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Harnessing Quantum Superposition in Soft Set Theory: Introducing Quantum Hypersoft and SuperHyperSoft Sets

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Abstract. This paper presents two novel extensions of the Quantum Soft Set framework by integrating the hierarchical structures of hypersoft and superhypersoft sets with quantum superposition principles. Soft sets offer a versatile approach to decision making by associating parameters with subsets of a universal set, effectively capturing uncertainty and imprecision [1, 2]. Hypersoft and superhypersoft sets further generalize this paradigm for increasingly complex scenarios. A Quantum Soft Set maps each parameter to a normalized quantum state [3], enabling probabilistic membership via amplitude coefficients. We rigorously define the Quantum Hypersoft Set and the Quantum SuperHypersoft Set, laying a foundation for future advances in quantum-enhanced decision analysis, topological modeling, and algebraic applications.

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1. Introduction

1.1. Soft Sets and Their Extensions

The challenge of modeling uncertainty has led to a variety of mathematical frameworks, including fuzzy sets [4], hyperfuzzy sets [5, 6], super-hyperfuzzy sets [7], rough sets [8], hyperrough sets [9], intuitionistic fuzzy sets [10, 11], picture fuzzy sets [12], bipolar fuzzy sets [13, 14], hesitant fuzzy sets [15, 16], neutrosophic sets [17, 18], and plithogenic sets [19, 20].

Among these, soft sets [1, 2] are notable for associating each parameter with a subset of a universal set, thereby offering a flexible tool for decision-making under imprecision. Building on this idea, researchers have proposed numerous generalizations:

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- *Hypersoft Sets* [21, 22] and *SuperHypersoft Sets* [23], which add layers of parameter interactions;
- *Expert Soft Sets* [24] and *Hypersoft Expert Sets* [25, 26], integrating expert judgments;
- *Fuzzy Soft Sets* [27] and *IndetermSoft Sets* [28–30], which blend fuzzy membership or indeterminacy with soft parameters;
- *TreeSoft Sets* [31, 32], organizing parameters into hierarchies, and *ForestSoft Sets* [33], which combine multiple trees into a cohesive framework.

Soft Sets have proven useful across decision analysis [34], topology [35], graph theory [36], risk assessment [25], and algebraic structures. Consequently, advancing soft set theory and its variants remains an important endeavor.

In this paper, we explore how Soft Sets, Hypersoft Sets, and SuperHypersoft Sets can be treated within the framework of quantum theory. As summarized in Table 1, the three frameworks differ in how they map parameters or attribute combinations to subsets of the universe and in the depth of their parameter interactions. Consequently, Hypersoft Sets and SuperHypersoft Sets are well suited for modeling real-world phenomena characterized by multi-layered, complex uncertainty.

Concept	**Mapping**	**Key Feature**
Soft Set	$\mathcal{F}\colon S \to \mathcal{P}(U)$	Associates each parameter $\alpha \in S$ with a subset $\mathcal{F}(\alpha) \subseteq U$ [1].
Hypersoft Set	$G\colon \mathcal{A}_1 \times \cdots \times \mathcal{A}_m \to \mathcal{P}(U)$	Maps each m-tuple of attribute values $(\gamma_1, \ldots, \gamma_m)$ to a subset $G(\gamma) \subseteq U$, enabling multi-attribute combinations[21].
SuperHypersoft Set	$F\colon \mathcal{P}(A_1) \times \cdots \times \mathcal{P}(A_n) \to \mathcal{P}(U)$	Extends Hypersoft by using power-set layers of each attribute domain, capturing interactions among arbitrary subsets of values[37].

Table 1: Overview of Soft Set, Hypersoft Set, and SuperHypersoft Set

1.2. Quantum Theory

Quantum Theory describes the behavior of matter and energy at atomic scales, employing superposition, wave–particle duality, and probabilistic measurement outcomes [38, 39]. These principles have garnered significant attention in recent years, particularly in contexts such as quantum computing [40, 41].

As noted above, this paper explores whether Soft Sets, Hypersoft Sets, and SuperHypersoft Sets can be extended within the framework of quantum theory. In the field of soft set theory, the Quantum-Soft Set [3] has already been introduced. The *Quantum-Soft*

Set enriches the classical soft set by mapping each parameter to a normalized quantum superposition in a Hilbert space, thereby encoding membership via amplitude coefficients and measurement probabilities. A related framework, the Quantum Fuzzy Set, has also been proposed (cf. [42–44]). In view of these quantum-enhanced approaches to uncertainty modeling, we anticipate that Quantum-Soft Sets will play a central role in future scientific and technological developments, and we therefore regard their rigorous study as both timely and essential.

1.3. Our Contributions

Research on Soft Sets and Quantum Theory is of considerable importance both in foundational mathematics and physics as well as in practical applications. Extending Soft Sets to richer frameworks is therefore a valuable and meaningful pursuit. While the Quantum-Soft Set has been defined, its integration into more advanced soft set models remains underexplored. In this paper, we introduce and formalize the *Quantum Hypersoft Set* and the *Quantum SuperHypersoft Set*, deriving their core properties and structural characteristics. By merging the quantum-soft paradigm with hypersoft and superhypersoft structures, we aim to stimulate new advances in set theory, decision-making methodologies, topological analysis, algebraic modeling, and quantum applications.

As summarized in Table 2, the three quantum-enhanced soft set frameworks differ in how they map parameters or attribute combinations to quantum states and in the scope of their attribute interactions.

Concept	**Mapping**	**Key Feature**
Quantum-Soft Set	$F\colon A \to \mathcal{H}(U)$	Assigns each attribute $a \in A$ to a normalized quantum state $\lvert\psi_a\rangle$, where $\lvert\alpha_{i,a}\rvert^2$ gives the membership probability of u_i[3].
Quantum-HyperSoft Set	$Q\colon \mathcal{A}_1 \times \cdots \times \mathcal{A}_m \to \mathcal{H}(U)$	Maps each tuple of values $(\gamma_1, \ldots, \gamma_m)$ to a quantum superposition, capturing joint multi-attribute influence via amplitude interference.
Quantum-SuperHypersoft Set	$Q\colon \mathcal{P}(A_1) \times \cdots \times \mathcal{P}(A_n) \to \mathcal{H}(U)$	Extends HyperSoft by mapping each subset-tuple of attribute values to a quantum state, modeling interactions among arbitrary subsets of values.

Table 2: Overview of Quantum Soft Set, Quantum HyperSoft Set, and Quantum SuperHyperSoft Set

1.4. Structure of this paper

The remainder of this paper is organized as follows. Section 2 reviews Soft Sets, Hypersoft Sets, SuperHypersoft Sets, and Quantum Soft Sets. Section 3 introduces and analyzes Quantum–HyperSoft Sets and Quantum–SuperHyperSoft Sets. Section 4 briefly examines the construction algorithms. Section 5 concludes the paper and discusses directions for future work.

2. Preliminaries

In this section, we present the foundational concepts of Soft Sets, Hypersoft Sets, and SuperHypersoft Sets. We also provide a concise explanation of the definition of Quantum Soft Sets.

2.1. Notation

Throughout this paper, all sets are assumed to be finite. The empty set is considered to be a subset of any set. For the basic operations related to each concept, the reader is referred to the corresponding references.

In this paper, we use the following notation.

- $U = \{u_1, \ldots, u_n\}$: a finite universe
- $\mathcal{P}(U)$: the power set of U
- $\mathcal{H}(U)$: the n-dimensional complex Hilbert space with orthonormal basis $\{|u_i\rangle\}_{i=1}^{n}$
- $\mathcal{C}$: the Cartesian product of the attribute domains (or of their power sets)
- General quantum state:

$$|\psi\rangle = \sum_{i=1}^{n} \alpha_i \, |u_i\rangle \,, \quad \sum_{i=1}^{n} \left|\alpha_i\right|^2 = 1$$

- Measurement probability:

$$P(u_i \mid \gamma) = \left|\alpha_{i,\gamma}\right|^2$$

2.2. Soft Sets

Let $U = \{u_1, \ldots, u_n\}$ be a finite universe and let A be a finite set of parameters. We denote by $\mathcal{P}(U)$ the power set of U. Given a subset $S \subseteq A$ of chosen parameters, a *soft set* over U (with respect to S) is defined as follows.

Definition 1 (Soft Set [2])**.** *A soft set over U parameterized by S is a pair $(\mathcal{F}, S)$ where*

$$\mathcal{F} \colon S \longrightarrow \mathcal{P}(U)$$

is a function assigning to each parameter $\alpha \in S$ *a subset* $\mathcal{F}(\alpha) \subseteq U$. *Equivalently,*

$$(\mathcal{F}, S) = \{(\alpha, \mathcal{F}(\alpha)) \mid \alpha \in S,\ \mathcal{F}(\alpha) \subseteq U\}.$$

Example 1 (Restaurant Selection)**.** *Let*

$$U = \{R_1, R_2, R_3, R_4, R_5\}$$

be a set of restaurants in a city, and let

$$A = \{\mathit{Italian},\ \mathit{Japanese},\ \mathit{Cheap},\ \mathit{FineDining},\ \mathit{FamilyFriendly}\}$$

be the full parameter set. Suppose a user selects

$$S = \{\mathit{Italian},\ \mathit{Cheap},\ \mathit{FamilyFriendly}\}.$$

Define the soft set $(\mathcal{F}, S)$ *by*

$$\begin{aligned} \mathcal{F}(\mathit{Italian}) &= \{R_1, R_3, R_5\}, \\ \mathcal{F}(\mathit{Cheap}) &= \{R_2, R_3, R_4\}, \\ \mathcal{F}(\mathit{FamilyFriendly}) &= \{R_1, R_4, R_5\}. \end{aligned}$$

Then:

- $\mathcal{F}(\mathit{Italian}) = \{R_1, R_3, R_5\}$ *is the set of Italian restaurants.*
- $\mathcal{F}(\mathit{Cheap}) = \{R_2, R_3, R_4\}$ *is the set of low-cost restaurants.*
- $\mathcal{F}(\mathit{FamilyFriendly}) = \{R_1, R_4, R_5\}$ *is the set of family-friendly restaurants.*

For instance,

$$\mathcal{F}(\mathit{Italian}) \cap \mathcal{F}(\mathit{Cheap}) = \{R_3\}$$

identifies R_3 *as the unique restaurant that is both Italian and inexpensive, while*

$$\mathcal{F}(\mathit{Italian}) \cap \mathcal{F}(\mathit{FamilyFriendly}) = \{R_1, R_5\}$$

identifies the Italian restaurants suitable for families. The soft set formalism thus enables flexible, parameter-driven filtering of U *under uncertainty.*

2.3. Hypersoft Sets

Let $U = \{u_1, \dots, u_n\}$ be a finite universe and let $\mathcal{P}(U)$ denote its power set. For a positive integer m, let

$$\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_m$$

be m disjoint attribute domains. We form their Cartesian product

$$\mathcal{C} = \mathcal{A}_1 \times \mathcal{A}_2 \times \cdots \times \mathcal{A}_m,$$

so that each $\gamma \in \mathcal{C}$ is an m-tuple

$$\gamma = (\gamma_1, \gamma_2, \dots, \gamma_m), \quad \gamma_i \in \mathcal{A}_i.$$

A *hypersoft set* over U is then defined as follows.

Definition 2 (Hypersoft Set [21])**.** *A hypersoft set over U with respect to the attribute product $\mathcal{C}$ is a pair $(G, \mathcal{C})$ where*

$$G \colon \mathcal{C} \longrightarrow \mathcal{P}(U)$$

is a function assigning to each $\gamma \in \mathcal{C}$ a subset $G(\gamma) \subseteq U$. Equivalently,

$$(G, \mathcal{C}) = \big\{ \big(\gamma,\, G(\gamma)\big) \mid \gamma \in \mathcal{C},\ G(\gamma) \subseteq U \big\}.$$

Example 2 (Laptop Purchase)**.** *Let*

$$U = \{\textit{XPS13},\ \textit{Envy13},\ \textit{MacBookAir},\ \textit{ZenBook},\ \textit{ThinkPad}\}$$

be a set of laptop models. Consider three attribute domains:

$$\mathcal{A}_1 = \{\textit{Dell},\ \textit{HP},\ \textit{Apple}\}, \quad \mathcal{A}_2 = \{\textit{Under1k},\ \textit{1k–2k},\ \textit{Over2k}\}, \quad \mathcal{A}_3 = \{\textit{SSD},\ \textit{HDD}\}.$$

Form

$$\mathcal{C} = \mathcal{A}_1 \times \mathcal{A}_2 \times \mathcal{A}_3,$$

so each $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ specifies a brand, price-range, and storage type. Define the hypersoft set $(G, \mathcal{C})$ by:

$$\begin{aligned}
G\big((\text{Dell},\ \text{1k}-\text{2k},\ \text{SSD})\big) &= \{\text{XPS13, Envy13}\},\\
G\big((\text{Apple},\ \text{Over2k},\ \text{SSD})\big) &= \{\text{MacBookAir}\},\\
G\big((\text{HP},\ \text{Under1k},\ \text{HDD})\big) &= \{\text{Envy13}\},\\
G\big((\text{Apple},\ \text{1k}-\text{2k},\ \text{HDD})\big) &= \{\text{MacBookAir, ZenBook}\},\\
G\big((\text{Dell},\ \text{Under1k},\ \text{SSD})\big) &= \{\text{ZenBook}\},\\
G\big((\text{HP},\ \text{Over2k},\ \text{SSD})\big) &= \{\text{ThinkPad}\}.
\end{aligned}$$

For example,

$$G\big((\textit{Dell},\ \textit{1k–2k},\ \textit{SSD})\big) = \{\textit{XPS13},\ \textit{Envy13}\}$$

lists all Dell laptops in the \$1k–\$2k range with SSD storage. This hypersoft set enables a buyer to filter models by any combination of brand, price, and storage type, even when some models lack complete attribute information.

2.4. SuperHypersoft Sets

Let $U = \{u_1, \ldots, u_n\}$ be a finite universe and let $\mathcal{P}(U)$ denote its power set. Let m be a positive integer, and let

$$A_1, A_2, \ldots, A_m$$

be m pairwise disjoint finite attribute value–sets:

$$A_i \cap A_j = \emptyset \quad (i \neq j).$$

For each i, write $\mathcal{P}(A_i)$ for the power set of A_i, and form the Cartesian product

$$\mathcal{C} = \mathcal{P}(A_1) \times \mathcal{P}(A_2) \times \cdots \times \mathcal{P}(A_m).$$

Each element $\gamma \in \mathcal{C}$ is an m-tuple

$$\gamma = (\alpha_1, \alpha_2, \ldots, \alpha_m), \quad \alpha_i \subseteq A_i.$$

A *SuperHypersoft Set* over U is defined as follows.

Definition 3 (SuperHypersoft Set [23]). *A SuperHypersoft Set over the universe U with respect to the attribute–power product $\mathcal{C}$ is a pair $(F, \mathcal{C})$ where*

$$F : \mathcal{C} \longrightarrow \mathcal{P}(U)$$

assigns to each $\gamma \in \mathcal{C}$ a subset $F(\gamma) \subseteq U$. Equivalently,

$$(F, \mathcal{C}) = \big\{ \big(\gamma, F(\gamma)\big) \mid \gamma \in \mathcal{C}, \ F(\gamma) \subseteq U \big\}.$$

Example 3 (Laptop Selection). *Let*

$$U = \{XPS13,\ Envy13,\ MacBookAir,\ MacBookPro\}.$$

Consider three attribute value–sets:

$$A_1 = \{Dell,\ HP,\ Apple\}, \quad A_2 = \{i5,\ i7,\ i9\}, \quad A_3 = \{Budget,\ Midrange,\ Premium\},$$

which are pairwise disjoint. Form

$$\mathcal{C} = \mathcal{P}(A_1) \times \mathcal{P}(A_2) \times \mathcal{P}(A_3).$$

Define the SuperHypersoft Set $(F, \mathcal{C})$ by, for example,

$$F\big(\{Dell, HP\}, \{i7\}, \{Midrange\}\big) = \{XPS13,\ Envy13\},$$
$$F\big(\{Apple\}, \{i5, i7\}, \{Premium\}\big) = \{MacBookAir,\ MacBookPro\}.$$

Here:

- $\gamma_1 = (\{Dell, HP\}, \{i7\}, \{Midrange\})$ *yields* $F(\gamma_1) = \{XPS13, Envy13\}$, *the midrange i7 models from Dell and HP.*
- $\gamma_2 = (\{Apple\}, \{i5, i7\}, \{Premium\})$ *yields* $F(\gamma_2) = \{MacBookAir, MacBookPro\}$, *the premium Apple models.*

Example 4 (Candidate Recruitment). *Let*

$$U = \{Alice,\ Bob,\ Carol\}$$

be a set of applicants. Consider three attributes:

$$A_1 = \{\textit{Developer, Manager}\}, \quad A_2 = \{\textit{Java, Python}\}, \quad A_3 = \{\textit{Remote, Onsite}\},$$

with $\mathcal{C} = \mathcal{P}(A_1) \times \mathcal{P}(A_2) \times \mathcal{P}(A_3)$. *Define*

$$F\big(\{\textit{Developer}\}, \{\textit{Python}\}, \{\textit{Remote, Onsite}\}\big) = \{\textit{Alice, Bob}\},$$
$$F\big(\{\textit{Developer, Manager}\}, \{\textit{Java}\}, \{\textit{Onsite}\}\big) = \{\textit{Bob, Carol}\}.$$

Thus:

- *For* $\gamma_1 = (\{\textit{Developer}\}, \{\textit{Python}\}, \{\textit{Remote, Onsite}\})$, $F(\gamma_1) = \{\textit{Alice, Bob}\}$.
- *For* $\gamma_2 = (\{\textit{Developer, Manager}\}, \{\textit{Java}\}, \{\textit{Onsite}\})$, $F(\gamma_2) = \{\textit{Bob, Carol}\}$.

2.5. Quantum-Soft Sets

A *complex Hilbert space* is a complete inner-product space over $\mathbb{C}$. In the finite-dimensional setting, let

$$U = \{u_1, u_2, \dots, u_n\}$$

be a finite universe, and denote by $\mathcal{H}(U)$ the n-dimensional complex Hilbert space with orthonormal basis $\{|u_i\rangle\}_{i=1}^n$.

Definition 4 (Quantum-Soft Set [3])**.** *Let* $A = \{a_1, a_2, \dots, a_m\}$ *be a finite set of parameters. A* Quantum-Soft Set *over* (U, A) *is a mapping*

$$F\colon A \longrightarrow \mathcal{H}(U), \quad a_j \mapsto |\psi_{a_j}\rangle = \sum_{i=1}^{n} \alpha_{i,j}\, |u_i\rangle\,,$$

subject to the normalization condition

$$\sum_{i=1}^{n} \big|\alpha_{i,j}\big|^2 = 1, \qquad j = 1, 2, \dots, m.$$

Measuring the state $|\psi_{a_j}\rangle$ *in the computational basis* $\{|u_i\rangle\}$ *yields outcome* $|u_i\rangle$ *with probability*

$$P\big(u_i \mid a_j\big) = \big|\alpha_{i,j}\big|^2.$$

Example 5 (Investment Portfolio Selection)**.** *Let*

$$U = \{\textit{Tech, Pharma, Energy}\}, \quad A = \{\textit{Growth, Stability}\}.$$

Identify $\mathcal{H}(U)$ *with* $\mathbb{C}^3$ *via the basis* $\{|\textit{Tech}\rangle, |\textit{Pharma}\rangle, |\textit{Energy}\rangle\}$. *Define the mapping* $F : A \to \mathcal{H}(U)$ *by*

$$|\psi_{\text{Growth}}\rangle = \sqrt{0.60}\ |\textit{Tech}\rangle + \sqrt{0.30}\ |\textit{Pharma}\rangle + \sqrt{0.10}\ |\textit{Energy}\rangle\,,$$

$$|\psi_{\text{Stability}}\rangle = \sqrt{0.20}\ |\mathit{Tech}\rangle + \sqrt{0.50}\ |\mathit{Pharma}\rangle + \sqrt{0.30}\ |\mathit{Energy}\rangle .$$

Each state is manifestly normalized (e.g. $0.60 + 0.30 + 0.10 = 1$*). Measuring in the basis* $\{|\mathit{Tech}\rangle, |\mathit{Pharma}\rangle, |\mathit{Energy}\rangle\}$ *yields*

$$P(\mathit{Tech} \mid \text{Growth}) = 0.60, \quad P(\mathit{Pharma} \mid \text{Growth}) = 0.30, \quad P(\mathit{Energy} \mid \text{Growth}) = 0.10,$$

$$P(\mathit{Tech} \mid \text{Stability}) = 0.20, \quad P(\mathit{Pharma} \mid \text{Stability}) = 0.50, \quad P(\mathit{Energy} \mid \text{Stability}) = 0.30.$$

This illustrates how a classical soft-set membership (uniform or weighted) can be embedded into a quantum state by choosing amplitudes whose squared moduli reproduce the desired probabilities.

We present below several mathematical theorems related to Quantum Soft Sets.

Theorem 1 (Soft Set Embedding)**.** *Let* $(\mathcal{F}, S)$ *be a classical Soft Set over the finite universe* $U = \{u_1, \ldots, u_n\}$ *with parameter set* $S = \{a_1, \ldots, a_m\}$ *and*

$$\mathcal{F} : S \longrightarrow \mathcal{P}(U), \qquad a_j \mapsto \mathcal{F}(a_j) \subseteq U.$$

Define a map

$$F_Q : S \longrightarrow \mathcal{H}(U)$$

by assigning to each $a_j \in S$ *the quantum state*

$$|\psi_j\rangle \;=\; \sum_{i=1}^{n} \alpha_{ij}\ |u_i\rangle\,, \quad \textit{where} \quad \alpha_{ij} = \begin{cases} \dfrac{1}{\sqrt{|\mathcal{F}(a_j)|}} & \textit{if } u_i \in \mathcal{F}(a_j), \\ 0 & \textit{otherwise.} \end{cases}$$

Then F_Q *is a valid Quantum-Soft Set over* (U, S)*. Moreover, measuring* $|\psi_j\rangle$ *yields exactly the classical Soft Set membership:*

$$P(u_i \mid a_j) = |\alpha_{ij}|^2 = \begin{cases} \dfrac{1}{|\mathcal{F}(a_j)|}, & u_i \in \mathcal{F}(a_j), \\ 0, & u_i \notin \mathcal{F}(a_j). \end{cases}$$

Proof. Fix $a_j \in S$. By construction, $\alpha_{ij} \neq 0$ precisely when $u_i \in \mathcal{F}(a_j)$, and there are $|\mathcal{F}(a_j)|$ such indices. Hence

$$\sum_{i=1}^{n} |\alpha_{ij}|^2 = |\mathcal{F}(a_j)| \times \frac{1}{|\mathcal{F}(a_j)|} = 1,$$

so $|\psi_j\rangle$ is normalized. Upon measurement in the basis $\{|u_i\rangle\}$, the Born rule gives

$$P(u_i \mid a_j) \;=\; |\alpha_{ij}|^2 = \begin{cases} \dfrac{1}{|\mathcal{F}(a_j)|}, & \text{if } u_i \in \mathcal{F}(a_j), \\ 0, & \text{otherwise,} \end{cases}$$

which matches exactly the uniform membership values of the classical Soft Set $\mathcal{F}$. Therefore, F_Q is a Quantum-Soft Set whose measurement reproduces $(\mathcal{F}, S)$.

Theorem 2 (Unitary Invariance). *Let $F : A \to \mathcal{H}(U)$ be any Quantum-Soft Set over (U, A). For each $a \in A$, let U_a be a unitary operator on $\mathcal{H}(U)$. Define a new map*

$$F' : A \longrightarrow \mathcal{H}(U), \quad a \mapsto |\psi'_a\rangle = U_a\ |\psi_a\rangle\,,$$

where $|\psi_a\rangle = F(a)$. Then F' is also a Quantum-Soft Set over (U, A).

Proof. Fix $a \in A$. Since U_a is unitary, it preserves inner products and norms. In particular,

$$\begin{aligned} \langle \psi_a \mid \psi_a \rangle = 1 \quad \Longrightarrow \quad & \langle \psi'_a \mid \psi'_a \rangle \\ &= \langle U_a\psi_a \mid U_a\psi_a \rangle \\ &= \langle \psi_a \mid U_a^\dagger U_a \mid \psi_a \rangle \\ &= \langle \psi_a \mid \psi_a \rangle = 1. \end{aligned}$$

Hence each $|\psi'_a\rangle$ is normalized and F' satisfies the definition of a Quantum-Soft Set.

Theorem 3 (Crisp State Characterization). *Let $F : A \to \mathcal{H}(U)$ be a Quantum-Soft Set. Then F induces a classical (crisp) Soft Set if and only if, for every $a \in A$, the quantum state $|\psi_a\rangle$ has exactly one nonzero amplitude (up to a global phase). Equivalently, there exists an index i_a such that*

$$\left|\alpha_{i_a,a}\right| = 1, \quad \text{and} \quad \alpha_{i,a} = 0 \quad \left(\forall\, i \neq i_a\right).$$

In that case, measuring $|\psi_a\rangle$ yields the deterministic outcome $|u_{i_a}\rangle$, reproducing the classical Soft Set assignment $\mathcal{F}(a) = \{\, u_{i_a}\}$.

Proof. $\Rightarrow$ Suppose F induces a crisp Soft Set $(\mathcal{F}, A)$. By definition, for each $a \in A$, measuring $|\psi_a\rangle$ must collapse deterministically to a single element u_{i_a}. The Born rule then implies $\left|\alpha_{i_a,a}\right|^2 = 1$ and $\left|\alpha_{i,a}\right|^2 = 0$ for $i \neq i_a$. Thus $\alpha_{i_a,a} = e^{i\theta}$ (global phase) and $\alpha_{i,a} = 0$ otherwise.

$\Leftarrow$ Conversely, if for each $a \in A$ there is a unique index i_a with $\left|\alpha_{i_a,a}\right| = 1$ and $\alpha_{i,a} = 0$ for $i \neq i_a$, then

$$\sum_{i=1}^{n} \left|\alpha_{i,a}\right|^2 = 1^2 = 1,$$

and measurement of $|\psi_a\rangle$ yields

$$P(u_i \mid a) = \left|\alpha_{i,a}\right|^2 = \begin{cases} 1, & i = i_a, \\ 0, & i \neq i_a, \end{cases}$$

deterministically collapsing to $|u_{i_a}\rangle$. Defining $\mathcal{F}(a) = \{\, u_{i_a}\}$ recovers a classical Soft Set.

Theorem 4 (Normalization Implies Probability Distribution)**.** *Let $F : A \to \mathcal{H}(U)$ be a Quantum-Soft Set with*

$$|\psi_a\rangle = \sum_{i=1}^{n} \alpha_{i,a}\ |u_i\rangle\,, \quad \sum_{i=1}^{n} \left|\alpha_{i,a}\right|^2 = 1.$$

Then for each $a \in A$, the collection $\{\ |\alpha_{i,a}|^2 \mid i = 1, \ldots, n\}$ forms a probability distribution on U. In particular,

$$0 \ \le\ |\alpha_{i,a}|^2 \ \le\ 1, \quad \sum_{i=1}^{n} |\alpha_{i,a}|^2 = 1.$$

Proof. Since $\alpha_{i,a} \in \mathbb{C}$, each $\left|\alpha_{i,a}\right|^2$ is a real number satisfying $0 \le \left|\alpha_{i,a}\right|^2 \le 1$. The normalization condition $\sum_{i=1}^{n} \left|\alpha_{i,a}\right|^2 = 1$ then implies $\{|\alpha_{i,a}|^2\}$ is a discrete probability distribution indexed by $i = 1, \ldots, n$. Hence the measurement outcomes $|u_i\rangle$ occur with probability $|\alpha_{i,a}|^2$.

3. Result: Quantum-HyperSoft Set and Quantum-SuperHyperSoft Set

In this section, we present the main results of this paper by providing the definitions, concrete examples, and mathematical theorems of the Quantum-HyperSoft Set and the Quantum-SuperHyperSoft Set.

3.1. Quantum-HyperSoft Sets

Quantum-HyperSoft Set assigns each attribute tuple to a normalized quantum superposition over universe. The definition is stated as follows.

Definition 5 (Quantum-HyperSoft Set)**.** *Let $U = \{u_1, u_2, \ldots, u_n\}$ be a finite universe and let $\mathcal{A}_1, \ldots, \mathcal{A}_m$ be m distinct attribute domains. Form the Cartesian product*

$$\mathcal{C} \ = \ \mathcal{A}_1 \times \cdots \times \mathcal{A}_m,$$

so that each $\gamma = (\gamma_1, \ldots, \gamma_m) \in \mathcal{C}$ is an m-tuple with $\gamma_i \in \mathcal{A}_i$. Let $\mathcal{H}(U)$ be the complex Hilbert space with orthonormal basis $\{|u_i\rangle\}_{i=1}^{n}$. A Quantum-HyperSoft Set *over $(U, \mathcal{C})$ is a map*

$$Q \colon \mathcal{C} \ \longrightarrow\ \mathcal{H}(U), \qquad \gamma\ \mapsto\ |\psi_\gamma\rangle = \sum_{i=1}^{n} \alpha_{i,\gamma}\ |u_i\rangle\,,$$

subject to the normalization condition

$$\sum_{i=1}^{n} \left|\alpha_{i,\gamma}\right|^2 \ = \ 1 \quad \textit{for each } \gamma \in \mathcal{C}.$$

Here $\alpha_{i,\gamma} \in \mathbb{C}$ *is the amplitude of element* u_i *under the combined attribute* γ*. Measuring* $|\psi_\gamma\rangle$ *in the computational basis yields*

$$P(u_i \mid \gamma) \;=\; \left|\alpha_{i,\gamma}\right|^2,$$

so that the post-measurement membership degrees $\mu_\gamma(u_i) = P(u_i \mid \gamma)$ *form a probability distribution on* U*.*

Example 6 (Quantum-HyperSoft Set for Medical Diagnosis)**.** *Let*

$$U = \{\mathit{Flu},\ \mathit{COVID\text{-}19},\ \mathit{CommonCold}\}$$

be the set of possible diagnoses. Consider three symptom-attribute domains:

$$\mathcal{A}_1 = \{\text{HighFever},\ \text{LowFever}\},\quad \mathcal{A}_2 = \{\text{DryCough},\ \text{WetCough}\},\quad \mathcal{A}_3 = \{\text{Fatigue},\ \text{NoFatigue}\}.$$

Form the Cartesian product

$$\mathcal{C} = \mathcal{A}_1 \times \mathcal{A}_2 \times \mathcal{A}_3,$$

so that each $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ *is a symptom-combination. Let* $\mathcal{H}(U)$ *be the complex Hilbert space with orthonormal basis* $\{|\text{Flu}\rangle, |\text{COVID-19}\rangle, |\text{CommonCold}\rangle\}$*.*

Case 1: $\gamma_1 = (\text{HighFever}, \text{DryCough}, \text{Fatigue})$*. Define the quantum state*

$$|\psi_{\gamma_1}\rangle = \sqrt{0.65}\ |\text{COVID-19}\rangle + \sqrt{0.25}\ |\text{Flu}\rangle + \sqrt{0.10}\ |\text{CommonCold}\rangle.$$

Normalization check:

$$0.65 + 0.25 + 0.10 = 1.00.$$

Measurement probabilities:

$$P\big(\text{COVID-19} \mid \gamma_1\big) = 0.65,\quad P\big(\text{Flu} \mid \gamma_1\big) = 0.25,\quad P\big(\text{CommonCold} \mid \gamma_1\big) = 0.10.$$

Case 2: $\gamma_2 = (\text{LowFever}, \text{WetCough}, \text{NoFatigue})$*. Define*

$$|\psi_{\gamma_2}\rangle = \sqrt{0.15}\ |\text{COVID-19}\rangle + \sqrt{0.30}\ |\text{Flu}\rangle + \sqrt{0.55}\ |\text{CommonCold}\rangle,$$

with

$$0.15 + 0.30 + 0.55 = 1.00.$$

Thus

$$P\big(\text{COVID-19} \mid \gamma_2\big) = 0.15,\quad P\big(\text{Flu} \mid \gamma_2\big) = 0.30,\quad P\big(\text{CommonCold} \mid \gamma_2\big) = 0.55.$$

Interpretation.

- *For* γ_1*, the "HighFever + DryCough + Fatigue" combination strongly suggests COVID-19 (65 %), moderately suggests Flu (25 %), and rarely Common Cold (10 %).*

- *For γ_2, the "LowFever + WetCough + NoFatigue" combination most likely corresponds to Common Cold (55 %), with lower likelihoods for Flu (30 %) and COVID-19 (15 %).*

This Quantum-HyperSoft Set $Q\colon \mathcal{C} \to \mathcal{H}(U)$ thus models uncertain diagnoses by encoding symptom combinations as quantum-like superpositions, whose measurement probabilities yield a soft membership degree of each disease given the observed symptoms.

Example 7 (Quantum-HyperSoft Set for Credit Risk Assessment)**.** *Let*

$$U = \{Approve,\ Review,\ Deny\}$$

be the set of possible credit decisions. Consider three financial-attribute domains:

$$\mathcal{A}_1 = \{HighIncome,\ LowIncome\},$$

$$\mathcal{A}_2 = \{GoodHistory,\ PoorHistory\},$$

$$\mathcal{A}_3 = \{LowDTI,\ HighDTI\},$$

where DTI = debt-to-income ratio. Form

$$\mathcal{C} = \mathcal{A}_1 \times \mathcal{A}_2 \times \mathcal{A}_3.$$

Let $\mathcal{H}(U)$ be the complex Hilbert space with orthonormal basis $\{|\text{Approve}\rangle, |\text{Review}\rangle, |\text{Deny}\rangle\}$.

Case 1: $\gamma_1 = (\text{HighIncome, GoodHistory, LowDTI})$*. Define*

$$|\psi_{\gamma_1}\rangle = \sqrt{0.80}\ |\text{Approve}\rangle + \sqrt{0.15}\ |\text{Review}\rangle + \sqrt{0.05}\ |\text{Deny}\rangle,$$

so that

$$P(\text{Approve} \mid \gamma_1) = 0.80, \quad P(\text{Review} \mid \gamma_1) = 0.15, \quad P(\text{Deny} \mid \gamma_1) = 0.05.$$

Case 2: $\gamma_2 = (\text{LowIncome, PoorHistory, HighDTI})$*. Define*

$$|\psi_{\gamma_2}\rangle = \sqrt{0.10}\ |\text{Approve}\rangle + \sqrt{0.20}\ |\text{Review}\rangle + \sqrt{0.70}\ |\text{Deny}\rangle,$$

with

$$P(\text{Approve} \mid \gamma_2) = 0.10, \quad P(\text{Review} \mid \gamma_2) = 0.20, \quad P(\text{Deny} \mid \gamma_2) = 0.70.$$

Interpretation.

- *Under γ_1, strong profile yields an 80*
- *Under γ_2, weak profile yields a 70*

Example 8 (Quantum-HyperSoft Set for Traffic Incident Prediction)**.** *Let*

$$U = \{NoIncident,\ MinorIncident,\ MajorIncident\}$$

be the set of traffic-incident outcomes. Consider three sensor-attribute domains:

$$\mathcal{A}_1 = \{HeavyFlow,\ LightFlow\},$$

$$\mathcal{A}_2 = \{AdverseWeather,\ ClearWeather\},$$

$$\mathcal{A}_3 = \{Daytime,\ Nighttime\}.$$

Form

$$\mathcal{C} = \mathcal{A}_1 \times \mathcal{A}_2 \times \mathcal{A}_3,$$

and let $\mathcal{H}(U)$ *have basis* $\{|\text{NoIncident}\rangle, |\text{MinorIncident}\rangle, |\text{MajorIncident}\rangle\}$.

Case 1: $\gamma_1 = (\text{HeavyFlow, AdverseWeather, Nighttime})$*. Define*

$$|\psi_{\gamma_1}\rangle = \sqrt{0.20}\ |\text{NoIncident}\rangle + \sqrt{0.50}\ |\text{MinorIncident}\rangle + \sqrt{0.30}\ |\text{MajorIncident}\rangle,$$

so that

$$P(\text{NoIncident} \mid \gamma_1) = 0.20, \quad P(\text{MinorIncident} \mid \gamma_1) = 0.50, \quad P(\text{MajorIncident} \mid \gamma_1) = 0.30.$$

Case 2: $\gamma_2 = (\text{LightFlow, ClearWeather, Daytime})$*. Define*

$$|\psi_{\gamma_2}\rangle = \sqrt{0.75}\ |\text{NoIncident}\rangle + \sqrt{0.20}\ |\text{MinorIncident}\rangle + \sqrt{0.05}\ |\text{MajorIncident}\rangle,$$

yielding

$$P(\text{NoIncident} \mid \gamma_2) = 0.75, \quad P(\text{MinorIncident} \mid \gamma_2) = 0.20, \quad P(\text{MajorIncident} \mid \gamma_2) = 0.05.$$

Interpretation.

- *Under* γ_1*, high risk conditions yield a 30 % chance of a major incident.*
- *Under* γ_2*, benign conditions yield a 75 % chance of no incident.*

Theorem 5. *Every Hypersoft Set and every Quantum-Soft Set arises as a special case of a Quantum-HyperSoft Set.*

Proof.

(i) *Reduction to Hypersoft Set.* Let $(G, \mathcal{C})$ be a classical Hypersoft Set over U, so $G\colon \mathcal{C} \to \mathcal{P}(U)$ with $G(\gamma) \subseteq U$ for each γ [21]. Define amplitudes

$$\alpha_{i,\gamma} = \begin{cases} \dfrac{1}{\sqrt{|G(\gamma)|}} & \text{if } u_i \in G(\gamma), \\ 0 & \text{otherwise.} \end{cases}$$

Then $\sum_i |\alpha_{i,\gamma}|^2 = |G(\gamma)| \cdot \frac{1}{|G(\gamma)|} = 1$, so $Q(\gamma) = |\psi_\gamma\rangle$ is a valid Quantum-HyperSoft state. Moreover, the measurement probabilities satisfy

$$P(u_i \mid \gamma) = |\alpha_{i,\gamma}|^2 = \begin{cases} \dfrac{1}{|G(\gamma)|}, & u_i \in G(\gamma), \\ 0, & \text{else}, \end{cases}$$

recovering exactly the classical Hypersoft membership $u_i \in G(\gamma)$.

(ii) *Reduction to Quantum-Soft Set.* If $m = 1$, then $\mathcal{C} = \mathcal{A}_1$ is a single attribute domain and the map

$$Q\colon \mathcal{A}_1 \to \mathcal{H}(U), \quad a \mapsto |\psi_a\rangle = \sum_{i=1}^{n} \alpha_{i,a} \, |u_i\rangle \, ,$$

matches precisely the definition of a Quantum-Soft Set, in which each attribute a is assigned a normalized superposition $|\psi_a\rangle$ over U [3].

Thus both classical Hypersoft and Quantum-Soft Sets embed naturally in the Quantum-HyperSoft framework.

Theorem 6 (Unitary Invariance)**.** *Let Q be a Quantum-HyperSoft Set over $(U, \mathcal{C})$. For each $\gamma \in \mathcal{C}$, let U_γ be a unitary operator on $\mathcal{H}(U)$, and define*

$$Q'(\gamma) \; = \; U_\gamma \, |\psi_\gamma\rangle \, .$$

Then Q' is also a Quantum-HyperSoft Set over $(U, \mathcal{C})$.

Proof. Since each U_γ is unitary, it preserves inner products and norms. In particular,

$$\langle \psi_\gamma \mid \psi_\gamma \rangle = 1 \quad \Longrightarrow \quad \langle U_\gamma \psi_\gamma \mid U_\gamma \psi_\gamma \rangle = \langle \psi_\gamma \mid U_\gamma^\dagger U_\gamma \mid \psi_\gamma \rangle = \langle \psi_\gamma \mid \psi_\gamma \rangle = 1.$$

Thus $Q'(\gamma) = U_\gamma \, |\psi_\gamma\rangle$ remains normalized for every γ, so Q' satisfies the definition of a Quantum-HyperSoft Set.

Theorem 7 (Crisp Hypersoft Embedding)**.** *A Quantum-HyperSoft Set Q induces a* crisp *Hypersoft Set if and only if for each $\gamma \in \mathcal{C}$ there exists a unique index i_γ such that*

$$|\alpha_{i_\gamma,\gamma}| = 1 \quad \text{and} \quad \alpha_{i,\gamma} = 0 \; (\forall \, i \neq i_\gamma).$$

In that case, measurement of $|\psi_\gamma\rangle$ yields $\mu_\gamma(u_{i_\gamma}) = 1$ and $\mu_\gamma(u_i) = 0$ for $i \neq i_\gamma$, corresponding exactly to the classical Hypersoft assignment $G(\gamma) = \{u_{i_\gamma}\}$.

Proof. ($\Rightarrow$) If Q is such that each $|\psi_\gamma\rangle$ collapses deterministically to a single element u_{i_γ}, then the Born rule implies $|\alpha_{i_\gamma,\gamma}|^2 = 1$ and $|\alpha_{i,\gamma}|^2 = 0$ for $i \neq i_\gamma$. Hence $\alpha_{i_\gamma,\gamma} = \pm 1$ (up to a global phase) and all other amplitudes vanish.

($\Leftarrow$) Conversely, if the amplitude condition holds, then $\sum_i |\alpha_{i,\gamma}|^2 = 1^2 = 1$ and measurement yields

$$P(u_i \mid \gamma) = \begin{cases} 1, & i = i_\gamma, \\ 0, & i \neq i_\gamma, \end{cases}$$

so the post-measurement membership is crisp, $G(\gamma) = \{u_{i_\gamma}\}$.

Theorem 8 (Maximum Entropy in Uniform Superposition). *Define the* Shannon entropy *of the measurement distribution at γ by*

$$H(\gamma) \;=\; -\sum_{i=1}^{n} P(u_i \mid \gamma)\, \log P(u_i \mid \gamma) = -\sum_{i=1}^{n} \big|\alpha_{i,\gamma}\big|^2 \log\big|\alpha_{i,\gamma}\big|^2.$$

Then $H(\gamma)$ is maximized if and only if $|\psi_\gamma\rangle$ is the uniform *superposition $\frac{1}{\sqrt{n}} \sum_{i=1}^{n} |u_i\rangle$, in which case $H(\gamma) = \log n$.*

Proof. By the concavity of the Shannon entropy on probability distributions, the maximum occurs at the uniform distribution $P(u_i \mid \gamma) = 1/n$ for all i. The only normalized pure state yielding this distribution is $|\psi_\gamma\rangle = \frac{1}{\sqrt{n}} \sum_i |u_i\rangle$. Hence

$$H(\gamma) \leq -n\,(1/n) \log(1/n) = \log n,$$

with equality if and only if $\alpha_{i,\gamma} = 1/\sqrt{n}$ up to phases.

3.2. Quantum-SuperHyperSoft Sets

Quantum-SuperHyperSoft Set maps each power-set combination of multi-valued attributes to normalized quantum states. The definition is stated as follows.

Definition 6 (Quantum–SuperHyperSoft Set). *Let $U = \{u_1, u_2, \ldots, u_n\}$ be a finite universe and let $a_1, a_2, \ldots, a_m$ be m distinct attributes with corresponding finite, pairwise disjoint value-sets $A_1, A_2, \ldots, A_m$:*

$$A_i \cap A_j = \emptyset \quad (i \neq j).$$

Form the Cartesian product of power-sets

$$\mathcal{C} \;=\; \mathcal{P}(A_1) \times \mathcal{P}(A_2) \times \cdots \times \mathcal{P}(A_m),$$

so each $\gamma = (\alpha_1, \alpha_2, \ldots, \alpha_m) \in \mathcal{C}$ with $\alpha_i \subseteq A_i$. Let $\mathcal{H}(U)$ be the complex Hilbert space with orthonormal basis $\{|u_i\rangle\}_{i=1}^{n}$. A Quantum–SuperHyperSoft Set *over $(U, \mathcal{C})$ is a map*

$$Q\colon \mathcal{C} \;\longrightarrow\; \mathcal{H}(U), \qquad \gamma \;\mapsto\; |\Psi_\gamma\rangle = \sum_{i=1}^{n} \alpha_{i,\gamma}\, |u_i\rangle\,,$$

subject to the normalization condition

$$\sum_{i=1}^{n} \left|\alpha_{i,\gamma}\right|^2 = 1 \quad \textit{for each } \gamma \in \mathcal{C}.$$

Here $\alpha_{i,\gamma} \in \mathbb{C}$ is the amplitude of element u_i under the combined (possibly multi-valued) attribute γ. Measuring $|\Psi_\gamma\rangle$ yields the probability distribution $\mu_\gamma(u_i) = |\alpha_{i,\gamma}|^2$ on U.

Example 9 (Quantum–SuperHyperSoft Set for Smartphone Recommendation)**.** *Let*

$$U = \{PhoneX,\ PhoneY,\ PhoneZ\}$$

be three candidate smartphones. Consider three attribute domains with disjoint value-sets:

$$A_1 = \{Apple,\ Samsung,\ Google\}, \quad A_2 = \{iOS,\ Android\}, \quad A_3 = \{4G,\ 5G\}.$$

Form the Cartesian product of power-sets

$$\mathcal{C} = \mathcal{P}(A_1) \times \mathcal{P}(A_2) \times \mathcal{P}(A_3),$$

so each element $\gamma = (\alpha_1, \alpha_2, \alpha_3)$ consists of $\alpha_1 \subseteq A_1,\ \alpha_2 \subseteq A_2,\ \alpha_3 \subseteq A_3$. Let $\mathcal{H}(U)$ be the complex Hilbert space with orthonormal basis $\{|PhoneX\rangle, |PhoneY\rangle, |PhoneZ\rangle\}$.

Case 1: Premium Apple 5G User

$$\gamma_1 = \big(\{Apple\},\ \{iOS\},\ \{5G\}\big).$$

Define the quantum state

$$|\Psi_{\gamma_1}\rangle = \sqrt{0.70}\ |PhoneX\rangle + \sqrt{0.20}\ |PhoneY\rangle + \sqrt{0.10}\ |PhoneZ\rangle .$$

Check normalization:

$$0.70 + 0.20 + 0.10 = 1.00.$$

Measurement probabilities:

$$P(PhoneX \mid \gamma_1) = 0.70, \quad P(PhoneY \mid \gamma_1) = 0.20, \quad P(PhoneZ \mid \gamma_1) = 0.10.$$

Case 2: Cross-Platform Budget 4G User

$$\gamma_2 = \big(\{Samsung, Google\},\ \{Android\},\ \{4G\}\big).$$

Define

$$|\Psi_{\gamma_2}\rangle = \sqrt{0.15}\ |PhoneX\rangle + \sqrt{0.55}\ |PhoneY\rangle + \sqrt{0.30}\ |PhoneZ\rangle ,$$

and verify

$$0.15 + 0.55 + 0.30 = 1.00.$$

Thus

$$P(PhoneX \mid \gamma_2) = 0.15, \quad P(PhoneY \mid \gamma_2) = 0.55, \quad P(PhoneZ \mid \gamma_2) = 0.30.$$

Interpretation.

- *Under γ_1 (Apple, iOS, 5G), PhoneX is strongly preferred (70%), PhoneY moderately (20%), and PhoneZ rarely (10%).*
- *Under γ_2 (Samsung or Google, Android, 4G), PhoneY is most likely (55%), PhoneZ next (30%), and PhoneX least (15%).*

This Quantum–SuperHyperSoft Set $Q\colon \mathcal{C} \to \mathcal{H}(U)$ encodes complex, overlapping user preferences (multi-brand, multi-network) as quantum superpositions. Measurement yields a soft ranking of the three phones for each preference combination.

Example 10 (Quantum–SuperHyperSoft Set for Movie Recommendation)**.** *Let*

$$U = \{MovieA,\ MovieB,\ MovieC\}$$

be three candidate films. Consider three attribute domains:

$$A_1 = \{Action,\ Drama\},$$

$$A_2 = \{Light,\ Dark\},$$

$$A_3 = \{Theater,\ Streaming\}.$$

Form the Cartesian product of power-sets

$$\mathcal{C} = \mathcal{P}(A_1) \times \mathcal{P}(A_2) \times \mathcal{P}(A_3),$$

so each $\gamma = (\alpha_1, \alpha_2, \alpha_3)$ with $\alpha_i \subseteq A_i$. Let $\mathcal{H}(U)$ be the Hilbert space with basis

$$\{|MovieA\rangle, |MovieB\rangle, |MovieC\rangle\}$$

.

Case 1: Action-Light-Streaming

$$\gamma_1 = \big(\{Action\},\ \{Light\},\ \{Streaming\}\big).$$

Define

$$|\Psi_{\gamma_1}\rangle$$
$$= \sqrt{0.80}\ |MovieA\rangle + \sqrt{0.15}\ |MovieB\rangle + \sqrt{0.05}\ |MovieC\rangle\,.$$

Check normalization:

$$0.80 + 0.15 + 0.05 = 1.00.$$

Thus

$$P(MovieA \mid \gamma_1) = 0.80, \quad P(MovieB \mid \gamma_1) = 0.15, \quad P(MovieC \mid \gamma_1) = 0.05.$$

Case 2: Drama-Dark-Theater

$$\gamma_2 = \big(\{Drama\},\ \{Dark\},\ \{Theater\}\big).$$

Define

$$|\Psi_{\gamma_2}\rangle = \sqrt{0.10}\ |MovieA\rangle + \sqrt{0.70}\ |MovieB\rangle + \sqrt{0.20}\ |MovieC\rangle\,,$$

with

$$0.10 + 0.70 + 0.20 = 1.00.$$

Hence

$$P(MovieA \mid \gamma_2) = 0.10, \quad P(MovieB \mid \gamma_2) = 0.70, \quad P(MovieC \mid \gamma_2) = 0.20.$$

This example shows how a Quantum–SuperHyperSoft Set models complex viewer preferences—genre, mood, and platform—as quantum superpositions, yielding soft rankings of films for each preference combination.

Example 11 (Quantum–SuperHyperSoft Set for Supplier Selection)**.** *Let*

$$U = \{S1,\ S2,\ S3\}$$

be three potential suppliers. Consider three attribute domains:

$$A_1 = \{Local,\ Overseas\}, \quad A_2 = \{LowCost,\ HighCost\}, \quad A_3 = \{ShortLead,\ LongLead\}.$$

Form

$$\mathcal{C} = \mathcal{P}(A_1) \times \mathcal{P}(A_2) \times \mathcal{P}(A_3),$$

and let the basis be $\{|S1\rangle\,, |S2\rangle\,, |S3\rangle\}$*.*

Case 1: Local, LowCost, ShortLead

$$\gamma_1 = \big(\{Local\},\ \{LowCost\},\ \{ShortLead\}\big).$$

Set

$$|\Psi_{\gamma_1}\rangle = \sqrt{0.60}\ |S1\rangle + \sqrt{0.30}\ |S2\rangle + \sqrt{0.10}\ |S3\rangle\,,$$

so

$$P(S1 \mid \gamma_1) = 0.60,\ P(S2 \mid \gamma_1) = 0.30,\ P(S3 \mid \gamma_1) = 0.10.$$

Case 2: Overseas, HighCost, LongLead

$$\gamma_2 = \big(\{Overseas\},\ \{HighCost\},\ \{LongLead\}\big).$$

Set

$$|\Psi_{\gamma_2}\rangle = \sqrt{0.05}\ |S1\rangle + \sqrt{0.25}\ |S2\rangle + \sqrt{0.70}\ |S3\rangle\,,$$

yielding

$$P(S1 \mid \gamma_2) = 0.05,\ P(S2 \mid \gamma_2) = 0.25,\ P(S3 \mid \gamma_2) = 0.70.$$

This shows how a Quantum–SuperHyperSoft Set can encode multi-criteria supplier characteristics as quantum-like states, with measurement probabilities giving soft selection weights.

Example 12 (Quantum–SuperHyperSoft Set for Algorithm Selection on Quantum Hardware)**.** *Let*

$$U = \{Grover,\ QAOA,\ VQE\}$$

be the set of candidate quantum algorithms. Consider three attribute domains with disjoint value-sets:

$$A_1 = \{IBM,\ Rigetti,\ IonQ\}, \quad A_2 = \{Superconducting,\ TrappedIon\}, \quad A_3 = \{AllToAll,\ Linear\}.$$

Form the Cartesian product of power-sets

$$\mathcal{C} = \mathcal{P}(A_1) \times \mathcal{P}(A_2) \times \mathcal{P}(A_3),$$

so each $\gamma = (\alpha_1, \alpha_2, \alpha_3)$ with $\alpha_i \subseteq A_i$. Let $\mathcal{H}(U)$ be the 3-dimensional complex Hilbert space with orthonormal basis $\{|\mathrm{Grover}\rangle, |\mathrm{QAOA}\rangle, |\mathrm{VQE}\rangle\}$.

Case 1: Hybrid Superconducting, All-to-All on IBM or Rigetti

$$\gamma_1 = \big(\{IBM, Rigetti\},\ \{Superconducting\},\ \{AllToAll\}\big).$$

Define the quantum state

$$|\Psi_{\gamma_1}\rangle = \sqrt{0.50}\,|\mathrm{QAOA}\rangle + \sqrt{0.30}\,|\mathrm{Grover}\rangle + \sqrt{0.20}\,|\mathrm{VQE}\rangle,$$

which satisfies

$$0.50 + 0.30 + 0.20 = 1.$$

Measuring in the basis $\{|\mathrm{Grover}\rangle, |\mathrm{QAOA}\rangle, |\mathrm{VQE}\rangle\}$ yields

$$P(\mathrm{QAOA} \mid \gamma_1) = 0.50, \quad P(\mathrm{Grover} \mid \gamma_1) = 0.30, \quad P(\mathrm{VQE} \mid \gamma_1) = 0.20.$$

Case 2: Ion-Trapped, Linear Connectivity on IonQ

$$\gamma_2 = \big(\{IonQ\},\ \{TrappedIon\},\ \{Linear\}\big).$$

Define

$$|\Psi_{\gamma_2}\rangle = \sqrt{0.10}\,|\mathrm{QAOA}\rangle + \sqrt{0.20}\,|\mathrm{Grover}\rangle + \sqrt{0.70}\,|\mathrm{VQE}\rangle,$$

with

$$0.10 + 0.20 + 0.70 = 1.$$

Thus

$$P(\mathrm{VQE} \mid \gamma_2) = 0.70, \quad P(\mathrm{Grover} \mid \gamma_2) = 0.20, \quad P(\mathrm{QAOA} \mid \gamma_2) = 0.10.$$

Interpretation.

- *Under γ_1, hybrid superconducting hardware with full connectivity favors QAOA (50 %), then Grover (30 %), then VQE (20 %).*

- *Under γ_2, trapped-ion hardware with linear connectivity strongly favors VQE (70 %) for chemistry simulations, with lower probabilities for Grover (20 %) and QAOA (10 %).*

Example 13 (Quantum–SuperHyperSoft Set for Dynamic Circuit Compilation)**.** *Let*

$$U = \{OptimizeDepth, MinimizeError, Balance\}$$

be compiler optimization strategies. Consider three attribute domains:

$$A_1 = \{HighGateError, LowGateError\},$$

$$A_2 = \{ShortQubitLifetime, LongQubitLifetime\},$$

$$A_3 = \{NoisyInterconnect, ReliableInterconnect\}.$$

Form

$$\mathcal{C} = \mathcal{P}(A_1) \times \mathcal{P}(A_2) \times \mathcal{P}(A_3),$$

and let $\mathcal{H}(U)$ have orthonormal basis $\{|\text{OptimizeDepth}\rangle, |\text{MinimizeError}\rangle, |\text{Balance}\rangle\}$.

Case 1: High Error, Short Lifetime, Noisy Links

$$\gamma_3 = \big(\{HighGateError\}, \{ShortQubitLifetime\}, \{NoisyInterconnect\}\big).$$

Set

$$|\Psi_{\gamma_3}\rangle = \sqrt{0.10}\ |\text{OptimizeDepth}\rangle + \sqrt{0.70}\ |\text{MinimizeError}\rangle + \sqrt{0.20}\ |\text{Balance}\rangle,$$

so

$$P(\text{MinimizeError} \mid \gamma_3) = 0.70,\ P(\text{Balance} \mid \gamma_3) = 0.20,\ P(\text{OptimizeDepth} \mid \gamma_3) = 0.10.$$

Case 2: Low Error, Long Lifetime, Reliable Links

$$\gamma_4 = \big(\{LowGateError\}, \{LongQubitLifetime\}, \{ReliableInterconnect\}\big).$$

Set

$$|\Psi_{\gamma_4}\rangle = \sqrt{0.40}\ |\text{OptimizeDepth}\rangle + \sqrt{0.30}\ |\text{Balance}\rangle + \sqrt{0.30}\ |\text{MinimizeError}\rangle,$$

giving

$$P(\text{OptimizeDepth} \mid \gamma_4) = 0.40,\ P(\text{Balance} \mid \gamma_4) = 0.30,\ P(\text{MinimizeError} \mid \gamma_4) = 0.30.$$

Interpretation.

- *Under γ_3, high-noise, short-lived hardware drives error-minimization (70 %) over depth optimization (10 %) or balanced trade-offs (20 %).*

- *Under γ_4, reliable hardware allows deeper circuits (40 %) with balanced (30 %) or error-focused (30 %) strategies.*

Theorem 9 (Generalization of SuperHypersoft Sets)**.** *Let $(F, \mathcal{C})$ be a classical SuperHypersoft Set $F: \mathcal{C} \to \mathcal{P}(U)$. Define amplitudes*

$$\alpha_{i,\gamma} = \begin{cases} \dfrac{1}{\sqrt{|F(\gamma)|}} & \text{if } u_i \in F(\gamma), \\ 0 & \text{otherwise.} \end{cases}$$

Then the map $Q: \mathcal{C} \to \mathcal{H}(U)$ with $|\Psi_\gamma\rangle = \sum_i \alpha_{i,\gamma} |u_i\rangle$ is a Quantum–SuperHyperSoft Set whose measurement-based membership $\mu_\gamma(u_i) = |\alpha_{i,\gamma}|^2$ recovers exactly the classical assignment $u_i \in F(\gamma)$.

Proof. For each γ, since $\alpha_{i,\gamma} \neq 0$ exactly when $u_i \in F(\gamma)$ and there are $|F(\gamma)|$ such i, we have

$$\sum_{i=1}^{n} \left|\alpha_{i,\gamma}\right|^2 = |F(\gamma)| \times \frac{1}{|F(\gamma)|} = 1.$$

Hence Q is normalized. Moreover

$$\mu_\gamma(u_i) = |\alpha_{i,\gamma}|^2 = \begin{cases} \frac{1}{|F(\gamma)|}, & u_i \in F(\gamma), \\ 0, & u_i \notin F(\gamma), \end{cases}$$

so measuring $|\Psi_\gamma\rangle$ yields a uniform distribution on the classical set $F(\gamma)$, thereby embedding the SuperHypersoft Set into the quantum framework.

Theorem 10 (Generalization of Quantum-Soft Sets)**.** *Let Q be a Quantum–SuperHyperSoft Set over $(U, \mathcal{C})$ with $m = 1$ and $A_1 = A$. Then restricting the domain to $\{\{a\} \mid a \in A\} \subseteq \mathcal{P}(A)$ gives a map*

$$Q' : A \longrightarrow \mathcal{H}(U),$$
$$a \mapsto \left|\Psi_{\{a\}}\right\rangle,$$

which is precisely a Quantum-Soft Set over A.

Proof. When $m = 1$, $\mathcal{C} = \mathcal{P}(A)$. Each singleton $\{a\} \subseteq A$ defines a quantum state $|\Psi_{\{a\}}\rangle$ with $\sum_i |\alpha_{i,\{a\}}|^2 = 1$. Identifying the label $\{a\} \leftrightarrow a$ and renaming $|\psi_a\rangle = |\Psi_{\{a\}}\rangle$ recovers the Quantum-Soft Set definition.

Theorem 11 (Generalization of Quantum–HyperSoft Sets)**.** *Let Q be a Quantum–SuperHyperSoft Set over $(U, \mathcal{C})$. Restrict the domain to $\{\, (\{a_1\}, \{a_2\}, \ldots, \{a_m\}) \mid a_i \in A_i\} \subseteq \mathcal{C}$. Then the induced map*

$$Q'' : A_1 \times \cdots \times A_m \longrightarrow \mathcal{H}(U),$$
$$(a_1, \ldots, a_m) \mapsto \left|\Psi_{(\{a_1\}, \ldots, \{a_m\})}\right\rangle$$

is exactly a Quantum–HyperSoft Set over $A_1, \ldots, A_m$.

Proof. Each tuple of singletons $(\{a_1\},\dots,\{a_m\})$ is an element of $\mathcal{P}(A_1)\times\cdots\times\mathcal{P}(A_m)$, and the associated state $|\Psi_{(\{a_1\},\dots,\{a_m\})}\rangle$ is normalized by assumption. Renaming indices to absorb the singleton braces yields a map on $A_1\times\cdots\times A_m$ exactly in the form of the Quantum–HyperSoft Set definition.

Theorem 12 (Unitary Invariance)**.** *Let Q be a Quantum–SuperHyperSoft Set over $(U,\mathcal{C})$. For each $\gamma\in\mathcal{C}$, let U_γ be a unitary operator on $\mathcal{H}(U)$, and define*

$$Q'(\gamma)=U_\gamma\;|\Psi_\gamma\rangle\;.$$

Then Q' is also a Quantum–SuperHyperSoft Set over $(U,\mathcal{C})$.

Proof. Since each U_γ is unitary, it preserves inner products and norms. In particular,

$$\big\langle\Psi_\gamma\mid\Psi_\gamma\big\rangle=1\quad\Longrightarrow\quad\big\langle U_\gamma\Psi_\gamma\mid U_\gamma\Psi_\gamma\big\rangle=\big\langle\Psi_\gamma\mid U_\gamma^\dagger U_\gamma\mid\Psi_\gamma\big\rangle=\big\langle\Psi_\gamma\mid\Psi_\gamma\big\rangle=1.$$

Thus each state $|\Psi'_\gamma\rangle=U_\gamma\,|\Psi_\gamma\rangle$ remains normalized, so Q' satisfies the definition of a Quantum–SuperHyperSoft Set.

Theorem 13 (Crisp Embedding)**.** *A Quantum–SuperHyperSoft Set Q induces a* crisp *SuperHypersoft Set if and only if for each $\gamma\in\mathcal{C}$ there is a unique index i_γ such that*

$$|\alpha_{i_\gamma,\gamma}|=1\quad\textit{and}\quad\alpha_{i,\gamma}=0\quad(\forall\,i\neq i_\gamma).$$

In that case, measurement of $|\Psi_\gamma\rangle$ yields $\mu_\gamma(u_{i_\gamma})=1$ and $\mu_\gamma(u_i)=0$ for $i\neq i_\gamma$, corresponding exactly to the classical SuperHypersoft assignment $F(\gamma)=\{u_{i_\gamma}\}$.

Proof. ($\Rightarrow$) If Q collapses deterministically to a single element u_{i_γ}, then the Born rule implies $|\alpha_{i_\gamma,\gamma}|^2=1$ and $|\alpha_{i,\gamma}|^2=0$ for $i\neq i_\gamma$. Thus $\alpha_{i_\gamma,\gamma}=\mathrm{e}^{i\theta}$ (global phase) and other amplitudes vanish.

($\Leftarrow$) Conversely, if the amplitude condition holds, then $\sum_i|\alpha_{i,\gamma}|^2=1$ and measurement yields

$$P(u_i\mid\gamma)=\begin{cases}1, & i=i_\gamma,\\ 0, & i\neq i_\gamma,\end{cases}$$

so the post-measurement membership is crisp, $F(\gamma)=\{u_{i_\gamma}\}$.

Theorem 14 (Entropy Bounds)**.** *Define the Shannon entropy of the measurement distribution at γ by*

$$H(\gamma)=-\sum_{i=1}^{n}\big|\alpha_{i,\gamma}\big|^2\,\log\big|\alpha_{i,\gamma}\big|^2.$$

Then

$$0\;\le\;H(\gamma)\;\le\;\log n,$$

with $H(\gamma)=0$ if and only if Q is crisp at γ, and $H(\gamma)=\log n$ if and only if $|\Psi_\gamma\rangle=\frac{1}{\sqrt{n}}\sum_{i=1}^{n}|u_i\rangle$.

Proof. By definition, Shannon entropy on a probability distribution is nonnegative, so $H(\gamma) \geq 0$, with equality exactly when one outcome has probability 1 (crisp case). Concavity of entropy on the simplex implies the maximum occurs at the uniform distribution $\left|\alpha_{i,\gamma}\right|^2 = 1/n$, yielding

$$H(\gamma) \leq -n\,(1/n)\log(1/n) = \log n,$$

with equality if and only if $\alpha_{i,\gamma} = 1/\sqrt{n}$ up to phase for all i.

Theorem 15 (Orthogonality of Disjoint Crisp States)**.** *Let Q be the crisp embedding of a SuperHypersoft Set F, as in Theorem 2. Then for any $\gamma, \gamma' \in \mathcal{C}$,*

$$F(\gamma) \cap F(\gamma') = \emptyset \implies \langle \Psi_\gamma \mid \Psi_{\gamma'} \rangle = 0.$$

Proof. By construction, $|\Psi_\gamma\rangle = \sum_{u_i \in F(\gamma)} \frac{1}{\sqrt{|F(\gamma)|}} |u_i\rangle$ and similarly for $|\Psi_{\gamma'}\rangle$. Hence

$$\langle \Psi_\gamma \mid \Psi_{\gamma'} \rangle = \sum_{i=1}^{n} \frac{\overline{\alpha_{i,\gamma}}\ \alpha_{i,\gamma'}}{\sqrt{|F(\gamma)|\,|F(\gamma')|}} = \sum_{u_i \in F(\gamma) \cap F(\gamma')} \frac{1}{\sqrt{|F(\gamma)|\,|F(\gamma')|}}.$$

If $F(\gamma) \cap F(\gamma') = \emptyset$, this sum is zero.

Theorem 16 (Fidelity Equals Classical Overlap for Crisp States)**.** *Under the same crisp embedding,*

$$\left|\langle \Psi_\gamma \mid \Psi_{\gamma'} \rangle\right|^2 = \frac{\left|F(\gamma) \cap F(\gamma')\right|^2}{|F(\gamma)|\,|F(\gamma')|}.$$

Proof. From the proof above, $\langle \Psi_\gamma | \Psi_{\gamma'} \rangle = \frac{|F(\gamma) \cap F(\gamma')|}{\sqrt{|F(\gamma)|\,|F(\gamma')|}}$. Squaring the modulus yields the stated formula.

Theorem 17 (Total Variation Distance Bound)**.** *For any two quantum states $|\Psi_\gamma\rangle, |\Psi_{\gamma'}\rangle$ in a Quantum–SuperHyperSoft Set, let $\{P(u_i \,|\, \gamma)\}$ and $\{P(u_i \,|\, \gamma')\}$ be their measurement distributions. Then the total variation distance* TV *satisfies*

$$\mathrm{TV}\big(P(\cdot\,|\,\gamma), P(\cdot\,|\,\gamma')\big) \;\leq\; \sqrt{1 - \left|\langle \Psi_\gamma \mid \Psi_{\gamma'} \rangle\right|^2}\,.$$

Proof. It is known that for pure states the fidelity $F = |\langle \Psi_\gamma | \Psi_{\gamma'} \rangle|^2$ and the classical total variation distance satisfy

$$\mathrm{TV}(P, P') \;\leq\; \sqrt{1 - F}.$$

Applying this to the two measurement distributions yields the bound.

Theorem 18 (Projector-Induced Sub-Set). *Let Q be a Quantum–SuperHyperSoft Set and let Π be any projector on $\mathcal{H}(U)$ that is diagonal in the $\{|u_i\rangle\}$ basis. Define*

$$|\Phi_\gamma\rangle = \frac{\Pi\,|\Psi_\gamma\rangle}{\big\|\Pi\,|\Psi_\gamma\rangle\big\|},$$

whenever $\Pi\,|\Psi_\gamma\rangle \neq 0$. Then the map $\gamma \mapsto |\Phi_\gamma\rangle$ is itself a Quantum–SuperHyperSoft Set (over the restricted universe corresponding to Π).

Proof. Because Π is diagonal in the measurement basis, it acts by zeroing some amplitudes and leaving others unchanged. Thus $\|\Pi\,|\Psi_\gamma\rangle\| \leq 1$, and normalizing restores unit norm:

$$\big\langle \Phi_\gamma \mid \Phi_\gamma \big\rangle = \frac{\langle \Psi_\gamma|\Pi^\dagger\Pi|\Psi_\gamma\rangle}{\langle \Psi_\gamma|\Pi^\dagger\Pi|\Psi_\gamma\rangle} = 1.$$

Hence $|\Phi_\gamma\rangle$ is a valid quantum state for each γ.

3.3. Algorithm for Constructing Quantum-HyperSoft Set and Quantum-SuperHyperSoft Set

With a view toward future computational experiments, we further examine the construction algorithms for both the Quantum-HyperSoft Set and the Quantum-SuperHyperSoft Set.

Algorithm 1 (ht). *Construct Quantum-HyperSoft Set Classical Hypersoft Set $G : \mathcal{C} \to \mathcal{P}(U)$, universe $U = \{u_1, \ldots, u_n\}$. Quantum-HyperSoft Set $Q : \mathcal{C} \to \mathcal{H}(U)$. $\gamma \in \mathcal{C}$ $S \leftarrow G(\gamma)$ $k \leftarrow |S|$ $i \leftarrow 1$ n $u_i \in S$ $\alpha_{i,\gamma} \leftarrow 1/\sqrt{k}$ $\alpha_{i,\gamma} \leftarrow 0$ $|\psi_\gamma\rangle \leftarrow \sum_{i=1}^{n} \alpha_{i,\gamma}\,|u_i\rangle$ $Q(\gamma) \leftarrow |\psi_\gamma\rangle$ Q*

Example 14 (Quantum-HyperSoft Set for Quantum Algorithm Selection). *Consider the problem of choosing among three quantum algorithms*

$$U = \{\text{Grover, QAOA, VQE}\}$$

based on hardware and noise characteristics. Define three attribute domains:

$$\mathcal{A}_1 = \{\text{IBM, IonQ}\}, \quad \mathcal{A}_2 = \{\text{AllToAll, Linear}\}, \quad \mathcal{A}_3 = \{\text{LowNoise, HighNoise}\}.$$

Form the Cartesian product $\mathcal{C} = \mathcal{A}_1 \times \mathcal{A}_2 \times \mathcal{A}_3$. Let $G : \mathcal{C} \to \mathcal{P}(U)$ be the classical Hypersoft Set defined by:

$$\begin{aligned} G\big((\mathrm{IBM}, \mathrm{AllToAll}, \mathrm{LowNoise})\big) &= \{\text{QAOA, VQE}\}, \\ G\big((\mathrm{IBM}, \mathrm{Linear}, \mathrm{HighNoise})\big) &= \{\text{VQE}\}, \\ G\big((\mathrm{IonQ}, \mathrm{AllToAll}, \mathrm{HighNoise})\big) &= \{\text{Grover, VQE}\}, \\ G\big((\mathrm{IonQ}, \mathrm{Linear}, \mathrm{LowNoise})\big) &= \{\text{QAOA}\}. \end{aligned}$$

Applying Algorithm 1 to G yields the Quantum-HyperSoft Set $Q : \mathcal{C} \to \mathcal{H}(U)$*. For example:*

$$|\psi_{(\mathrm{IBM,AllToAll,LowNoise})}\rangle = \frac{1}{\sqrt{2}}\,|\mathrm{QAOA}\rangle + \frac{1}{\sqrt{2}}\,|\mathrm{VQE}\rangle\,,$$

$$|\psi_{(\mathrm{IBM,Linear,HighNoise})}\rangle = 1 \cdot |\mathrm{VQE}\rangle\,,$$

$$|\psi_{(\mathrm{IonQ,AllToAll,HighNoise})}\rangle = \frac{1}{\sqrt{2}}\,|\mathrm{Grover}\rangle + \frac{1}{\sqrt{2}}\,|\mathrm{VQE}\rangle\,,$$

$$|\psi_{(\mathrm{IonQ,Linear,LowNoise})}\rangle = 1 \cdot |\mathrm{QAOA}\rangle\,.$$

Each state is normalized, and measuring $|\psi_\gamma\rangle$ *in the basis* $\{|\mathrm{Grover}\rangle, |\mathrm{QAOA}\rangle, |\mathrm{VQE}\rangle\}$ *yields the recommended algorithm with the appropriate probabilities.*

Algorithm 2 (ht)**.** *Construct Quantum-SuperHyperSoft Set Classical SuperHypersoft Set* $F : \mathcal{C} \to \mathcal{P}(U)$*, universe* $U = \{u_1, \ldots, u_n\}$*. Quantum-SuperHyperSoft Set* $Q : \mathcal{C} \to \mathcal{H}(U)$*.* $\gamma \in \mathcal{C}$ $S \leftarrow F(\gamma)$ $k \leftarrow |S|$ $i \leftarrow 1$ n $u_i \in S$ $\alpha_{i,\gamma} \leftarrow 1/\sqrt{k}$ $\alpha_{i,\gamma} \leftarrow 0$ $|\Psi_\gamma\rangle \leftarrow \sum_{i=1}^{n} \alpha_{i,\gamma}\,|u_i\rangle$ $Q(\gamma) \leftarrow |\Psi_\gamma\rangle$ Q

Example 15 (Quantum–SuperHyperSoft Set for Quantum Algorithm Deployment)**.** *Consider a pool of quantum algorithms*

$$U = \{\mathrm{Grover},\ \mathrm{QAOA},\ \mathrm{VQE}\}.$$

Suppose we have three attribute domains with disjoint value-sets:

$$A_1 = \{\mathrm{IBM,\ IonQ,\ Rigetti}\}, \quad A_2 = \{\mathrm{AllToAll,\ Linear}\}, \quad A_3 = \{\mathrm{LowNoise,\ HighNoise}\}.$$

Form the Cartesian product of power-sets:

$$\mathcal{C} = \mathcal{P}(A_1) \times \mathcal{P}(A_2) \times \mathcal{P}(A_3).$$

Define the classical SuperHypersoft Set $F : \mathcal{C} \to \mathcal{P}(U)$ *by:*

$$\begin{aligned} F\big(\{\mathrm{IBM,Rigetti}\},\ \{\mathrm{AllToAll,Linear}\},\ \{\mathrm{LowNoise}\}\big) &= \{\mathrm{QAOA},\ \mathrm{VQE}\}, \\ F\big(\{\mathrm{IonQ}\},\ \{\mathrm{AllToAll}\},\ \{\mathrm{HighNoise}\}\big) &= \{\mathrm{Grover},\ \mathrm{VQE}\}, \\ F\big(\{\mathrm{IBM,IonQ}\},\ \{\mathrm{Linear}\},\ \{\mathrm{LowNoise,HighNoise}\}\big) &= \{\mathrm{Grover},\ \mathrm{QAOA}\}. \end{aligned}$$

Applying Algorithm 2 to F yields the Quantum–SuperHyperSoft Set $Q : \mathcal{C} \to \mathcal{H}(U)$*. For example:*

$$|\Psi_{\{\mathrm{IBM,Rigetti}\},\{\mathrm{AllToAll,Linear}\},\{\mathrm{LowNoise}\}}\rangle = \frac{1}{\sqrt{2}}\,|\mathrm{QAOA}\rangle + \frac{1}{\sqrt{2}}\,|\mathrm{VQE}\rangle\,,$$

$$|\Psi_{\{\mathrm{IonQ}\},\{\mathrm{AllToAll}\},\{\mathrm{HighNoise}\}}\rangle = \frac{1}{\sqrt{2}}\,|\mathrm{Grover}\rangle + \frac{1}{\sqrt{2}}\,|\mathrm{VQE}\rangle\,,$$

$$|\Psi_{\{\text{IBM,IonQ}\},\{\text{Linear}\},\{\text{LowNoise,HighNoise}\}}\rangle = \frac{1}{\sqrt{2}}\,|\text{Grover}\rangle + \frac{1}{\sqrt{2}}\,|\text{QAOA}\rangle\,.$$

Each state is manifestly normalized. Measuring any $|\Psi_\gamma\rangle$ in the computational basis $\{|\text{Grover}\rangle, |\text{QAOA}\rangle, |\text{VQE}\rangle\}$ yields a soft ranking of algorithms adapted to multiple-vendor, connectivity, and noise-level preferences.

Theorem 19 (Correctness of Construct Quantum-HyperSoft Set)**.** *Let $G : \mathcal{C} \to \mathcal{P}(U)$ be any classical Hypersoft Set over the finite universe $U = \{u_1, \dots, u_n\}$. Algorithm 1 produces a map*

$$Q : \mathcal{C} \longrightarrow \mathcal{H}(U), \quad \gamma \mapsto |\psi_\gamma\rangle$$

such that each $|\psi_\gamma\rangle$ is a unit vector in the Hilbert space $\mathcal{H}(U)$. Hence Q is a valid Quantum-HyperSoft Set.

Proof. Fix an arbitrary $\gamma \in \mathcal{C}$ and let $S = G(\gamma) \subseteq U$ with $|S| = k$. In the inner loop, the algorithm sets

$$\alpha_{i,\gamma} = \begin{cases} 1/\sqrt{k}, & \text{if } u_i \in S, \\ 0, & \text{otherwise.} \end{cases}$$

Thus

$$\sum_{i=1}^{n} \left|\alpha_{i,\gamma}\right|^2 = \sum_{u_i \in S} \frac{1}{k} = \frac{k}{k} = 1.$$

By construction, $|\psi_\gamma\rangle = \sum_{i=1}^{n} \alpha_{i,\gamma}\,|u_i\rangle$ is normalized. Since this holds for every γ, the output map Q satisfies the definition of a Quantum-HyperSoft Set.

Theorem 20 (Correctness of Construct Quantum-SuperHyperSoft Set)**.** *Let $F : \mathcal{C} \to \mathcal{P}(U)$ be any classical SuperHypersoft Set over $U = \{u_1, \dots, u_n\}$. Algorithm 2 produces a map*

$$Q : \mathcal{C} \longrightarrow \mathcal{H}(U), \quad \gamma \mapsto |\Psi_\gamma\rangle$$

with each $|\Psi_\gamma\rangle$ normalized, so Q is a valid Quantum–SuperHyperSoft Set.

Proof. Identical to the proof of Theorem 19, replacing $S = G(\gamma)$ by $S = F(\gamma)$. The same calculation shows $\sum_i |\alpha_{i,\gamma}|^2 = 1$ and hence each $|\Psi_\gamma\rangle$ is unit-norm.

Theorem 21 (Time and Space Complexity)**.** *Assume membership in each $S = G(\gamma)$ or $S = F(\gamma)$ can be tested in $O(1)$ time (e.g. using a hash set). Then both Algorithm 1 and Algorithm 2 run in $O(|\mathcal{C}| \cdot n)$ time and use $O(n \cdot |\mathcal{C}|)$ space.*

Proof. Let $|\mathcal{C}|$ be the number of attribute combinations and $n = |U|$. For each γ, the algorithm:

(i) Retrieves $S \subseteq U$ in $O(1)$.

(ii) Computes $k = |S|$ in $O(1)$.

(iii) Executes an n-iteration loop where each membership test $u_i \in S$ takes $O(1)$. Thus the inner loop is $O(n)$.

(iv) Forms the vector $|\psi_\gamma\rangle$ or $|\Psi_\gamma\rangle$ in $O(n)$ by summing n amplitudes.

Steps 3–4 dominate, yielding $O(n)$ per γ. Over all $\gamma \in \mathcal{C}$, the total time is

$$O\big(|\mathcal{C}| \times n\big).$$

Space usage arises from storing one length-n complex vector per γ, for total

$$O\big(n \cdot |\mathcal{C}|\big)$$

complex numbers. This completes the complexity analysis.

4. Conclusion and Future Works

4.1. Conclusion

In this paper, we introduced and rigorously formalized the concepts of the *Quantum Hypersoft Set* and the *Quantum SuperHyperSoft Set*, thereby extending the existing theory of Quantum Soft Sets. We also provided initial algorithmic formulations for their construction. These newly proposed structures offer a promising foundation for representing and reasoning about complex uncertainty in both Quantum Theory and Soft Computing. However, since this work is purely theoretical, further validation through computational experiments and practical implementations is required in future research.

4.2. Future Works

In future work, we plan to extend these quantum-set-based models to more general structures such as Graphs [45], Hypergraphs [46, 47], Bidirected Graphs [48, 49], and SuperHypergraphs [50, 51]. Moreover, we aim to incorporate advanced soft computing paradigms such as the Soft Expert Set [52, 53], IndetermSoft Sets[54, 55], IndetermHyperSoft Sets[55, 56], Fuzzy Soft Sets[57], and IndetermSuperHyperSoft Sets[33]. Potential extensions based on Fuzzy Sets[4, 58], HyperRough Sets[59, 60], Intuitionistic Fuzzy Sets[61, 62], Neutrosophic Sets[18, 63], and Plithogenic Sets[20, 64] also offer rich avenues for further development. We hope that future research will pursue computational implementations and practical applications of these models in real-world decision-making and uncertainty reasoning scenarios.

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Data Availability

This paper is purely theoretical and does not involve any empirical data. We welcome future empirical studies that build upon and test the concepts presented here.

Ethical Approval

As this work is entirely conceptual and involves no human or animal subjects, ethical approval was not required.

Conflicts of Interest

The authors declare no conflicts of interest in connection with this study or its publication.

Author Contributions

Conceptualization, Takaaki Fujita and Florentin Smarandache; Investigation, Takaaki Fujita; Methodology, Takaaki Fujita; Writing – original draft, Takaaki Fujita; Writing – review & editing, Takaaki Fujita and Florentin Smarandache.

Disclaimer (Limitations and Claims)

The theoretical concepts presented in this paper have not yet been subject to practical implementation or empirical validation. Future researchers are invited to explore these ideas in applied or experimental settings. Although every effort has been made to ensure the accuracy of the content and the proper citation of sources, unintentional errors or omissions may persist. Readers should independently verify any referenced materials.

To the best of the authors' knowledge, all mathematical statements and proofs contained herein are correct and have been thoroughly vetted. Should you identify any potential errors or ambiguities, please feel free to contact the authors for clarification.

The results presented are valid only under the specific assumptions and conditions detailed in the manuscript. Extending these findings to broader mathematical structures may require additional research. The opinions and conclusions expressed in this work are those of the authors alone and do not necessarily reflect the official positions of their affiliated institutions.

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