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Tensor Product of Spaces with Generalized 2-Inner Product

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Abstract. In this work, we introduce the notion of tensor product of spaces with a generalized 2-inner product (see Definition 5), and we establish several interesting properties (see Proposition 5), thereby generalizing the classical properties of the tensor product of inner product spaces. Moreover, we equip this tensor product with a mapping that defines a generalized 2-inner product (see Theorem 3) and, consequently, endow it with a generalized 2-norm (see Theorem 1). In this context, we also define the tensor product of linear operators (see Definition 9) and prove a series of results for example, that the tensor product of two 2-bounded linear operators is again 2-bounded under the tensor product (see Proposition 10).

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1. Introduction

The ideas that gave rise to the concept of the tensor product of vector spaces were developed by various researchers throughout the nineteenth century; however, it was not until the work carried out by the mathematician Hassler Whitney (1938) that the notion of the tensor product of abelian groups and more generally of modules was firmly established [1]. Consequently, the tensor product of vector spaces was cast in the language of universal properties in works such as those of Bourbaki (1943) in the article [2] and in the book by Artin, Nesbitt, and Thrall (1944) [3]. These developments, in turn, spurred rapid advancement and further elaboration of the concept in other mathematical contexts.

The notion of the tensor product of modules has been extensively developed in various mathematical contexts for example, within Homological Algebra and Differential Geometry (see [4–8]). It is a concept of great significance, since, broadly speaking, it provides a

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method for constructing new spaces with the desired structure and, in the case of vector spaces, offers a way to pass from bilinear operators to linear operators. Moreover, it arises naturally in areas of physics such as quantum physics and quantum computing (see [9, 10]).

In recent years, the concept of the tensor product of complex vector spaces equipped with a positive definite sesquilinear form [11] has been extended to the tensor product of spaces endowed with an arbitrary sesquilinear form [12], defining simple tensors using such sesquilinear forms in an appropriate manner. Moreover, since the concept of the classical inner product can be extended to what is known as the 2-inner product and the generalized 2-inner product in the senses of Gähler [13] and Lewandoska [14], respectively, it is natural to consider defining the tensor product of spaces endowed with a 2-inner product of this kind, which in this work will be referred to as the 2-tensor product of spaces with a generalized 2-inner product. The theory of spaces with a 2-inner product has been extensively studied in articles by Diminnie, Gähler and White [15–18].

Thus, the present work is organized as follows: in Section 2, we include the necessary preliminaries to introduce the notion of the algebraic 2-tensor product of vector spaces endowed with a generalized 2-inner product, such as the concept of a generalized 2-inner product and some of its properties. In Section 3, we introduce the notion of the 2-tensor product of elements in spaces with a generalized 2-inner product, based on the ideas of the simple tensor product of elements in spaces with a classical inner product; consequently, we establish the notion of the 2-tensor product of spaces with a generalized 2-inner product, define a generalized 2-inner product on it, and prove some interesting properties. Section 4 contains the notion of the 2-tensor product of linear operators and some of their satisfied properties. Finally, Section 5 presents the conclusions of this research.

2. Preliminaries

In this section, we study the concepts of the generalized 2-norm, the generalized 2-inner product, and some of the most important properties that are satisfied in spaces endowed with these structures.

Definition 1. *[14] A* ***generalized 2-norm*** *on $\mathcal{X}$ is a map $\|\cdot,\cdot\| : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ such that for every $w, y, z \in \mathcal{X}$ is true*

i) $\|w, z\| = \|z, w\|$;

ii) $\|w, \alpha z\| = |\alpha| \|w, z\|$ *for all* $\alpha \in \mathbb{C}$;

iii) $\|w, y + z\| \leq \|w, y\| + \|w, z\|$.

The pair $(\mathcal{X}, \|\cdot,\cdot\|)$ is called a ***space with generalized 2-norm****.*

In the following example we show how to define a generalized 2-norm from an indefinite sesquilinear form.

Example 1. *[19] We define on the vector space $\mathbb{C}^n$ the function $\|\cdot,\cdot\|: \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{R}$ by means of the rule $\|x, y\| := \left|\sum_{i=1}^{n}(-1)^i x_i \overline{y}_i\right|$ for all $x = (x_1, \cdots, x_n), y = (y_1, \cdots, y_n) \in \mathbb{C}^n$. What defines a generalized 2-norm. In general, it can be shown that, given a classical inner product space $(\mathcal{X}, \langle\cdot,\cdot\rangle)$, the mapping*

$$\begin{aligned} \|\cdot,\cdot\| : \mathcal{X} \times \mathcal{X} &\to \mathbb{R} \\ (w, z) &\mapsto \|w, z\| = |\langle w, z\rangle|, \end{aligned}$$

defines a generalized 2-norm on $\mathcal{X}$.

Below we present the concept of generalized 2-inner product, a notion of great importance for this work.

Definition 2. *[14] A generalized 2-inner product is a map $\langle\cdot,\cdot|\cdot\rangle : \mathcal{X} \times \mathcal{X} \times \mathcal{X} \longrightarrow \mathbb{C}$ such that for all $w, y, z, w_1, w_2 \in \mathcal{X}$ and for all $\alpha \in \mathbb{C}$*

i) $\langle y, w|z\rangle = \overline{\langle w, y|z\rangle}$;

ii) $\langle w_1 + w_2, y|z\rangle = \langle w_1, y|z\rangle + \langle w_2, y|z\rangle$;

iii) $\langle \alpha w, y|z\rangle = \alpha\langle w, y|z\rangle$;

iv) $\langle w, w|z\rangle = \langle z, z|w\rangle$;

v) $\langle w, w|z\rangle \geq 0$.

A complex vector space $\mathcal{X}$ together a generalized 2-inner product $\langle\cdot,\cdot|\cdot\rangle$, denoted by $(\mathcal{X}, \langle\cdot,\cdot|\cdot\rangle)$, it is said to be a generalized 2-inner product space.

Proposition 1. *[20]It is easy to check that the following function is a generalized 2-inner product on $\mathcal{X}$*

$$\langle w, y|z\rangle = \begin{vmatrix} \langle w, y\rangle & \langle w, z\rangle \\ \langle z, y\rangle & \langle z, z\rangle \end{vmatrix} = \langle w, y\rangle \langle z, z\rangle - \langle w, z\rangle \langle z, y\rangle \ \text{ for all } w, y, z \in \mathcal{X}.$$

This 2-inner product is called generalized standard 2-inner product and it is denoted by $\langle\cdot,\cdot|\cdot\rangle_{stand}$.

A consequence of [21, Prop 3.1] when $(\mathcal{X}, \langle\cdot,\cdot\rangle)$ is a classical inner product space is:

Proposition 2. *[21] Let $(\mathcal{X}, \langle\cdot,\cdot\rangle)$ be a classical inner product space. Then the mapping*

$$\langle\cdot,\cdot|\cdot\rangle : \mathcal{X} \times \mathcal{X} \times \mathcal{X} \longrightarrow \mathbb{C}$$

defined by

$$\langle x, y|z\rangle = \langle x, y\rangle \|z\|^2 \quad \text{for all } x, y, z \in \mathcal{X},$$

is a generalized 2-inner product on $\mathcal{X}$.

Proposition 3. *[19] Let $(\mathcal{X}, \langle \cdot, \cdot | \cdot \rangle)$ be a generalized 2-inner product space. Then, for all $w, y, z \in \mathcal{X}$ it holds that:*

$$|\langle w, y|z\rangle|^2 \leq \langle w, w|z\rangle \langle y, y|z\rangle.$$

Remark 1. *The function*

$$\|w, z\| := \sqrt{\langle w, w|z\rangle}, \ w, z \in \mathcal{X},$$

sometimes it is called **induced generalized 2-norm** *by the generalized 2-inner product.*

Definition 3 (2-bounded operator). *[14] Let $\mathcal{X}$ be a vector space endowed with two generalized 2-norms $\|\cdot, \cdot\|_1$ and $\|\cdot, \cdot\|_2$. An operator $T : (X, \|\cdot, \cdot\|_1) \to (X, \|\cdot, \cdot\|_2)$ is said to be 2-bounded if there $\alpha \geq 0$ such that*

$$\|T(x), y\|_2 + \|x, T(y)\|_2 \leq \alpha \|x, y\|_1 \text{ for all } x, y \in \mathcal{X}.$$

The symbol $\overset{2}{\mathcal{B}}(\mathcal{X})$ will denote the set of 2-bounded linear operator on $\mathcal{X}$, that is,

$$\overset{2}{\mathcal{B}}(\mathcal{X}) := \{T : \mathcal{X} \to \mathcal{X} : T \text{ is linear and 2-bounded}\}.$$

Remark 2. *It is clear that the set $\overset{2}{\mathcal{B}}(\mathcal{X})$ can be endowed with a vector space structure over $\mathbb{C}$, by means of the pointwise operations of operators.*

By virtue of the properties of generalized 2-normed spaces, in the following theorem we establish an equivalent result for the Definition 3.

Theorem 1. *Note that a linear operator $T \colon (\mathcal{X}, \|\cdot, \cdot\|_1) \to (\mathcal{X}, \|\cdot, \cdot\|_2)$ is 2-bounded according to Definition 3 if and only if there exists a constant $\beta > 0$ such that*

$$\|Tz, w\|_2 \leq \beta \|z, w\|_1 \quad \text{for all } z, w \in \mathcal{X}.$$

Proof. Suppose first that there exists $\alpha > 0$ such that

$$\|Tx, y\|_2 + \|x, Ty\|_2 \leq \alpha \|x, y\|_1 \quad \text{for all } x, y \in \mathcal{X}.$$

Since each 2-norm is nonnegative, it follows that

$$\|Tx, y\|_2 \leq \|Tx, y\|_2 + \|x, Ty\|_2 \leq \alpha \|x, y\|_1,$$

hence T satisfies the above inequality with $\beta = \alpha$. Conversely, assume there exists $\beta > 0$ such that

$$\|Tz, w\|_2 \leq \beta \|z, w\|_1 \quad \text{for all } z, w \in \mathcal{X}.$$

Then for any $x, y \in \mathcal{X}$ we have

$$\|Tx, y\|_2 + \|x, Ty\|_2 = \|Tx, y\|_2 + \|Ty, x\|_2 \leq \beta \|x, y\|_1 + \beta \|y, x\|_1 = 2\beta \|x, y\|_1.$$

If we set $\alpha = 2\beta$, the desired inequality follows.

Definition 4. *If* $T \in \overset{2}{\mathcal{B}}(\mathcal{X})$*, we define* $\|T\|$*, by*

$$\|T\| = \inf\{a \geq 0 : \|T(w), z\| \leq a\|w, z\| \text{ for all } w, z \in \mathcal{X}\}.$$

Note that the identity operator in $\mathcal{X}$*,* $I : \mathcal{X} \to \mathcal{X}$*, fulfills* $\|I\| = 1$.

Theorem 2. *In the context of the Definition 4, for all* $T \in \overset{2}{\mathcal{B}}(\mathcal{X})$ *it is true that*

$$\|T\| = \sup\{\|Tw, z\| : w, z \in \mathcal{X} \text{ and } \|w, z\| = 1\}$$
$$\|T\| = \sup\{\|Tw, z\| : w, z \in \mathcal{X} \text{ and } \|w, z\| \leq 1\}$$
$$\|T\| = \sup\left\{\frac{\|Tw, z\|}{\|w, z\|} : w, z \in \mathcal{X} \text{ and } \|w, z\| \neq 0\right\}$$

Moreover, thanks to the Definition 4 we prove that given a generalized 2-normed space $\mathcal{X}$, it is possible to endow the vector space $\overset{2}{\mathcal{B}}(\mathcal{X})$ with the structure of a semi-normed space.

Proposition 4. *[14] For all* $T \in \overset{2}{\mathcal{B}}(\mathcal{X})$ *it is true that*

$$\|T(w), y\| \leq \|T\|\|w, y\|$$

for all $w, y, z \in \mathcal{X}$.

3. Main Results

3.1. 2-Tensor Product

In this section, we introduce the notion of the 2-tensor product of elements in vector spaces equipped with a generalized 2-inner product.

Definition 5. *Let* $(\mathcal{X}_1, \langle \cdot, \cdot | \cdot \rangle_1)$ *and* $(\mathcal{X}_2, \langle \cdot, \cdot | \cdot \rangle_2)$ *be spaces with a generalized 2-inner product. Given* $x_1 \in \mathcal{X}_1$ *and* $x_2 \in \mathcal{X}_2$*, the 2-tensor product of* x_1 *and* x_2*, denoted by* $x_1 \overset{2}{\odot} x_2$*, is the function*

$$x_1 \overset{2}{\odot} x_2 \colon (\mathcal{X}_1 \times \mathcal{X}_2) \times (\mathcal{X}_1 \times \mathcal{X}_2) \longrightarrow \mathbb{C}$$

defined by

$$(x_1 \overset{2}{\odot} x_2)\big((t_1, t_2), (r_1, r_2)\big) = \langle x_1, t_1 \mid r_1 \rangle_1 \, \langle x_2, t_2 \mid r_2 \rangle_2.$$

Proposition 5. *Let* $(\mathcal{X}_1, \langle \cdot, \cdot | \cdot \rangle_1)$ *and* $(\mathcal{X}_2, \langle \cdot, \cdot | \cdot \rangle_2)$ *be spaces with a generalized 2-inner product. For all* $x, x_1, y_1 \in \mathcal{X}_1$*,* $x_2, y_2, y \in \mathcal{X}_2$*, and all* $\alpha, \beta \in \mathbb{C}$*, the following properties hold:*

i) $x \overset{2}{\odot} \boldsymbol{0}_2 = 0$

ii) $\boldsymbol{0}_1 \overset{2}{\odot} y = 0$

iii) $(\alpha x_1) \overset{2}{\odot} x_2 = \alpha(x_1 \overset{2}{\odot} x_2) = x_1 \overset{2}{\odot} (\alpha x_2)$

iv) $\alpha\beta(x_1 \overset{2}{\odot} x_2) = (\alpha x_1 \overset{2}{\odot} \beta x_2)$

v) $(x_1 + y_1) \overset{2}{\odot} x_2 = (x_1 \overset{2}{\odot} x_2) + (y_1 \overset{2}{\odot} x_2)$

vi) $x_1 \overset{2}{\odot} (x_2 + y_2) = (x_1 \overset{2}{\odot} x_2) + (x_1 \overset{2}{\odot} y_2)$

Proof. Let $t_1, r_1 \in \mathcal{X}_1$ be and $t_2, r_2 \in \mathcal{X}_2$ be.

i) $(x_1 \overset{2}{\odot} \mathbf{0}_2)((t_1,t_2),(r_1,r_2)) = \langle x_1, t_1|r_1\rangle_1 \langle \mathbf{0}_2, t_2|r_2\rangle_2 = \langle x_1, t_1|r_1\rangle_1 \cdot 0 = 0$

ii) $(\mathbf{0}_1 \overset{2}{\odot} x_2)((t_1,t_2),(r_1,r_2)) = \langle \mathbf{0}_1, t_1|r_1\rangle_1 \langle x_2, t_2|r_2\rangle_2 = 0 \cdot \langle x_2, t_2|r_2\rangle_2 = 0$

iii)
$$\begin{aligned}((\alpha x_1) \overset{2}{\odot} x_2)((t_1,t_2),(r_1,r_2)) &= \langle \alpha x_1, t_1|r_1\rangle_1 \langle x_2, t_2|r_2\rangle_2 = \alpha\langle x_1, t_1|r_1\rangle_1 \langle x_2, t_2|r_2\rangle_2\\ &= \langle x_1, t_1|r_1\rangle_1 \langle (\alpha x_2), t_2|r_2\rangle_2 = (x_1 \overset{2}{\odot} (\alpha x_2))((t_1,t_2),(r_1,r_2))\\ &= \alpha(x_1 \overset{2}{\odot} x_2)((t_1,t_2),(r_1,r_2))\end{aligned}$$

iv)
$$\begin{aligned}(\alpha x_1 \overset{2}{\odot} \beta x_2)((t_1,t_2),(r_1,r_2)) &= \langle (\alpha x_1), t_1|r_1\rangle_1 \langle (\beta x_2), t_2|r_2\rangle_2 = \alpha\langle x_1, t_1|r_1\rangle_1 \beta\langle x_2, t_2|r_2\rangle_2\\ &= \alpha\beta\langle x_1, t_1|r_1\rangle_1 \langle x_2, t_2|r_2\rangle_2 = \alpha\beta(x_1 \overset{2}{\odot} x_2)((t_1,t_2),(r_1,r_2))\end{aligned}$$

v)
$$\begin{aligned}((x_1 + y_1) \overset{2}{\odot} x_2)((t_1,t_2),(r_1,r_2)) &= \langle (x_1 + y_1), t_1|r_1\rangle_1 \langle x_2, t_2|r_2\rangle_2\\ &= (\langle x_1, t_1|r_1\rangle_1 + \langle y_1, t_1|r_1\rangle_1)\langle x_2, t_2|r_2\rangle_2\\ &= \langle x_1, t_1|r_1\rangle_1 \langle x_2, t_2|r_2\rangle_2 + \langle y_1, t_1|r_1\rangle_1 \langle x_2, t_2|r_2\rangle_2\\ &= (x_1 \overset{2}{\odot} x_2)((t_1,t_2),(r_1,r_2)) + (y_1 \overset{2}{\odot} x_2)((t_1,t_2),(r_1,r_2))\end{aligned}$$

vi)
$$\begin{aligned}(x_1 \overset{2}{\odot} (x_2 + y_2))((t_1,t_2),(r_1,r_2)) &= \langle x_1, t_1|r_1\rangle_1 \langle (x_2 + y_2), t_2|r_2\rangle_2\\ &= \langle x_1, t_1|r_1\rangle_1 (\langle x_2, t_2|r_2\rangle_2 + \langle y_2, t_2|r_2\rangle_2)\\ &= \langle x_1, t_1|r_1\rangle_1 \langle x_2, t_2|r_2\rangle_2 + \langle x_1, t_1|r_1\rangle_1 \langle y_2, t_2|r_2\rangle_2\\ &= (x_1 \overset{2}{\odot} x_2)((t_1,t_2),(r_1,r_2)) + (x_1 \overset{2}{\odot} y_2)((t_1,t_2),(r_1,r_2))\end{aligned}$$

3.2. Algebraic Tensor Product

Next, we introduce the notion of the algebraic tensor product between two vector spaces endowed with a generalized 2-inner product.

Definition 6. *Let $(\mathcal{X}_1, \langle \cdot, \cdot | \cdot \rangle_1)$ and $(\mathcal{X}_2, \langle \cdot, \cdot | \cdot \rangle_2)$ be spaces with a generalized 2-inner product. We define their algebraic tensor product, denoted by $\mathcal{X}_1 \overset{2}{\odot} \mathcal{X}_2$, as*

$$\mathcal{X}_1 \overset{2}{\odot} \mathcal{X}_2 := \Big\{ \sum_{i=1}^{n} \beta_i \, (x_i \overset{2}{\odot} y_i) \;\Big|\; n \in \mathbb{N}^*, \, \beta_i \in \mathbb{C}, \, x_i \in \mathcal{X}_1, \, y_i \in \mathcal{X}_2 \Big\}.$$

Note that $\mathcal{X}_1 \overset{2}{\odot} \mathcal{X}_2$ is a complex vector space.

Note that by the properties of the mapping $\overset{2}{\odot}$ every element of the vector space $X_1 \overset{2}{\odot} X_2$ can be written simply as $\sum_{i=1}^{n} x_i \overset{2}{\odot} y_i$ with $x_i \in X_1$, $y_i \in X_2$ and $n \in \mathbb{N}$.

In the following theorem, we equip the vector space from Definition 6 with a generalized 2-inner product, which we call the generalized 2-inner tensor product.

Theorem 3. *Let $(\mathcal{X}_1, \langle \cdot, \cdot | \cdot \rangle_1)$ and $(\mathcal{X}_2, \langle \cdot, \cdot | \cdot \rangle_2)$ be spaces with a generalized 2-inner product. Then the mapping*

$$\langle \xi, \eta \mid \lambda \rangle_{\overset{2}{\odot}} \; : \; (\mathcal{X}_1 \overset{2}{\odot} \mathcal{X}_2) \times (\mathcal{X}_1 \overset{2}{\odot} \mathcal{X}_2) \times (\mathcal{X}_1 \overset{2}{\odot} \mathcal{X}_2) \longrightarrow \mathbb{C}$$

defined by

$$\langle \xi, \eta \mid \lambda \rangle_{\overset{2}{\odot}} = \Big\langle \sum_{i=1}^{n} x_i \overset{2}{\odot} y_i, \sum_{j=1}^{m} z_j \overset{2}{\odot} w_j \;\Big|\; \sum_{t=1}^{p} r_t \overset{2}{\odot} s_t \Big\rangle := \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{t=1}^{p} \delta_{i,j} \, \langle x_i, z_j \mid r_t \rangle_1 \, \langle y_i, w_j \mid s_t \rangle_2,$$

where

$$\xi = \sum_{i=1}^{n} x_i \overset{2}{\odot} y_i, \quad \eta = \sum_{j=1}^{m} z_j \overset{2}{\odot} w_j, \quad \lambda = \sum_{t=1}^{p} r_t \overset{2}{\odot} s_t,$$

is a generalized 2-inner product.

Proof. Let

$$\xi = \sum_{i=1}^{n} x_i \overset{2}{\odot} y_i, \quad \eta = \sum_{j=1}^{m} z_j \overset{2}{\odot} w_j, \quad \lambda = \sum_{t=1}^{p} r_t \overset{2}{\odot} s_t, \quad \sigma = \sum_{k=1}^{g} l_k \overset{2}{\odot} v_k$$

be elements of $\mathcal{X}_1 \overset{2}{\odot} \mathcal{X}_2$ and let $\alpha \in \mathbb{C}$. To verify linearity in the first slot, first observe that

$$\lambda = \sum_{t=1}^{p} r_t \overset{2}{\odot} s_t = \sum_{i=n+1}^{n+p} x_i \overset{2}{\odot} y_i,$$

where we set $x_{n+t} \overset{2}{\odot} y_{n+t} := r_t \overset{2}{\odot} s_t$ for each $1 \leq t \leq p$. Hence

$$\xi + \lambda = \sum_{i=1}^{n+p} x_i \overset{2}{\odot} y_i.$$

Therefore:

$$\begin{aligned}
\langle \xi + \lambda, \eta | \sigma \rangle_{\overset{2}{\odot}} &= \left\langle \sum_{i=1}^{n+p} x_i \overset{2}{\odot} y_i, \sum_{j=1}^{m} z_j \overset{2}{\odot} w_j | \sum_{k=1}^{g} l_k \overset{2}{\odot} v_k \right\rangle_{\overset{2}{\odot}} = \sum_{i=1}^{n+p} \sum_{j=1}^{m} \sum_{k=1}^{g} \delta_{i,j} \langle x_i, z_j | l_k \rangle_1 \langle y_i, w_j | v_k \rangle_2 \\
&= \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{k=1}^{g} \delta_{i,j} \langle x_i, z_j | l_k \rangle_1 \langle y_i, w_j | v_k \rangle_2 + \sum_{i=n+1}^{n+p} \sum_{j=1}^{m} \sum_{k=1}^{g} \delta_{i,j} \langle x_i, z_j | l_k \rangle_1 \langle y_i, w_j | v_k \rangle_2 \\
&= \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{k=1}^{g} \delta_{i,j} \langle x_i, z_j | l_k \rangle_1 \langle y_i, w_j | v_k \rangle_2 + \sum_{t=1}^{p} \sum_{j=1}^{m} \sum_{k=1}^{g} \delta_{n+t,j} \langle x_{n+t}, z_j | l_k \rangle_1 \langle y_{n+t}, w_j | v_k \rangle_2 \\
&= \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{k=1}^{g} \delta_{i,j} \langle x_i, z_j | l_k \rangle_1 \langle y_i, w_j | v_k \rangle_2 + \sum_{t=1}^{p} \sum_{j=1}^{m} \sum_{k=1}^{g} \delta_{i,j} \langle r_i, z_j | l_k \rangle_1 \langle s_i, w_j | v_k \rangle_2 \\
&= \langle \xi, \eta | \sigma \rangle_{\overset{2}{\odot}} + \langle \lambda, \eta | \sigma \rangle_{\overset{2}{\odot}}
\end{aligned}$$

In addition,

$$\begin{aligned}
\langle \alpha\xi, \eta | \lambda \rangle_{\overset{2}{\odot}} &= \left\langle \alpha \sum_{i=1}^{n} x_i \overset{2}{\odot} y_i, \sum_{j=1}^{r} z_j \overset{2}{\odot} w_j | \sum_{t=1}^{p} r_t \overset{2}{\odot} s_t \right\rangle_{\overset{2}{\odot}} = \left\langle \sum_{i=1}^{n} \alpha (x_i \overset{2}{\odot} y_i), \sum_{j=1}^{r} z_j \overset{2}{\odot} w_j | \sum_{t=1}^{p} r_t \overset{2}{\odot} s_t \right\rangle_{\overset{2}{\odot}} \\
&= \left\langle \sum_{i=1}^{n} (\alpha x_i) \overset{2}{\odot} y_i, \sum_{j=1}^{r} z_j \overset{2}{\odot} w_j | \sum_{t=1}^{p} r_t \overset{2}{\odot} s_t \right\rangle_{\overset{2}{\odot}} = \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{t=1}^{p} \delta_{i,j} \langle (\alpha x_i), z_j | r_t \rangle_1 \langle y_i, w_j | s_t \rangle_2 \\
&= \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{t=1}^{p} \alpha \delta_{i,j} \langle x_i, z_j | r_t \rangle_1 \langle y_i, w_j | s_t \rangle_2 = \alpha \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{t=1}^{p} \delta_{i,j} \langle x_i, z_j | r_t \rangle_1 \langle y_i, w_j | s_t \rangle_2 \\
&= \alpha \langle \xi, \eta | \lambda \rangle_{\overset{2}{\odot}}.
\end{aligned}$$

Let us now show that this mapping is Hermitian.

$$\begin{aligned}
\langle \xi, \eta | \lambda \rangle_{\overset{2}{\odot}} &= \sum_{i=1}^{n} \sum_{j=1}^{r} \sum_{t=1}^{p} \delta_{i,j} \langle x_i, z_j | r_t \rangle_1 \langle y_i, w_j | s_t \rangle_2 = \sum_{i=1}^{n} \sum_{j=1}^{r} \sum_{t=1}^{p} \overline{\delta_{i,j} \langle z_j, x_i | r_t \rangle_1} \, \overline{\langle w_j, y_i | s_t \rangle_2} \\
&= \overline{\sum_{j=1}^{r} \sum_{i=1}^{n} \sum_{t=1}^{p} \delta_{i,j} \langle z_j, x_i | r_t \rangle_1 \langle w_j, y_i | s_t \rangle_2} = \overline{\langle \eta, \xi | \lambda \rangle_{\overset{2}{\odot}}}.
\end{aligned}$$

Therefore the mapping $\langle\cdot,\cdot|\cdot\rangle_{\overset{2}{\odot}}$ is Hermitian. Moreover, we have

$$\begin{aligned}\langle\xi,\xi|\lambda\rangle_{\overset{2}{\odot}} &= \left\langle \sum_{i=1}^{n} x_i \overset{2}{\odot} y_i, \sum_{l=1}^{n} x_l \overset{2}{\odot} y_l \,\Big|\, \sum_{t=1}^{p} r_t \overset{2}{\odot} s_t \right\rangle_{\overset{2}{\odot}} \\ &= \sum_{i=1}^{n}\sum_{l=1}^{n}\sum_{t=1}^{p} \delta_{i,l} \langle x_i, x_l|r_t\rangle_1 \langle y_i, y_l|s_t\rangle_2 = \sum_{i=1}^{n}\sum_{t=1}^{p}\langle x_i, x_i|r_t\rangle_1 \langle y_i, y_i|s_t\rangle_2 \\ &= \sum_{i=1}^{n}\sum_{t=1}^{p}\langle r_t, r_t|x_i\rangle_1 \langle s_t, s_t|y_i\rangle_2 = \sum_{j=1}^{p}\sum_{t=1}^{p}\sum_{i=1}^{n} \delta_{j,t} \langle r_j, r_t|x_i\rangle_1 \langle s_j, s_t|y_i\rangle_2 \\ &= \left\langle \sum_{j=1}^{p} r_j \overset{2}{\odot} s_j, \sum_{t=1}^{p} r_t \overset{2}{\odot} s_t \,\Big|\, \sum_{i=1}^{n} x_i \overset{2}{\odot} y_i \right\rangle_{\overset{2}{\odot}} = \langle\lambda,\lambda|\xi\rangle_{\overset{2}{\odot}}.\end{aligned}$$

Finally, it is clear that $\langle\xi,\xi|\lambda\rangle_{\overset{2}{\odot}} \geq 0$ for all $\xi,\lambda \in \mathcal{X}_1 \overset{2}{\odot} \mathcal{X}_2$. Hence $\langle\cdot,\cdot|\cdot\rangle_{\overset{2}{\odot}}$ defines a generalized 2-inner product on $\mathcal{X}_1 \overset{2}{\odot} \mathcal{X}_2$.

Example 2. *Consider $\mathbb{C}^2$ equipped with the application $\langle\cdot,\cdot|\cdot\rangle_{\mathbb{C}^2} : \mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{C}^2 \longrightarrow \mathbb{C}$ given by*

$$\langle(x_1,x_2),(y_1,y_2)|(z_1,z_2)\rangle_{\mathbb{C}^2} := x_1\overline{y}_1|z_1|^2 + x_2\overline{y}_2|z_2|^2, \;\; (x_1,x_2),(y_1,y_2),(z_1,z_2) \in \mathbb{C}^2.$$

In addition, let us consider $\mathbb{C}$ with the application $\langle\cdot,\cdot|\cdot\rangle_{\mathbb{C}} : \mathbb{C}\times\mathbb{C}\times\mathbb{C} \longrightarrow \mathbb{C}$ given by

$$\langle x,y|z\rangle_{\mathbb{C}} := x\overline{y}|z|^2, \;\; x,y,z \in \mathbb{C}.$$

Thus, given $(a,b) \in \mathbb{C}^2$ y $c \in \mathbb{C}$, the function $(a,b) \overset{2}{\odot} c : (\mathbb{C}^2 \times \mathbb{C}) \times (\mathbb{C}^2 \times \mathbb{C}) \to \mathbb{C}$

$$\begin{aligned}\left((a,b) \overset{2}{\odot} c\right)(((x_1,x_2),d),((y_1,y_2),e)) &:= \langle(a,b),(x_1,x_2)|(y_1,y_2)\rangle_{\mathbb{C}^2} \cdot \langle c,d|e\rangle_{\mathbb{C}} \\ &= \left(a\overline{x_1}|y_1|^2 + b\overline{x_2}|y_2|^2\right)\left(c\overline{d}|e|^2\right) = ac\overline{x_1}\overline{d}|y_1e|^2 + bc\overline{x_2}\overline{d}|y_2e|^2 \\ &= \langle(ac,bc),(x_1d,x_2d)|(y_1e,y_2e)\rangle_{\mathbb{C}^2},\end{aligned}$$

for all $((x_1,x_2),d),((y_1,y_2),e) \in \mathbb{C}^2 \times \mathbb{C}$. Now, the tensor product of the spaces with generalized 2-inner product $(\mathbb{C}^2, \langle\cdot,\cdot|\cdot\rangle_{\mathbb{C}^2})$ and $(\mathbb{C}, \langle\cdot,\cdot|\cdot\rangle_{\mathbb{C}})$, in accordance with the Definition 6, is given by $\mathbb{C}^2 \overset{2}{\odot} \mathbb{C} = \left\{\sum_{i=1}^{n}(x_i,y_i) \overset{2}{\odot} c_i : n \in \mathbb{N}, (x_i,y_i) \in \mathbb{C}^2, c_i \in \mathbb{C}, 1 \leq i \leq n\right\}$, and the generalized 2-inner product of Theorem 3 $\langle\cdot,\cdot|\cdot\rangle : (\mathbb{C}^2 \overset{2}{\odot} \mathbb{C}) \times (\mathbb{C}^2 \overset{2}{\odot} \mathbb{C}) \times (\mathbb{C}^2 \overset{2}{\odot} \mathbb{C}) \to \mathbb{C}$, has the form

$$\begin{aligned}
\langle \xi, \eta | \lambda \rangle_{\overset{2}{\odot}} &:= \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{p} \delta_{i,j} \langle (x_i, y_i), (z_j, w_j) | (r_k, s_k) \rangle_{\mathbb{C}^2} \langle c_i, d_j | e_k \rangle_{\mathbb{C}} \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{p} \delta_{i,j} \langle (x_i c_i, y_i c_i), (z_i d_i, w_i d_i) | (r_k e_k, s_k e_k) \rangle_{\mathbb{C}^2} \\
&= \sum_{i=1}^{n}\sum_{k=1}^{p} \langle (x_i c_i, y_i c_i), (z_i d_i, w_i d_i) | (r_k e_k, s_k e_k) \rangle_{\mathbb{C}^2} \\
&= \sum_{i=1}^{n}\sum_{k=1}^{p} x_i c_i \overline{z_i d_i} |r_k e_k|^2 + y_i c_i \overline{w_i d_i} |s_k e_k|^2
\end{aligned}$$

for all $\xi = \sum_{i=1}^{n} (x_i, y_i) \overset{2}{\odot} c_i, \eta = \sum_{j=1}^{m} (z_j, w_j) \overset{2}{\odot} d_j, \lambda = \sum_{k=1}^{p} (r_k, s_k) \overset{2}{\odot} e_k \in \mathbb{C}^2 \overset{2}{\odot} \mathbb{C}.$

Next, we equip the algebraic tensor product space from Definition 6 with a generalized 2-norm, which is induced by the mapping defined in Theorem 3.

Theorem 4. *Let* $(\mathcal{X}_1, \langle \cdot, \cdot | \cdot \rangle_1)$ *and* $(\mathcal{X}_2, \langle \cdot, \cdot | \cdot \rangle_2)$ *be spaces with a generalized 2-inner product. We define a generalized 2-norm on* $\mathcal{X}_1 \overset{2}{\odot} \mathcal{X}_2$*, called the* induced 2-tensor norm *by the generalized 2-inner product of Theorem 3, as the mapping*

$$\|\cdot, \cdot\|_{\overset{2}{\odot}} : (\mathcal{X}_1 \overset{2}{\odot} \mathcal{X}_2) \times (\mathcal{X}_1 \overset{2}{\odot} \mathcal{X}_2) \longrightarrow \mathbb{R}$$

given by

$$\|\xi, \lambda\|_{\overset{2}{\odot}} := \sqrt{\langle \xi, \xi | \lambda \rangle_{\overset{2}{\odot}}}, \qquad \xi, \lambda \in \mathcal{X}_1 \overset{2}{\odot} \mathcal{X}_2.$$

Proof. Since $\langle \cdot, \cdot | \cdot \rangle_{\overset{2}{\odot}}$ is a generalized 2-inner product, it follows that $\|\cdot, \cdot\|_{\overset{2}{\odot}}$ defines a generalized 2-norm. The proof is straightforward and analogous to the classical case.

Following the work of Lewandoska [22], we introduce the notion of a 2-bounded linear operator on the 2-tensor product of spaces with a generalized 2-inner product.

Definition 7 (**2-bounded operator**). *Let* $\left(\mathcal{X}_1 \overset{2}{\odot} \mathcal{X}_2, \|\cdot, \cdot\|_{\overset{2}{\odot}}\right)$ *be the generalized 2-normed space from Theorem 1, and let* T *be a linear operator* T *a linear operator on* $\mathcal{X}_1 \overset{2}{\odot} \mathcal{X}_2$*. Then we said that* T *is a 2-bounded linear operator if there exists a positive number* $\alpha > 0$ *such that*

$$\|Tw, z\|_{\overset{2}{\odot}} \leq \alpha \, \|w, z\|_{\overset{2}{\odot}} \text{ for all } w, z \in \mathcal{X}_1 \overset{2}{\odot} \mathcal{X}_2.$$

The symbol $\mathcal{B}(\mathcal{X}_1 \overset{2}{\odot} \mathcal{X}_2)$ *will denote the set of 2-bounded linear operator on* $\mathcal{X}_1 \overset{2}{\odot} \mathcal{X}_2$*, that is,*

$$\mathcal{B}(\mathcal{X}_1 \overset{2}{\odot} \mathcal{X}_2) := \{T : \mathcal{X}_1 \overset{2}{\odot} \mathcal{X}_2 \to \mathcal{X}_1 \overset{2}{\odot} \mathcal{X}_2 : T \text{ is linear and 2-bounded}\}.$$

Remark 3. *It is clear that the set* $\mathcal{B}(\mathcal{X}_1 \overset{2}{\odot} \mathcal{X}_2)$ *can be endowed with a vector space structure over* $\mathbb{C}$*, by means of the pointwise operations of operators.*

Definition 8. *[14] If* $T \in \mathcal{B}(\mathcal{X}_1 \overset{2}{\odot} \mathcal{X}_2)$*, we define* $\|T\|_{\overset{2}{\odot}}$*, by*

$$\|T\|_{\overset{2}{\odot}} = \inf\{a \geq 0 : \|T(w), z\|_{\overset{2}{\odot}} \leq a\|w, z\|_{\overset{2}{\odot}} \text{ for all } w, z \in \mathcal{X}\}.$$

Theorem 5. *[14] In the context of the Definition 8, for all* $T \in \overset{2}{\mathcal{B}}(\mathcal{X})$ *it is true that*

$$\|T\|_{\overset{2}{\odot}} = \sup\{\|Tw, z\|_{\overset{2}{\odot}} : w, z \in \mathcal{X} \text{ and } \|w, z\|_{\overset{2}{\odot}} = 1\}$$
$$\|T\|_{\overset{2}{\odot}} = \sup\{\|Tw, z\|_{\overset{2}{\odot}} : w, z \in \mathcal{X} \text{ and } \|w, z\|_{\overset{2}{\odot}} \leq 1\}$$
$$\|T\|_{\overset{2}{\odot}} = \sup\left\{\frac{\|Tw, z\|_{\overset{2}{\odot}}}{\|w, z\|_{\overset{2}{\odot}}} : w, z \in \mathcal{X} \text{ and } \|w, z\|_{\overset{2}{\odot}} \neq 0\right\}$$

Moreover, thanks to the Definition 8 we prove that given a generalized 2-normed space $\mathcal{X}_1 \overset{2}{\odot} \mathcal{X}_2$, it is possible to endow the vector space $\mathcal{B}(\mathcal{X}_1 \overset{2}{\odot} \mathcal{X}_2)$ with the structure of a semi-normed space.

Proposition 6. *The mapping* $\|\cdot\|_{\overset{2}{\odot}} : \mathcal{B}(\mathcal{X}_1 \overset{2}{\odot} \mathcal{X}_2) \to \mathbb{R}$ *given by*

$$\|T\|_{\overset{2}{\odot}} = \sup\left\{\frac{\|T(w), z\|_{\overset{2}{\odot}}}{\|w, z\|_{\overset{2}{\odot}}} : w, z \in \mathcal{X}_1 \overset{2}{\odot} \mathcal{X}_2 \text{ and } \|w, z\|_{\overset{2}{\odot}} \neq 0\right\}$$

defines a semi-norm in $\mathcal{B}(\mathcal{X}_1 \overset{2}{\odot} \mathcal{X}_2)$*.*

The proofs are obtained from [14] using Theorem 1.

Proposition 7. *[14] For all* $T \in \mathcal{B}(\mathcal{X}_1 \overset{2}{\odot} \mathcal{X}_2)$ *it is true that* $\|T(w), y\|_{\overset{2}{\odot}} \leq \|T\|\|w, y\|_{\overset{2}{\odot}}$ *for all* $w, y, z \in \mathcal{X}_1 \overset{2}{\odot} \mathcal{X}_2$*.*

4. Tensor Product of Linear Operators

In the following definition, we establish the notion of the 2-tensor product of linear operators on spaces with a generalized 2-inner product.

Definition 9. *Let* $(\mathcal{X}_1, \langle\cdot,\cdot|\cdot\rangle_1)$ *and* $(\mathcal{X}_2, \langle\cdot,\cdot|\cdot\rangle_2)$ *be spaces with a generalized 2-inner product, and let* $T\colon \mathcal{X}_1 \to \mathcal{X}_1$*,* $S\colon \mathcal{X}_2 \to \mathcal{X}_2$ *be linear operators on* $\mathcal{X}_1$ *and* $\mathcal{X}_2$*, respectively. Then the 2-tensor product of* T *and* S *on* $\mathcal{X}_1 \overset{2}{\odot} \mathcal{X}_2$*, denoted by* $T \overset{2}{\odot} S$*, is the linear operator*

$$T \overset{2}{\odot} S\colon \mathcal{X}_1 \overset{2}{\odot} \mathcal{X}_2 \longrightarrow \mathcal{X}_1 \overset{2}{\odot} \mathcal{X}_2$$

defined by

$$\left(T \overset{2}{\odot} S\right)\Big(\sum_{i=1}^{n} x_i \overset{2}{\odot} y_i\Big) := \sum_{i=1}^{n}\left(Tx_i\right) \overset{2}{\odot} \left(Sy_i\right),$$

for each $\sum_{i=1}^{n} x_i \overset{2}{\odot} y_i \in \mathcal{X}_1 \overset{2}{\odot} \mathcal{X}_2$.

Remark 4. *Let* $(\mathcal{X}_1, \langle \cdot, \cdot | \cdot \rangle_1)$ *and* $(\mathcal{X}_2, \langle \cdot, \cdot | \cdot \rangle_2)$ *be spaces with a generalized 2-inner product. We denote the tensor product* $(I_1 \overset{2}{\odot} I_2)$ *on* $\mathcal{X}_1 \overset{2}{\odot} \mathcal{X}_2$ *by* $I_\odot$*; this is the identity mapping on the vector space* $\mathcal{X}_1 \overset{2}{\odot} \mathcal{X}_2$*. Indeed, for every* $\xi = \sum_{i=1}^{n} x_i \overset{2}{\odot} y_i \in \mathcal{X}_1 \overset{2}{\odot} \mathcal{X}_2$,

$$I_\odot(\xi) = (I_1 \overset{2}{\odot} I_2)\Big(\sum_{i=1}^{n} x_i \overset{2}{\odot} y_i\Big) = \sum_{i=1}^{n} I_1 x_i \overset{2}{\odot} I_2 y_i = \sum_{i=1}^{n} x_i \overset{2}{\odot} y_i = \xi.$$

Note also that $\|I_\odot\|_{\overset{2}{\odot}} = 1$.

Proposition 8. *Let* $(\mathcal{X}_1, \langle \cdot, \cdot \mid \cdot \rangle_1)$ *and* $(\mathcal{X}_2, \langle \cdot, \cdot \mid \cdot \rangle_2)$ *be spaces with a generalized 2-inner product, let* $T_1, S_1 \colon \mathcal{X}_1 \to \mathcal{X}_1$, $T_2, S_2 \colon \mathcal{X}_2 \to \mathcal{X}_2$ *be linear operators, and let* $\alpha, \beta \in \mathbb{C}$*. Then the following identities hold:*

i) $T_1 S_1 \overset{2}{\odot} T_2 S_2 = \left(T_1 \overset{2}{\odot} T_2\right) \circ \left(S_1 \overset{2}{\odot} S_2\right)$;

ii) $\alpha\beta\,(T_1 \overset{2}{\odot} T_2) = (\alpha T_1) \overset{2}{\odot} (\beta T_2)$;

iii) $T_1 \overset{2}{\odot} (T_2 + S_2) = (T_1 \overset{2}{\odot} T_2) + (T_1 \overset{2}{\odot} S_2)$;

iv) $(T_1 + S_1) \overset{2}{\odot} T_2 = (T_1 \overset{2}{\odot} T_2) + (S_1 \overset{2}{\odot} T_2)$;

v) $(T_1 + S_1) \overset{2}{\odot} (T_2 + S_2) = T_1 \overset{2}{\odot} T_2 + T_1 \overset{2}{\odot} S_2 + S_1 \overset{2}{\odot} T_2 + S_1 \overset{2}{\odot} S_2$;

vi) $T_1 \overset{2}{\odot} T_2$ *is invertible if and only if both* T_1 *and* T_2 *are invertible, and in that case*

$$(T_1 \overset{2}{\odot} T_2)^{-1} = T_1^{-1} \overset{2}{\odot} T_2^{-1}.$$

Proof. The proofs follow directly by applying each operator definition to a simple tensor sum $\xi = \sum_{i=1}^{n} x_i \overset{2}{\odot} y_i$ and grouping like terms, using associativity and distributivity of sums and scalars, as well as the definition of the tensor-product operator. The invertibility statement in (vi) uses the fact that a tensor-product of invertible operators is itself invertible, with inverse given by the tensor product of the individual inverses.

Example 3. *If we consider* $(\mathbb{C}^2, \langle \cdot, \cdot | \cdot \rangle_{\mathbb{C}^2})$ *and* $(\mathbb{C}, \langle \cdot, \cdot | \cdot \rangle_{\mathbb{C}})$ *as in Example 2, then the following mappings are linear operators*

$$T : \mathbb{C}^2 \to \mathbb{C}^2,\ T(a, b) = (0, a),\ \ (a, b) \in \mathbb{C}^2.$$

$$S : \mathbb{C} \to \mathbb{C}, \ S(c) = ic, \ \ c \in \mathbb{C},$$

and consequently, for each $\sum_{k=1}^{n}(a_k, b_k) \overset{2}{\odot} c_k \in \mathbb{C}^2 \overset{2}{\odot} \mathbb{C}$ *the map* $T \overset{2}{\odot} S : \mathbb{C}^2 \overset{2}{\odot} \mathbb{C} \to \mathbb{C}^2 \overset{2}{\odot} \mathbb{C}$ *is given by*

$$(T \overset{2}{\odot} S)\Big(\sum_{k=1}^{n}(a_k, b_k) \overset{2}{\odot} c_k\Big) = \sum_{k=1}^{n} T(a_k, b_k) \overset{2}{\odot} Sc_k = \sum_{i=1}^{n}(0, a_k) \overset{2}{\odot} ic_k = i\sum_{i=1}^{n}(0, a_k) \overset{2}{\odot} c_k.$$

Now, for all $((x_1, x_2), d), ((y_1, y_2), e) \in \mathbb{C}^2 \times \mathbb{C}$ *we have*

$$\begin{aligned}\left(i\sum_{i=1}^{n}(0, a_k) \overset{2}{\odot} c_k\right)(((x_1, x_2), d), ((y_1, y_2), e)) &= i\sum_{k=1}^{n}\langle (0, a_k), (x_1, x_2)|(y_1, y_2)\rangle_{\mathbb{C}^2}\langle c_k, d|e\rangle_{\mathbb{C}} \\ &= i\sum_{k=1}^{n} a_k c_k \overline{x_2 d}|y_2 e|^2 = i\overline{x_2 d}|y_2 e|^2 \sum_{k=1}^{n} a_k c_k \\ &= i\overline{x_2 d}|y_2 e|^2 \langle a, \bar{c}\rangle_{\mathbb{C}^n}\end{aligned}$$

where $a = (a_1, \cdots, a_n)$, $\bar{c} = (\bar{c}_1, \cdots, \bar{c}_n) \in \mathbb{C}^n$ *and* $\langle \cdot, \cdot \rangle_{\mathbb{C}^n}$ *denotes the classical inner product on* $\mathbb{C}^n$.

Remark 5. *Given two spaces with classical inner product* $(\mathcal{X}_1, \langle \cdot, \cdot \rangle_{X_1})$ *and* $(\mathcal{X}_2, \langle \cdot, \cdot \rangle_{X_2})$, *and* $x \in \mathcal{X}_1, y \in \mathcal{X}_2$, *we know that* $x \odot y$ *is defined as the mapping* $x \odot y : \mathcal{X}_1 \times \mathcal{X}_2 \to \mathbb{C}$, *given by* $(x \odot y)(r, s) = \langle x, r\rangle_X \langle y, s\rangle_Y$ *for all* $(r, s) \in \mathcal{X}_1 \times \mathcal{X}_2$ *(see [11]). Now, we can consider the application* $x\tilde{\odot}y : (\mathcal{X}_1 \times \mathcal{X}_2) \times (\mathcal{X}_1 \times \mathcal{X}_2) \to \mathbb{C}$, *defined by* $(x\tilde{\odot}y)((x_1, y_1), (x_2, y_2)) := (x \odot y)(x_1, y_1)$. *Thus we obtain the following complex vector spaces*

$$\mathcal{X}_1 \odot \mathcal{X}_2 = \left\{\sum_{i=1}^{n} \alpha_i (x_i \odot y_i) : \alpha_i \in \mathbb{C}, x_i \in \mathcal{X}_1, y_i \in \mathcal{X}_2, n \in \mathbb{N}\right\}$$
$$\mathcal{X}_1 \tilde{\odot} \mathcal{X}_2 = \left\{\sum_{j=1}^{m} \beta_j (x_j \tilde{\odot} y_j) : \beta_j \in \mathbb{C}, x_j \in \mathcal{X}_1, y_j \in \mathcal{X}_2, m \in \mathbb{N}\right\}$$

whose relationship is expressed in the following proposition:

Proposition 9. *Let* $(\mathcal{X}_1, \langle \cdot, \cdot \rangle_1)$ *and* $(\mathcal{X}_2, \langle \cdot, \cdot \rangle_2)$ *spaces with a classical inner product. Then* $\mathcal{X}_1 \odot \mathcal{X}_2 \cong \mathcal{X}_1 \tilde{\odot} \mathcal{X}_2$.

Proof. Consider the linear transformation $\varphi : \mathcal{X}_1 \odot \mathcal{X}_2 \to \mathcal{X}_1 \tilde{\odot} \mathcal{X}_2$ given by

$$\varphi(\sum_{i=1}^{n} x_i \odot y_i) = \sum_{i=1}^{n} x_i \tilde{\odot} y_i$$

note that clearly φ is bijective.

Theorem 6. *Let* $(\mathcal{X}_1, \langle\cdot,\cdot\rangle_1)$ *and* $(\mathcal{X}_2, \langle\cdot,\cdot\rangle_2)$ *spaces with a classical inner product. Then,* $\mathcal{X}_1 \overset{2}{\odot} \mathcal{X}_2$ *is a vector subspace of* $\mathcal{X}_1 \tilde{\odot} \mathcal{X}_2$*, where the vector space* $\mathcal{X}_1 \overset{2}{\odot} \mathcal{X}_2$ *is defined as in the Definition 5.*

Proof. It is enough to see that $\mathcal{X}_1 \overset{2}{\odot} \mathcal{X}_2 \subseteq \mathcal{X}_1 \tilde{\odot} \mathcal{X}_2$. In fact, whether $\xi \in \mathcal{X}_1 \overset{2}{\odot} \mathcal{X}_2$ then there is a natural number n, and there are $\{\alpha_i\}_{i=1}^n \subseteq \mathbb{C}, \{x_i\}_{i=1}^n \subseteq \mathcal{X}_1, \{y_i\}_{i=1}^n \subseteq \mathcal{X}_2$ such that $\xi = \sum_{i=1}^n \alpha_i (x_i \overset{2}{\odot} y_i)$. However, for all $(x, y) \in \mathcal{X}_1 \times \mathcal{X}_2$ and all $(r, s) \in \mathcal{X}_1 \times \mathcal{X}_2$ with $\|r\|_X = 1, \|s\|_Y = 1$ it is necessary to,

$$\begin{aligned}
\xi((x,y),(r,s)) &= \Big(\sum_{i=1}^n \alpha_i (x_i \overset{2}{\odot} y_i)\Big)((x,y),(r,s)) = \sum_{i=1}^n (\alpha_i (x_i \overset{2}{\odot} y_i))((x,y),(r,s)) \\
&= \sum_{i=1}^n \alpha_i \langle x_i, x|r\rangle_X \langle y_i, y|s\rangle_Y = \sum_{i=1}^n \alpha_i \langle x_i, x\rangle_X \|r\|_X^2 \langle y_i, y\rangle_Y \|s\|_Y^2 \\
&= \sum_{i=1}^n \alpha_i \langle x_i, x\rangle_X \langle y_i, y\rangle_Y = \sum_{i=1}^n \alpha_i (x_i \tilde{\odot} y_i)((x,y),(r,s)) \\
&= \Big(\sum_{i=1}^n \alpha_i (x_i \tilde{\odot} y_i)\Big)((x,y),(r,s))
\end{aligned}$$

Then, $\xi = \sum_{i=1}^n \alpha_i (x_i \tilde{\odot} y_i) \in \mathcal{X}_1 \tilde{\odot} \mathcal{X}_2$. Therefore $\xi \in \mathcal{X}_1 \tilde{\odot} \mathcal{X}_2$.

Proposition 10. *Let* $(\mathcal{X}_1, \langle\cdot,\cdot \mid \cdot\rangle_1)$ *and* $(\mathcal{X}_2, \langle\cdot,\cdot \mid \cdot\rangle_2)$ *be spaces with a generalized 2-inner product, if* $T_1 \in \overset{2}{\mathcal{B}}(\mathcal{X}_1)$, $T_2 \in \overset{2}{\mathcal{B}}(\mathcal{X}_2)$ *then* $T_1 \overset{2}{\odot} T_2 \in \overset{2}{\mathcal{B}}(\mathcal{X}_1 \overset{2}{\odot} \mathcal{X}_2)$.

Proof. Let $\xi_1 = \sum_{i=1}^n x_i \overset{2}{\odot} y_i, \xi_2 = \sum_{k=1}^m r_k \overset{2}{\odot} s_k \in \mathcal{X}_1 \overset{2}{\odot} \mathcal{X}_2$ with $\|\xi_1, \xi_2\|_{\overset{2}{\odot}} = 1$. Thus, we have

$$\begin{aligned}
\left\|(T_1 \overset{2}{\odot} T_2)(\xi_1), \xi_2\right\|_{\overset{2}{\odot}}^2 &= \left\langle (T_1 \overset{2}{\odot} T_2)(\xi_1), (T_1 \overset{2}{\odot} T_2)(\xi_1) | \xi_2 \right\rangle_{\overset{2}{\odot}} \\
&= \left\langle \sum_{i=1}^n T_1(x_i) \overset{2}{\odot} T_2(y_i), \sum_{j=1}^n T_1(x_j) \overset{2}{\odot} T_2(y_j) | \sum_{k=1}^m r_k \overset{2}{\odot} s_k \right\rangle_{\overset{2}{\odot}} \\
&= \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^m \delta_{i,j} \left\langle T_1(x_i), T_1(x_j) | r_k \right\rangle_1 \left\langle T_2(y_i), T_2(y_j) | s_k \right\rangle_2 \\
&= \sum_{i=1}^n \sum_{k=1}^m \left\langle T_1(x_i), T_1(x_j) | r_k \right\rangle_1 \left\langle T_2(y_i), T_2(y_j) | s_k \right\rangle_2 \\
&= \sum_{i=1}^n \sum_{k=1}^m \|T_1(x_i), r_k\|_1^2 \, \|T_2(y_i), s_k\|_2^2
\end{aligned}$$

$$\begin{aligned}
&\leq \sum_{i=1}^{n}\sum_{k=1}^{m} \|T_1\|^2 \|x_i, r_k\|_1^2 \|T_2\|^2 \|y_i, s_k\|_2^2 \\
&= \|T_1\|^2 \|T_2\|^2 \sum_{i=1}^{n}\sum_{k=1}^{m} \|x_i, r_k\|_1^2 \|y_i, s_k\|_2^2 \\
&= \|T_1\|^2 \|T_2\|^2 \|\xi_1, \xi_2\|_{\overset{2}{\odot}}^2 = \|T_1\|^2 \|T_2\|^2
\end{aligned}$$

Therefore $\left\|(T_1 \overset{2}{\odot} T_2)(\xi_1), \xi_2\right\|_{\overset{2}{\odot}} \leq \|T_1\| \|T_2\|$.

Proposition 11. *Let $(\mathcal{X}_1, \langle\cdot,\cdot|\cdot\rangle_{\mathcal{X}_1}), (\mathcal{X}_2, \langle\cdot,\cdot|\cdot\rangle_{\mathcal{X}_2}), (\mathcal{Y}_1, \langle\cdot,\cdot|\cdot\rangle_{\mathcal{Y}_1}), (\mathcal{Y}_2, \langle\cdot,\cdot|\cdot\rangle_{\mathcal{Y}_2})$ spaces with generalized 2-inner product and $T_1 : \mathcal{X}_1 \to \mathcal{X}_2$ and $T_2 : \mathcal{Y}_1 \to \mathcal{Y}_2$ linear operators, then*

$$\begin{aligned}
T_1 \times T_2 : \mathcal{X}_1 \times \mathcal{Y}_1 &\to \mathcal{X}_2 \times \mathcal{Y}_2 \\
(x_1, y_1) &\to (T_1 \times T_2)(x_1, y_1) = T_1(x_1) \odot T_2(y_2)
\end{aligned}$$

is a bilinear operator.

Proof. In fact, note that for all $x_1, x_1' \in \mathcal{X}_1$ and all $y_1, y_1' \in \mathcal{Y}_1$ is met

$$\begin{aligned}
(T_1 \times T_2)(\alpha x_1 + x_1', y_1) &= T_1(\alpha x_1 + x_1') \odot T_2(y_1) = (\alpha T_1(x_1) + T_1(x_1')) \odot T_2(y_1) \\
&= (\alpha T_1(x_1)) \odot T_2(y_1) + T_1(x_1') \odot T_2(y_1) \\
&= \alpha(T_1(x_1) \odot T_2(y_1)) + T_1(x_1') \odot T_2(y_1) \\
&= \alpha(T_1 \times T_2)(x_1, y_1) + (T_1 \times T_2)(x_1', y_1)
\end{aligned}$$

$$\begin{aligned}
(T_1 \times T_2)(x_1, y_1 + y_1') &= T_1(x_1) \odot T_2(y_1 + y_1') = T_1(x_1) \odot ((T_2(\alpha y_1) + T_2(y_1')) \\
&= T_1(x_1) \odot (\alpha T_2(y_1)) + T_1(x_1) \odot T_2(y_1') \\
&= \alpha(T_1(x_1) \odot T_2(y_1)) + T_1(x_1) \odot T_2(y_1') \\
&= \alpha(T_1 \times T_2)(x_1, y_1) + (T_1 \times T_2)(x_1, y_1')
\end{aligned}$$

5. Conclusions

In this research, we introduced the notion of tensor product of elements of spaces equipped with a generalized 2-inner product, calling it the 2-tensor product (see Definition 5), which turns out to be a bilinear mapping in the sense of Proposition 5, as in the classic case. We also define the algebraic tensor product of spaces endowed with a generalized 2-inner product and prove that it satisfies all fundamental properties (see Proposition 5). Furthermore, we define a generalized 2-inner product on the algebraic tensor product (see Definition 6) and induced a 2-norm, which we call the induced 2-tensor norm (see

Theorem 1). We also established the notion of the 2-tensor product of linear operators on spaces with a generalized 2-inner product (see Definition 9), and demonstrated that several well-known identities for operators in inner product spaces remain valid in this new structure (see Proposition 8). Finally, the notions established in this work have been illustrated with examples (see Example 2 and Example 3). In this context, other theories can be developed, such as frame theory, soft set theory, the theory of functions of bounded variation, the study of the numerical range of operators, and others.

Conflict of interest

The authors declare that they have no conflict of interest in this work.

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