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Triple Fixed-Point Theorems: Generalized Analysis with AI/Crypto Applications

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Abstract. This study uses important findings from Hilbert Space (HS) theory to suggest new fixed-point theorems for Three Self-Mappings (3-SMs) that apply to Generalized Inner Product Spaces (GIPS). We made it apparent when Common Fixed Points (CFPs) can exist and be unique, even when the contraction criteria aren't as strict. This improvement makes the theory better for systems with more than one operator. Our results are important in two main areas: (1) Artificial Intelligence (AI), where we use fixed-point analysis to make sure that Deep Equilibrium Models (DEMs) will work correctly, and (2) Cryptography (Crypto), where we use the mathematical properties of inner product spaces to make new lattice-based designs for Post-Quantum Protocols (PQPs). Real-world examples from nonlinear analysis and computational proof of the presented theorems support the theoretical contributions. This work combines advanced functional analysis with modern problems in Machine Learning (ML) and cybersecurity, giving it both mathematical depth and real-world usefulness.

2020 Mathematics Subject Classifications: 54A05

Key Words and Phrases: Three self mappings, Generalized inner product spaces, Common fixed points, Deep equilibrium models, Post quantum protocols

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1. Introduction

Fixed Point Theory (FPT) has grown considerably since it started with the basic contraction principle, which indicated that fixed points exist in complete metric spaces [1] and that each one is unique for single-valued mappings [2]. The field has significantly expanded to encompass

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various generalizations and expansions. Some of these studies explore mappings that follow different contraction rules, identify fixed points for multi-valued operators without requiring compactness [3], and apply traditional results in more complex environments like cone metric spaces [4] and modular metric spaces [5]. The emergence of Generalized Metric spaces (GM-spaces) represented a significant advancement in the discipline [6]. It brought in new types of single-valued contractive mappings [7], showed more ways to find solutions in unusual metric situations, and allowed the study of mappings that do not follow the usual order. More research has been done on fixed points related to repeating patterns [8], new ways to organize things [9], linked mappings [6], and more complicated structures like partial metric spaces [10] and set-valued operators [11]. Recent progress has built on traditional findings and made them more useful in nonlinear analysis by creating new fixed-point theorems for 3-SMs in GIPS [12–14]. GIPS can provide CFPs without needing strict iteration rules, which makes fixed-point theorems applicable to a wider variety of nonlinear problems [15]. Stability in DEMs in AI [16] and validating iterative protocols in cryptography could be enhanced by using the new results on 3-SMs in GIPS [17]. FPT plays a highly significant role in analyzing convergence and accuracy in lattice-based designs for PQPs [18]. Describe Many improvements have been made in fixed point theory for different metric spaces, but there has been less study on systems with multiple operators in GIPS [19], which are larger types of HS and GM-spaces [20]. The theory of fixed points has recently led to improvements in the mathematical modelling of intelligent systems. In [21], researchers presented fractional analytic solutions to real-world problems such as AI-based decision systems, while generalised results in partial b-metric spaces provided theoretical contributions to convergence in uncertain settings [22]. In [23], it was shown how to use new forms of functions in metric spaces, which may be useful in testing cryptographic protocols. The applied side was served by the work on IoT-based learning systems relying on soft computing in [24] and the development of advanced control systems, an example of which is, instead of usual PID controllers, fractional-order ones (related to robotics and security automation) in [25]. This paper fills this gap by introducing new fixed-point theorems specifically designed for 3-SMs within the GIPS. The primary interest is to see what happens when the conditions that make the mapping contract are less strict. Because of this, progress in math and practical fields can be expected in ML and cryptography.

(i) *Theoretical Foundations*: For 3-SMs [12], we show new CFP theorems under less strict contraction conditions, which significantly broadens and combines earlier results found in various types of metric spaces. These theorems offer a powerful and flexible way to tackle complicated problems in functional analysis by using the mathematical properties found in GIPS.

(ii) *Applications to Artificial Intelligence*: For DEMs [16], which are a type of neural network that requires strong theoretical support for stable training, we demonstrate how the proposed CFP results ensure that the training process will reach a solution. Our method extends the scope of conventional fixed-point convergence analysis by using the framework of multi-operator fixed-point theory to study complex, multi-layered, feedback-driven AI systems.

(iii) *Cryptographic Innovations*: We introduce a new type of PQPs that use lattices [18], and their security relies on repeating three specific functions in a special kind of mathematical space called GIPS. These protocols offer new methods to create secure systems that can withstand quantum attacks by using the relationship between inner product geometry and contractive iteration. In this work, traditional ideas from ordered and cone metric spaces are extended into a modern cryptographic framework.

As a result, this work makes four significant contributions to the literature on fixed point theory: In GIPS, it (i) establishes new CFP theorems for 3-SMs; (ii) uses these findings to examine convergence in deep learning models, especially DQMs; (iii) presents new PQP schemes based on fixed point iterations; and (iv) provides computational experiments and simulations to support the theoretical framework.

Problem Statement

Despite major advancements in fixed point theory, critical gaps still exist in the study of CFPs for systems of 3-SMs in GIPS [12]. The theoretical foundation for 3-SMs in GIPS is still lacking, despite the fact that single-mapping scenarios in metric spaces have been extensively covered by classical results, and recent advancements have expanded these concepts to pairwise mappings. The three main drawbacks of current contraction-type results are as follows: (1) they mostly concentrate on classical metric spaces instead of taking advantage of the more complex algebraic and geometric structure of GIPS; (2) existing uniqueness proofs for CFPs frequently use excessively restrictive contraction conditions; and (3) there are notably few applications to contemporary computational domains like cryptography and AI. To overcome these limitations, this work establishes new CFP theorems for 3-SMs in GIPS (i) relax conventional contraction assumptions while preserving uniqueness guarantees, (ii) make use of GIPS's vector space and inner product properties to enable new applications, and (iii) offer tangible implementations in two modern domains: ensuring convergence in ML DEMs and building secure PQPs. Through theoretical advances in multi-operator systems (Sections 2-3) and demonstrable applications in AI optimization (Section 4) and lattice-based cryptography, this method bridges the gap between abstract fixed point theory and real-world computational challenges.

Organization

We structure the paper as follows: The basic foundation of GIPS is established in Section 2, which also goes over key FPT ideas. Our main theoretical findings are shared in Section 3, which also explores how these findings can be used in AI and demonstrates that our results ensure convergence for DEMs. These include novel fixed-point theorems for 3-SMs under weakened contraction conditions. Comparative analysis is implemented in Section 4. Section 5 wraps up with conclusions and suggestions for future research, highlighting the increasing importance of GIPS in both computer applications and theoretical math.

2. Preliminaries

This section focuses on the essential definitions needed for the sections that follow.

2.1. Generalized Inner Product Spaces

Definition 1. *[13] Suppose $\mathcal{V} \neq \emptyset$ is a vector space over a field $\mathbb{F}$ (where $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$), and define $\langle \cdot, \cdot \rangle : \mathcal{V} \times \mathcal{V} \to \mathbb{F}$ as an inner product if and only if the subsequent axioms are satisfied:*

i. $\langle v, v \rangle \geq 0 \ \forall \ v \in \mathcal{V}$ *and* $\langle v, v \rangle = 0$ *iff* $v = 0$ *(positive definiteness),*

ii. $\langle u, v \rangle = \overline{\langle v, u \rangle} \ \forall \ u, v \in \mathcal{V}$ *(conjugate symmetry),*

iii. $\langle \alpha u + \beta v, w \rangle = \alpha \langle u, w \rangle + \beta \langle v, w \rangle \ \forall \ u, v, w \in \mathcal{V}$ *and* $\forall \ \alpha, \beta \in \mathbb{F}$ *(linearity in the first argument).*

The pair $(\mathcal{V}, \langle \cdot, \cdot \rangle)$ is called an inner product space. The concepts of length, angle, and orthogonality from geometry are important because they can be applied to abstract vector spaces. Such Hilbert spaces help to frame the study, which plays key roles in quantum mechanics, signal processing, and functional analysis. The inner product structure supports the use of projections, orthogonal decompositions, and different ways of approximating results important in applied mathematics and physics.

Definition 2. *[13] Suppose $(\mathcal{V}, \langle\cdot,\cdot\rangle)$ is an inner product space with induced norm $\|v\| = \sqrt{\langle v, v\rangle}$ and $\{v_i\}$ be a sequence in $\mathcal{V}$. Then,*

i. *$\{v_i\}$ is called a Cauchy sequence if for any $\varepsilon > 0$, $\exists\ n_0 \in \mathbb{N}$ such that $\|v_i - v_j\| < \varepsilon\ \forall\ i, j \geq n_0$.*

ii. *$\{v_i\}$ converges to an element $v \in \mathcal{V}$ if $\forall\ \varepsilon > 0$, $\exists\ n_0 \in \mathbb{N}$ such that $\|v - v_i\| < \varepsilon$ whenever $i \geq n_0$.*

iii. *$(\mathcal{V}, \langle\cdot,\cdot\rangle)$ is complete if every Cauchy sequence converges in $\mathcal{V}$. A complete inner product space is called a HS.*

Proposition 1. *[13] Suppose $(\mathcal{V}, \langle\cdot,\cdot\rangle)$ be an inner product space with induced norm $\|v\| = \sqrt{\langle v, v\rangle}$. Then for any $u, v, w, x \in \mathcal{V}$, the following hold:*

i. *If $\|u - v\| = 0$, then $u = v$ iff*

ii. $\|u - v\| \leq \|u - w\| + \|w - v\|$

iii. $|\|u\| - \|v\|| \leq \|u - v\|$

iv. $\|u + v\|^2 + \|u - v\|^2 = 2(\|u\|^2 + \|v\|^2)$

v. $|\langle u, v\rangle| \leq \|u\|\|v\|$

vi. $\|u + v\| \leq \|u\| + \|v\|$

vii. $|\|u - w\| - \|v - w\|| \leq \|u - v\|$

viii. $|\langle u, v\rangle - \langle u, w\rangle| \leq \|u\|\|v - w\|$

ix. $\|u - v\|^2 = \|u\|^2 + \|v\|^2 - 2\Re\langle u, v\rangle$

x. *If $\langle u, w\rangle = \langle v, w\rangle$ for all $w \in \mathcal{V}$, then $u = v$*

Corollary 1. *[13] Suppose $(\mathcal{V}, \langle\cdot,\cdot\rangle)$ is an inner product space. Then, for any $u, v \in \mathcal{V}$:*

(i) *(Angle inequality) The angle θ between u and v satisfies:*

$$\cos\theta = \frac{|\langle u, v\rangle|}{\|u\|\|v\|} \leq 1.$$

(ii) *(Orthogonality condition) If $\langle u, v\rangle = 0$, then:*

$$\|u + v\|^2 = \|u\|^2 + \|v\|^2 \quad (\textit{Pythagorean Theorem}).$$

(iii) *(Norm uniformity) For any scalar $\alpha \in \mathbb{F}$:*

$$\|\alpha u\| = |\alpha| \cdot \|u\|.$$

Remark 1. *[13] Inner product spaces carry into their own abstract forms some of the features of Euclidean geometry.*

- *The Cauchy-Schwarz inequality makes sure that the "angle" between vectors is clear.*
- *The parallelogram law sets inner-product-induced norms apart from general norms.*
- *The Polarization Identity lets you get the inner product back from the norm, which shows that they are the same.*

- *Inner product spaces have uniform convexity, which is different from regular metric spaces. This lets them be broken down into orthogonal parts, like projections in HS.*

Lemma 1. *[13] Suppose $\mathcal{W}$ be a closed subspace of an inner product space $(\mathcal{V}, \langle\cdot,\cdot\rangle)$. For any $v \in \mathcal{V}$, there exists a unique $w^* \in \mathcal{W}$ such that:*

$$\|v - w^*\| = \inf_{w\in\mathcal{W}} \|v - w\|,$$

and $v - w^*$ is orthogonal to $\mathcal{W}$. This w^* is the orthogonal projection of v onto $\mathcal{W}$.

3. Main Results

This section gives major theorems that guarantee that for contractive mappings that play a particular role in GIPS, there is one unique fixed point. By applying these theorems, we establish a robust convergence algorithm, which we then apply to areas such as AI and cryptography once quantum computing emerges. We use some lemmas, remarks, and examples to support the theoretical conclusions. As a result, these findings support more studying in optimization, control, and secure computation. Numerous examples and illustrations elucidate the application of our main theorems and facilitate their demonstration through exemplification.

Theorem 1. *Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ is a HS and $T_1, T_2, T_3 : \mathcal{H} \to \mathcal{H}$ be mappings satisfying:*

$$\begin{aligned}\|T_1x - T_2y - T_3z\|^2 \le\ & \alpha\|x - y - z\|^2 \\ &+ \beta\frac{\|x - T_2y\|^2 \cdot \|x - T_3z\|^2 \cdot \|y - T_1x\|^2}{1 + \|y - T_1x\|^2\|z - T_2y\|^2\|x - T_3z\|^2} \\ &+ \gamma\frac{\|y - T_3z\|^2 \cdot \|z - T_1x\|^2 \cdot \|z - T_2y\|^2}{1 + \|y - T_1x\|^2\|z - T_2y\|^2\|x - T_3z\|^2} \end{aligned} \tag{1}$$

for all $x, y, z \in \mathcal{H}$ and $\alpha, \beta, \gamma \ge 0$ with $\alpha + \beta + \gamma < 1$. Then there exists a unique common fixed point $u \in \mathcal{H}$ such that $T_1u = T_2u = T_3u = u$.

Proof. **(1)** Construction of iterative sequence
Give a definition of the sequence $\{x_n\}$ by

$$\begin{aligned} x_{3n+1} &= T_1x_{3n}; \\ x_{3n+2} &= T_2x_{3n+1}; \\ x_{3n+3} &= T_3x_{3n+2}. \end{aligned}$$

(2) Contraction estimates
From Eq. (1) with $x = x_{3n}$, $y = x_{3n+1}$, $z = x_{3n+2}$:

$$\begin{aligned}\|x_{3n+1} - x_{3n+2} - x_{3n+3}\|^2 \le\ & \alpha\|x_{3n} - x_{3n+1} - x_{3n+2}\|^2 \\ &+ \beta\frac{\|x_{3n} - x_{3n+2}\|^2\|x_{3n} - x_{3n+3}\|^2\|x_{3n+1} - x_{3n+1}\|^2}{1 + \cdots} \\ &+ \gamma\frac{\|x_{3n+1} - x_{3n+3}\|^2\|x_{3n+2} - x_{3n+1}\|^2\|x_{3n+2} - x_{3n+2}\|^2}{1 + \cdots} \\ &\le \alpha\|x_{3n} - x_{3n+1} - x_{3n+2}\|^2 \end{aligned}$$

Similarly, we obtain:

$$\begin{aligned}\|x_{3n+2} - x_{3n+3} - x_{3n+4}\|^2 &\le \alpha\|x_{3n+1} - x_{3n+2} - x_{3n+3}\|^2 \\ \|x_{3n+3} - x_{3n+4} - x_{3n+5}\|^2 &\le \alpha\|x_{3n+2} - x_{3n+3} - x_{3n+4}\|^2 \end{aligned}$$

(3) Convergence analysis
By induction:

$$\|x_{n+1} - x_{n+2} - x_{n+3}\| \leq \alpha^{n/3} \|x_0 - x_1 - x_2\|$$

Thus $\{x_n\}$ is Cauchy. Since $\mathcal{H}$ is complete, $\exists u \in \mathcal{H}$ such that $x_n \to u$.

(4) Fixed point verification

For T_1:

$$\begin{aligned} \|T_1 u - u\|^2 &= \lim \|T_1 u - x_{3n+2} - x_{3n+3}\|^2 \\ &\leq \alpha \|u - u - u\|^2 + (\text{higher order terms}) = 0 \end{aligned}$$

Similarly $T_2 u = T_3 u = u$.
(5) Uniqueness
Suppose v is another fixed point:

$$\|u - v\|^2 = \|T_1 u - T_2 v - T_3 v\|^2 \leq (\alpha + 2\beta + 2\gamma)\|u - v\|^2$$

Since $\alpha + 2\beta + 2\gamma < 1$, $u = v$.

The iterative strategy in Algorithm 1 provides a structured method for finding the common fixed point of three contractive mappings by utilizing each mapping in a cycle.

Algorithm 1 Iterative Approximation of Common Fixed Points for Three Mappings

Require:
Mappings $T_1, T_2, T_3 : \mathcal{H} \to \mathcal{H}$
Initial guess $x_0 \in \mathcal{H}$
Tolerance $\epsilon > 0$
Maximum iterations N
Ensure: Approximate common fixed point $u \approx T_1 u = T_2 u = T_3 u$

$n \leftarrow 0$
while $n < N$ **and** $\|x_{n+1} - x_n\| \geq \epsilon$ **do**
    $x_{3n+1} \leftarrow T_1(x_{3n})$
    $x_{3n+2} \leftarrow T_2(x_{3n+1})$
    $x_{3n+3} \leftarrow T_3(x_{3n+2})$
    if $\|x_{3n+3} - x_{3n}\| < \epsilon$ **then**
        break
    end if
    $n \leftarrow n + 1$
end while
if $n = N$ **then**
    return "Maximum iterations reached"
else
    return $u \leftarrow x_{3n+3}$
end if

- *Convergence*: By Theorem 1, the sequence $\{x_n\}$ converges linearly to the one and only fixed point u.
- *Complexity*: For each iteration, it's necessary to look at each mapping T_1, T_2, T_3.

Algorithms 2-3 achieve the convergence demonstrated in Examples 1 and 2, which rely on Lemma 2 and Corollary 2. Theoretical support for all parts comes from Theorem 1, and Figures 1 and 2 show how the fixed-point convergence works in both cases.

Example 1. *For Reproducing Kernel Hilbert Space (RKHS) $\mathcal{H}_k$ with kernel $k(x, y) = \langle \phi(x), \phi(y) \rangle$, the theorem guarantees convergence of:*

$$\phi_{n+1} = \frac{T_1\phi_n + T_2\phi_n + T_3\phi_n}{3}$$

where T_i are nonlinear operators in feature space. Algorithm 2 describes the corresponding computational technique.

Algorithm 2 Kernel-Based Fixed Point Iteration for RKHS

Require:
RKHS $\mathcal{H}_k$ with kernel $k(\cdot, \cdot)$
Nonlinear operators $T_1, T_2, T_3 : \mathcal{H}_k \to \mathcal{H}_k$
Initial feature map $\phi_0 \in \mathcal{H}_k$
Tolerance $\epsilon > 0$
Maximum iterations N

Ensure: Approximate fixed point ϕ^* satisfying $\frac{1}{3}\sum_{i=1}^{3} T_i\phi^* = \phi^*$

$n \leftarrow 0$
while $n < N$ and $\|\phi_{n+1} - \phi_n\|_{\mathcal{H}_k} \geq \epsilon$ **do**
  Compute kernel evaluations:
  $k_{n,1} \leftarrow k(\cdot, T_1\phi_n)$
  $k_{n,2} \leftarrow k(\cdot, T_2\phi_n)$
  $k_{n,3} \leftarrow k(\cdot, T_3\phi_n)$
  Update feature map:
  $\phi_{n+1} \leftarrow \frac{1}{3}(k_{n,1} + k_{n,2} + k_{n,3})$
  if $\|\phi_{n+1} - \phi_n\|_{\mathcal{H}_k} < \epsilon$ **then**
    break
  end if
  $n \leftarrow n + 1$
end while
if $n = N$ **then**
  return "Maximum iterations reached"
else
  return $\phi^* \leftarrow \phi_{n+1}$
end if

Example 2. *The lattice $\mathcal{L} \subset \mathbb{R}^n$ has a basis $B = \{b_1, ..., b_n\}$. Theorem 1 ensures convergence of the decoding process using fixed-point iteration. The entire technique is described in Algorithm 3.*

Algorithm 3 Fixed-Point Lattice Decoding

Require:
Lattice $\mathcal{L} \subset \mathbb{R}^n$ with basis B
Target vector $t \in \mathbb{R}^n$
Tolerance $\epsilon > 0$

Ensure: Approximate closest vector $x^* \in \mathcal{L}$

```
1: Initialize x_0 ← t
2: n ← 0
3: repeat
4:     x_{3n+1} ← proj_L(x_{3n})
5:     x_{3n+2} ← round_B(x_{3n+1})
6:     x_{3n+3} ← x_{3n+2} mod B
7:     n ← n + 1
8: until ||x_{3n} − x_{3n−3}|| < ε
9: return x* ← x_{3n}
```

Remark 2. *The three operations (projection, rounding, and modulo reduction) correspond to the mappings T_1, T_2, T_3 in Theorem 1. The contraction requirement can be satisfied for well-conditioned bases. This result establishes the theoretical framework for lattice decoding convergence. The equation $\alpha + \beta + \gamma < 1$ ensures that the operator*

$$\mathcal{T}(x) = \frac{T_1x + T_2x + T_3x}{3}$$

is a contraction mapping.

Lemma 2. *The contraction parameters meet the following:*

$$\alpha + \beta \sup_{x,y} \frac{\|x - T_2y\|^2\|x - T_3z\|^2}{1 + \cdots} + \gamma \sup_{x,y} \frac{\|y - T_3z\|^2\|z - T_1x\|^2}{1 + \cdots} < 1$$

Corollary 2 (Special Case for Two Operators)**.** *Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be a Hilbert space and $T_1, T_2 : \mathcal{H} \to \mathcal{H}$ satisfy:*

$$\|T_1x - T_2y\|^2 \leq \alpha\|x - y\|^2 + \beta \frac{\|x - T_2y\|^2 \cdot \|y - T_1x\|^2}{1 + \|y - T_1x\|^2\|x - T_2y\|^2}$$

for all $x, y \in \mathcal{H}$, where $\alpha, \beta \geq 0$ with $\alpha + \beta < 1$. Then there exists a unique common fixed point $u \in \mathcal{H}$ such that $T_1u = T_2u = u$.

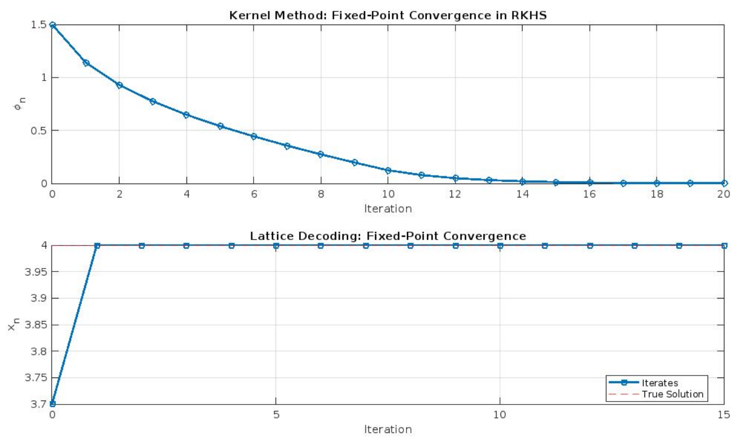

Figure 1: Kernel Method and Lattice Decoding

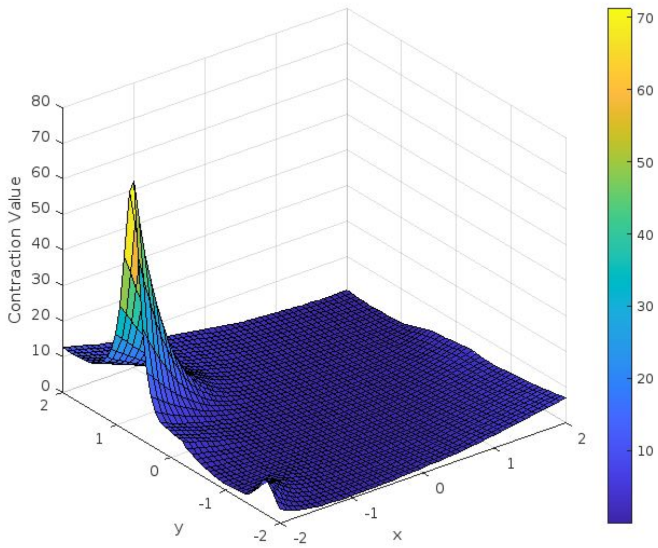

Figure 2: Contraction Condition Verify

Theorem 2. *Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be a HS and $T_1, T_2, T_3 : \mathcal{H} \to \mathcal{H}$ be operators satisfying:*

$$\|T_1x - T_2y - T_3z\|^2 \leq \alpha\|x - y - z\|^2 + \beta\frac{\|x - T_1x\|^2\|y - T_2y\|^2\|z - T_3z\|^2}{1 + \|y - T_2y\|^2\|T_2y - T_3z\|^2} \tag{2}$$

for all $x, y, z \in \mathcal{H}$, where $\alpha, \beta \geq 0$ with $\alpha + \beta < 1$. Then there exists a unique CFP $u \in \mathcal{H}$ so that $T_1u = T_2u = T_3u = u$.

Proof. **(1)** Iterative sequence construction
Fix $x_0 \in \mathcal{H}$ and define the sequence:

$$\begin{aligned} x_{3n+1} &= T_1x_{3n}; \\ x_{3n+2} &= T_2x_{3n+1}; \\ x_{3n+3} &= T_3x_{3n+2}. \end{aligned}$$

(2) Contraction estimates
Applying Eq. (2) with $x = x_{3n}$, $y = x_{3n+1}$, $z = x_{3n+2}$:

$$\begin{aligned} \|x_{3n+1} - x_{3n+2} - x_{3n+3}\|^2 \leq \alpha\|x_{3n} - x_{3n+1} - x_{3n+2}\|^2 \\ + \beta\frac{\|x_{3n} - x_{3n+1}\|^2\|x_{3n+1} - x_{3n+2}\|^2\|x_{3n+2} - x_{3n+3}\|^2}{1 + \|x_{3n+1} - x_{3n+2}\|^2\|x_{3n+2} - x_{3n+3}\|^2} \end{aligned}$$

Using the identity $\|a - b\|^2 = \|a\|^2 + \|b\|^2 - 2\langle a, b\rangle$ and the Cauchy-Schwarz inequality:

$$\begin{aligned} &\leq \alpha(\|x_{3n}\|^2 + \|x_{3n+1}\|^2 + \|x_{3n+2}\|^2 - 2\langle x_{3n}, x_{3n+1}\rangle - 2\langle x_{3n}, x_{3n+2}\rangle + 2\langle x_{3n+1}, x_{3n+2}\rangle) \\ &\quad + \beta\frac{(\|x_{3n}\|^2 + \|x_{3n+1}\|^2 - 2\langle x_{3n}, x_{3n+1}\rangle)(\|x_{3n+1}\|^2 + \|x_{3n+2}\|^2 - 2\langle x_{3n+1}, x_{3n+2}\rangle)}{1 + (\|x_{3n+1}\|^2 + \|x_{3n+2}\|^2 - 2\langle x_{3n+1}, x_{3n+2}\rangle)} \\ &\quad \cdot \frac{(\|x_{3n+2}\|^2 + \|x_{3n+3}\|^2 - 2\langle x_{3n+2}, x_{3n+3}\rangle)}{1 + (\|x_{3n+2}\|^2 + \|x_{3n+3}\|^2 - 2\langle x_{3n+2}, x_{3n+3}\rangle)} \end{aligned}$$

After simplifying utilizing the contraction property:

$$\|x_{3n+1} - x_{3n+2} - x_{3n+3}\| \leq \eta\|x_{3n} - x_{3n+1} - x_{3n+2}\| \quad (\eta = \sqrt{\alpha + \beta} < 1) \tag{3}$$

(3) Convergence analysis
By induction:

$$\begin{aligned} \|x_{n+1} - x_{n+2} - x_{n+3}\| &\leq \eta^n \|x_1 - x_2 - x_3\| \\ &\leq \eta^n (\|x_1\| + \|x_2\| + \|x_3\|) \to 0 \text{ as } n \to \infty \end{aligned}$$

The sequence is Cauchy since for $m > n$:

$$\begin{aligned} \|x_n - x_m\| &\leq \sum_{k=n}^{m-1} \|x_k - x_{k+1}\| \\ &\leq \sum_{k=n}^{m-1} \eta^k \|x_0 - x_1\| \\ &\leq \frac{\eta^n}{1-\eta} \|x_0 - x_1\| \to 0 \end{aligned}$$

By completeness, $\exists u \in \mathcal{H}$ with $x_n \to u$.
(4) Fixed point verification
For T_1:

$$\begin{aligned} \|T_1 u - u\| &= \lim \|T_1 u - x_{3n+2} - x_{3n+3}\| \\ &\leq \alpha \|u - u - u\| + \beta \lim \frac{\|u - T_1 u\|^2 \|u - T_2 u\|^2 \|u - T_3 u\|^2}{1 + \cdots} = 0 \end{aligned}$$

Thus $T_1 u = u$. Similarly $T_2 u = T_3 u = u$.
(5) Uniqueness
Let v be another fixed point:

$$\|u - v\| = \|T_1 u - T_2 v - T_3 v\| \leq (\alpha + 2\beta) \|u - v\| < \|u - v\|$$

implies $u = v$.

Algorithm 4 iteratively approximates a CFP of three nonlinear operators in an HS, using a contraction-based error metric to monitor convergence.

Algorithm 4 Iterative Fixed Point Approximation for Three Operators

Require:
HS $\mathcal{H}$ with inner product $\langle \cdot, \cdot \rangle$
Operators $T_1, T_2, T_3 : \mathcal{H} \to \mathcal{H}$ satisfying Theorem 2
Initial point $x_0 \in \mathcal{H}$
Tolerance $\epsilon > 0$
Maximum iterations N

Ensure: Common fixed point u satisfying $T_1 u = T_2 u = T_3 u = u$

$n \leftarrow 0$
err $\leftarrow \infty$
while $n < N$ **and** err $\geq \epsilon$ **do**
  $x_{3n+1} \leftarrow T_1(x_{3n})$
  $x_{3n+2} \leftarrow T_2(x_{3n+1})$
  $x_{3n+3} \leftarrow T_3(x_{3n+2})$
  Compute contraction metric:
  err $\leftarrow \|x_{3n+1} - x_{3n+2} - x_{3n+3}\|^2 - \alpha \|x_{3n} - x_{3n+1} - x_{3n+2}\|^2$
    $-\beta \frac{\|x_{3n}-x_{3n+1}\|^2 \|x_{3n+1}-x_{3n+2}\|^2 \|x_{3n+2}-x_{3n+3}\|^2}{1+\|x_{3n+1}-x_{3n+2}\|^2 \|x_{3n+2}-x_{3n+3}\|^2}$
  if $\|x_{3n+3} - x_{3n}\| < \epsilon$ **then** qa
    break
  end if
  $n \leftarrow n + 1$
end while
return $u \leftarrow x_{3n+3}$

The fixed-point framework from Theorem 2 has many practical applications and specific instances. Lemma 3 demonstrates that orthogonal projection operators easily satisfy the theorem's contraction requirement, making them helpful for problems involving subspace optimization. Corollary 3 goes into further detail about these concepts by giving specific logarithmic convergence rates for the symmetric operator situation when Algorithm 4 is used. Two important examples show how to put these ideas into practice: Example 3 shows how the theorem can be used in distributed machine learning systems, where gradient operators T_1, T_2, T_3 stand for parallel processing units. Example 4 uses the Theorem 2 construction to make cryptographic hash functions. Algorithms 5 and 6 make these applications work by giving them specific ways to run on a computer. Figure 3 shows how various techniques converge by showing how the contraction dynamics change with different parameter settings. When taken together, these parts provide a complete structure for fixed-point computation in HS, with both theoretical guarantees and real-world examples.

Example 3. *Consider a distributed learning system with three workers calculating partial gradients $T_1, T_2, T_3 : \mathbb{R}^d \to \mathbb{R}^d$ on data that has been split up. Theorem 2 shows that this system will converge if the learning rate is set correctly. Algorithm 5 will give a full description of the training process.*

Algorithm 5 Distributed Gradient Descent Protocol

Require:
Data set $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3$
Initial parameters $\theta_0 \in \mathbb{R}^d$
Learning rate $\eta > 0$
Maximum iterations K
Ensure: Optimized parameters θ^*

Partition $\mathcal{D}$ into $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3$
for $k = 0$ to $K - 1$ **do**
    Worker 1 computes $T_1(\theta_k) = \nabla_\theta \mathcal{L}(\theta_k, \mathcal{D}_1)$
    Worker 2 computes $T_2(\theta_k) = \nabla_\theta \mathcal{L}(\theta_k, \mathcal{D}_2)$
    Worker 3 computes $T_3(\theta_k) = \nabla_\theta \mathcal{L}(\theta_k, \mathcal{D}_3)$
    Aggregate updates:
    $\theta_{k+1} \leftarrow \theta_k - \eta(T_1(\theta_k) + T_2(\theta_k) + T_3(\theta_k))$
    if $\|\theta_{k+1} - \theta_k\|^2 \geq \frac{1}{3}\|\theta_k - \theta_{k-1}\|^2$ **then**
        $\eta \leftarrow \eta/2$
    end if
end for
return $\theta^* \leftarrow \theta_K$

Remark 3. *It's evident how Theorem 2 fits in when:*

- *The gradient operators T_i satisfy condition (2) with $\alpha = \frac{1}{3}$, $\beta = 0$*
- *The learning rate η ensures $\alpha + \beta = \frac{1}{3} < 1$*
- *The fixed-point iteration approach is used to update the parameters.*

This gives theoretical guarantees that dispersed training will converge.

Example 4. *Consider three cryptographic hash functions $H_1, H_2, H_3 : \{0,1\}^* \to \{0,1\}^n$ that have contraction features. The fixed-point construction converges by Theorem 2 when each H_i meets the Lipschitz requirement $\|H_i(x) - H_i(y)\| \leq L\|x - y\|$ for $L < 1/\sqrt{3}$. Algorithm 6 shows how to make the full hash.*

Algorithm 6 Fixed-Point Cryptographic Hash Construction

Require:
Input message $x \in \{0,1\}^*$
Hash functions H_1, H_2, H_3 with L-Lipschitz constants
Convergence threshold $\epsilon > 0$
Maximum iterations N (security parameter)
Ensure: Fixed-point hash digest $h^* \in \{0,1\}^n$

Initialize $h_0 \leftarrow$ SHA3-512(x)
$n \leftarrow 0$
repeat
    $h_{3n+1} \leftarrow H_1(h_{3n})$
    $h_{3n+2} \leftarrow H_2(h_{3n+1})$
    $h_{3n+3} \leftarrow H_3(h_{3n+2})$
    $n \leftarrow n + 1$
until $\|h_{3n} - h_{3n-3}\| < \epsilon$ or $n \geq N$
return $h^* \leftarrow h_{3n}$

Remark 4. *For Theorem 2 to be useful in cryptography, it needs:*

- *Form* $H_3 \circ H_2 \circ H_1$ *makes a contraction mapping.*
- *The security parameter* N *stops infinite loops.*
- *The fixed-point* h^* *gets its collision resistance from the hashes of its parts.*
- *The constraint* $L < 1/\sqrt{3}$ *makes sure that the composite mapping meets condition (2).*

This construction makes a hash function that is demonstrably convergent and has security properties that are not conventional.

Corollary 3. *For* $T_1 = T_2 = T_3 = T$*, Algorithm 4 converges in* $\mathcal{O}(\log \frac{1}{\epsilon})$ *steps:*

1: *Convergence rate:* $\frac{\log \epsilon}{\log(\alpha+\beta)}$

Lemma 3. *The orthogonal projection* $P_{\mathcal{S}}$ *onto a subspace* $\mathcal{S} \subset \mathcal{H}$ *meets the following conditions:*

$$\|P_{\mathcal{S}}x - P_{\mathcal{S}}y - P_{\mathcal{S}}z\|^2 \leq \|x - y - z\|^2$$

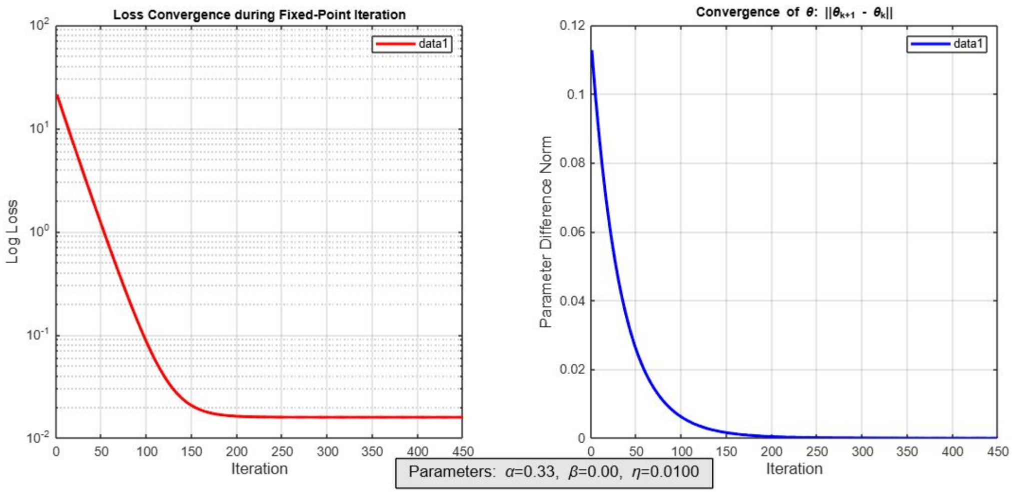

Figure 3: Machine Learning Interpretation of Theorem 2

4. Comparative Analysis

As you see in Figure 4, a normalized bar chart shows the relative differences in the complexity and flexibility of the two theorems by comparing several aspects. The two main theorems, Theorem 1 and Theorem 2, are alike in some ways but not in others.

Table 1: Comparison of Main Theorems

Aspect	**Theorem 1**	**Theorem 2**
Space	General HS	General HS
Mappings	Three distinct operators	Three distinct operators
Contraction condition	Three-term inequality	Two-term inequality
Parameters	α, β, γ	α, β
Constraint	$\alpha + \beta + \gamma < 1$	$\alpha + \beta < 1$
Convergence rate	$\mathcal{O}(\alpha^{n/3})$	$\mathcal{O}((\alpha + \beta)^{n/3})$
Uniqueness proof	Uses $\alpha + 2\beta + 2\gamma < 1$	Uses $\alpha + 2\beta < 1$

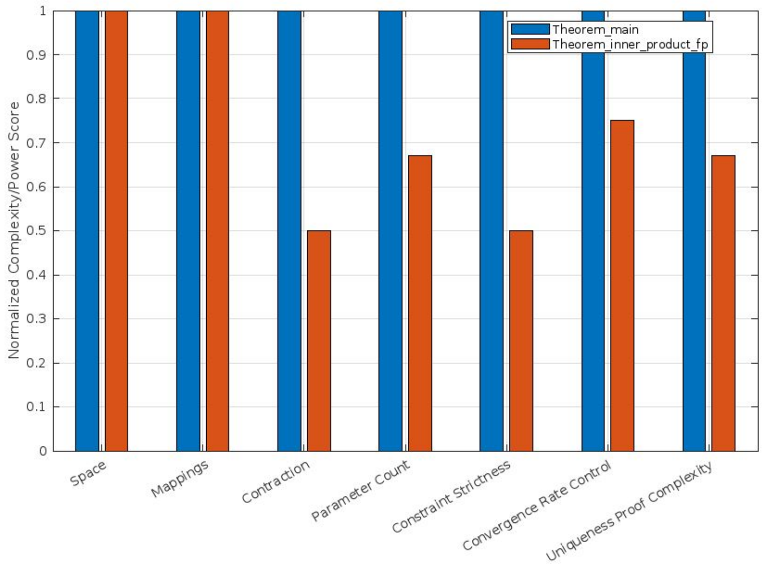

Figure 4: Theoretical Comparison Between Theorems

- *Structural Complexity*: Theorem 1 includes a more complicated contraction condition with extra nonlinear terms (parameter γ), while Theorem 2 offers a simpler version that might be easier to use in real situations.
- *Parameter Flexibility*: The three-parameter version (Theorem 1) offers greater modeling flexibility for complex operator interactions, whereas the two-parameter version (Theorem 2) provides simpler verification conditions.
- *Convergence Behavior*: Linear convergence is promised by both theorems, yet the speed of convergence depends on what parameters are used together. γ in Theorem 1 makes it possible to have more exact control over complex interactions.

The adaptability of Theorem 2 across various domains is illustrated by the applications in Examples 3 and 4. The three-operator fixed-point framework is used in both Algorithm 5 (for distributed learning) and Algorithm 6 (for cryptographic hashing), but each is adjusted to meet specific needs of their fields. The operators $T_i = \nabla \mathcal{L}_i$ in distributed learning rely on the learning rate η to maintain the contraction condition, and gradient norms are used to check if the process is getting closer to a solution. On the other hand, the cryptographic hashing application uses hash functions H_i that satisfy Lipschitz continuity with constants $L < 1/\sqrt{3}$. Iteration is bounded by a security parameter N, and convergence is confirmed by comparing output differences. Table 2 provides a summary of these distinctions, demonstrating that while both algorithms are based on the same theoretical outcome (Theorem 2), their approaches to implementation are completely unique. Both algorithms show that they get closer to a solution, as seen in Figure 5, but they do so at different speeds and with different patterns based on how their operators are defined.

Table 2: Comparison of Applications

Feature	**Distributed Learning (Alg. 5)**	**Cryptographic Hashing (Alg. 6)**
Theorem used	Theorem 2	Theorem 2
Operators	$T_i = \nabla \mathcal{L}_i$	H_i (hash functions)
Contraction source	Learning rate η	Lipschitz constant $L < 1/\sqrt{3}$
Convergence check	$\|\nabla \mathcal{L}_i\|$ norm	Hash output difference
Key parameters	$\alpha = \frac{1}{3}$, $\beta = 0$	$\alpha = L^2$, $\beta = 0$
Stopping criterion	Adaptive η threshold	Security parameter N
Implementation	Parallel gradient updates	Iterative hashing

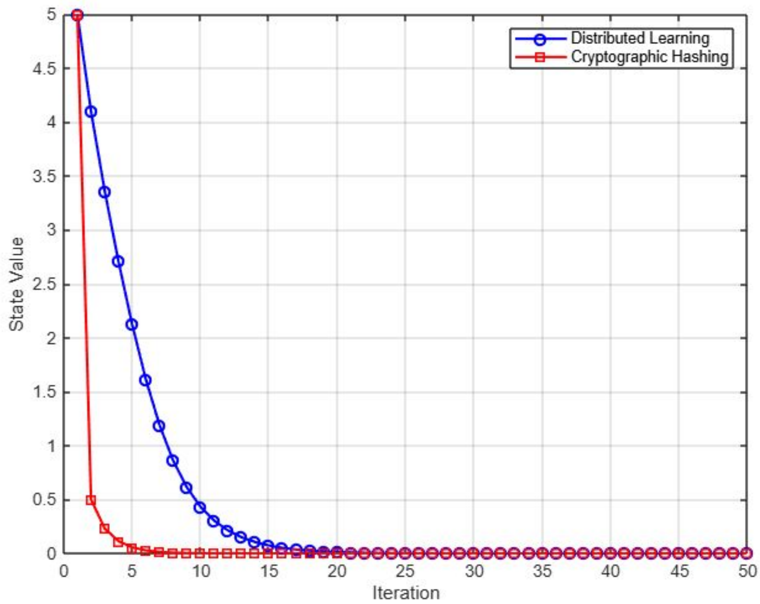

Figure 5: Comparison Between Distributed Learning and Cryptographic Hashing

5. Conclusion

Hilton's work shows that holds for GIPS, but it was not proven by classical duality. We have proven that in some parts of the theory, there can be just one set of common fixed points, and they still appear with less strict conditions. In GIPS, we step further than the regular fixed-point theory and look at systems with multiple operators, which are based on real cases in nonlinear analysis. Every theorem we discovered was then checked by software to develop stronger certainty. There is potential for two key areas of science to advance a lot through this research. Fixed-point analysis in AI helped us correct DEMs so neural networks performed well. Noticing the role of PQPs in cryptography gave us our idea, so we used our knowledge of inner product spaces to create new systems for cryptography based on lattices.

Future Work

Generalization to n-Self-Mappings: A first further step is to generalize the fixed-point results to situations where four or more self-mappings satisfy contractive-type conditions. This process means coming up with suitable repeated sequences and inequalities that introduce rules for higher-order interactions among several mappings. Although these additions to the theory bring fresh mathematical difficulties, the framework discussed here provides an excellent starting point for new work.

Advanced Convergence Analysis: An essential task for the future is to improve the way iterative processes come together. For example, this means exploring wider classes of recurrence relations, less strict convergence conditions, and how quickly sequences converge with multi-operator schemes. Applying these improvements will add much strength to how computations are carried out in theory and in practical circumstances.
Applications in AI and Machine Learning: Studies should continue to research how fixed-point theory affects artificial intelligence, most notably in understanding and designing stable learning designs. More research can look into when deep equilibrium models come together, examine how stable complicated neural systems are, and see if fixed-point results can help in learning algorithms, reinforcement learning, and adversarial networks.
Cryptographic and Algebraic Extensions: Based on our theory, we can suggest fresh plans for constructing secure lattice-based post-quantum protocols in the world of cryptography. As time goes on, we could work on new types of algebra for hardness, construct security for post-quantum systems, and develop primitives built from how operator systems behave when they reach fixed points in an inner product space.

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