not\equiv 2, 3 \pmod 6$ and $k = \left\lceil \frac{2nm+5n-4}{3} \right\rceil + 1$ for $2nm + 5n - 4 \equiv 2, 3 \pmod 6$.

Vertex label u_i for $i = 1, 2, \cdots, n$:

$$\theta(u_i) = \begin{cases} 0, & i = 1, \\ \frac{4m+2a}{3}, & i = 2,\ m \equiv a \pmod 3,\ a = 0,1,2, \\ \frac{2im+5i}{3}, & i \equiv 0 \pmod 6,\ m \geq 1 \\ & i \equiv 4 \pmod 6,\ m \equiv 2 \pmod 3, \\ \frac{2im+5i-2a+1}{3}, & i \equiv 1 \pmod 6,\ i \neq 1,\ m \equiv a \pmod 3,\ a = 1,2,3, \\ \frac{2im+5i+2a-4}{3}, & i \equiv 2 \pmod 6,\ i \neq 2,\ m \equiv a \pmod 3,\ a = 0,1,2, \\ \frac{2im+5i-3}{3}, & i \equiv 3 \pmod 6,\ m \geq 1, \\ \frac{2im+5i-2}{3}, & i \equiv 4 \pmod 6,\ m \equiv 0 \pmod 3, \\ \frac{2im+5i-4}{3}, & i \equiv 4 \pmod 6,\ m \equiv 1 \pmod 3, \\ \frac{2im+5i+2a-7}{3}, & i \equiv 5 \pmod 6,\ m \equiv a \pmod 3,\ a = 1,2,3. \end{cases}$$

Vertex label v_i with $1 \leq i \leq n$:

$$\theta(v_i) = \begin{cases} 0, & i = 1, \\ \frac{2im+5i}{3}, & i \equiv 0 \pmod 6,\ m \geq 1 \\ & i \equiv 4 \pmod 6,\ m \equiv 2 \pmod 3, \\ \frac{2im+5i-2a+1}{3}, & i \equiv 1 \pmod 6,\ i \neq 1,\ m \equiv a \pmod 3,\ a = 1,2,3, \\ \frac{2im+5i+2a-4}{3}, & i \equiv 2 \pmod 6,\ m \equiv a \pmod 3,\ a = 0,1,2, \\ \frac{2im+5i-3}{3}, & i \equiv 3 \pmod 6,\ m \geq 1, \\ \frac{2im+5i-2}{3}, & i \equiv 4 \pmod 6,\ m \equiv 0 \pmod 3, \\ \frac{2im+5i-4}{3}, & i \equiv 4 \pmod 6,\ m \equiv 1 \pmod 3, \\ \frac{2im+5i+2a-7}{3}, & i \equiv 5 \pmod 6,\ m \equiv a \pmod 3,\ a = 1,2,3. \end{cases}$$

Vertex label $u_{i,j}$ for $i = 1, 2, \cdots, n$ and $j = 1, 2, \cdots, m$:

$$\theta(u_{i,j}) = \begin{cases} 0, & i = 1, \\ \frac{2m+6}{3}, & i = 2,\ m \equiv 0 \pmod 3, \\ \frac{2m+4}{3}, & i = 2,\ m \equiv 1 \pmod 3, \\ \frac{2m+8}{3}, & i = 2,\ m \equiv 2 \pmod 3, \\ \frac{2im+5i}{3}, & i \equiv 0 \pmod 6,\ m \geq 1 \\ & i \equiv 4 \pmod 6,\ m \equiv 2 \pmod 3, \\ \frac{2im+5i-2a+1}{3}, & i \equiv 1 \pmod 6,\ i \neq 1,\ m \equiv a \pmod 3,\ a = 1,2,3, \\ \frac{2im+5i+2a-4}{3}, & i \equiv 2 \pmod 6,\ i \neq 2,\ m \equiv a \pmod 3,\ a = 0,1,2, \\ \frac{2im+5i-3}{3}, & i \equiv 3 \pmod 6,\ m \geq 1, \\ \frac{2im+5i-2}{3}, & i \equiv 4 \pmod 6,\ m \equiv 0 \pmod 3, \\ \frac{2im+5i-4}{3}, & i \equiv 4 \pmod 6,\ m \equiv 1 \pmod 3, \\ \frac{2im+5i+2a-7}{3}, & i \equiv 5 \pmod 6,\ m \equiv a \pmod 3,\ a = 1,2,3. \end{cases}$$

Vertex label $v_{i,j}$ for $i = 1, 2, \cdots, n$ and $j = 1, 2, \cdots, m$:

$$\theta(v_{i,j}) = \begin{cases} \frac{2m-2}{3}, & i = 1,\ m \equiv 1 \pmod 3, \\ 4, & i = 2,\ m = 1, \\ \frac{4m+2}{3}, & i = 2,\ m \equiv 1 \pmod 3,\ m \neq 1 \\ \frac{2im+5i}{3}, & i \equiv 0 \pmod 6,\ m \geq 1 \\ & i \equiv 4 \pmod 6,\ m \equiv 2 \pmod 3, \\ \frac{2im+5i-2a+1}{3}, & i \equiv 1 \pmod 6,\ m \equiv a \pmod 3,\ a = 1, 2, 3, \\ \frac{2im+5i+2a-4}{3}, & i \equiv 2 \pmod 6,\ m \equiv a \pmod 3,\ a = 0, 2, 1 \text{ and } i \neq 2, \\ \frac{2im+5i-3}{3}, & i \equiv 3 \pmod 6,\ m \geq 1, \\ \frac{2im+5i-2}{3}, & i \equiv 4 \pmod 6,\ m \equiv 0 \pmod 3, \\ \frac{2im+5i-4}{3}, & i \equiv 4 \pmod 6,\ m \equiv 1 \pmod 3, \\ \frac{2im+5i+2a-7}{3}, & i \equiv 5 \pmod 6,\ m \equiv a \pmod 3,\ a = 1, 2, 3. \end{cases}$$

Edge label u_iu_{i+1} for $i = 1, 2, \cdots, n-1$:

$$\theta(u_iu_{i+1}) = \begin{cases} \frac{2m-2a+6}{3}, & i = 1,\ m \equiv a \pmod 3,\ a = 0, 1, 2, \\ \frac{2m-2a+9}{3}, & i = 2,\ m \equiv a \pmod 3,\ a = 0, 1, 2, \\ \frac{(2i-2)m+5i+2a-15}{3}, & i \equiv 0 \pmod 6,\ m \equiv a \pmod 3,\ a = 1, 2, 3, \\ \frac{(2i-2)m+5i-5}{3}, & i \equiv 1 \pmod 6,\ i \neq 1,\ m \equiv 0 \pmod 3 \\ & i \equiv 4 \pmod 6,\ m \equiv 1 \pmod 3, \\ \frac{(2i-2)m+5i-11}{3}, & i \equiv 1 \pmod 6,\ m \equiv 1, 2 \pmod 3,\ i \neq 1, \\ & i \equiv 3 \pmod 6,\ m \equiv 2 \pmod 3, \\ & i \equiv 4 \pmod 6,\ m \equiv 0, 2 \pmod 3, \\ \frac{(2i-2)m+5i-2a-7}{3}, & i \equiv 2 \pmod 6,\ i \neq 2,\ m \equiv a \pmod 3,\ a = 0, 1, 2 \\ & i \equiv 5 \pmod 6,\ m \equiv a \pmod 3,\ a = 1, 2, 3, \\ \frac{(2i-2)m+5i-9}{3}, & i \equiv 3 \pmod 6,\ m \equiv 0 \pmod 3, \\ \frac{(2i-2)m+5i-7}{3}, & i \equiv 3 \pmod 6,\ m \equiv 1 \pmod 3. \end{cases}$$

Edge label v_iv_{i+1} for $i = 1, 2, \cdots, n-1$:

$$\theta(v_iv_{i+1}) = \begin{cases} \frac{2m-2a+9}{3}, & i = 1,\ m \equiv a \pmod 3,\ a = 0,1,2, \\ \frac{(2i-2)m+5i+2a-6}{3}, & i \equiv 0 \pmod 6,\ m \equiv a \pmod 3,\ a = 1,2,3, \\ \frac{(2i-2)m+5i+4}{3}, & i \equiv 1 \pmod 6,\ i \neq 1,\ m \equiv 0 \pmod 3) \\ & i \equiv 4 \pmod 6,\ m \equiv 1 \pmod 3, \\ \frac{(2i-2)m+5i-2}{3}, & i \equiv 1 \pmod 6,\ i \neq 1,\ m \equiv 1,2 \pmod 3, \\ & i \equiv 3 \pmod 6,\ m \equiv 2 \pmod 3, \\ & i \equiv 4 \pmod 6,\ m \equiv 0,2 \pmod 3, \\ \frac{(2i-2)m+5i-2a+2}{3}, & i \equiv 2 \pmod 6,\ m \equiv a \pmod 3,\ a = 0,1,2, \\ & i \equiv 5 \pmod 6,\ m \equiv a \pmod 3,\ a = 1,2,3, \\ \frac{(2i-2)m+5i}{3}, & i \equiv 3 \pmod 6,\ m \equiv 0 \pmod 3, \\ \frac{(2i-2)m+5i+2}{3}, & i \equiv 3 \pmod 6,\ m \equiv 1 \pmod 3. \end{cases}$$

Edge label u_iv_i for $i = 1, 2, \cdots, n$:

$$\theta(u_iv_i) = \begin{cases} m+1, & i = 1, \\ \frac{m-4a+12}{3}, & i = 2,\ m \equiv a \pmod 3,\ a = 0,1,2, \\ \frac{(2i-3)m+5i-12}{3}, & i \equiv 0 \pmod 6,\ m \geq 1 \\ & i \equiv 4 \pmod 6,\ m \equiv 2 \pmod 3, \\ \frac{(2i-3)m+5i+4a-14}{3}, & i \equiv 1 \pmod 6,\ i \neq 1,\ m \equiv a \pmod 3,\ a = 1,2,3, \\ \frac{(2i-3)m+5i-4a-4}{3}, & i \equiv 2 \pmod 6,\ i \neq 2,\ m \equiv a \pmod 3,\ a = 0,1,2, \\ \frac{(2i-3)m+5i-6}{3}, & i \equiv 3 \pmod 6,\ m \geq 1, \\ \frac{(2i-3)m+5i-8}{3}, & i \equiv 4 \pmod 6,\ m \equiv 0 \pmod 3), \\ \frac{(2i-3)m+5i-4}{3}, & i \equiv 4 \pmod 6,\ m \equiv 1 \pmod 3), \\ \frac{(2i-3)m+5i-4a+2}{3}, & i \equiv 5 \pmod 6,\ m \equiv a \pmod 3,\ a = 1,2,3. \end{cases}$$

Edge label u_iv_{i+1} for $i = 1, 2, \cdots, n-1$:

$$\theta(u_iv_{i+1}) = \begin{cases} \frac{2m-2a+3}{3}, & i = 1,\ m \equiv a \pmod 3,\ a = 0,1,2, \\ \frac{2m-2a+12}{3}, & i = 2,\ m \equiv a \pmod 3,\ a = 0,1,2, \\ \frac{(2i-2)m+5i+2a-12}{3}, & i \equiv 0 \pmod 6,\ m \equiv a \pmod 3,\ a = 1,2,3, \\ \frac{(2i-2)m+5i-2}{3}, & i \equiv 1 \pmod 6,\ i \neq 1,\ m \equiv 0 \pmod 3) \\ & i \equiv 4 \pmod 6,\ m \equiv 1 \pmod 3, \\ \frac{(2i-2)m+5i-8}{3}, & i \equiv 1 \pmod 6,\ i \neq 1,\ m \equiv 1,2 \pmod 3, \\ & i \equiv 3 \pmod 6,\ m \equiv 2 \pmod 3, \\ & i \equiv 4 \pmod 6,\ m \equiv 0,2 \pmod 3, \end{cases}$$

Continue of edge label u_iv_{i+1}:

$$\theta(u_iv_{i+1}) = \begin{cases} \frac{(2i-2)m+5i-2a-4}{3}, & i \equiv 2 \pmod 6,\ i \neq 2,\ m \equiv a \pmod 3,\ a = 0,1,2, \\ & i \equiv 5 \pmod 6,\ m \equiv a \pmod 3,\ a = 1,2,3, \\ \frac{(2i-2)m+5i-6}{3}, & i \equiv 3 \pmod 6,\ m \equiv 0 \pmod 3, \\ \frac{(2i-2)m+5i-4}{3}, & i \equiv 3 \pmod 6,\ m \equiv 1 \pmod 3. \end{cases}$$

Edge label v_iu_{i+1} for $i = 1, 2, \cdots, n-1$:

$$\theta(v_iu_{i+1}) = \begin{cases} \frac{2m-2a+12}{3}, & i = 1,\ m \equiv a \pmod 3,\ a = 0,1,2, \\ \frac{(2i-2)m+5i+2a-9}{3}, & i \equiv 0 \pmod 6,\ m \equiv a \pmod 3,\ a = 1,2,3, \\ \frac{(2i-2)m+5i+1}{3}, & i \equiv 1 \pmod 6,\ i \neq 1,\ m \equiv 0 \pmod 3 \\ & i \equiv 4 \pmod 6,\ m \equiv 1 \pmod 3, \\ \frac{(2i-2)m+5i-5}{3}, & i \equiv 1 \pmod 6,\ m \equiv 1,2 \pmod 3,\ i \neq 1, \\ & i \equiv 3 \pmod 6,\ m \equiv 2 \pmod 3, \\ & i \equiv 4 \pmod 6,\ m \equiv 0,2 \pmod 3, \\ \frac{(2i-2)m+5i-2a-1}{3}, & i \equiv 2 \pmod 6,\ m \equiv a \pmod 3,\ a = 0,1,2 \\ & i \equiv 5 \pmod 6,\ m \equiv a \pmod 3,\ a = 1,2,3, \\ \frac{(2i-2)m+5i-3}{3}, & i \equiv 3 \pmod 6,\ m \equiv 0 \pmod 3, \\ \frac{(2i-2)m+5i-1}{3}, & i \equiv 3 \pmod 6,\ m \equiv 1 \pmod 3. \end{cases}$$

Edge label $u_iu_{i,j}$ for $i = 1, 2, \cdots, n$ and $j = 1, 2, \cdots, m$:

$$\theta(u_iu_{i,j}) = \begin{cases} j, & i = 1, \\ j+3, & i = 2,\ m \equiv 0,1 \pmod 3, \\ j+1, & i = 2,\ m \equiv 2 \pmod 3, \\ j+2, & i = 3, \\ (\frac{(2i-6)m+5i-15}{3}) + j, & i \equiv 0 \pmod 6,\ m \geq 1, \\ & i \equiv 4 \pmod 6,\ m \equiv 2 \pmod 3, \\ (\frac{(2i-6)m+5i+4a-17}{3}) + j, & i \equiv 1 \pmod 6,\ m \equiv a \pmod 3,\ a = 1,2,3,\ i \neq 1, \\ (\frac{(2i-6)m+5i-4a-7}{3}) + j, & i \equiv 2 \pmod 6,\ m \equiv a \pmod 3,\ a = 0,1,2,\ i \neq 2, \\ (\frac{(2i-6)m+5i-9}{3}) + j, & i \equiv 3 \pmod 6,\ i \neq 3,\ m \geq 1. \\ (\frac{(2i-6)m+5i-11}{3}) + j, & i \equiv 4 \pmod 6,\ m \equiv 0 \pmod 3, \\ (\frac{(2i-6)m+5i-7}{3}) + j, & i \equiv 4 \pmod 6,\ m \equiv 1 \pmod 3, \\ (\frac{(2i-6)m+5i-4a-1}{3}) + j, & i \equiv 5 \pmod 6,\ m \equiv a \pmod 3,\ a = 1,2,3. \end{cases}$$

Edge label $v_iv_{i,j}$ with $i = 1, 2, \cdots, n$ and $j = 1, 2, \cdots, m$:

$$\theta(v_iv_{i,j}) = \begin{cases} (\frac{m+3}{3}) + j, & i = 1,\, m \equiv 0 \pmod 3, \\ (\frac{m+5}{3}) + j, & i = 1,\, m \equiv 1 \pmod 3, \\ (\frac{m+1}{3}) + j, & i = 1,\, m \equiv 2 \pmod 3, \\ 2, & i = 2,\, m = 1, \\ (\frac{m+8}{3}) + j, & i = 2,\, m \equiv 1 \pmod 3,\, m \neq 1 \\ (\frac{(2i-3)m+5i-12}{3}) + j, & i \equiv 0 \pmod 6,\, m \geq 1 \\ & i \equiv 4 \pmod 6,\, m \equiv 2 \pmod 3, \\ (\frac{(2i-3)m+5i+4a-14}{3}) + j, & i \equiv 1 \pmod 6,\, m \equiv a \pmod 3,\, a = 1, 2, 3,\, i \neq 1, \\ (\frac{(2i-3)m+5i-4a-4}{3}) + j, & i \equiv 2 \pmod 6,\, m \equiv a \pmod 3,\, a = 0, 1, 2,,\, i \neq 2, \\ (\frac{(2i-3)m+5i-6}{3}) + j, & i \equiv 3 \pmod 6,\, m \geq 1, \\ (\frac{(2i-3)m+5i-8}{3}) + j, & i \equiv 4 \pmod 6,\, m \equiv 0 \pmod 3, \\ (\frac{(2i-3)m+5i-4}{3}) + j, & i \equiv 4 \pmod 6,\, m \equiv 1 \pmod 3, \\ (\frac{(2i-3)m+5i-4a+2}{3}) + j, & i \equiv 5 \pmod 6,\, m \equiv a \pmod 3,\, a = 1, 2, 3. \end{cases}$$

The weight of an edge uv is $wt_\theta(uv) = \theta(u) + \theta(v) + \theta(uv)$. According to the above labeling, the edge weights are as follows:

$$\begin{aligned} wt_\theta(u_iu_{i+1}) &= 2im + 5i - 3, \text{ for } 1 \leq i \leq n-1. \\ wt_\theta(v_iv_{i+1}) &= 2im + 5i, \text{ for } 1 \leq i \leq n-1. \\ wt_\theta(u_iv_i) &= (2i-1)m + 5i - 4, \text{for } 2 \leq i \leq n. \\ wt_\theta(u_iv_{i+1}) &= 2im + 5i - 2, \text{ for } 1 \leq i \leq n-1. \\ wt_\theta(v_iu_{i+1}) &= 2im + 5i - 1, \text{ for } 1 \leq i \leq n-1. \\ wt_\theta(u_iu_{i,j}) &= \begin{cases} j, & \text{for } i = 1,\, 1 \leq j \leq m, \\ (2i-2)m + 5i - 5 + j, & \text{for } 2 \leq i \leq n,\, 1 \leq j \leq m. \end{cases} \\ wt_\theta(v_iv_{i,j}) &= (2i-1)m + 5i - 4 + j, \text{ for } 1 \leq i \leq n,\, 1 \leq j \leq m. \end{aligned}$$

It could be seen that all edge weights are different, the lower bound and upper bound are the same as $res(DL_n \odot N_m)$. Therefore θ is the "edge irregular reflexive k-labeling" and the formula of the res according to Theorem 2. Thus, the theorem is proven.

Figure 4 gives an example of this labeling. The label of each edge is indicated by blue colour and each vertex label is indicated by green colour. The weight of each edge is indicated by red colour.

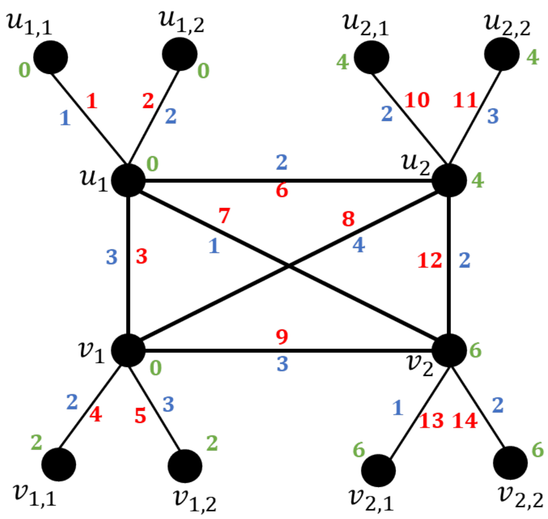

Figure 4: The edge irregular reflexive 6-labeling of $DL_2 \odot N_2$

3. Conclusion

We have proved the "reflexive edge strength" of ladder graph corona null graph families $SL_n \odot N_m$ and $DL_n \odot N_m$ as follows:

$$res(SL_n \odot N_m) = \begin{cases} \left\lceil \frac{2nm+3n-3}{3} \right\rceil, \text{if } 2nm + 3n - 3 \not\equiv 2, 3 \pmod 6, \\ \left\lceil \frac{2nm+3n-3}{3} \right\rceil + 1, \text{if } 2nm + 3n - 3 \equiv 2, 3 \pmod 6. \end{cases}$$

Moreover,

$$res(DL_n \odot N_m) = \begin{cases} \left\lceil \frac{2nm+5n-4}{3} \right\rceil, & 2nm + 5n - 4 \not\equiv 2, 3 \pmod 6, \\ \left\lceil \frac{2nm+5n-4}{3} \right\rceil + 1, & 2nm + 5n - 4 \equiv 2, 3 \pmod 6. \end{cases}$$

For the future research, we propose the open problem as follows: how is the reflexive edge strength for other graph corona null graph.

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