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Folding on Topological Graphs and Their Fundamental Group

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Abstract. We introduce the effect of folding on the fundamental groups of a connected topological graph and its dual graph. We will deduce the limit of folding on the fundamental group of a topological graph and its duality. We obtain the relations between the induced folding and the induced retraction on the fundamental group. We apply retraction and folding operations to reduce the size of the graph while preserving its essential structural properties, which can assist us in developing mathematical software.

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Key Words and Phrases: Topological graph, Dual topological graph, Folding, Fundamental group

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1. Introduction

Graph theory serves as a mathematical representation that effectively investigates several tangible, real-world problems. Many different concerns related to the fields of chemistry, physics, communication, computer science, the genetic code, social science, psychology, and linguistics could potentially be articulated as problems related to graph theory. Many different areas of mathematics, including matrix theory, group theory, and topological structures, have strong relationships with graph theory [1]. Any graph $\mathcal{G}$ could indicate a topological space such that every vertex corresponds to only one point and every edge corresponds to a distinct arc, homeomorphic to the closed interval. The boundary points of an arc denote the endpoints of the associated edge, the interiors of the arcs are mutually disjoint and do not intersect the points representing vertices, and this configuration is referred to as a topological representation of $\mathcal{G}$ [2, 3].

A graph is defined as an ordered pair $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G}))$, in which $V(\mathcal{G}) \neq \Phi$ is a set and $E(\mathcal{G})$ is a set that is disjoint from $V(\mathcal{G})$, which are referred to as the vertices of $\mathcal{G}$. The elements of $E(\mathcal{G})$ are referred to as the edges of $\mathcal{G}$ [1, 4, 5]. A graph $\mathcal{H}$ is called a subgraph of a graph $\mathcal{G}$, denoted $\mathcal{H} \subseteq \mathcal{G}$, if $V(\mathcal{H}) \subseteq V(\mathcal{G})$ and $E(\mathcal{H}) \subseteq E(\mathcal{G})$ [6]. A graph

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$\mathcal{G}$ is planar if its edges meet only at their endpoints in the plane. We refer to a drawing of this form as a planar embedding of the graph $\mathcal{G}$ [7]. Let $\mathcal{G}$ be a plane graph. To construct a new graph $d(\mathcal{G})$ of $\mathcal{G}$, for all faces f of $\mathcal{G}$, select a vertex $\hat{f}$, with respect to all edges e of $\mathcal{G}$, choose an edge $\hat{e}$. Then edge $\hat{e}$ joins vertices $\hat{f}$ and $\hat{g}$ in $d(\mathcal{G})$ iff edge e is common to the boundaries of faces f and g in $\mathcal{G}$, this graph is called the dual of $\mathcal{G}$ [1]. We refer to an edge with coinciding vertices as a loop, while we refer to two vertices connected by more than one edge as multiple edges. An infinite graph is defined as a graph where both the edge set and the vertex set have infinite cardinality. A cycle graph is defined by the existence of a singular cycle; we will denote the cycle graph with one vertex by C, and the cycle graph with n- vertices is represented as C_n. The path graph is a graph that consists of a single path; the path graph with n-vertices is represented as P_n [1, 4]. The cobblestone path J_n is the graph formed by duplicating each edge of the n-vertex path P_n. A graph with one vertex and n self-loops is called an n-bouquet B_n [2, 3]. Let Y be a space, and let y_0 be an element of Y. The set $\pi_1(Y,\ y_0)$ consists of the homotopy classes of loops in $(Y,\ y_0)$, and it is equipped with the product operation $[u][v] = [u.v]$ referred to as the fundamental group [8, 9]. The fundamental groups with particular spaces were looked at [10, 11]. Let Y_1 and Y_2 be spaces such that $y_1 \in Y_1$ and $y_2 \in Y_2$, with $Y_1 \cap Y_2$. The wedge sum $Y_1 \curlyvee Y_2$ is defined as the quotient of $Y_1 \cup Y_2$ by the identification $y_1 \sim y_2$. [8]. A map $\Phi : V(\mathcal{G}_1) \longrightarrow V(\mathcal{G}_2)$ constitutes a homomorphism from $\mathcal{G}_1$ to $\mathcal{G}_2$ if it maintains edge preservation; specifically, for each edge $[u_1, u_2]$ in $\mathcal{G}_1$, $[\Phi(u_1), \Phi(u_2)]$ must be an edge in $\mathcal{G}_1$ [3]. A retract of a graph $\mathcal{G}_1$ is a subgraph $\mathcal{G}_2$ of $\mathcal{G}_1$ for which there is a homomorphism $r : \mathcal{G}_1 \to \mathcal{G}_2$, denoted retraction, providing $r(u) = u$ for every vertex u in $\mathcal{G}_2$ [8]. The folding is a continuous map $\digamma : \mathcal{G}_1 \to \mathcal{G}_2$ in which $\digamma(v) \in V(\mathcal{G}_2)$ for every $v \in V(\mathcal{G}_1)$ and $\digamma(e) \in E(\mathcal{G}_2)$ for every $e \in E(\mathcal{G}_1)$ [12, 13]. The topological and Banhatti indices for various silicate and oxide networks, using fuzzy set extensions, were presented in [14, 15]. For further information about the folding and retraction on manifolds and graphs, see [12, 16–19].

The central problem of this paper is to investigate the effects of folding and retraction on various classes of graphs and to characterize how retraction and folding impact key graph invariants. Transformations mainly consist of folding, limit folding, and rewrite rules that help sort graphs, find unchanging features, and enhance complex networks while keeping important information intact. Folding theory is the important transformation used in structural topology. Its simplest form merges and eliminates edges and vertices. This operation may remove multi-edges or loops by using folding and limit folding mapping in a special family of graph theory. This study will improve our insight into graph theory by looking at folding transformations, which will help us better understand graph structure and classification.

2. The main result

Theorem 1. *Given a connected graph* $\mathcal{G}$, *then*
(i) The folding $\digamma : \mathcal{G} \longrightarrow \mathcal{G}$, *induces* $\widehat{\digamma} : \pi_1(\mathcal{G}) \longrightarrow \pi_1(\mathcal{G})$ *for which* $\widehat{\digamma}(\pi_1(\mathcal{G})) = \pi_1(\digamma(\mathcal{G}))$.
(ii) There exists a specific type of folding $\digamma : \mathcal{G} \longrightarrow \tilde{\mathcal{G}}$ *that induces* $\hat{\digamma} : \pi_1(\mathcal{G}) \longrightarrow \pi_1(\tilde{\mathcal{G}})$,

and for this type, rank $(\widehat{F}(\pi_1(\mathcal{G}))) \geq$ *rank* $(\pi_1(\mathcal{G}))$.

Proof. (i) Let $\mathcal{G}$ *be a connected graph. Then,*
$\widehat{F}(\pi_1(\mathcal{G})) = \widehat{F}\{[C]$ *:where* C *is a cycle based at one vertex* $v_0 \in \mathcal{G}\} = \{[F(C)]$ *:where* $F(C)$ *is a cycle based at* $F(v_0) \in F(\mathcal{G})\} = \pi_1(F(\mathcal{G}))$.
(ii) Let $F : \mathcal{G} \longrightarrow \tilde{\mathcal{G}}$ *be a folding map in such a way that an edge is folded into another edge, then* $F(\mathcal{G})$ *contains a new multiple edge that induces* $\widehat{F} : \pi_1(\mathcal{G}) \longrightarrow \pi_1(\tilde{\mathcal{G}}\ )$ *such that* $\widehat{F}(\pi_1(\mathcal{G}) = \pi_1(F(\mathcal{G}))$. *Since rank* $(\pi_1(F(\mathcal{G}))) \geq$*rank* $(\pi_1(\mathcal{G}))$, *it follows that rank* $(\widehat{F}(\pi_1(\mathcal{G}))) \geq$*rank* $(\pi_1(\mathcal{G}))$.

Definition 1. *Let* $\{F_i : \mathcal{G}_{i-1} \to \mathcal{G}_i : i = 1, 2, \ldots m\}$ *be a sequence of folding maps on a connected graph, then we define the limit folding map as* $\lim\limits_{m\to\infty} F_m(\mathcal{G}_{m-1}) = \lim\limits_{m\to\infty} F_m(F_{m-1}(\ldots(F_1(\mathcal{G}_0)\ldots)$.

Theorem 2. *Let* J_n *be a cobblestone path, then*
(i) There is folding $F : J_n \to J_n$ *which induces* $\widehat{F} : \pi_1(J_n) \to \pi_1(J_n)$ *such that rank* $(\widehat{F}(\pi_1(J_n))) \leq n-1$.
(ii) The folding $F : dual(J_n) \to dual(J_n)$ *induces* $\widehat{F} : \pi_1(dual(J_n)) \to \pi_1(dual(J_n))$ *for which rank* $(\widehat{F}(\pi_1(J_n))) = 0$.

Proof. (i) Consider the folding $F : J_n \to J_n$ such that $F(J_n) = J_m$, $m \leq n-2$, $n \geq 3$, as in Figure 1, for $n = 5$, which induces $\widehat{F} : \pi_1(J_n) \to \pi_1(J_n)$ for which rank $(\widehat{F}(\pi_1(J_n))) =$ rank $(\pi_1(J_{n-2})) = n-1$.

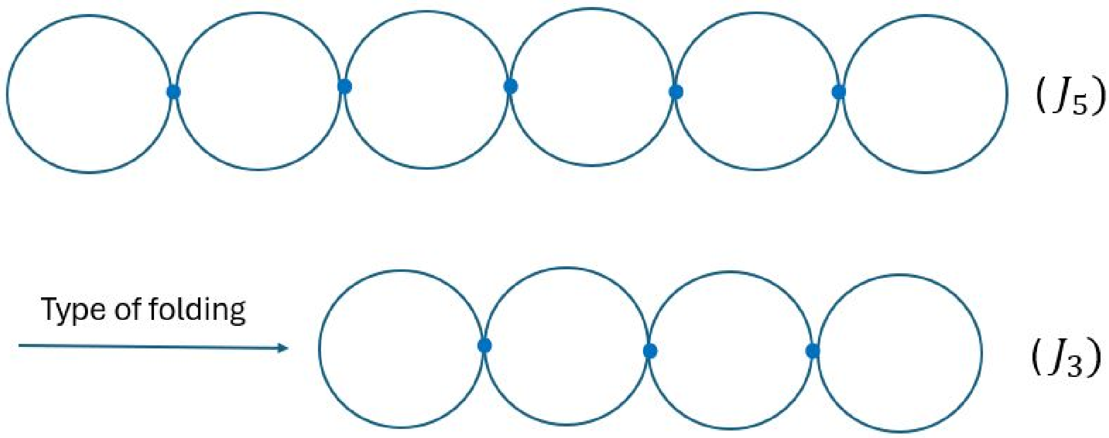

Figure 1:

(ii) Let $F : dual(J_n) \to dual(J_n)$ be a folding such that $F(dual(J_n)) = F(\ P_{n+2}) = P_m$ and $m \leq n+2$ as in Figure 2, for $n = 5$, which induces $\widehat{F} : \pi_1(dual(J_n)) \to \pi_1(dual(J_n))$ for which rank ($\widehat{F}(\pi_1(J_n))) = 0$.

Theorem 3. *Let* J_∞ *be an infinite cobblestone path. Then, there are two types of foldings* $F : J_\infty\ \to J_\infty$ *induce* $\widehat{F} : \pi_1(J_\infty\ ) \to \pi_1(J_\infty)$ *for which rank* $(\widehat{F}(\pi_1(J_\infty))) \leq 1$.

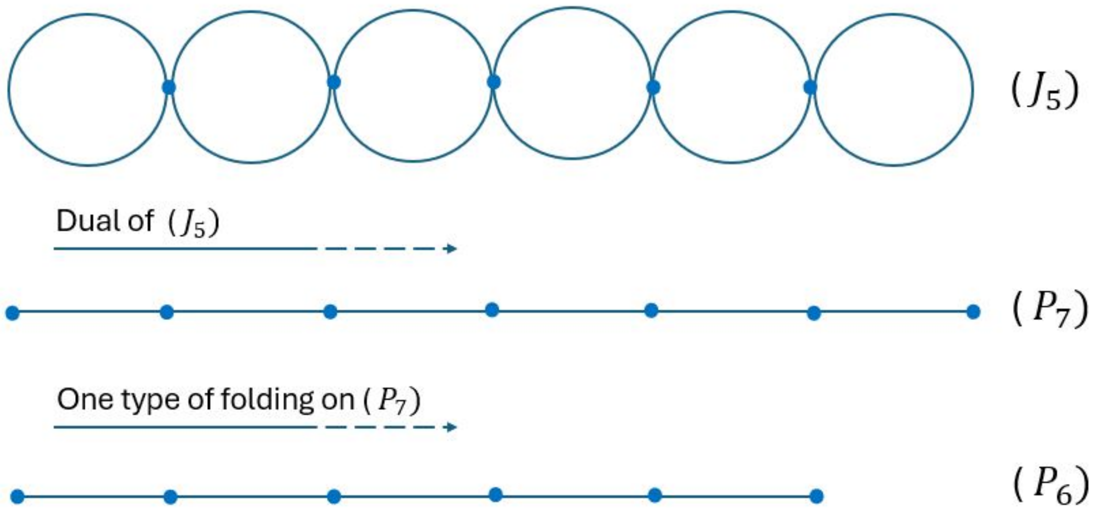

Figure 2:

Proof. Let J_∞ be an infinite cobblestone path, and consider $\digamma : J_\infty \to J_\infty$ to be a folding such that $\digamma(J_\infty) = C_2$ as in FIGURE 3(a), which induces $\widehat{\digamma} : \pi_1(J_\infty) \to \pi_1(J_\infty)$ in which rank $(\widehat{\digamma}(\pi_1(J_\infty))) = 1$. Furthermore, let $\digamma : J_\infty \to J_\infty$ be a folding such that $\digamma(J_\infty) = I_1 \curlyvee p_\infty \curlyvee I_2$, where I_1 and I_2 are arcs homeomorphic to the closed interval $[a, b]$, as shown in FIGURE 3(b), which induces $\widehat{\digamma} : \pi_1(J_\infty) \to \pi_1(J_\infty)$ for which rank $(\widehat{\digamma}(\pi_1(J_\infty))) = 0$.

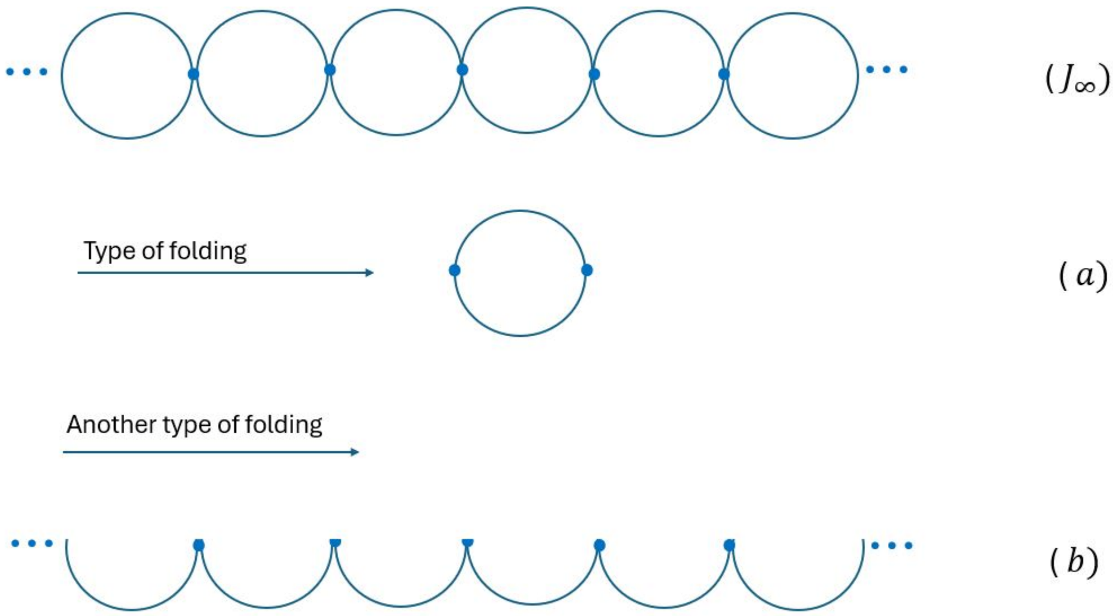

Figure 3:

Theorem 4. *Given an m-bouquet graph B_m, then the sequence of folding maps $\{F_i : B_{m_{i-1}} \to B_{m_i} : i = 1, 2, \ldots n\}$ induces $\{\widehat{F}_i : \pi_1 \left(B_{m_{i-1}}\right) \to \pi_1(B_{m_i}) : i = 1, 2, \ldots n\}$ such that rank $(\lim\limits_{n\to\infty}(\widehat{F}_n(\pi_1\ (B_{m_{n-1}})))) =\ 0$.*

Proof. *Given an m-bouquet graph B_m .*
Now, consider the following sequence of folding maps:
$F_1 : B_{m_0} \to B_{m_1}, F_2 : B_{m_1} \to B_{m_2}, \ldots, F_n : B_{m_{n-1}} \to B_{m_n}$, *for which* $\lim\limits_{n\to\infty} F_n(B_{m_{n-1}}) =$ v *(one vertex) as in* FIGURE 4, *which induces*
$\widehat{F}_1 : \pi_1\ (B_{m_0}) \to \pi_1\ (B_{m_1}),\ \widehat{F}_2 : \pi_1\ (B_1) \to \pi_1\ (B_2), \ldots,\ \widehat{F}_n : \pi_1 \left(B_{m_{n-1}}\right) \to \pi_1(B_{m_n})$, *for which*
$\lim\limits_{n\to\infty}(\widehat{F}_n(\pi_1\ (B_{m_{n-1}}))) = \pi_1(\ \lim\limits_{n\to\infty}(F_n\ (B_{m_{n-1}}))) = \pi_1\ (v)$.
Hence, rank $(\lim\limits_{n\to\infty}(\widehat{F}_n(\pi_1\ (B_{m_{n-1}})))) =\ 0$.

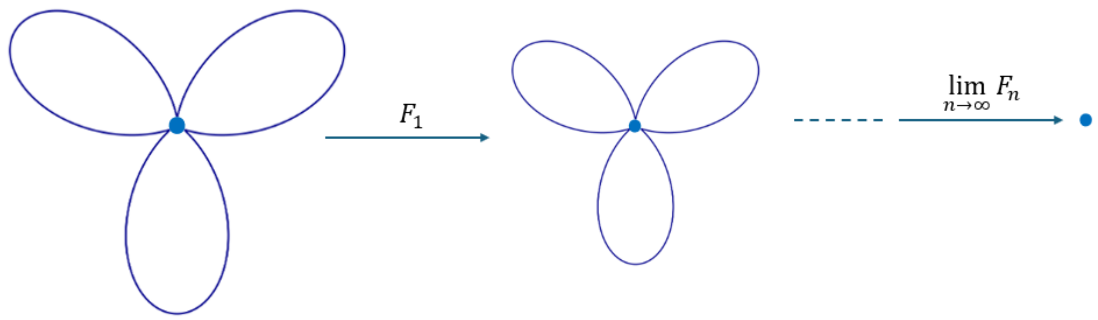

Figure 4:

Theorem 5. *Given a connected graph $\mathcal{G}_k$ that is homeomorphic to k-bouquet graph B_k for $k = 1,\ 2, \ldots, n$, then for any h, there is a folding $F_h : \overset{n}{\underset{k=1}{\Upsilon}} \mathcal{G}_k \longrightarrow \overset{n}{\underset{k=1}{\Upsilon}} \mathcal{G}_k$ which induces a folding $\widehat{F}_h :\ \overset{n}{\underset{k=1}{*}} \pi_1(\mathcal{G}_k) \to \overset{n}{\underset{k=1}{*}} \pi_1(\mathcal{G}_k)$ such that $\widehat{F}_h(\overset{n}{\underset{k=1}{*}} \pi_1(\mathcal{G}_k))$ is a free group of rank $m - h$, for $h = 1,\ 2, \ldots, n$ where, $m = \frac{n(n+1)}{2}$.*

Proof. Let $F_1 : \overset{n}{\underset{k=1}{\Upsilon}} \mathcal{G}_k \longrightarrow \overset{n}{\underset{k=1}{\Upsilon}} \mathcal{G}_k$ be a folding such that $F_1(\overset{n}{\underset{k=1}{\Upsilon}} \mathcal{G}_k) = \mathcal{G}_1 \Upsilon \mathcal{G}_2 \Upsilon \ldots \Upsilon F_1(\mathcal{G}_{r_1}) \Upsilon \ldots \Upsilon \mathcal{G}_n$ for $r_1 = 1, 2, \ldots, n$ where $F_1(\mathcal{G}_t) = F_1(B_t) = B_{t-1}$, folding one loop into another, then we obtain the induced folding $\widehat{F}_1 : \overset{n}{\underset{k=1}{*}} \pi_1(\mathcal{G}_k) \longrightarrow \overset{n}{\underset{k=1}{*}} \pi_1(\mathcal{G}_k)$ such that $\widehat{F}_1(\overset{n}{\underset{k=1}{*}} \pi_1(\mathcal{G}_k)) = \pi_1(F_1(\overset{n}{\underset{i=1}{\Upsilon}} \mathcal{G}_k))$ and so $\widehat{F}_1(\overset{n}{\underset{k=1}{*}} \pi_1(\mathcal{G}_i)\,) \approx \pi_1(\mathcal{G}_1) * \pi_1(\mathcal{G}_2) * \cdots * \pi_1(F_1(\mathcal{G}_{r_1})) * \cdots * \pi_1(\mathcal{G}_n)$. Since $\pi_1(F_1(\mathcal{G}_{r_1})) = \pi_1(B_{r_1}-1)$, it follows that rank $(\pi_1(F_1(\mathcal{G}_{r_1})))$ =rank $(\pi_1(\mathcal{G}_{r_1})) - 1$. Hence, rank $(\widehat{F}_1(\overset{n}{\underset{i=1}{*}} \pi_1(\mathcal{G}_i)\,)) = n - 1$. Moreover, let $F_2 : \overset{n}{\underset{k=1}{\Upsilon}} \mathcal{G}_i \longrightarrow \overset{n}{\underset{k=1}{\Upsilon}} \mathcal{G}_i$ be folding such that $F_2(\overset{n}{\underset{k=1}{\Upsilon}} \mathcal{G}_k) = \mathcal{G}_1 \Upsilon \mathcal{G}_2 \Upsilon \ldots \Upsilon F_2(\mathcal{G}_{r_1}) \Upsilon \ldots \Upsilon F_2(\mathcal{G}_{r_2}) \Upsilon \ldots \Upsilon \mathcal{G}_n$ for $r_1, r_2 = 1, 2, \ldots, n,\ r_1 \prec r_2$, and $F_2(\mathcal{G}_{r_1}) = F_2(B_{r_1}) = B_{r_1-1},\ F_2(\mathcal{G}_{r_2}) = F_2(B_{r_2}) = B_{r_2-1}$, then we get the induced folding $\widehat{\digamma}_2 : \overset{n}{\underset{k=1}{*}}\pi_1(\mathcal{G}_k) \longrightarrow \overset{n}{\underset{k=1}{*}}\pi_1(\mathcal{G}_k)$ such that rank $(\widehat{\digamma}_2(\overset{n}{\underset{k=1}{*}}\pi_1(\mathcal{G}_k)) = n-2$. We follow this approach, we have $\digamma_n : \overset{n}{\underset{k=1}{\curlyvee}}\mathcal{G}_k \longrightarrow \overset{n}{\underset{k=1}{\curlyvee}}\mathcal{G}_k$ such that $\digamma_n(\overset{n}{\underset{k=1}{\curlyvee}}\mathcal{G}_k) = \overset{n}{\underset{k=1}{\curlyvee}}\digamma_n(\mathcal{G}_k)$ and $\digamma_n(\mathcal{G}_k) = \digamma_n(B_k) = B_{k-1}$, $\forall k = 1,\ 2, \ldots, n$, which induces a folding $\widehat{\digamma}_n : \overset{n}{\underset{k=1}{*}}\pi_1(\mathcal{G}_k) \longrightarrow \overset{n}{\underset{k=1}{*}}\pi_1(\mathcal{G}_k)$ such that $\widehat{\digamma}_n(\overset{n}{\underset{k=1}{*}}\pi_1(\mathcal{G}_k))$ is a free group of rank $m-n$. Consequently, rank $(\widehat{\digamma}_h(\overset{n}{\underset{k=1}{*}}\pi_1(\mathcal{G}_k)\ )) = m-h$, for $h = 1,\ 2, \ldots, n$.

Theorem 6. *Assume that $\mathcal{G}$ is a graph and $\mathcal{H}$ is a subgraph of it. For $i = 1, 2, \ldots n$, let $\digamma_i$ and r_i be chains of folding and retraction functions, respectively. Then, there is a chain of a commutative diagram of a graph that induces a chain of a commutative diagram of fundamental groups.*

Proof. *Given a commutative diagram:*

$$\begin{array}{ccccccc}
\mathcal{G} & \xrightarrow{\digamma_1} & \mathcal{G}_1 & \xrightarrow{\digamma_2} & \mathcal{G}_2 & ---\xrightarrow{\lim_{m\to\infty} \digamma_m} & \text{only one vertex or one edge} \\
\downarrow r_1 & & \downarrow r_2 & & \downarrow r_3 & & \downarrow \lim\limits_{m\to\infty} r_m \\
\mathcal{H} & \xrightarrow{\digamma_1} & \mathcal{H}_1 & \xrightarrow{\digamma_2} & \mathcal{H}_2 & ---\xrightarrow{\lim_{m\to\infty} \digamma_m} & \text{only one vertex or one edge.}
\end{array}$$

As the fundamental group constitutes a functor, we get the following chain of a commutative diagram of fundamental groups:

$$\begin{array}{ccccccc}
\pi_1\ (\mathcal{G}) & \xrightarrow{\widehat{\digamma}_1} & \pi_1\ (\mathcal{G}_1\ ) & \xrightarrow{\widehat{\digamma}_1} & \pi_1\ (\mathcal{G}_2) & ---\xrightarrow{\lim_{m\to\infty} \widehat{\digamma}_m} & \{0\} \\
\downarrow \hat{r}_1 & & \downarrow \hat{r}_2 & & \downarrow \hat{r}_3 & & \downarrow \lim\limits_{m\to\infty} \hat{r}_m \\
\pi_1\ (\mathcal{H}) & \xrightarrow{\widehat{\digamma}_1} & \pi_1\ (\mathcal{H}_1) & \xrightarrow{\widehat{\digamma}_2} & \pi_1\ (\mathcal{H}_2) & ---\xrightarrow{\lim_{m\to\infty} \widehat{\digamma}_m} & \{0\}.
\end{array}$$

3. Conclusion

In topological graph theory, the effect of folding on special types of graphs and their duals is introduced. Also, the folding induced by the fundamental groups is obtained. The limits of folding on the induced fundamental group are presented. The relations between the induced folding and the induced retraction on the fundamental group are deduced.

References

[1] R. Balakrishnan and K. Ranganathan. *A Textbook of Graph Theory.* Springer, New York, Heidelberg, Dordrecht, London, 2012.

[2] L. W. Beineke, R. J. Wilson, J. L. Gross, and T. W. Tucker. *Topics in Topological Graph Theory.* Cambridge University Press, New York, 2009.

[3] J. L. Gross and T. W. Tucker. *Topological Graph Theory.* Courier Corporation, 2001.

[4] A. T. White. *Graphs, Groups and Surfaces*, volume 8. Elsevier, 1985.

[5] R. J. Wilson and J. J. Watkins. *Graphs: An Introductory Approach. A First Course in Discrete Mathematics.* John Wiley & Sons, Inc., Canada, 1990.

[6] G. Chartrand and P. Zhang. *A First Course in Graph Theory.* Dover Publications, New York, 2012.

[7] J. A. Bondy and U. S. R. Murty. *Graph Theory.* Springer, 2008.

[8] A. Hatcher. *Algebraic Topology.* Cambridge University Press, Cambridge, 2002.

[9] W. S. Massey. *Algebraic Topology: An Introduction.* Harcourt Brace and World, New York, 1967.

[10] J. Brazas. The fundamental group as a topological group. *Topology and its Applications*, 160:70–188, 2013.

[11] O. Neto and P. C. Silva. The fundamental group of an algebraic link. *Comptes Rendus Mathematique*, 340(2):141–146, 2005.

[12] P. Hell and J. Nešetřil. *Graphs and Homomorphisms*, volume 28 of *Oxford Lecture Series in Mathematics and Its Applications.* Oxford University Press, 2004.

[13] S. I. Nada and E. Hamouda. On the folding of graphs-theory and application. *Chaos, Solitons & Fractals*, 42(2):669–675, 2009.

[14] F. Dayan, M. Javaid, M. Zulqarnain, M. T. Ali, and B. Ahmad. Computing banhatti indices of hexagonal, honeycomb and derived networks. *American Journal of Mathematical and Computer Modelling*, 3(2):38–45, 2018.

[15] R. M. Zulqarnain, X. L. Xin, and Y. B. Jun. Fuzzy axiom of choice, fuzzy zorn's lemma and fuzzy hausdorff maximal principle. *Soft Computing*, 25(25):11421–11428, 2021.

[16] M. Abu-Saleem. The folded map on an identification graph and its application. *Afrika Matematika*, 36(1):23, 2025.

[17] M. Abu-Saleem. Retractions and homomorphisms on some operations of graphs. *Journal of Mathematics*, 2018(1):7328065, 2018.

[18] M. Abu-Saleem. A neutrosophic folding and retraction on a single-valued neutrosophic graph. *Journal of Intelligent & Fuzzy Systems*, 40(3):5207–5213, 2021.

[19] M. Abu-Saleem. Folding on the wedge sum of graphs and their fundamental group. *APPS. Applied Sciences*, 12:14–19, 2010.