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On Π-Property of p-Subgroups and the p-Supersolvability of Finite Groups

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Abstract. A subgroup H of a group G is said to satisfy the Π-property in G such that for any G-chief factor U/V, $|G/V : N_{G/V}(HV/V \cap U/V)|$ is a $\pi(HV/V \cap U/V)$-number. In this paper, we present a new criterion for p-supersolvability of finite groups by using a small quantity of maximal subgroups of a Sylow p-subgroup satisfying the Π-property of G. As applications, we obtain some necessary and sufficient conditions for a finite group to be p-nilpotent and supersolvable. A number of known results are improved and extended.

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1. Introduction

All groups in the present paper are supposed to be finite. Let G be a group, by $\pi(G)$, we denote the set of prime divisors of $|G|$. An integer n is called a π-number if all prime divisors of n belong to π. In particular, an integer n is called a p-number if it is a power of p. Other notations involved are standard (refer [1]).

Let H be a subgroup of a group G. Recall that H is said to satisfy the Π-property in G if $|G/V : N_{G/V}(HV/V \cap U/V)|$ is a $\pi(HV/V \cap U/V)$-number for any G-chief factor U/V [2]. The concept of Π-property unifies many known embedding properties. By using the Π-property of subgroups, many scholars have investigated the structure of finite groups. For example, Li in [2] presented some sufficient conditions of p-nilpotency and supersolvability of finite groups; Su, Li and Wang in [3] obtained some sufficient conditions of p-supersolvability of finite groups.

We remark that the involved subgroups in the above results are precisely all maximal subgroups of a Sylow subgroup. A natural question is whether the structure of a finite

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group can still be characterized based on only a small number of such maximal subgroups. For example, Qiu, Liu and Chen in [4] have explored the p-supersolvability of finite groups by utilizing a limited number of maximal subgroups with the Π-property, under the condition that their intersection equals the Frattini subgroup $\Phi(P)$ of a Sylow p-subgroup P. If this restriction can not be satisfied, then the above results need not be true (see the following Remark 1). Building upon this framework, we extends the investigation into how certain maximal subgroups of a Sylow p-subgroup with the Π-property influence the structure of finite groups. As well known, a p-group has $\frac{p^d-1}{p-1}$ maximal subgroups, where d is the minimum number of generators of the p-group. Noticing that

$$\lim_{p\to\infty}\frac{p^d-1}{p-1}\div\frac{p^{d-1}-1}{p-1}=\lim_{p\to\infty,d\to\infty}\frac{p^d-1}{p-1}\div\frac{p^{d-1}-1}{p-1}=\infty,$$

hence $\frac{p^d-1}{p-1}\gg\frac{p^{d-1}-1}{p-1}$ while $p\to\infty$ and $\frac{p^{d-1}-1}{p-1}$ is also a good number to be used to minimize the number of maximal subgroups of a p-group.

In this paper, we establish a new criterion for the p-supersolvability of finite groups, by using m_p (greater than $\frac{p^{d-1}-1}{p-1}$) maximal subgroups of a Sylow p-subgroup P that satisfy the Π-property. Their intersection need not equal $\Phi(P)$. Using this criterion, we further explore sufficient conditions for p-nilpotency and supersolvability in finite groups.

2. Preliminaries

In this section, we give some useful lemmas which will be used in proofs of our main results.

Lemma 1. ([2, Lemma 2.1]) *Let G be a group and H a subgroup of G and N a normal subgroup of G. If H satisfies the Π-property in G, then HN/N satisfies the Π-property in G/N.*

Lemma 2. ([5, Lemma 2.4]) *Suppose that N is a non-abelian simple group. If the Sylow p-subgroups N_p of N are of order p, then the outer automorphism group $Out(N)$ of N is a p'-group.*

Lemma 3. ([6, I, Satz 17.4]) *Let N be a normal abelian subgroup of a group G and let $N\le M\le G$ such that $(|N|,|G:M|)=1$. If a complement subgroup of N in M exists, then N possesses a complement subgroup in G.*

Lemma 4. ([7]) *If A is a subnormal subgroup of a group G and A is a π-group, then $A\le O_\pi(G)$.*

Lemma 5. ([8, Lemma 2.6]) *Let N be a solvable normal subgroup of G ($N\ne 1$). If every minimal normal subgroup of G that is contained in N is not contained in $\Phi(G)$, then the Fitting subgroup $F(N)$ of N is the direct product of the minimal normal subgroups of G that are contained in N.*

Lemma 6. ([4, Lemma 2.5]) *Let H be a p-subgroup of G, where $p \in \pi(G)$. If H satisfies the Π-property in G, $H \leq L \trianglelefteq G$, then H satisfies the Π-property in L.*

Lemma 7. ([4, Lemma 2.6]) *Let H be a p-subgroup of G, where $p \in \pi(G)$. If G is p-supersolvable, then H satisfies the Π-property in G.*

Lemma 8. *Let G be a group with a cyclic Sylow p-subgroup P and K be a non-trivial p-subgroup of G, where $p \in \pi(G)$. Suppose K satisfies the Π-property in G with p dividing $|G : K|$, then G is p-supersolvable.*

Proof. We use induction on $|G|$. We may assume $O_{p'}(G) = 1$ by induction. Let N be a minimal normal subgroup of G. It follows that $1 < P \cap N \leq P$. If $P \cap N < P$, then p divides $|N|$ and $|G : N|$. We deduce that G is p-solvable by [9, Theorem2.1]. Furthermore, $P \leq C_G(O_p(G)) \leq O_p(G)$ and so $P = C_G(O_p(G))$. Now G/P is a p'-group and every G-chief factor under P is of order p, thereby G is p-supersolvable. If $P \leq N$, since K is a p-subgroup of G, there exists $x \in G$ such that $K \leq P^x \leq N$. By hypothesis, K satisfies the Π-property in G. Then $|G : N_G(K)|$ is a p-number. Since $K \trianglelefteq P^x$, it follows that $K \trianglelefteq G$. This yields that $K = N = P$, a contradiction.

Lemma 9. ([10, Lemma 2.2]) *Let G be a group and let p be a prime number dividing $|G|$ with $(|G|, p - 1) = 1$. Then*

(1) *If N is normal in G of order p, then N lies in $Z(G)$,*

(2) *If G has cyclic Sylow p-subgroups, then G is p-nilpotent,*

(3) *If M is a subgroup of G with index p, then M is normal in G.*

Recall that *the generalized Fitting subgroup* $F^*(G)$ of a group G is the unique maximal normal quasinilpotent subgroup of G (see [8]). For a prime $p \in \pi(G)$, *the generalized p-Fitting subgroup* $F_p^*(G)$ is defined as: $F_p^*(G)/O_{p'}(G) = F^*(G/O_{p'}(G))$ (see [11, Proposition 2.9]). In the subsequent discussion, we establish fundamental properties of $F^*(G)$ and $F_p^*(G)$.

Lemma 10. ([8, Lemma 2.3]) *Let G be a group and N a subgroup of G.*

(1) *If N is normal in G, then $F^*(N) \leq F^*(G)$.*

(2) *$F^*(G) \neq 1$ if $G \neq 1$; in fact, $F^*(G)/F(G) = Soc(F(G)C_G(F(G))/F(G))$.*

(3) *$F^*(F^*(G)) = F^*(G) \geq F(G)$; if $F^*(G)$ is soluble, then $F^*(G) = F(G)$.*

(4) *$C_G(F^*(G)) \leq F(G)$.*

Lemma 11. ([11, Lemma 2.10]) *Let G be a group. Then*

(1) *$Soc(G) \leq F_p^*(G)$.*

(2) *$O_{p'}(G) \leq F_p^*(G)$. In fact, $F^*(G/O_{p'}(G)) = F_p^*(G/O_{p'}(G)) = F_p^*(G)/O_{p'}(G)$.*

(3) *If $F_p^*(G)$ is p-solvable, then $F_p^*(G) = F_p(G)$.*

(4) *If $C = C_G(F_p(G)/O_{p'}(G))$, then $F_p^*(G)/F_p(G) = Soc(CF_p(G)/F_p(G))$.*

3. Main results

In the present section, we will prove the following criterion for p-supersolvability of groups. Furthermore, we will establish the p-nilpotency and supersolvability of finite groups.

Theorem 1. *Let P be a Sylow p-subgroup of a group G, where $p \in \pi(G)$. Then either G is p-supersolvable or P is of order p if and only if P has m_p maximal subgroups which satisfy the Π-property in G, where $m_p > \frac{p^{d-1}-1}{p-1}$ and $|P/\Phi(P)| = p^d$.*

Proof. By Lemma 7, we only need to prove the sufficiency. Assume that G is a counterexample of minimal order. Then G is not p-supersolvable and $|P| \geq p^2$. Let $P_1, P_2, \ldots, P_{m_p}$ be the maximal subgroups of P which satisfy the Π-property in G. We divide the proof into the following five steps.

(1) $O_{p'}(G) = 1$.

Write $N = O_{p'}(G)$. If $N \neq 1$, we consider G/N. Clearly, PN/N is a Sylow p-subgroup of G/N. Moreover, PN/N has the same smallest generator number as P. Since P_i satisfies the Π-property in G, P_iN/N satisfies the Π-property in G/N by Lemma 1. Thus G/N satisfies the hypotheses of the theorem. The minimal choice of G implies that either G/N is p-supersolvable or $|PN/N| = p$. It follows that either G is p-supersolvable or $|P| = p$, a contradiction. Therefore, $O_{p'}(G) = 1$.

(2) $\Phi(P)_G = 1$.

Assume that $\Phi(P)_G > 1$. Let N be a minimal normal subgroup of G contained in $\Phi(P)_G$. We consider the quotient group $\overline{G} = G/N$. Evidently, $\overline{P} = P/N \in \mathrm{Syl}_p(\overline{G})$ and $\overline{P}/\Phi(\overline{P}) \cong P/\Phi(P)$, so $|\overline{P}/\Phi(\overline{P})| = p^d$. By Lemma 1, $\overline{G}$ satisfies the hypotheses of the theorem. The minimality of G implies that $\overline{G}$ is p-supersolvable or $|\overline{P}| = p$. If $|\overline{P}| = p$, then $N = \Phi(P)$ and P is cyclic, so G is p-supersolvable by Lemma 8, a contradiction. If $\overline{G}$ is p-supersolvable, then G is p-supersolvable as $N \leq \Phi(P)$ implying $N \leq \Phi(G)$, also a contradiction.

(3) Every minimal normal subgroup of G contained in $O_p(G)$ is of order p.

Let N be a minimal normal subgroups of G contained in $O_p(G)$. In view of (2), $\Phi(P) \cap N < N$. Assume $N \leq P_i$ for each i. Since $\Phi(P)N/N \leq \Phi(PN/N)$, we have

$$|(P/N)/\Phi(P/N)| \leq \frac{|P/N|}{|\Phi(P)N/N|} = \frac{|P/\Phi(P)|}{|N/\Phi(P) \cap N|} \leq \frac{|P/\Phi(P)|}{p} = p^{d-1}.$$

It follows that the number of maximal subgroups of P containing N is not greater that $\frac{p^{d-1}-1}{p-1}$, which is contrary to $m_p > \frac{p^{d-1}-1}{p-1}$. Hence, there exists P_j such that $P = P_jN$. By hypothesis, P_j satisfies the Π-property in G. Hence $|G : N_G(P_j \cap N)|$ is a p-number. Since $P_j \cap N \trianglelefteq P$, it follows that $P_j \cap N \trianglelefteq G$ and so $P_j \cap N = N$ or $P_j \cap N = 1$. If the former holds, then $N \leq P_j$, so $P = P_j$, a contradiction. If the latter holds, then $N = |P : P_j| = p$ and (3) follows.

(4) All minimal normal subgroups of G are contained in $O_p(G)$.

Let N be a minimal normal subgroup of G which is not a p-subgroup. Then N is a direct product of some non-abelian simple groups which are isomorphic from each other and $p||N|$ by (1). We have

(4.1) $P_i \cap N = 1$ for each i and $|P \cap N| = p$. In particular, N is a non-abelian simple group.

By hypothesis, P_i satisfies the Π-property in G for each i. Then $|G : N_G(P_i \cap N)|$ is a p-number. Since $P_i \cap N \trianglelefteq P$, it follows that $P_i \cap N \trianglelefteq G$. Noticing that N is not a p-group, we have $P_i \cap N = 1$ and $|P \cap N| \leq p$. By (1), $P \cap N \neq 1$, hence $|P \cap N| = p$ and N is a non-abelian simple group.

(4.2) $O_p(G) = 1$.

Assume $O_p(G) > 1$. Let T be a minimal normal subgroup of G contained in $O_p(G)$. By (3), $|T| = p$. Since $N \cap T = 1$, $NT = N \times T$. By hypothesis, P_i satisfies the Π-property in G for each i. Then $|G : N_G(P_iT \cap NT)|$ is a p-number for the G-chief factor NT/T. Since $P_iT \cap NT \trianglelefteq P$, $P_iT \cap NT \trianglelefteq G$ and $P_iT \cap NT = T$ or $P_iT \cap NT = NT$. Again, N is not a p-group, so $NT \nleq P_iT$. Thus, $P_iT \cap NT = T$. If $T \nleq P_i$ for some i, then $P_iT = P$. This implies that

$$P_iT \cap NT = P \cap NT = T(P \cap N) > T,$$

a contradiction. Therefore $T \leq P_i$ for each i. Similarly to the proof of (3), we obtain a contradiction.

(4.3) $C_G(N) = 1$.

Assume $C_G(N) > 1$. Then we can pick a minimal normal subgroup K of G contained in $C_G(N)$. By (4.1), $N \cap K = 1$. Since P_i satisfies the Π-property in G, we see that $|G : N_G(P_iN \cap KN)|$ is a p-number for the G-chief factor KN/N. Hence, $P_iN \cap KN \trianglelefteq G$, and thus $P_iN \cap KN = KN$ or $P_iN \cap KN = N$. If the former holds, then $KN \leq P_iN$ and KN/N is a p-group. Furthermore, K is a non-identity p-group by $K \cong KN/N$, which is contrary to (4.2). If the latter holds, then $P_i \cap KN \leq N$. This forces that $P_i \cap KN = P_i \cap N = 1$ by (4.1). Thus,

$$|P_iKN|_p = |P_i||KN|_p = |P_i||K|_p|N|_p > |P|,$$

which is impossible.

(4.4) Finish the proof of (4).

Since $C_G(N) = 1$ by (4.3), G is isomorphic to a subgroup of $\mathrm{Aut}(N)$, and thus $|\mathrm{Aut}(N)|$ is divided by $|G|$. Since $\mathrm{Inn}(N) \cong N$, $|\mathrm{Out}(N)| = |\mathrm{Aut}(N)/\mathrm{Inn}(N)|$ is divided by $|G/N|$. Consequently, p divides $|\mathrm{Out}(N)|$. However, $(|\mathrm{Out}(N)|, p) = 1$ by Lemma 2. This is a contradiction and (4) follows.

(5) The final contradiction.

From (4), we have $O_p(G) \neq 1$. Let N be a minimal normal subgroup of G contained in $O_p(G)$. By (3), $|N| = p$. By (2), N has a complement in P. Hence N has a complement in G by Lemma 3. It follows that $O_p(G) \cap \Phi(G) = 1$. In view of Lemma 5, we obtain $O_p(G) = N_1 \times N_2 \times \cdots \times N_s$, where $N_i (i = 1, 2, \ldots s)$ are minimal normal in G with order p. Moreover, $G = O_p(G) \rtimes M$, the semidirect product of $O_p(G)$ with a subgroup M of G.

Since $N_i \leq Z(P)$, $P \leq C_G(O_p(G))$. Note that $C_G(O_p(G)) \cap M \trianglelefteq O_p(G)M = G$. So by (4), we have $C_G(O_p(G)) \cap M = 1$. Then $P \cap M = 1$. This implies that

$$P = P \cap O_p(G)M = O_p(G)(P \cap M) = O_p(G).$$

It follows that $G/O_p(G)$ is p-supersolvable. However, every G-chief factor under $O_p(G)$ is of order p, thereby G is p-supersolvable. This is the final contradiction.

As an immediate consequence of Theorem 1, we obtain the following Corollary:

Corollary 1. *Let G be a p-solvable group and let P be a Sylow p-subgroup of G, where $p \in \pi(G)$. Then G is p-supersolvable if and only if P has m_p maximal subgroups which satisfy the Π-property in G, where $m_p > \frac{p^{d-1}-1}{p-1}$ and $|P/\Phi(P)| = p^d$.*

Remark 1. *The condition $m_p > \frac{p^{d-1}-1}{p-1}$ in Theorem 1 is sharp. It cannot be replaced by $m_p \geq \frac{p^{d-1}-1}{p-1}$ or d. For instance, let $G = S_4 \times Z_2$, $P = P_1 \times Z_2 \in Syl_2(G)$, where $P_1 \in Syl_2(S_4)$. Then*

$$P_1 = \langle a, b | a^4 = b^2 = 1, b^{-1}ab = a^{-1} \rangle \cong D_8,$$

$Z_2 = \langle c \rangle$ and $d = 3 = \frac{2^{d-1}-1}{2-1}$. Clearly, P_1, $P_2 = \langle ac, bc \rangle = \langle a^2, ab \rangle \times \langle bc \rangle$ and $P_3 = \langle a^2, ab \rangle \times \langle c \rangle$ are maximal subgroups of P. Evidently, P_3 is normal in G and $P_1Q = S_4, P_2Q = A_4\langle bc \rangle$, for all $Q \in Syl_3(G) = Syl_3(S_4)$. Let U/V be a chief factor of G. Since G is solvable, $|U/V| = 2^\alpha$ or 3, where $1 \leq \alpha \leq 4$. Obviously, $|G/V : N_{G/V}(P_iV/V \cap U/V)| = 1$ if $|U/V| = 3$, where $i = 1, 2, 3$. If $|U/V| = 2^\alpha$, then

$$P_iV/V \cap U/V = (P_iV/V)(QV/V) \cap U/V \trianglelefteq (P_iV/V)(QV/V),$$

hence $QV/V \leq N_{G/V}(P_iV/V \cap U/V)$ and $|G/V : N_{G/V}(P_iV/V \cap U/V)|$ is a 2-number, namely P_i satisfies the Π-property in G, where $i = 1, 2, 3$. It is clear that G is not 2-supersolvable and $P_1 \cap P_2 \cap P_3 = \langle a^2, ab \rangle \neq \Phi(P)$.

Theorem 2. *Let G be a group with a normal subgroup H and a subnormal subgroup L such that G/H is p-nilpotent and $F_p^*(H) \leq L \leq H$, where $p \in \pi(G)$. Then G is p-nilpotent if and only if $N_G(P)$ is p-nilpotent and P has m_p maximal subgroups which satisfy the Π-property in G, where P be a Sylow p-subgroup of L, $m_p > \frac{p^{d-1}-1}{p-1}$ and $|P/\Phi(P)| = p^d$.*

Proof. We only need to prove the sufficiency. Let G be a counter-example with minimal order. Let $P_1, P_2, \ldots, P_{m_p}$ be the maximal subgroups of P which satisfy the Π-property in G. Then

(1) $O_{p'}(H) = 1$.

Write $K = O_{p'}(H)$ and $\overline{G} = G/K$. If $T \neq 1$, we claim that $\overline{G}$ satisfies the hypotheses of the theorem. In fact,

$$F_p^*(\overline{H}) = F^*(\overline{H}) = F_p^*(H)/K \leq L/K = \overline{L}.$$

Obviously, $\overline{L} \lhd \lhd \overline{G}$ and $\overline{P} = PN/N \in \mathrm{Syl}_p(\overline{L})$. Noticing that $P \cong \overline{P}$, so $|\overline{P}/\Phi(\overline{P})| = p^d$. Again,

$$N_{\overline{G}}(\overline{P}) = N_{G/K}(PK/K) = N_G(P)K/K \cong N_G(P)/(N_G(P) \cap K)$$

is p-nilpotent. By Lemma 1, $\overline{G}$ satisfies the hypotheses of the theorem. The choice of G implies that $\overline{G}$ is p-nilpotent, thus G is p-nilpotent, a contradiction.

(2) $H = G$.

If not, then $H < G$. Evidently, $L \lhd \lhd H$ and $N_H(P) = N_G(P) \cap H$ is p-nilpotent. By Lemma 6, P_i satisfies the Π-property in H. Hence H satisfies the hypotheses of the theorem and H is p-nilpotent by the minimal choice of G. In particular, L is also p-nilpotent. Let T be the normal p-complement of L. Since T char L and $L \lhd \lhd H$, $T \lhd \lhd H$ and $T \leq O_{p'}(H)$ by Lemma 4. Thus $T = 1$ by (1), namely $L = P$. Again by Lemma 4, $L \leq O_p(H)$. Applying Lemma 10 and Lemma 11, we have

$$F_p^*(H) = F^*(H) = F(H) = O_p(H) \leq L.$$

Thereby $P = L = O_p(H) \lhd G$ and $G = N_G(P)$ is p-nilpotent. This is contrary to the choice of G and (2) follows.

(3) The final contradiction.

By applying Theorem 1, we see that either G is p-supersolvable or $|P| = p$. If the latter holds, then P is an abelian group. Since $N_G(P)$ is p-nilpotent, $N_G(P) = P \times T$, where T is a normal p-complement of $N_G(P)$, and so $N_G(P) = C_G(P)$. By Burnside's theorem, G is p-nilpotent, a contradiction. If the former holds, then p-length of G is at most 1 by [6, Kapitel VI, Hauptsatz 6.6]. So $PO_{p'}(G)$ is normal in G. Further, P is normal in G by (1). Thus $G = N_G(P)$ is p-nilpotent. This is the final contradiction.

If p is a special prime divisor of $|G|$, then the assumption that $N_G(P)$ is p-nilpotent in Theorem 2 can be removed.

Theorem 3. *Let H be a normal subgroup of a group G such that G/H is p-nilpotent and let P be a Sylow p-subgroup of H, where $p \in \pi(G)$ and $(|G|, p-1) = 1$. Then G is p-nilpotent if and only if P has m_p maximal subgroups which satisfy the* Π*-property in G, where $m_p > \frac{p^{d-1}-1}{p-1}$ and $|P/\Phi(P)| = p^d$.*

Proof. We only need to prove the sufficiency. Suppose that $P_1, P_2, \ldots, P_{m_p}$ are the maximal subgroups of P which satisfy the Π-property in G. Since $H \trianglelefteq G$, P_i satisfies the Π-property in H by Lemma 6, where $i = 1, 2, \ldots, m_p$. By Theorem 1, either H is p-supersolvable or $|P| = p$. If the former holds, then every p-chief factor U/V of H is of order p and so it is in $Z(H/V)$ by Lemma 9, thus H is p-nilpotent, namely H has a normal p-complement T. Since T char $H \trianglelefteq G$, $T \trianglelefteq G$. If $T > 1$, then G/T is p-nilpotent by induction. Thus, G is p-nilpotent. If $T = 1$, then $H = P$. Let L/P be the normal p-complement of G/P. Then $L \trianglelefteq G$ and $P \in \mathrm{Syl}_p(L)$. Furthermore, P_i satisfies the Π-property in L by Lemma 6. Similarly, we know that L has a normal p-complement L_1. Of course, L_1 is also the normal p-complement of G, namely G is p-nilpotent. The proof is complete.

Theorem 4. *Let G be a group. Then G is supersolvable if and only if every non-cyclic Sylow p-subgroup P of G has m_p maximal subgroups which satisfy the* Π*-property in G for every $p \in \pi(G)$, where $m_p > \frac{p^{d-1}-1}{p-1}$, $|P/\Phi(P)| = p^d$.*

Proof. We only need to prove the sufficiency. Suppose that q is the smallest prime divisor of $|G|$ and Q is a Sylow q-subgroup of G. If Q is cyclic, then G is q-nilpotent by Lemma 9. If Q is non-cyclic, then G is q-nilpotent by Theorem 3. By the same arguments and induction, we know that G has a Sylow tower of supersolvable type, of course, G is solvable. By Theorem 1, we conclude that G is supersolvable. The proof is complete.

4. Some Applications

As we know, all normal subgroups, s-permutable subgroups [12], CAP-subgroups [1, Chapter A, Definition 10.8] and s-semipermutable p-subgroups [13] and SS-quasinormal p-subgroups [14] satisfy the Π-property (refer to [2, Proposition 2.2 and Proposition 2.3] and [4, Proposition 4.2]). Hence the following results are respectively direct corollaries of Theorem 2, Theorem 3 and Theorem 4.

Corollary 2. ([15, Theorem 1]) *If all maximal subgroups of Sylow subgroups of G are normal in G, then G is supersoluble.*

Corollary 3. ([15, Theorem 2]) *If all maximal subgroups of Sylow subgroups of G are s-permutable in G, then G is supersoluble.*

Corollary 4. ([16, Coroollary 1]) *Let p be a prime and G a p-soluble group and P be a Sylow p-subgroup of G. If all maximal subgroups of P are CAP-subgroups of G, then G is p-supersolvable.*

Corollary 5. ([16, Coroollary 2]) *Let G be a group. Suppose that all maximal subgroups of Sylow subgroups of G are CAP-subgroups of G. Then G is supersolvable.*

Corollary 6. ([17, Theorem 3.3]) *Let p be the smallest prime dividing the order of a group G and let P be a Sylow p-subgroup of G. If all maximal subgroups of P are CAP-subgroups of G. Then G is p-nilpotent.*

Corollary 7. ([17, Theorem 3.4]) *Let p be a prime dividing the order of a group G and let P be a Sylow p-subgroup of G. If $N_G(P)$ is p-nilpotent and all maximal subgroups of P are CAP-subgroups of G. Then G is p-nilpotent.*

Corollary 8. ([18, Theorem 3.1]) *Let p be an odd prime dividing the order of a group G and P be a Sylow p-subgroup of G. If $N_G(P)$ is p-nilpotent and every maximal subgroup of P is s-semipermutable in G, then G is p-nilpotent.*

Corollary 9. ([18, Theorem 3.3]) *Let p be the smallest prime number dividing the order of a group G and P a Sylow p-subgroup of G. If every maximal subgroup of P is s-semipermutable in G, then G is p-nilpotent.*

Corollary 10. ([19, Lemma 2.5]) *Suppose that N is a normal subgroup of a group G such that G/N is p-nilpotent and P is a Sylow p-subgroup of N, where $p \in \pi(G)$ with $(|G|, p-1) = 1$. If all maximal subgroup of P are SS-quasinormal in G, then G is p-nilpotent.*

5. Conclusion

This study investigates the structure of finite groups through the embedding properties of certain maximal subgroups within Sylow p-subgroups. We establish the following criterion: a finite group G is p-supersolvable provided that one of its Sylow p-subgroups has m_p maximal subgroups which satisfy the Π-property. As applications, we obtain some necessary and sufficient conditions for the p-nilpotency and supersolvability of a finite group. These results unify and generalize some classical embedding theorems, refining the understanding of how subgroup embedding controls global group structure. Future work may explore extensions of this approach to other local embedding properties or to invariant properties under coprime actions in the context of finite group theory.

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Conflicts of Interest

All authors declare no conflict of interest.

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