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A Fractional Calculus Approach to Interval-Valued Variational Programming Problems

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Abstract. This study explores a class of fractional interval-valued variational programming problems involving the Caputo-Fabrizio (C-F) fractional derivative. By employing the concepts of invex and generalized invex functions, we establish sufficient optimality conditions for these problems. Additionally, we develop a Wolfe-type dual formulation and investigate the corresponding duality relationships. In particular, we derive and prove the weak, strong, and converse duality theorems to establish a connection between the primal and dual problems. The theoretical findings are further illustrated through carefully constructed numerical examples, demonstrating the applicability and effectiveness of the proposed approach.

2020 Mathematics Subject Classifications: 26A51, 49J40, 49K99, 90C46

Key Words and Phrases: Variational programming problem, sufficient optimality conditions, LU-optimality, Caputo-Fabrizio fractional derivative, Wolfe-type duality.

1. Introduction

The optimization theory acknowledges interval-valued programming as a crucial component. In various scientific and mathematical domains, interval-valued optimization has recently gained popularity. Due to the uncertainty of the theory underpinning the parameters, estimating a physical world system's parameters is challenging. We can see that

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there are several applications in many different fields, including decision-making [1], diagnostic [2], portfolio optimization [3], financial planning, business planning, healthcare, production, hospital planning and management, and many more.

In the very recent work, [4] studied a new class of optimization problems governed by interval-valued variational programming and inequalities. Its results work in applications of Control and Optimization problems. And at the same time, [5] discussed some results on solutions associated with interval-valued optimal control problems driven by generalized invariant convex (invex) functionals and also investigated necessary and sufficient optimality conditions for the considered optimization problem. Before them,[6] developed a framework for analyzing problems involving interval-valued optimization. Reading the books [7–9] and some recent articles [4–6, 10, 11] will help to learn the basics of interval-valued optimization.

Problems with variational programming start with the calculus of variations. Recent advances in variational calculus and optimization theory have given us a cogent framework for analyzing a range of issues in numerous other fields of pure and applied mathematics. The dynamics of rigid bodies, orbit optimization [12], flight design [13, 14], and other problems are among the areas where the calculus of variations is utilized to address problems. Using invexity assumptions, [15] developed some optimality requirements and theorems of duality for interval-valued optimization problems. Later, [11] modified the definitions of pre-invexity and generalized invexity in interval-valued functions and additionally developed Karush-Kuhn-Tucker optimality requirements for the optimization problem under consideration, where the objective function was assumed to be interval-valued. Recently, [16] used extended $(p, r) - \rho - (\aleph, \theta)$-invexity for a problem of interval-valued optimization to study optimality and duality. [17] established duality conclusions for cases involving variational programming. Jimenez et al., [18], developed some duality solutions for the multiobjective variational issue utilizing the pseudo-invexity notion. Treanctua et al. [19] investigated efficiency conditions in interval-valued control models using a modified objective functional and saddle-point criteria. Numerous additional studies have discussed the issues with variational programming (see, for example, [20, 21]).

Fractional Calculus (FC) is currently recognized as a fascinating subject by the community of practical engineers. It is a generalization of conventional calculus because derivatives and integrals are employed outside of integer orders. Fractional calculus has numerous uses in science and engineering [22], and it has grown in popularity in recent years as a tool for researching the dynamics of practical problems. [23, 24] looked into a few common variational issues by incorporating fractional derivatives of Riemann-Liouville [25], Caputo, and Riesz types. In other research work [26], variational problems pertaining to a Lagrangian function given fractional derivatives have been analyzed to determine optimality conditions. Shalini et al. [27] investigate dynamical control systems using a fuzzy modeling framework, wherein the system dynamics are governed by a fuzzy stochastic process (FSP) driven by fuzzy Brownian motion (FBM).

Fuzzy fractional variational problems (FVPs) have been explored in the literature for necessary optimality conditions, as seen in the works of [28] and [29]. These studies provide foundational insights into the interplay between fuzzy logic and fractional calculus, particularly in variational settings. Caputo and Fabrizio [30] introduced a novel fractional derivative, characterized by the order $\theta^{\bullet} \in (0,1)$, defined using an exponential kernel. This derivative avoids singular kernels and is well-suited for modeling systems with memory effects, which marks a significant departure from classical Riemann–Liouville and Caputo definitions that employ singular kernels. Jayswal and Uniyal [31] presented necessary and sufficient optimality conditions and Mond-Weir duality results for semi-infinite variational programming (SIVP) problems involving Caputo-Fabrizio fractional derivatives. Their analysis leveraged Slater-type constraint qualifications and generalized convexity assumptions. In a follow-up work [32], they extended this framework to investigate optimality conditions for broader classes of semi-infinite fractional variational problems with similar structural features. The present study advances the existing literature by considering interval-valued variational programming problems (P) governed by Caputo-Fabrizio fractional derivatives. In contrast to previous works that primarily address crisp-valued problems, the incorporation of interval-valued objective and constraint functions introduces a layer of uncertainty and imprecision, which is more reflective of real-world systems. Moreover, unlike earlier criteria that focus on classical convexity, this work applies LU-optimality conditions and generalized-invexity criteria that offer a broader, more flexible framework for establishing optimality in uncertain fractional variational environments. The proposed approach also contributes by establishing Karush-Kuhn-Tucker-type sufficient optimality conditions and analyzing Wolfe-type duality in the context of interval-valued programming problems with Caputo-Fabrizio derivatives. These enhancements allow for a more generalized efficiency framework than those discussed in [28] [31], and [29], and facilitate a better understanding of solution robustness under fuzzy uncertainty and memory effects introduced by the exponential kernel.

We will now continue to discuss this article's contents. Several fractional calculus ideas, concepts, and features were reviewed in preliminary section 2. We also go through the LU-Optimal approach to the programming problem of variational calculus. We examine the fractional derivative C-F for the KKT-type sufficient optimality conditions in Section 3. We establish and verify weak, strong, and strict converse duality theorems for the Wolfe-type dual model in section 4 and section 5 contains the article's conclusion and future direction.

2. Preliminaries

This section recollects some notations, symbols, and definitions that will be important in the follow-up to this work. For any number of intervals $\mathbb{A} = \left[a^L, a^U\right]$ and $\mathbb{B} = \left[b^L, b^U\right]$ where a^L, a^U, b^L, $b^U \in \mathbb{R}$, we define the following partial ordering relations on the lines of [33] and [21]:

(*i*) $\mathbb{A} \preceq_{LU} \mathbb{B} \iff a^L \leq\ b^L and\ a^U \leq b^U.$

(*ii*) $\mathbb{A} \prec_{LU} \mathbb{B} \iff a^L < b^L, a^U \leq b^U, or\ a^L \leq b^L, a^U < b^U, or\ a^L < b^L,\ a^U < b^U.$

Throughout the paper, $\varkappa : [a_1, a_2] \to R$ is a function of class C^1 and $\theta^\bullet \in (0, 1)$.

Definition 1. [8] *Left and right Fractional derivatives of Riemann-Liouville of order $\theta^\bullet$ are defined by*

$$ {}_{a1}\mathbb{D}_{\varsigma}^{\theta^\bullet} \varkappa(\varsigma) = \frac{1}{\Gamma(1-\theta^\bullet)} \frac{d}{d\varsigma} \int_{a_1}^{\varsigma} (\varsigma - \nu)^{-\theta^\bullet} \varkappa(\nu) d\nu, $$

$$ {}_{\varsigma}\mathbb{D}_{a_2}^{\theta^\bullet} \varkappa(\theta^\bullet) = -\frac{1}{\Gamma(1-\theta^\bullet)} \frac{d}{d\varsigma} \int_{\varsigma}^{a_2} (\nu - \varsigma)^{-\theta^\bullet} \varkappa(\nu) d\nu, \theta^\bullet \in (0, 1). $$

Definition 2. [34] *The fractional derivative of Caputo, $\varkappa(\varsigma) : [a_1, a_2] \to \mathbb{R}$ of order $\theta^\bullet \in (0, 1)$ is stated to be*

$$ {}^{c}\mathbb{D}_{a_1+}^{\theta^\bullet} \varkappa(\varsigma) = \frac{1}{\Gamma(1-\theta^\bullet)} \frac{d}{d\varsigma} \int_{a_1}^{\varsigma} \frac{1}{(\varsigma - \nu)^{\theta^\bullet}} [\varkappa(\nu) - \varkappa(a)] d\nu. $$

If $\varkappa \in C^1$, then

$$ {}^{c}\mathbb{D}_{a_1+}^{\theta^\bullet} \varkappa(\varsigma) = \frac{1}{\Gamma(1-\theta^\bullet)} \frac{d}{d\varsigma} \int_{a_1}^{\varsigma} \frac{1}{(\varsigma - \nu)^{\theta^\bullet}} \varkappa'(\nu) d\nu. $$

As $\theta^\bullet \to 1, {}^{c}\mathbb{D}_{a_1+}^{\theta^\bullet} \varkappa(\varsigma)$ approaches to $\varkappa'(\varsigma)$.

Definition 3. [30] *The operator of new Caputo-Fabrizio (CF) fractional derivative is described as*

$$ {}^{CF}\mathbb{D}_{a_1+}^{\theta^\bullet} \varkappa(\varsigma) = \frac{\kappa(\theta^\bullet)}{(1-\theta^\bullet)} \int_{a_1}^{\varsigma} exp\left(-\frac{\theta^\bullet(\varsigma - \nu)}{(1-\theta^\bullet)}\right) \varkappa'(\nu) d\nu,\ \ \theta^\bullet \in (0, 1), $$

where $\kappa(\theta^\bullet)$ signifies the normalization function $(1-\theta^\bullet) + \dfrac{\theta^\bullet}{\Gamma(\theta^\bullet)}$ with the property $\kappa(0) = \kappa(1) = 1$. Clearly ${}^{CF}\mathbb{D}_{a_1+}^{\theta^\bullet} \varkappa(\varsigma) = 0$, if $\varkappa(\varsigma)$ is a constant function, i.e, a constant function's CF derivative equals zero, but the CF derivative lacks a unique kernel for $\varsigma = \nu$, like the Caputo derivative.

Remark 1. *As $\theta^\bullet \to 1$, ${}^{CF}\mathbb{D}_{a_1+}^{\theta^\bullet} \varkappa(\varsigma) \Rightarrow \varkappa'(\varsigma)$ and as $\theta^\bullet \to 0$, ${}^{CF}\mathbb{D}_{a_1+}^{\theta^\bullet} \varkappa(\varsigma) \Rightarrow \varkappa(\varsigma) - \varkappa(a_1)$.*

Definition 4. [1] *The right Caputo-Fabrizio fractional derivative is defined as*

$$ {}^{CF}\mathbb{D}_{a_2-}^{\theta^\bullet} \varkappa(\varsigma) = \frac{\kappa(\theta^\bullet)}{(1-\theta^\bullet)} \int_{\varsigma}^{a_2} exp\left(-\frac{\theta^\bullet(\varsigma - \nu)}{(1-\theta^\bullet)}\right) \varkappa'(\nu) d\nu, \theta^\bullet \in (0, 1). $$

Definition 5. *The order of Sobolev space $1 \in (a_1,\ a_2)$ is defined: $H^1(a_1, a_2) = \{y \in L^2(a_1, a_2) \mid y^{'} \in L^2(a_1,\ a_2)\}$, y' is the weak derivative of y.*

Definition 6. [30] *Let $\varkappa \in H^1(a_1, a_2), a_2 > a_1, 0 < \theta^\bullet < 1$. The fractional derivative of CF is therefore given as in Definition 3, where $\kappa(\theta^\bullet)$ specifies the function of normalization characterized by $\kappa(0) = \kappa(1) = 1$. If the function is $\varkappa \notin H^1(a_1, a_2)$, then the derivative is written in the following manner:*

$$^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\varkappa(\varsigma) = \frac{\theta^\bullet \kappa(\theta^\bullet)}{(1-\theta^\bullet)} \int_{a_1}^{\varsigma} exp\left(-\frac{\theta^\bullet(\varsigma-\nu)}{(1-\theta^\bullet)}\right)[\varkappa(\varsigma) - \varkappa(\nu)]\,d\nu,$$

where CF has an exponential kernel.

Definition 7. [1, 35] *Let $\varkappa$ be a function with the property that $\varkappa \in H^{'}(a_1, a_2), a_1 < a_2$. In the Caputo-Fabrizio sense, the order of the left Riemann fractional derivative $\theta^\bullet$ is expressed as*

$$^{CFR}\mathbb{D}^{\theta^\bullet}_{a_1+}\varkappa(\varsigma) = \frac{\kappa(\theta^\bullet)}{(1-\theta^\bullet)} \frac{d}{d\varsigma} \int_{a_1}^{\varsigma} exp\left(-\frac{\theta^\bullet(\varsigma-\nu)}{(1-\theta^\bullet)}\right)\varkappa(\nu)d\nu, \tag{1}$$

where $a_1 \leq \varsigma$, $\theta^\bullet$ $(0 < \theta^\bullet < 1)$ is a real number and $\kappa(\theta^\bullet)$ is a normalization function that depends on $\theta^\bullet$ such that $\kappa(0) = \kappa(1) = 1$. Similarly, in the Caputo-Fabrizio sense, the order of right Riemann fractional derivative $\theta^\bullet$ can be stated as follows:

$$^{CFR}\mathbb{D}^{\theta^\bullet}_{a_2-}\varkappa(\varsigma) = \frac{\kappa(\theta^\bullet)}{(1-\theta^\bullet)} \frac{d}{d\varsigma} \int_{\varsigma}^{a_2} exp\left(-\frac{\theta^\bullet(\varsigma-\nu)}{(1-\theta^\bullet)}\right)\varkappa(\nu)d\nu, \text{ where } \varsigma \leq a_2.$$

Remark 2. *When $\theta^\bullet \to 0$, (1) becomes*

$$\lim_{\theta^\bullet \to 0} {}^{CFR}\mathbb{D}^{\theta^\bullet}_{a_1+}\varkappa(\varsigma) = \frac{d}{d\varsigma}\int_{a_1}^{\varsigma}\varkappa(\nu)d\nu = \varkappa(\varsigma).$$

Proposition 1 (Abdeljawad and Baleanu [1]). *Let $\theta^\bullet \in (0, 1)$ and $\varkappa, z : [a_1, a_2] \to \mathbb{R}$ be two continuous functions of class $C^1[a_1, a_2]$. Then the following integration by parts formula holds true:*

$$\int_{a_1}^{a_2} \varkappa(\varsigma)\,{}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}z(\varsigma)d\varsigma = [z(\varsigma)I^{1-\theta^\bullet}_{a_2-}\varkappa(\varsigma)]\Big|_{\varsigma=a_1}^{\varsigma=a_2} + \int_{a_1}^{a_2} z(\varsigma)\,{}^{CFR}\mathbb{D}^{\theta^\bullet}_{a_2-}\varkappa(\varsigma)d\varsigma,$$

where $I^{1-\theta^\bullet}_{a_2-}\varkappa(\varsigma) = \frac{K(\theta^\bullet)}{(1-\theta^\bullet)}\int_{\varsigma}^{a_2} exp\left(-\frac{\theta^\bullet}{(1-\theta^\bullet)}(\nu-\varsigma)\right)\varkappa(\nu)d\nu.$

Let $\digamma : \Im \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function where $\Im = [a_1, a_2]$ is real valued interval. Now, we are dealing with the function $\digamma(\varsigma, \varkappa(\varsigma), {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\varkappa(\varsigma))$, where $\varkappa : \Im \to \mathbb{R}^n$ is an n-dimensional function of class $C^1[a_1, a_2]$ and ${}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\varkappa$ represents the Caputo-Fabrizio derivative of order $0 < \theta^\bullet < 1$ for a function $\varkappa$. The partial derivatives of $\digamma$ are represented by

$$\digamma_\varsigma = \frac{\partial \digamma}{\partial \varsigma}, \digamma_\varkappa = \left[\frac{\partial \digamma}{\partial \varkappa_1}, \frac{\partial \digamma}{\partial \varkappa_2}, \frac{\partial \digamma}{\partial \varkappa_3}, ..., \frac{\partial \digamma}{\partial \varkappa_n}\right],$$

$F_{{}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\varkappa} = \left[\frac{\partial F}{\partial({}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\varkappa_1)}, \frac{\partial F}{\partial({}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\varkappa_2)}, \frac{\partial F}{\partial({}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\varkappa_3)}, ..., \frac{\partial F}{\partial({}^{\mathrm{CF}}D^{\theta^\bullet}_{a_1+}\varkappa_n)}\right].$

Where $\varkappa_1, \varkappa_2, \varkappa_3, ..., \varkappa_n$ are components of $\varkappa$. Consider the space of piecewise smooth functions to be $\varkappa : \Im \to R^n$ along with the norm $\|\varkappa\| = \|\varkappa\|_\infty + \|D\varkappa\|_\infty$, where the differential operator $\mathbb{D}$ is described as follows:

$$v = \mathbb{D}\varkappa \iff \varkappa(\varsigma) = \varkappa_0 + \int_{a_1}^{\varsigma} v(s)ds, \text{ where } \varkappa_0 \text{ signifies the boundary value.}$$

Let $F : X \to \mathbb{R}$ defined by $F(\varkappa) = \int_{a_1}^{a_2} F(\varsigma, \varkappa(\varsigma), {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\varkappa(\varsigma))d\varsigma$ be differentiable. For notational convenience, $F(\varsigma, \varkappa(\varsigma), {}^{\mathrm{CF}}D^{\theta^\bullet}_{a_1+}\varkappa(\varsigma))$ will be written as $F(\varsigma, \varkappa, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\varkappa)$.
Now, we define the concept of invex and generalized invex functions by using the Caputo-Fabrizio (CF) fractional derivative of order $0 < \theta^\bullet < 1$, in the following way:

Definition 8. *The functional F is stated as invex (strictly invex) with respect to $\aleph$ if a differentiable vector function $\aleph(\varsigma, \varkappa, \bar{\varkappa}) \in C^1[a_1, a_2]$ with $\aleph(\varsigma, \varkappa, \varkappa) = 0$ occurs such that forall $\varkappa, \bar{\varkappa} \in X$,*

$$\int_{a_1}^{a_2} F(\varsigma, \varkappa, {}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\varkappa)d\varsigma - \int_{a_1}^{a_2} F(\varsigma, \bar{\varkappa}, {}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})d\varsigma$$
$$\geq (>) \int_{a_1}^{a_2} \Big[\aleph(\varsigma, \varkappa, \bar{\varkappa})F_{\bar{\varkappa}}(\varsigma, \bar{\varkappa}, {}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})$$
$$+ ({}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\aleph(\varsigma, \varkappa, \bar{\varkappa}))F_{{}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa}}(\varsigma, \bar{\varkappa}, {}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})\Big] d\varsigma.$$

The following example demonstrates that, the function F is invex under Caputo-Fabrizio fractional derivative.

Example 1. *: Let $F(\varkappa) : X = [0, 1] \to \mathbb{R}$ be defined by $F(\varkappa) = \int_0^1 \{-\varkappa(\varsigma) + 6.5514\varsigma - 16.3785e^{-\frac{\varsigma}{3}} + 22.9299)\}d\varsigma$ and $\varkappa(\varsigma) = -\varsigma^2 + \varsigma + 1 \in X$, forall $\varsigma \in [0, 1]$ and note that $(\varkappa \neq \bar{\varkappa})$ $\bar{\varkappa}(\varsigma) = 1 \in X$ in such way that,*

$$\int_{a_1}^{a_2} F(\varsigma, \varkappa, {}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\varkappa)d\varsigma - \int_{a_1}^{a_2} F(\varsigma, \bar{\varkappa}, {}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})d\varsigma$$
$$> \int_{a_1}^{a_2} \{\aleph(\varsigma, \varkappa, \bar{\varkappa})F_{\bar{\varkappa}}(\varsigma, \bar{\varkappa}, {}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa}) + {}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}(\aleph(\varsigma, \varkappa, \bar{\varkappa}))$$
$$F_{{}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa}}(\varsigma, \bar{\varkappa}, {}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})\}d\varsigma.$$

In this example, $\theta^\bullet = \frac{1}{4}$, $a_1 = 0$, $a_2 = 1$, $\aleph(\varsigma, \varkappa, \bar{\varkappa}) = \varkappa - \bar{\varkappa}$ and $\varkappa(\varsigma) = -\varsigma^2 + \varsigma + 1$ are

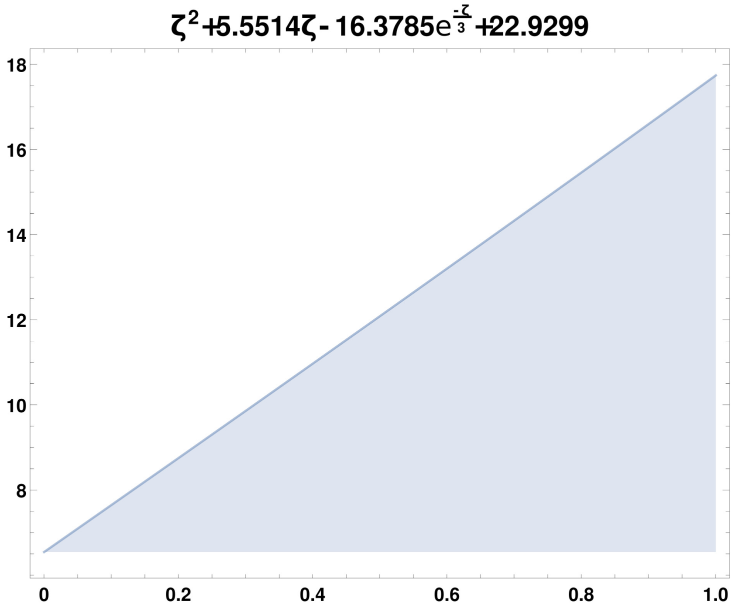

Figure 1: Graphical view of the function $\digamma = -\varkappa(\varsigma) + 6.5514\varsigma - 16.3785e^{-\frac{\varsigma}{3}} + 22.9299$

taken relevantly and deduce ${}^{CF}\mathbb{D}^{\theta^{\bullet}}_{a_1+}\varkappa(\varsigma) = -6.5514\varsigma - 22.9299e^{-\varsigma/3} + 22.9299$. *But in the given example, the function is not convex for the differentiable function* $(\varkappa - \bar{\varkappa})^T = \varkappa + \bar{\varkappa}$.

Definition 9. *The functional* $\digamma$ *is stated as pseudo-invex with regard to* $\aleph$ *if a differentiable function* $\aleph(\varsigma, \varkappa, \bar{\varkappa}) \in C^1[a_1, a_2]$ *with* $\aleph(\varsigma, \varkappa, \varkappa) = 0$ *occurs such that* $\forall \varkappa, \bar{\varkappa} \in X$,

$$\int_{a_1}^{a_2} \Big[\aleph(\varsigma, \varkappa, \bar{\varkappa})\digamma_{\bar{\varkappa}}(\varsigma, \bar{\varkappa}, {}^{CF}\mathbb{D}^{\theta^{\bullet}}_{a_1+}\bar{\varkappa})$$

$$+ ({}^{CF}\mathbb{D}^{\theta^{\bullet}}_{a_1}\aleph(\varsigma, \varkappa, \bar{\varkappa}))\digamma_{{}^{CF}\mathbb{D}^{\theta^{\bullet}}_{a+}\bar{\varkappa}}(\varsigma, \bar{\varkappa}, {}^{CF}\mathbb{D}^{\theta^{\bullet}}_{a_1}\bar{\varkappa})\Big] d\varsigma \geq 0$$

$$\Rightarrow \int_{a_1}^{a_2} \digamma(\varsigma, \varkappa, {}^{CF}\mathbb{D}^{\theta^{\bullet}}_{a_1+}\varkappa)d\varsigma \geq \int_{a_1}^{a_2} \digamma(\varsigma, \bar{\varkappa}, {}^{CF}\mathbb{D}^{\theta^{\bullet}}_{a_1+}\bar{\varkappa})d\varsigma,$$

or equivalently,

$$\int_{a_1}^{a_2} \digamma(\varsigma, \varkappa, {}^{CF}\mathbb{D}^{\theta^{\bullet}}_{a_1+}\varkappa)d\varsigma < \int_{a_1}^{a_2} g(\varsigma, \bar{\varkappa}, {}^{CF}\mathbb{D}^{\theta^{\bullet}}_{a_1+}\bar{\varkappa})d\varsigma$$

$$\Rightarrow \int_{a_1}^{a_2} \Big[\aleph(\varsigma, \varkappa, \bar{\varkappa})\digamma_{\bar{\varkappa}}(\varsigma, \bar{\varkappa}, {}^{CF}\mathbb{D}^{\theta^{\bullet}}_{a_1+}\bar{\varkappa})$$

$$+ {}^{CF}\mathbb{D}^{\theta^{\bullet}}_{a_1+}\aleph(\varsigma, \varkappa, \bar{\varkappa}))\digamma_{{}^{CF}\mathbb{D}^{\theta^{\bullet}}_{a_1+}\bar{\varkappa}}(\varsigma, \bar{\varkappa}, {}^{CF}\mathbb{D}^{\theta^{\bullet}}_{a_1+}\bar{\varkappa})\Big] d\varsigma < 0.$$

The following example illustrates that, the function G is pseudo-invex, but not a invex under Caputo-Fabrizio fractional derivative.

Example 2. *: Let* $F(\varkappa) : X = [0,1] \to \mathbb{R}$ *be defined by* $F(\varkappa) = \int_0^1 \{-\varkappa(\varsigma) + 2.1838\varsigma + 9.8272e^{-\frac{\varsigma}{3}} - 7.6459)\}d\varsigma$ *and* $\varkappa(\varsigma) = \frac{1}{3}\varsigma^2 - \frac{3.6666667}{11}\varsigma + 1 \in X$, *forall* $\varsigma \in [0,1]$ *and note that* $(\varkappa \neq \bar{\varkappa})$ $\bar{\varkappa}(\varsigma) = 1 \in X$ *in such way that,*
$\int_{a_1}^{a_2} \{\aleph(\varsigma,\varkappa,\bar{\varkappa}) F_{\bar{\varkappa}}(\varsigma,\bar{\varkappa}, {}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa}) + {}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}(\aleph(\varsigma,\varkappa,\bar{\varkappa}))\, F_{{}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa}}(\varsigma,\bar{\varkappa}, {}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})\}\, d\varsigma > 0 \Rightarrow \int_{a_1}^{a_2} F(\varsigma,\varkappa, {}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\varkappa)d\varsigma > \int_{a_1}^{a_2} F(\varsigma,\bar{\varkappa}, {}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})d\varsigma$. *But,* $F = \int_0^1 \{-\varkappa(\varsigma) + 2.1838\varsigma +$

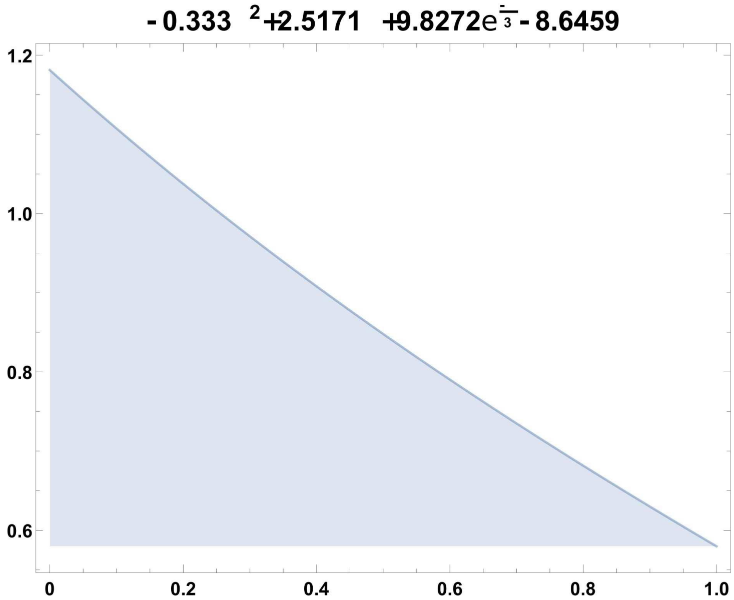

Figure 2: Graphical view of the function $F = -\varkappa(\varsigma) + 2.1838\varsigma + 9.8272e^{-\frac{\varsigma}{3}} - 7.6459$

$9.8272e^{-\frac{\varsigma}{3}} - 7.6459\}d\varsigma$ *is not invex at* $\bar{\varkappa} = 1 \in X$, *that is*

$$\int_{a_1}^{a_2} F(\varsigma,\varkappa, {}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\varkappa)d\varsigma - \int_{a_1}^{a_2} F(\varsigma,\bar{\varkappa}, {}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})d\varsigma$$
$$\ngtr \int_{a_1}^{a_2} \{\aleph(\varsigma,\varkappa,\bar{\varkappa}) F_{\bar{\varkappa}}(\varsigma,\bar{\varkappa}, {}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})$$
$$+ {}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}(\aleph(\varsigma,\varkappa,\bar{\varkappa})) F_{{}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa}}(\varsigma,\bar{\varkappa}, {}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})\}d\varsigma.$$

This example provided, on taking $\theta^\bullet = \frac{1}{4}$, $a_1 = 0$, $a_2 = 1$, $\aleph(\varsigma, \varkappa, \bar{\varkappa}) = \varkappa - \bar{\varkappa}$, $\varkappa(\varsigma) =$

$\frac{1}{3}\varsigma^2 - \frac{3.6666667}{11}\varsigma + 1$ *and deduce* ${}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\varkappa(\varsigma) = 2.1838\varsigma + 10.919e^{-\varsigma/3} - 7.6459.$

Definition 10. *The functional F is stated as strictly pseudo-invex with regards to $\aleph$ if a differentiable function $\aleph(\varsigma,\varkappa,\bar{\varkappa}) \in C^1[a_1,a_2]$ with $\aleph(\varsigma,\varkappa,\varkappa) = 0$ occurs such that $\forall\ \varkappa,\bar{\varkappa} \in X$,*

$$\int_{a_1}^{a_2}\Big[\aleph(\varsigma,\varkappa,\bar{\varkappa})F_{\bar{\varkappa}}(\varsigma,\bar{\varkappa},{}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})$$
$$+{}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\aleph(\varsigma,\varkappa,\bar{\varkappa}))F_{{}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa}}(\varsigma,\bar{\varkappa},{}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})\Big]d\varsigma \geq 0$$
$$\Rightarrow \int_{a_1}^{a_2} F(\varsigma,\varkappa,{}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\varkappa)d\varsigma > \int_{a_1}^{a_2} F(\varsigma,\bar{\varkappa},{}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})d\varsigma,$$

or, alternatively

$$\int_{a_1}^{a_2} F(\varsigma,\varkappa,{}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}F)d\varsigma \leq \int_{a_1}^{a_2} F(\varsigma,\bar{\varkappa},{}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})d\varsigma$$
$$\Rightarrow \int_{a_1}^{a_2}\Big[\aleph(\varsigma,\varkappa,\bar{\varkappa})F_{\bar{\varkappa}}(\varsigma,\bar{\varkappa},{}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})$$
$$+{}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\aleph(\varsigma,\varkappa,\bar{\varkappa}))F_{{}^{CF}\mathbb{D}^{\theta^\bullet}_{a+}\bar{\varkappa}}(\varsigma,\bar{\varkappa},{}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})\Big]d\varsigma < 0.$$

Definition 11. *The functional F is stated as quasi-invex in respect to $\aleph$ if a differentiable function $\aleph(\varsigma,\varkappa,\bar{\varkappa}) \in C^1[a_1,a_2]$ with $\aleph(\varsigma,\varkappa,\varkappa) = 0$ occurs in such way that $\forall\ \varkappa,\bar{\varkappa} \in X$,*

$$\int_{a_1}^{a_2}\Big[\aleph(\varsigma,\varkappa,\bar{\varkappa})F_{\bar{\varkappa}}(\varsigma,\bar{\varkappa},{}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})$$
$$+{}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\aleph(\varsigma,\varkappa,\bar{\varkappa}))F_{{}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa}}(\varsigma,\bar{\varkappa},{}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})\Big]d\varsigma > 0$$
$$\Rightarrow \int_{a_1}^{a_2} F(\varsigma,\varkappa,{}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\varkappa)d\varsigma > \int_{a_1}^{a_2} F(\varsigma,\bar{\varkappa},{}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})d\varsigma,$$

or equivalently,

$$\int_{a_1}^{a_2} F(\varsigma,\varkappa,{}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\varkappa)d\varsigma \leq \int_{a_1}^{a_2} F(\varsigma,\bar{\varkappa},{}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})d\varsigma$$
$$\Rightarrow \int_{a_1}^{a_2}\Big[\aleph(\varsigma,\varkappa,\bar{\varkappa})F_{\bar{\varkappa}}(\varsigma,\bar{\varkappa},{}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})$$
$$+{}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\aleph(\varsigma,\varkappa,\bar{\varkappa}))F_{{}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa}}(\varsigma,\bar{\varkappa},{}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})\Big]d\varsigma \leq 0.$$

The following example shows that G is a quasi-invex function but neither invex nor pseudo-invex.

Example 3. *: Let* $F(\varkappa) : X = [0,1] \to \mathbb{R}$ *be defined by* $F(\varkappa) = \int_0^1 \{-\varkappa(\varsigma) + 1.871822\varsigma + 6.5514e^{-\frac{\varsigma}{3}} - 5.6154\} d\varsigma$ *and* $\varkappa(\varsigma) = \frac{2}{7}\varsigma^2 - \frac{3.1428}{11}\varsigma + 1 \in X, forall\ \varsigma \in [0,1]$ *and note that* $(\varkappa \neq \bar{\varkappa})$ $\bar{\varkappa}(\varsigma) = 1 \in X$ *such that,*

$$\int_{a_1}^{a_2} \{\aleph(\varsigma,\varkappa,\bar{\varkappa}) F_{\bar{\varkappa}}(\varsigma,\bar{\varkappa},{}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})$$

$$+{}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}(\aleph(\varsigma,\varkappa,\bar{\varkappa})) F_{{}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa}}(\varsigma,\bar{\varkappa},{}^{CF}\mathbb{D}^{\gamma}_{a+}\bar{\varkappa})\} d\varsigma > 0$$

$$\Rightarrow \int_{a_1}^{a_2} F(\varsigma,\varkappa,{}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\varkappa) d\varsigma > \int_{a_1}^{a_2} F(\varsigma,\bar{\varkappa},{}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa}) d\varsigma.$$

But, F *is neither invex nor pseudo-invex, that is*

$$\int_{a_1}^{a_2} F(\varsigma,\varkappa,{}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\varkappa) d\varsigma - \int_{a_1}^{a_2} F(\varsigma,\bar{\varkappa},{}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa}) d\varsigma$$

$$\ngtr \int_{a_1}^{a_2} \{\aleph(\varsigma,\varkappa,\bar{\varkappa}) F_{\bar{\varkappa}}(\varsigma,\bar{\varkappa},{}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})$$

$$+{}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}(\aleph(\varsigma,\varkappa,\bar{\varkappa})) F_{{}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa}}(\varsigma,\bar{\varkappa},{}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})\} d\varsigma.$$

and

$$\int_{a_1}^{a_2} F(\varsigma,\varkappa,{}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\varkappa) d\varsigma \nless \int_{a_1}^{a_2} F(\varsigma,\bar{\varkappa},{}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa}) d\varsigma$$

$$\Rightarrow \int_{a_1}^{a_2} \{\aleph(\varsigma,\varkappa,\bar{\varkappa}) F_{\bar{\varkappa}}(\varsigma,\bar{\varkappa},{}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})$$

$$+{}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}(\aleph(\varsigma,\varkappa,\bar{\varkappa})) F_{{}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa}}(\varsigma,\bar{\varkappa},{}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})\} d\varsigma \nless 0.$$

In construction of this example, relevantly taken $\theta^\bullet = \frac{1}{4}$, $a_1 = 0$, $a_2 = 1$, $\aleph(\varsigma,\varkappa,\bar{\varkappa}) = \varkappa - \bar{\varkappa}$, $\varkappa(\varsigma) = \frac{2}{7}\varsigma^2 - \frac{3.1428}{11}\varsigma + 1$ *and deduce* ${}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\varkappa(\varsigma) = 1.871822\varsigma + 6.5514e^{-\frac{\varsigma}{3}} - 5.6154.$

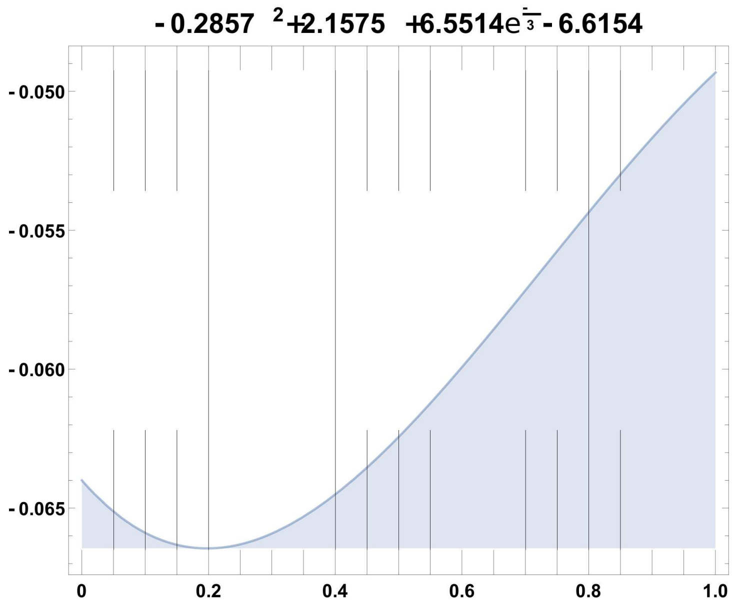

Figure 3: Graphical view of the function $F = -\varkappa(\varsigma) + 1.871822\varsigma + 6.5514e^{-\frac{\varsigma}{3}} - 5.6154$

In the above Definitions (8) - (11), ${}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\aleph(\varsigma, \varkappa, \bar{\varkappa})$ is the vector whose i^{th} component is $\frac{d^{\theta^\bullet}}{d\varsigma^{\theta^\bullet}}\aleph^i(\varsigma, \varkappa, \bar{\varkappa})$. Let $\phi(\varsigma, \varkappa, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\varkappa(\varsigma)) = \left[\phi^L(\varsigma, \varkappa,\ {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}y(\varsigma)),\quad \phi^U(\varsigma, \varkappa, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\varkappa(\varsigma))\right]$ be an interval-valued function and $h(\varsigma,\ \varkappa,\ {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}y(\varsigma))$ be a m-dimensional function having continuous derivatives up to the second order with regard to each of its parameters. Here, $\varkappa$ is a n-dimensional function of ς, and ${}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}y(\varsigma)$ is the CF fractional derivative of order $\theta^\bullet$ with respect to ς, where $0 < \theta^\bullet < 1$.

Let us Consider the following problem (P) under Caputo-Fabrizio fractional derivative:
$(P) : \min \varphi(\varkappa)_{\varkappa \in X}$

$$= \left[\int_{a_1}^{a_2} \phi^L(\varsigma, \varkappa(\varsigma), {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\varkappa(\varsigma))d\varsigma, \int_{a_1}^{a_2} \phi^U(\varsigma, y(\varsigma), {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\varkappa(\varsigma))d\varsigma\right]$$

subject to,

$$\varkappa(a_1) = \alpha, \quad \varkappa(a_2) = \beta, \tag{2}$$

$$h(\varsigma, \varkappa(\varsigma), {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\varkappa(\varsigma)) \leq 0,\ \varsigma \in \Im. \tag{3}$$

The region (feasibility region), where the restrictions are satisfied, is provided by $\Phi = \{\varkappa \in X : \varkappa(a_1) = \alpha, \varkappa(a_2) = \beta, h(\varsigma, \varkappa(\varsigma), {}^{\mathrm{CF}}D^{\theta^\bullet}_{a_1+}\varkappa(\varsigma)) \leq 0, \varsigma \in \Im = [a_1, a_2]\}$.

Definition 12. *A feasible point $\bar{\varkappa}$ is said to be a LU optimal solution of the problem (P),* *if there is no feasible point $\varkappa \in \mathbb{F}$ such that,*

$$\left[\int_{a_1}^{a_2} \phi^L(\varsigma, \varkappa(\varsigma), {}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\varkappa(\varsigma)d\varsigma, \int_{a_1}^{a_2} \phi^U(\varsigma, \varkappa(\varsigma), {}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\varkappa(\varsigma)d\varsigma\right]$$

$$\prec_{LU} \left[\int_{a_1}^{a_2} \phi^L(\varsigma, \bar{\varkappa}(\varsigma), {}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa}(\varsigma)d\varsigma, \int_{a_1}^{a_2} \phi^U(\varsigma, \bar{\varkappa}(\varsigma), {}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa}(\varsigma)d\varsigma\right].$$

For convenience we write as, $\phi(\varsigma, \bar{\varkappa}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})$ in the place of $\phi(\varsigma,\ \bar{\varkappa}(\varsigma),\ {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\ \bar{\varkappa}(\varsigma))$.

3. Optimality Conditions

The following KKT necessary optimality conditions were established in [36] for the problem(P), which will be used to demonstrate the sufficient conditions and strong duality in the subsequent sections of the paper.

Theorem 1 (Karush-Kuhn-Tucker Necessary Optimaliy Conditions)**.** *Let $\bar{\varkappa}$ be the LU optimum solution of (P) with Slater's constraint qualification satisfied at $\bar{\varkappa}$. Then, $\exists$ a function that is piecewise smooth, $\bar{\Theta} : \Im \to R^m$, such that $(\bar{\varkappa}, \bar{\Theta})$ satisfies,*

$$\begin{aligned}
&\phi^L_{\bar{\varkappa}}(\varsigma, \bar{\varkappa}, {}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa}) + \phi^U_{\bar{\varkappa}}(\varsigma, \bar{\varkappa}, {}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa}) + \bar{\Theta}(\varsigma)h_{\bar{\varkappa}}(\varsigma, \bar{\varkappa}, {}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})\\
&= -{}^{CFR}\mathbb{D}^{\theta^\bullet}_{b-}\Big[\phi^L_{{}^{CF}\mathbb{D}^{\theta^\bullet}_{a+}\bar{\varkappa}(\varsigma)}(\varsigma, \bar{\varkappa}, {}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa}) + \phi^U_{{}^{CF}\mathbb{D}^{\theta^\bullet}_{a+}\bar{\varkappa}(\varsigma)}(\varsigma, \bar{\varkappa}, {}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})\\
&\qquad + \bar{\Theta}(\varsigma)h_{{}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa}(\varsigma)}(\varsigma, \bar{\varkappa}, {}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})\Big],
\end{aligned} \tag{4}$$

$$\bar{\Theta}(\varsigma)h(\varsigma, \bar{\varkappa}, {}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa}) = 0, \bar{\Theta}(\varsigma) \geq 0. \tag{5}$$

Theorem 2 (Sufficient Optimality Conditions)**.** *Let $\bar{\varkappa} \in X$ be a feasible solution of (P) and there is a piecewise smooth function $\bar{\Theta}$: $\Im \to R^m$, $\bar{\Theta}(\varsigma) \geq 0$ such that equations (4) and (5) are satisfied at $(\bar{\varkappa}, \bar{\Theta})$. Also, assume that*

(i) the functional $\int_{a_1}^{a_2}(\phi^L + \phi^U)(\varsigma, \varkappa(\varsigma), {}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\varkappa(\varsigma))d\varsigma$ is invex at $\bar{\varkappa}$ on X,

(ii) the functional $\int_{a_1}^{a_2} \bar{\Theta}(\varsigma)h(\varsigma, \bar{\varkappa}, {}^{CF}\mathbb{D}^{\theta^\bullet}_{a+}\bar{\varkappa})d\varsigma$ is invex at $\bar{\varkappa}$ on X, then $\bar{\varkappa}$ is a LU optimal solution for (P).

Proof. If $\bar{\varkappa}$ is not a LU optimum solution for (P), then by Definition 12 another feasible solution $\varkappa$ for (P) exists, such that

$$\left[\int_{a_1}^{a_2} \phi^L(\varsigma, \varkappa, {}^{\mathrm{CF}}D^{\theta^\bullet}_{a_1+}\varkappa)d\varsigma, \int_{a_1}^{a_2} \phi^U(\varsigma, \varkappa, {}^{\mathrm{CF}}\mathbb{D}^{\gamma}_{a_1+}\varkappa)d\varsigma\right]$$

$$\prec_{LU} \left[\int_{a_1}^{a_2} \phi^L(\varsigma, \bar{\varkappa}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa}) d\varsigma, \int_{a_1}^{a_2} \phi^U(\varsigma, \bar{\varkappa}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa}) d\varsigma \right].$$

Thus, we have

$$\begin{cases} \int_{a_1}^{a_2} \phi^L(\varsigma, \varkappa, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\varkappa) d\varsigma < \int_{a_1}^{a_2} \phi^L(\varsigma, \bar{\varkappa}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa}) d\varsigma \\ \int_{a_1}^{a_2} \phi^U(\varsigma, \varkappa, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\varkappa) d\varsigma \le \int_{a_1}^{a_2} \phi^U(\varsigma, \bar{\varkappa}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa}) d\varsigma, \end{cases}$$

or

$$\begin{cases} \int_{a_1}^{a_2} \phi^L(\varsigma, \varkappa, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\varkappa) d\varsigma \le \int_{a_1}^{a_2} \phi^L(\varsigma, \bar{\varkappa}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa}) d\varsigma \\ \int_{a_1}^{a_2} \phi^U(\varsigma, \varkappa, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\varkappa) d\varsigma < \int_{a_1}^{a_2} \phi^U(\varsigma, \bar{\varkappa}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa}) d\varsigma, \end{cases}$$

or

$$\begin{cases} \int_{a_1}^{a_2} \phi^L(\varsigma, \varkappa, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\varkappa) d\varsigma < \int_{a_1}^{a_2} \phi^L(\varsigma, \bar{\varkappa}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa}) d\varsigma \\ \int_{a_1}^{a_2} \phi^U(\varsigma, \varkappa, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\varkappa) d\varsigma < \int_{a_1}^{a_2} \phi^U(\varsigma, \bar{\varkappa}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa}) d\varsigma. \end{cases}$$

From the above inequalities, we get

$$\int_{a_1}^{a_2} \left[\phi^L + \phi^U\right](\varsigma, \varkappa, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\varkappa) d\varsigma < \int_{a_1}^{a_2} \left[\phi^L + \phi^U\right](\varsigma, \bar{\varkappa}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa}) d\varsigma \tag{6}$$

By hypothesis(i), the functional $\int_{a_1}^{a_2}(\phi^L + \phi^U)(\varsigma, \varkappa, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\varkappa) d\varsigma$ is invex at $\bar{\varkappa} \in X$. Thus, from the Definition 8, we have

$$\begin{aligned} &\int_{a_1}^{a_2} F(\varsigma, \varkappa, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\varkappa) d\varsigma - \int_{a_1}^{a_2} F(\varsigma, \bar{\varkappa}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa}) d\varsigma \\ &\quad \ge (>) \int_{a_1}^{a_2} \Big[\aleph(\varsigma, \varkappa, \bar{\varkappa}) F_{\bar{\varkappa}}(\varsigma, \bar{\varkappa}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa}) \\ &\qquad + ({}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\aleph(\varsigma, \varkappa, \bar{\varkappa})) F_{{}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa}}(\varsigma, \bar{\varkappa}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa}) \Big] d\varsigma, \end{aligned}$$

which along with (6), gives

$$\int_{a_1}^{a_2} \{\aleph(\varsigma, \varkappa, \bar{\varkappa}) \left[\phi^L_{\bar{\varkappa}} + \phi^U_{\bar{\varkappa}}\right](\varsigma, \bar{\varkappa}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})$$

$$+{}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\aleph(\varsigma,\varkappa,\bar{\varkappa})\left[\phi^L_{\mathrm{CF}_{\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa}}}+\phi^U_{\mathrm{CF}_{\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa}}}\right](\varsigma,\bar{\varkappa},{}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})\}d\varsigma<0 \tag{7}$$

On the other hand, from (4) together with Proposition 1, yields

$$\int_{a_1}^{a_2}\aleph(\varsigma,\varkappa,\bar{\varkappa})[\phi^L_{\bar{\varkappa}}(\varsigma,\bar{\varkappa},{}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})+\phi^U_{\bar{\varkappa}}(\varsigma,\bar{\varkappa},{}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})$$
$$+\bar{\Theta}(\varsigma)h_{\bar{\varkappa}}(\varsigma,\bar{\varkappa},{}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})]d\varsigma$$
$$=\int_{a_1}^{a_2}\aleph(\varsigma,\varkappa,\bar{\varkappa})(-{}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{b-})[\phi^L_{\mathrm{CF}_{\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa}}}(\varsigma,\bar{\varkappa},{}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})$$
$$+\phi^U_{\mathrm{CF}_{\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa}}}(\varsigma,\bar{\varkappa},{}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})+\bar{\Theta}(\varsigma)h_{\mathrm{CF}_{\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa}}}(\varsigma,\bar{\varkappa},{}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})]d\varsigma$$

$$=\{\aleph(\varsigma,\varkappa,\bar{\varkappa})I^{1-\theta^\bullet}_{b-}[\phi^L_{\bar{\varkappa}}(\varsigma,\bar{\varkappa},{}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})+\phi^U_{\bar{\varkappa}}(\varsigma,\bar{\varkappa},{}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})$$
$$+\bar{\Theta}(\varsigma)h_{\bar{\varkappa}}(\varsigma,\bar{\varkappa},{}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})]\}|^b_a-\int_{a_1}^{a_2}\{\phi^L_{\mathrm{CF}_{\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa}}}(\varsigma,\bar{\varkappa},{}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})$$
$$+\phi^U_{\mathrm{CF}_{\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa}}}(\varsigma,\bar{\varkappa},{}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})+\bar{\Theta}(\varsigma)h_{\mathrm{CF}_{\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa}}}(\varsigma,\bar{\varkappa},{}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})\}.$$

By using (2), we get

$$\int_{a_1}^{a_2}\aleph(\varsigma,\varkappa,\bar{\varkappa})[\phi^L_{\bar{\varkappa}}(\varsigma,\bar{\varkappa},{}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})+\phi^U_{\bar{\varkappa}}(\varsigma,\bar{\varkappa},{}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})$$
$$+\bar{\Theta}(\varsigma)h_{\bar{\varkappa}}(\varsigma,\bar{\varkappa},{}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})]d\varsigma$$
$$=-\int_{a_1}^{a_2}\left\{\phi^L_{\mathrm{CF}_{\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa}}}(\varsigma,\bar{\varkappa},{}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})+\phi^U_{\mathrm{CF}_{\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa}}}(\varsigma,\bar{\varkappa},{}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})\right.$$
$$\left.+\bar{\Theta}(\varsigma)h_{\mathrm{CF}_{\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa}}}(\varsigma,\bar{\varkappa},{}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})\right\}{}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\aleph(\varsigma,\varkappa,\bar{\varkappa}))d\varsigma,$$

that is

$$\int_{a_1}^{a_2}\aleph(\varsigma,\varkappa,\bar{\varkappa})\left[\phi^L_{\bar{\varkappa}}(\varsigma,\bar{\varkappa},{}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})+\phi^U_{\bar{\varkappa}}(\varsigma,\bar{\varkappa},{}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})\right.$$
$$\left.+\bar{\Theta}(\varsigma)h_{\bar{\varkappa}}(\varsigma,\bar{\varkappa},{}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})\right]d\varsigma$$
$$+\int_{a_1}^{a_2}\left\{\phi^L_{\mathrm{CF}_{\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa}}}(\varsigma,\bar{\varkappa},{}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})+\phi^U_{\mathrm{CF}_{\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa}}}(\varsigma,\bar{\varkappa},{}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})\right.$$

$$+\bar{\Theta}(\varsigma)h_{{}^{\text{CF}}\mathbb{D}^{\theta^{\bullet}}_{a_1+}\bar{\varkappa}}(\varsigma,\bar{\varkappa},{}^{\text{CF}}\mathbb{D}^{\theta^{\bullet}}_{a_1+}\bar{\varkappa})\Bigg\}\,{}^{\text{CF}}\mathbb{D}^{\theta^{\bullet}}_{a_1+}\aleph(\varsigma,\varkappa,\bar{\varkappa})d\varsigma=0. \tag{8}$$

For the feasibility of $\varkappa$ of (P), we have $h(\varsigma,\varkappa(\varsigma),{}^{\text{CF}}D^{\theta^{\bullet}}_{a+}\varkappa(\varsigma))\leq 0$, $\varsigma\in\Im$, wherein, by utilising the fact $\bar{\Theta}(\varsigma)\in R^m,\bar{\Theta}(\varsigma)\geq 0$ and (5), we have

$$\int_{a_1}^{a_2}\bar{\Theta}(\varsigma)h(\varsigma,\varkappa,{}^{\text{CF}}\mathbb{D}^{\theta^{\bullet}}_{a_1+})d\varsigma-\int_{a_1}^{a_2}\bar{\Theta}(\varsigma)h(\varsigma,\bar{\varkappa},{}^{\text{CF}}\mathbb{D}^{\theta^{\bullet}}_{a_1+}\bar{\varkappa})d\varsigma\leq 0,$$

which along with the hypothesis(ii), invexity of $\int_{a_1}^{a_2}\bar{\Theta}(\varsigma)h(\varsigma,\bar{\varkappa},{}^{\text{CF}}\mathbb{D}^{\theta^{\bullet}}_{a+}\bar{\varkappa})d\varsigma$ at $\bar{\varkappa}\in X$, yeilds

$$0\geq\int_{a_1}^{a_2}\bar{\Theta}(\varsigma)\Big[\aleph(\varsigma,\varkappa,\bar{\varkappa})h_{\bar{\varkappa}}(\varsigma,\bar{\varkappa},{}^{\text{CF}}\mathbb{D}^{\theta^{\bullet}}_{a_1+}\bar{\varkappa})+$$

$${}^{\text{CF}}\mathbb{D}^{\theta^{\bullet}}_{a_1+}\aleph(\varsigma,\varkappa,\bar{\varkappa})h_{{}^{\text{CF}}\mathbb{D}^{\theta^{\bullet}}_{a_1+}\bar{\varkappa}}(\varsigma,\bar{\varkappa}(\varsigma),{}^{\text{CF}}\mathbb{D}^{\theta^{\bullet}}_{a_1+}\bar{\varkappa}(\varsigma))\Big]d\varsigma. \tag{9}$$

On adding (7) and (9), we get

$$\int_{a_1}^{a_2}\Big\{\aleph(\varsigma,\varkappa,\bar{\varkappa})\Big[\phi^L_{\bar{\varkappa}}(\varsigma,\bar{\varkappa},{}^{\text{CF}}D^{\theta^{\bullet}}_{a+}\bar{\varkappa})+\phi^U_{\bar{\varkappa}}(\varsigma,\bar{\varkappa},{}^{\text{CF}}D^{\theta^{\bullet}}_{a_1+}\bar{\varkappa})$$

$$+\bar{\Theta}(\varsigma)h_{\bar{\varkappa}}(\varsigma,\bar{\varkappa},{}^{\text{CF}}\mathbb{D}^{\theta^{\bullet}}_{a_1+}\bar{\varkappa})\Big]$$

$$+({}^{\text{CF}}D^{\theta^{\bullet}}_{a_1+}\aleph(\varsigma,\varkappa,\bar{\varkappa}))[\phi^L_{{}^{\text{CF}}D^{\theta^{\bullet}}_{a_1+}\bar{\varkappa}}(\varsigma,\bar{\varkappa},{}^{\text{CF}}D^{\theta^{\bullet}}_{a_1+}\bar{\varkappa})$$

$$+\phi^U_{{}^{\text{CF}}D^{\theta^{\bullet}}_{a+}\bar{\varkappa}}(\varsigma,\bar{\varkappa},{}^{\text{CF}}D^{\theta^{\bullet}}_{a_1+}\bar{\varkappa})+\bar{\Theta}(\varsigma)h_{{}^{\text{CF}}\mathbb{D}^{\theta^{\bullet}}_{a_1+}\bar{\varkappa}}(\varsigma,\bar{\varkappa},{}^{\text{CF}}\mathbb{D}^{\theta^{\bullet}}_{a_1+}\bar{\varkappa})]\Big\}d\varsigma<0,$$

which is a contradiction to (8). Hence the theorem.

We provide an algorithm for the Theorem 3.2 in the following way:

•Algorithm:

Input:

• Primal objective functional:

$$\text{(P-2):}\quad\min\left[\int_{a_1}^{a_2}\phi^L(\varsigma,\varkappa(\varsigma),{}^{\text{CF}}\mathbb{D}^{\theta^{\bullet}}_{a_1+}\varkappa(\varsigma))d\varsigma,\int_{a_1}^{a_2}\phi^U(\varsigma,\varkappa(\varsigma),{}^{\text{CF}}\mathbb{D}^{\theta^{\bullet}}_{a_1+}\varkappa(\varsigma))d\varsigma\right]$$

• Set of constraints:

$$h(\varsigma, \varkappa(\varsigma), {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\varkappa(\varsigma)) \leq 0,$$
$$\varkappa(a_1) = \alpha, \quad \varkappa(a_2) = \beta, \ \varsigma \in [a_1, \ a_2].$$

• Set of feasible point:

$$\Phi_2 = \{\varkappa \in X : h(\varsigma, \varkappa(\varsigma), {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\varkappa(\varsigma)) \leq 0, \varkappa(a_1) = \alpha, \varkappa(a_2) = \beta\}.$$

• Set of self data:

$$\phi^L(\varsigma, \varkappa(\varsigma), {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\varkappa(\varsigma)),$$
$$\phi^U(\varsigma, \varkappa(\varsigma), {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\varkappa(\varsigma)),$$
$$h(\varsigma, \varkappa(\varsigma), {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\varkappa(\varsigma)).$$

The above functions are continuous differetiable andare invexity for $\varsigma \in \Phi_2$.

output:

• Verification of invexity:

$$\phi^L(\varsigma, \varkappa(\varsigma), {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\varkappa(\varsigma)),$$
$$\phi^U(\varsigma, \varkappa(\varsigma), {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\varkappa(\varsigma)),$$
$$h(\varsigma, \varkappa(\varsigma), {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\varkappa(\varsigma)), \varsigma \in \Phi_2.$$

the point $\bar{\varkappa}$ is satisfying the Definition12.
Set of self data.

Begin:

• Selective stage:

select a point that $\bar{\varkappa} \in \Phi_2$
If slater's constraint qualification satisfied at$\bar{\varkappa}$;
then continue to the next, $if\ KKT$
necessary optimality conditions (4) and (5) holds at $\bar{\varkappa}$;

else stop;
end if;

- Screening stage:

detecting piece-wise smooth function $\bar{\Theta}$
If self data holds at $\bar{\varkappa} for\ (P-2)$;
else stop;
end if;

- Conclusive stage:

detecting the point $\bar{\varkappa} = 0,$ is optimal solution for the problem (P-2).
else stop;
end;

The following example illustrates Theorem 3.2.

Example 4. *Consider the following interval-valued variational programming problem under Caputo-Fabrizio fractional derivative:*

$$(P\text{-}2): \min \left[\int_{a_1}^{a_2} \phi^L(\varsigma, \varkappa(\varsigma), {}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\varkappa(\varsigma))d\varsigma, \int_{a_1}^{a_2} \phi^U(\varsigma, \varkappa(\varsigma), {}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\varkappa(\varsigma))d\varsigma \right]$$

subject to,
$ln2 - ln(\varsigma + 2) \leq 0,$
$\varkappa(0) = 0, \quad \varkappa(1) = 1, \ \varsigma \in [0, 1],$

where,
$$\phi^L(\varsigma, \varkappa(\varsigma), {}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\varkappa(\varsigma)) = \varsigma^3 + \varsigma,$$
$$\phi^U(\varsigma, \varkappa(\varsigma), {}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\varkappa(\varsigma)) = 3\varsigma^3 + 3.$$

Take a note that $\bar{\varkappa} = 0$ is a feasible solution of (P-2) and it can be easily observed that there is $\Theta \in R$ and $\bar{\Theta} = 0$, such that the relations (4) and (5) hold for the problem (P-2). Also, it is observed that $\int_{a_1}^{a_2} (\phi^L + \phi^U)(\varsigma, \varkappa(\varsigma), {}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\varkappa(\varsigma))d\varsigma$ is invex at $\bar{\varkappa}$ on Φ_2 and $\int_{a_1}^{a_2} \bar{\Theta}(\varsigma)h(\varsigma, \bar{\varkappa}, {}^{CF}\mathbb{D}^{\theta^\bullet}_{a+}\bar{\varkappa})d\varsigma$ is invex at $\bar{\varkappa}$ on Φ_2. Since all of the observations of Theorem 3.2 are satisfied, then $(\bar{\Theta} = 0, \bar{\varkappa} = 0)$ is the LU optimum solution to the problem (P-2).

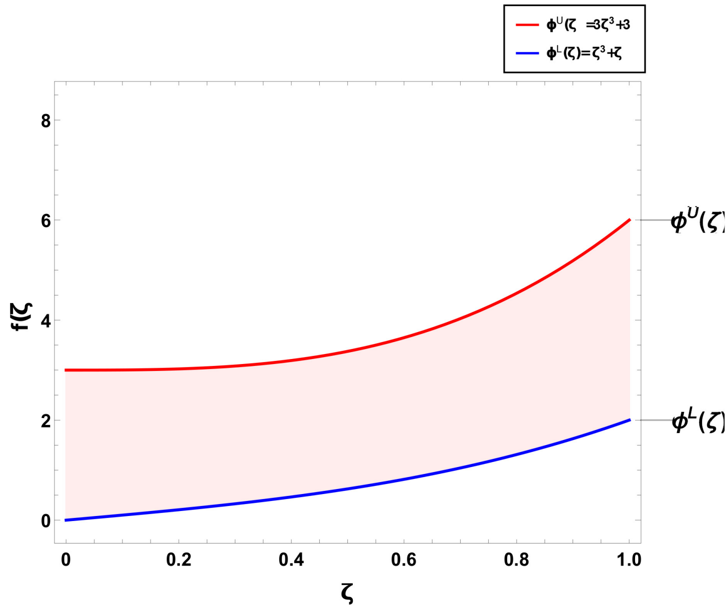

Figure 4: Graphical view of the example problem (P-2)

The problem (P-2) has a feasible region, $\Phi_2 = \{\varkappa \in X : ln2 - ln(\varsigma + 2) \leq 0, \varkappa(0) = 0, \varkappa(1) = 1\}$.

Theorem 3 (Sufficient Optimality Conditions)**.** *Let $\bar{\varkappa} \in X$ be a feasible solution of (P) and there is a function that is piecewise smooth, $\bar{\Theta}\colon \Im \to R^m$, $\bar{\Theta}(\varsigma) \geq 0$ in such way that (4) and (5) are satisfied at $(\bar{\varkappa}, \bar{\Theta})$. Also, assume that*

(i) The functional $\int_{a_1}^{a_2} (\phi^L + \phi^U)(\varsigma, \varkappa(\varsigma), {}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\varkappa(\varsigma))d\varsigma$ is pseudo-invex at $\bar{\varkappa}$ on X,

(ii) The functional $\int_{a_1}^{a_2} \bar{\Theta}(\varsigma)h(\varsigma, \bar{\varkappa}, {}^{CF}\mathbb{D}^{\theta^\bullet}_{a+}\bar{\varkappa})d\varsigma$ is quasi-invex at $\bar{\varkappa}$ on X, then $\bar{\varkappa}$ is a LU optimal solution for (P).

Proof.

If $\bar{\varkappa}$ is not a LU optimal solution for (P), then by Definition 12 there is another feasible solution $\varkappa$ for (P), such that

$$\left[\int_{a_1}^{a_2} \phi^L(\varsigma, \varkappa, {}^{\text{CF}}D^{\theta^\bullet}_{a_1+}\varkappa)d\varsigma, \int_{a_1}^{a_2} \phi^U(\varsigma, \varkappa, {}^{\text{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\varkappa)d\varsigma\right]$$

$$\prec_{LU} \left[\int_{a_1}^{a_2} \phi^L(\varsigma, \bar{\varkappa}, {}^{\text{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})d\varsigma, \int_{a_1}^{a_2} \phi^U(\varsigma, \bar{\varkappa}, {}^{\text{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})d\varsigma\right].$$

Thus, we have

$$\begin{cases} \int\limits_{a_1}^{a_2} \phi^L(\varsigma, \varkappa, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\varkappa)d\varsigma < \int\limits_{a_1}^{a_2} \phi^L(\varsigma, \bar{\varkappa}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})d\varsigma \\ \int\limits_{a_1}^{a_2} \phi^U(\varsigma, \varkappa, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\varkappa)d\varsigma \leq \int\limits_{a_1}^{a_2} \phi^U(\varsigma, \bar{\varkappa}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})d\varsigma, \end{cases}$$

or

$$\begin{cases} \int\limits_{a_1}^{a_2} \phi^L(\varsigma, \varkappa, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\varkappa)d\varsigma \leq \int\limits_{a_1}^{a_2} \phi^L(\varsigma, \bar{\varkappa}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})d\varsigma \\ \int\limits_{a_1}^{a_2} \phi^U(\varsigma, \varkappa, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\varkappa)d\varsigma < \int\limits_{a_1}^{a_2} \phi^U(\varsigma, \bar{\varkappa}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})d\varsigma, \end{cases}$$

or

$$\begin{cases} \int\limits_{a_1}^{a_2} \phi^L(\varsigma, \varkappa, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\varkappa)d\varsigma < \int\limits_{a_1}^{a_2} \phi^L(\varsigma, \bar{\varkappa}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})d\varsigma \\ \int\limits_{a_1}^{a_2} \phi^U(\varsigma, \varkappa, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\varkappa)d\varsigma < \int\limits_{a_1}^{a_2} \phi^U(\varsigma, \bar{\varkappa}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})d\varsigma. \end{cases}$$

From the above inequalities, we get

$$\int\limits_{a_1}^{a_2} \left[\phi^L + \phi^U\right](\varsigma, \varkappa(\varsigma), {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\varkappa(\varsigma))d\varsigma$$

$$< \int\limits_{a_1}^{a_2} \left[\phi^L + \phi^U\right](\varsigma, \bar{\varkappa}(\varsigma), {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa}(\varsigma))d\varsigma,$$

which, according to the hypothesis (i), there exists $\aleph(\varsigma, \varkappa, \bar{\varkappa}) \in C^1[a, b]$ such that

$$\int\limits_{a_1}^{a_2} \Big\{\aleph(\varsigma, \varkappa, \bar{\varkappa})\left[\phi^L_{\bar{\varkappa}} + \phi^U_{\bar{\varkappa}}\right](\varsigma, \bar{\varkappa}(\varsigma), {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa}(\varsigma))$$

$$+ {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\aleph(\varsigma, \varkappa, \bar{\varkappa})(\phi^L_{{}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa}} + \phi^U_{{}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa}})(\varsigma, \bar{\varkappa}(\varsigma), {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa}(\varsigma))\Big\} d\varsigma < 0. \tag{10}$$

On the other hand, from (4), we have

$$\int\limits_{a_1}^{a_2} \aleph(\varsigma, \varkappa, \bar{\varkappa})\Big[\phi^L_{\bar{\varkappa}}(\varsigma, \bar{\varkappa}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa}) + \phi^U_{\bar{\varkappa}}(\varsigma, \bar{\varkappa}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})$$

$$+ \bar{\Theta}(\varsigma)h_{\bar{\varkappa}}(\varsigma, \bar{\varkappa}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})\Big] d\varsigma$$

$$= \int_{a_1}^{a_2} \aleph(\varsigma, \varkappa, \bar{\varkappa})(-{}^{\mathrm{CF}}\mathbb{D}_{a_2-}^{\theta^\bullet})\{\phi^L_{\mathrm{CF}_{\mathbb{D}^{\theta^\bullet}_{a_1+}}\bar{\varkappa}(\varsigma)}(\varsigma, \bar{\varkappa}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})$$

$$+\phi^U_{\mathrm{CF}_{\mathbb{D}^{\theta^\bullet}_{a_1+}}\bar{\varkappa}(\varsigma)}(\varsigma, \bar{\varkappa}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa}) + \bar{\Theta}(\varsigma)h_{\mathrm{CF}_{\mathbb{D}^{\theta^\bullet}_{a_1+}}\bar{\varkappa}(\varsigma)}(\varsigma, \bar{\varkappa}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})\}d\varsigma$$

$$= \{\aleph(\varsigma, \varkappa, \bar{\varkappa})I^{1-\theta^\bullet}_{b-}\left[\phi^L_{\mathrm{CF}_{\mathbb{D}^{\theta^\bullet}_{a_1+}}\bar{\varkappa}}(\varsigma, \bar{\varkappa}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa}) + \phi^U_{\mathrm{CF}_{\mathbb{D}^{\theta^\bullet}_{a_1+}}\bar{\varkappa}}(\varsigma, \bar{\varkappa}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})\right.$$

$$\left.+\bar{\Theta}(\varsigma)h_{\bar{\varkappa}}(\varsigma, \bar{\varkappa}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})\right]\}\Big|_{a_1}^{a_2}$$

$$-\int_{a_1}^{a_2}\left\{\phi^L_{\mathrm{CF}_{\mathbb{D}^{\theta^\bullet}_{a_1+}}\bar{\varkappa}}(\varsigma, \bar{\varkappa}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa}) + \phi^U_{\mathrm{CF}_{\mathbb{D}^{\theta^\bullet}_{a_1+}}\bar{\varkappa}}(\varsigma, \bar{\varkappa}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})\right.$$

$$\left.+\bar{\Theta}(\varsigma)h_{\mathrm{CF}_{\mathbb{D}^{\theta^\bullet}_{a_1+}}\bar{\varkappa}}(\varsigma, \bar{\varkappa}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})\right\}{}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\aleph(\varsigma, \varkappa, \bar{\varkappa})d\varsigma.$$

(by Proposition 1)

By using (2), we get

$$\int_{a_1}^{a_2}\aleph(\varsigma, \varkappa, \bar{\varkappa})[\phi^L_{\bar{\varkappa}}(\varsigma, \bar{\varkappa}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa}) + \phi^U_{\bar{\varkappa}}(\varsigma, \bar{\varkappa}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})$$

$$+\bar{\Theta}(\varsigma)h_{\bar{\varkappa}}(\varsigma, \bar{\varkappa}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})]d\varsigma$$

$$= -\int_{a_1}^{a_2}\left\{\phi^L_{\mathrm{CF}_{\mathbb{D}^{\theta^\bullet}_{a_1+}}\bar{\varkappa}}(\varsigma, \bar{\varkappa}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa}) + \phi^U_{\mathrm{CF}_{\mathbb{D}^{\theta^\bullet}_{a_1+}}\bar{\varkappa}}(\varsigma, \bar{\varkappa}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})\right.$$

$$\left.+\bar{\Theta}(\varsigma)h_{\mathrm{CF}_{\mathbb{D}^{\theta^\bullet}_{a_1+}}\bar{\varkappa}}(\varsigma, \bar{\varkappa}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})\right\}{}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\aleph(\varsigma, \varkappa, \bar{\varkappa}))d\varsigma,$$

that is

$$\int_{a_1}^{a_2}\aleph(\varsigma, \varkappa, \bar{\varkappa})\left[\phi^L_{\bar{\varkappa}}(\varsigma, \bar{\varkappa}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa}) + \phi^U_{\bar{\varkappa}}(\varsigma, \bar{\varkappa}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})\right.$$

$$\left.+\bar{\Theta}(\varsigma)h_{\bar{\varkappa}}(\varsigma, \bar{\varkappa}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})\right]d\varsigma$$

$$+\int_{a_1}^{a_2}\left\{\phi^L_{\mathrm{CF}_{\mathbb{D}^{\theta^\bullet}_{a_1+}}\bar{\varkappa}}(\varsigma, \bar{\varkappa}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa}) + \phi^U_{\mathrm{CF}_{\mathbb{D}^{\varsigma}_{a+}}\bar{\varkappa}}(\varsigma, \bar{\varkappa}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})\right.$$

$$\left.+\bar{\Theta}(\varkappa)h_{\mathrm{CF}_{\mathbb{D}^{\theta^\bullet}_{a_1+}}\bar{\varkappa}}(\varsigma, \bar{\varkappa}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})\right\}{}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\aleph(\varsigma, \varkappa, \bar{\varkappa})d\xi = 0. \tag{11}$$

In order to determine the feasibility of $\varkappa$ in the problem (P), we have $h(\varsigma, \varkappa(\varsigma), {}^{\mathrm{CF}}D^{\theta^\bullet}_{a+}\varkappa(\varsigma)) \leq 0$, $\varsigma \in \Im$, which by using the fact $\bar{\Theta}(\varsigma) \in R^m, \bar{\Theta}(\varsigma) \geq 0$ and (5) we have

$$\int_{a_1}^{a_2} \bar{\Theta}(\varsigma) h(\varsigma, \varkappa, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}) d\varsigma \leq \int_{a_1}^{a_2} \bar{\Theta}(\varsigma) h(\varsigma, \bar{\varsigma}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa}) d\varsigma.$$

On the basis of hypothesis (ii), there exists $\aleph(\varsigma, \varkappa, \bar{\varkappa}) \in C^1[a, b]$ such that

$$\int_{a_1}^{a_2} \bar{\Theta}(\varsigma) \Big[\aleph(\varsigma, \varkappa, \bar{\varkappa}) h_{\bar{\varkappa}}(\varsigma, \bar{\varkappa}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa}) + {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\aleph(\varsigma, \varkappa, \bar{\varkappa}) h_{{}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa}}(\varsigma, \bar{\varkappa}(\varsigma), {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa}(\varsigma)) \Big] d\varsigma \leq 0. \tag{12}$$

On adding (10) and (12), we get

$$\begin{aligned} \int_{a_1}^{a_2} \Big\{ \aleph(\varsigma, \varkappa, \bar{\varkappa}) \Big[\phi^L_{\bar{\varkappa}}(\varsigma, \bar{\varkappa}, {}^{\mathrm{CF}}D^{\theta^\bullet}_{a_1+}\bar{\varkappa}) + \phi^U_{\bar{\varkappa}}(\varsigma, \bar{\varkappa}, {}^{\mathrm{CF}}D^{\theta^\bullet}_{a_1+}\bar{\varkappa}) \\ + \bar{\Theta}(\varsigma) h_{\bar{\varkappa}}(\varsigma, \bar{\varkappa}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa}) \Big] \\ + ({}^{\mathrm{CF}}D^{\theta^\bullet}_{a_1+}\aleph(\varsigma, \varkappa, \bar{\varkappa}))[\phi^L_{{}^{\mathrm{CF}}D^{\theta^\bullet}_{a_1+}\bar{\varkappa}}(\varsigma, \bar{\varkappa}, {}^{\mathrm{CF}}D^{\theta^\bullet}_{a_1+}\bar{\varkappa}) + \phi^U_{{}^{\mathrm{CF}}D^{\theta^\bullet}_{a+}\bar{\varkappa}}(\varsigma, \bar{\varkappa}, {}^{\mathrm{CF}}D^{\theta^\bullet}_{a+}\bar{\varkappa}) + \\ \bar{\Theta}(\varsigma) h_{{}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa}}(\varsigma, \bar{\varkappa}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})] \Big\} d\varsigma < 0, \end{aligned}$$

which is a contradiction to (11). Hence the theorem.

The following example illustrates Theorem 3.3.

Example 5. *Consider the following problem (P-3):*

$$(P\text{-}3) = \min \left[\int_{a_1}^{a_2} \phi^L(\varsigma, y(\varsigma), {}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\varkappa(\varsigma)) d\varsigma, \int_{a_1}^{a_2} \phi^U(\varsigma, \varkappa(\varsigma), {}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\varkappa(\varsigma)) d\varsigma \right]$$

subject to,

$$\begin{aligned} &- y(\varsigma) + 1.871822\varsigma + 6.5514 e^{-\frac{\varsigma}{3}} - 5.6154 \leq 0, \\ &\varkappa(0) = 1, \quad \varkappa(1) = 1, \ \varsigma \in [0, 1], \end{aligned}$$

where,

$$\phi^L(\varsigma, \varkappa(\varsigma), {}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\varkappa(\varsigma)) = -\varkappa(\varsigma) + 6.5514\varsigma - 16.37859 e^{-\frac{\varsigma}{3}} + 23.9299,$$

$$\phi^U(\varsigma, \varkappa(\varsigma), {}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\varkappa(\varsigma)) = -\varkappa^2(\varsigma) + 6.5514\varsigma - 16.3785e^{-\dfrac{\varsigma}{3}} + 22.9299),$$

and

$$\varkappa(\varsigma) = -\varsigma^2 + \varsigma + 1 \in X.$$

The feasible region of (P-3) is $\Phi_3 = \{\varkappa \in X : -\varkappa(\varsigma) + 1.871822\varsigma + 6.5514e^{-\dfrac{\varsigma}{3}} - 5.6154) \leq 0, \varkappa(0) = 1, \varkappa(1) = 1\}$. *Note that* $\bar{\varkappa} = 1$ *is a feasible solution of (P-3) and it can be easily observe that there is* $\Theta \in R$ *and* $\bar{\Theta} = 0$, *such that the relations (4) and (5) holds for (P-3). Also it is observed that* $\int_{a_1}^{a_2}(\phi^L + \phi^U)(\varsigma, \varkappa(\varsigma), {}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\varkappa(\varsigma))d\varsigma$ *is pseudo-convex at* $\bar{\varkappa}$ *on* Φ_3 *and* $\int_{a_1}^{a_2} \bar{\Theta}(\varsigma)h(\varsigma, \bar{\varkappa}, {}^{CF}\mathbb{D}^{\theta^\bullet}_{a+}\bar{\varkappa})d\varsigma$ *is quasi-convex at* $\bar{\varkappa}$ *on* Φ_3. *Since all the observations of Theorem 3 are satisfied, then* $(\bar{\Theta} = 0, \bar{\varkappa} = 1)$ *is a LU optimal solution (P-3).*

4. Wolfe-Type Dual Model

We are concerned in this part with the Wolfe-type dual problem (WD) with the CF derivative operator in relation to the primary problem (P), it is as follows:

$$\text{(WD)} \qquad \max G(\varepsilon, \bar{\Theta}) = \int_{a_1}^{a_2} \Bigg[\Big[\phi^L(\varsigma, \varepsilon, {}^{\text{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\varepsilon), \phi^U(\varsigma, \varepsilon, {}^{\text{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\varepsilon)\Big] + (\bar{\Theta})^T h(\varsigma, \varepsilon, {}^{\text{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\varepsilon(\varsigma))\Bigg] d\varsigma,$$

subject to,

$$\varepsilon(a_1) = \alpha, \varepsilon(a_2) = \beta, \tag{13}$$

$$\begin{aligned}&\phi^L_\varepsilon(\varsigma, \varepsilon, {}^{\text{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\varepsilon) + \phi^U_\varepsilon(\varsigma, \varepsilon, {}^{\text{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\varepsilon) + (\bar{\Theta})^T(\varsigma)h_\varepsilon(\varsigma, \varepsilon, {}^{\text{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\varepsilon(\varsigma))\\ &= -{}^{\text{CF}}\mathbb{D}^{\theta^\bullet}_{a_2-}\Bigg\{\phi^L_{{}^{\text{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\varepsilon(\varsigma)}(\varsigma, \varepsilon, {}^{\text{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\varepsilon) + \phi^U_{{}^{\text{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\varepsilon(\varsigma)}(\varsigma, \varepsilon, {}^{\text{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\varepsilon)\\ &\qquad + (\bar{\Theta})^T(\varsigma)h_{{}^{\text{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\varepsilon(\varsigma)}(\varsigma, \varepsilon, {}^{\text{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\varepsilon(\varsigma))\Bigg\},\end{aligned} \tag{14}$$

$$\int_{a_1}^{a_2}(\bar{\Theta})^T(\varsigma)h(\varsigma, \varepsilon(\varsigma), {}^{\text{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\varepsilon(\varsigma))d\varsigma \leq 0, \tag{15}$$

$$\bar{\Theta}(\xi) \geq 0, \varsigma \in \Im. \tag{16}$$

Here, $\varepsilon(\varsigma)$ signifies an n-dimensional function, $\bar{\Theta}(\varsigma)$ denotes an m-dimensional function. Let $\Theta(\bar{\varepsilon}) = \{(\bar{\Theta}, \bar{\varepsilon}) : \bar{\Theta} \in \mathbb{R}^m, \bar{\varepsilon} \in X$: satisfying the constraints of (WD), forall $\varsigma \in \Im\}$ be the collection of all feasible points to (WD).

Definition 13. *A feasible point* $(\bar{\varepsilon}, \bar{\Theta})$ *is stated to be an LU optimal point of a maximum type for (WD), if there is no feasible point* (ε, Θ)*, such that*

$$\left[\int_{a_1}^{a_2} \phi^L(\varsigma, \bar{\varepsilon}, {}^{CF}D^{\theta^\bullet}_{a+}\bar{\varepsilon})d\varsigma, \int_{a_1}^{a_2} \phi^U(\varsigma, \bar{\varepsilon}, {}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varepsilon})d\theta^\bullet\right] + \int_{a_1}^{a_2} (\bar{\Theta})^T h(\varsigma, \bar{\varepsilon}, {}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varepsilon})d\varsigma$$

$$\prec_{LU} \left[\int_{a_1}^{a_2} \phi^L(\varsigma, \varepsilon, {}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\varepsilon)d\varsigma, \int_{a_1}^{a_2} \phi^U(\varsigma, \nu, {}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\varepsilon)d\varsigma\right]$$

$$+ \int_{a_1}^{a_2} (\bar{\Theta})^T h(\varsigma, \varepsilon, {}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\varepsilon)d\varsigma.$$

The weak, strong, and strict converse duality theorems are studied for (WD) from the standpoint of the CF fractional derivative operator:

Theorem 4 (Weak Duality)**.** *Let* $(\bar{\Theta}, \bar{\varkappa})$ *and* $(\bar{\Theta}, \bar{\varepsilon})$ *be the feasible points for (P) and (WD), respectively. Suppose that,* $(\bar{\Theta})^T > 0$ *and the functional* $(\phi^L + \phi^U + (\bar{\Theta})^T h)(\varsigma, \bar{\varepsilon}, {}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varepsilon})$ *is invex at* $\bar{\varepsilon}$ *on* X*, then the following can't hold*

$$\left[\int_{a_1}^{a_2} \phi^L(\varsigma, \bar{\varkappa}, {}^{CF}D^{\theta^\bullet}_{a_1+}\bar{\varkappa})d\varsigma, \int_{a_1}^{a_2} \phi^U(\varsigma, \bar{\varkappa}, {}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})d\varsigma\right]$$

$$\prec_{LU} \left[\int_{a_1}^{a_2} (\phi^L + (\bar{\Theta})^T h)(\varsigma, \bar{\varepsilon}, {}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varepsilon})d\varsigma, \int_{a_1}^{a_2} (\phi^U + (\bar{\Theta})^T h)(\varsigma, \bar{\varepsilon}, {}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varepsilon})d\varsigma\right].$$

Proof. Assume, contrary to the outcome,

$$\left[\int_{a_1}^{a_2} \phi^L(\varsigma, \bar{\varkappa}, {}^{\mathrm{CF}}D^{\theta^\bullet}_{a_1+}\bar{\varkappa})d\varsigma, \int_{a_1}^{a_2} \phi^U(\varsigma, \bar{\varkappa}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})d\varsigma\right]$$

$$\prec_{LU} \left[\int_{a_1}^{a_2} (\phi^L + (\bar{\Theta})^T h)(\varsigma, \bar{\varepsilon}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varepsilon})d\varsigma, \int_{a_1}^{a_2} (\phi^U + (\bar{\Theta})^T h)(\varsigma, \bar{\varepsilon}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varepsilon})d\varsigma\right].$$

From (P), $h(\varsigma, \varkappa(\varsigma), {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\varkappa(\varsigma)) \leq 0$ and h is continuously differential function. Moreover $(\bar{\Theta})^T \geq 0$. Then, it follows that

$$\begin{cases} \int_{a_1}^{a_2}(\phi^L + (\bar{\Theta})^T h)(\varsigma, \bar{\varkappa}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})d\varsigma < \int_{a_1}^{a_2}(\phi^L + (\bar{\Theta})^T h)(\varsigma, \bar{\varepsilon}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varepsilon})d\varsigma \\ \int_{a_1}^{a_2}(\phi^U + (\bar{\Theta})^T h)(\varsigma, \bar{\varkappa}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})d\varsigma \leq \int_{a_1}^{a_2}(\phi^U + (\bar{\Theta})^T h)(\varsigma, \bar{\varepsilon}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varepsilon})d\varsigma, \end{cases}$$

or

$$\begin{cases} \int_{a_1}^{a_2}(\phi^L + (\bar{\Theta})^T h)(\varsigma, \bar{\varkappa}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})d\varsigma \leq \int_{a_1}^{a_2}(\phi^L + (\bar{\Theta})^T h)(\varsigma, \bar{\varepsilon}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\nu})d\varsigma \\ \int_{a_1}^{a_2}(\phi^U + (\bar{\Theta})^T h)(\varsigma, \bar{\varkappa}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})d\varsigma < \int_{a_1}^{a_2}(\phi^U + (\bar{\Theta})^T h)(\varsigma, \bar{\varepsilon}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varepsilon})d\varsigma, \end{cases}$$

or

$$\begin{cases} \int_{a_1}^{a_2}(\phi^L + (\bar{\Theta})^T h)(\varsigma, \bar{\varkappa}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})d\varsigma < \int_{a_1}^{a_2}(\phi^L + (\bar{\Theta})^T h)(\varsigma, \bar{\varepsilon}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varepsilon})d\varsigma \\ \int_{a_1}^{a_2}(\phi^U + (\bar{\Theta})^T h)(\varsigma, \bar{\varkappa}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})d\varsigma < \int_{a_1}^{a_2}(\phi^U + (\bar{\Theta})^T h)(\varsigma, \bar{\varepsilon}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varepsilon})d\varsigma. \end{cases}$$

From the above inequalities, we get

$$\int_{a_1}^{a_2}(\phi^L + \phi^U + (\bar{\Theta})^T h))(\varsigma, \bar{\varkappa}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})d\varsigma$$

$$-\int_{a_1}^{a_2}(\phi^L + \phi^U + (\bar{\Theta})^T h)(\varsigma, \bar{\varepsilon}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varepsilon})d\varsigma < 0,$$

Hence, taking the invexity assumption on $\int_{a_1}^{a_2}(\phi^L + \phi^U + (\bar{\Theta})^T h))(\varsigma, ., .)d\varsigma$ at $\bar{\varepsilon}$ on X, there exists $\aleph(\varsigma, \bar{\varkappa}, \bar{\varepsilon}) \in C^1[a_1, a_2]$ such that

$$\begin{aligned} \int_{a_1}^{a_2} & \{\aleph(\varsigma, \bar{\varkappa}, \bar{\varepsilon})^T \left[\phi^L_{\bar{\varepsilon}} + \phi^U_{\bar{\varepsilon}} + (\bar{\Theta})^T h\right](\varsigma, \bar{\varepsilon}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varepsilon}) \\ & + {}^{\mathrm{CF}}\mathbb{D}^{\gamma}_{a_1+}\aleph(\varsigma, \bar{\varkappa}, \bar{\varepsilon})^T \Big[\phi^L_{{}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varepsilon}} + \phi^U_{{}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varepsilon}} \\ & + \aleph(\varsigma, \bar{\varkappa}, \bar{\varepsilon})^T h_{{}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varepsilon}}\Big](\varsigma, \bar{\varepsilon}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varepsilon})\}d\varsigma < 0. \end{aligned} \tag{17}$$

Further, from the dual constraint (14) and the Proposition 2.1, we get

$$\begin{aligned} \int_{a_1}^{a_2} & \Big\{\aleph(\varsigma, \bar{\varkappa}, \bar{\varepsilon})^T \Big[\phi^L_{\bar{\varepsilon}}(\varsigma, \bar{\varepsilon}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varepsilon}) + \phi^U_{\bar{\varepsilon}}(\varsigma, \bar{\varepsilon}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varepsilon}) \\ & + (\bar{\Theta})^T(\varsigma) h_{\bar{\varepsilon}}(\varsigma, \bar{\varepsilon}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varepsilon})\Big]\Big\} d\varsigma \\ = \int_{a_1}^{a_2} & \aleph(\varsigma, \bar{\varkappa}, \bar{\varepsilon})^T \Big[- {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1-}\Big\{\phi^L_{{}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varepsilon}}(\varsigma, \bar{\varepsilon}, {}^{\mathrm{CF}}\mathbb{D}^{\varsigma}_{a_1+}\bar{\varepsilon}) \end{aligned} \tag{18}$$

$$+ \phi^{U}_{\mathrm{CF}_{\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varepsilon}}}(\varsigma, \bar{\varepsilon}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varepsilon})\Big\} + (\bar{\Theta})^T(\varsigma) h_{\mathrm{CF}_{\mathbb{D}^{\gamma}_{a+}\bar{\varepsilon}}}(\varsigma, \bar{\varepsilon}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varepsilon})\Big] d\varsigma$$

$$= \Big[\aleph(\varsigma, \bar{\varkappa}, \bar{\varepsilon})^T I^{1-\theta^\bullet}_{b-} \Big\{\phi^{L}_{\mathrm{CF}_{\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varepsilon}}} + \phi^{U}_{\mathrm{CF}_{\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varepsilon}}}\Big\}(\varsigma, \bar{\varepsilon}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varepsilon})$$

$$+ (\bar{\Theta})^T(\varsigma) h_{\mathrm{CF}_{\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varepsilon}}}(\varsigma, \bar{\varepsilon}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varepsilon})\Big]_a^b$$

$$- \int_{a_1}^{a_2} \aleph(\varsigma, \bar{\varkappa}, \bar{\varepsilon})^T \Big[{}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+} \Big\{\phi^{L}_{\mathrm{CF}_{\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varepsilon}}} + +\phi^{U}_{\mathrm{CF}_{\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varepsilon}}}\Big\}(\varsigma, \bar{\varepsilon}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varepsilon})$$

$$+ (\bar{\Theta})^T(\varsigma) h_{\mathrm{CF}_{\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varepsilon}}}(\varsigma, \bar{\varepsilon}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varepsilon})\Big] d\varsigma.$$

By using (13), it gives

$$\int_{a_1}^{a_2} \Big[\aleph(\varsigma, \bar{\varkappa}, \bar{\varepsilon})^T \Big\{\phi^{L}_{\bar{\varepsilon}}(\varsigma, \bar{\varepsilon}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varepsilon}) + \phi^{U}_{\bar{\varepsilon}}(\varsigma, \bar{\varepsilon}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varepsilon})$$

$$+(\bar{\Theta})^T(\varsigma) h_{\bar{\varepsilon}}(\varsigma, \bar{\varepsilon}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varepsilon})\Big\}\Big] d\varsigma$$

$$= -\int_{a_1}^{a_2} \Big[{}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}(\varepsilon - \bar{\varepsilon}) \Big\{\phi^{L}_{\mathrm{CF}_{\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varepsilon}}} + \phi^{U}_{\mathrm{CF}_{\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varepsilon}}}\Big\}(\varsigma, \bar{\varepsilon}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varepsilon})$$

$$+(\bar{\Theta})^T(\varsigma) h_{\mathrm{CF}_{\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varepsilon}}}(\xi, \bar{\varepsilon}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varepsilon})\Big] d\varsigma.$$

That is

$$\int_{a_1}^{a_2} \aleph(\varsigma, \bar{\varkappa}, \bar{\varepsilon})^T \Big\{\phi^{L}_{\bar{\varepsilon}}(\varsigma, \bar{\varepsilon}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varepsilon}) + \phi^{U}_{\bar{\varepsilon}}(\varsigma, \bar{\varepsilon}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varepsilon})$$

$$+(\bar{\Theta})^T(\varsigma) h_{\bar{\varepsilon}}(\varsigma, \bar{\varepsilon}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varepsilon})\Big\} d\varsigma$$

$$+ \int_{a_1}^{a_2} {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}(\varepsilon - \bar{\varepsilon})^T \Big\{\phi^{L}_{\mathrm{CF}_{\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varepsilon}}}(\varsigma, \bar{\varepsilon}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varepsilon}) + \phi^{U}_{\mathrm{CF}_{\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varepsilon}}}(\varsigma, \bar{\varepsilon}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varepsilon})$$

$$+(\bar{\Theta})^T(\varsigma) h_{\mathrm{CF}_{\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varepsilon}}}(\varsigma, \bar{\varepsilon}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varepsilon})\Big\} d\varsigma = 0, \tag{19}$$

which contradicts (17). Hence the theorem.

We provide an algorithm for the weak duality theorem in the following way:

Algorithm of weak duality

Input:

- Primal objective functional:

$$\text{(P-4):}\quad \min\left[\int_{a_1}^{a_2}\phi^L(\varsigma,\varkappa(\varsigma),{}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\varkappa(\varsigma))d\varsigma,\int_{a_1}^{a_2}\phi^U(\varsigma,\varkappa(\varsigma),{}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\varkappa(\varsigma))d\varsigma\right]$$

- Set of constraints:

$$\begin{aligned}&h(\varsigma,\varkappa(\varsigma),{}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\varkappa(\varsigma))\leq 0,\\&\varkappa(a_1)=\alpha,\quad \varkappa(a_2)=\beta,\ \varsigma\in[a_1,a_2].\end{aligned}$$

- Set of feasible point for (P-4):

$$\Phi_3=\{h(\varsigma,\varkappa(\varsigma),{}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\varkappa(\varsigma))\leq 0,\varkappa(a_1)=\alpha,\varkappa(a_2)=\beta\}.$$

- Wolfe dual objective functional corresponding to the primal (P-4):

$$\begin{aligned}\text{(WD-1)}=\max\int_{a_1}^{a_2}\Bigg[&\Big[\phi^L(\varsigma,\varepsilon,{}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\varepsilon),\ \phi^U(\varsigma,\varepsilon,{}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\varepsilon)\Big]\\&+(\bar{\Theta})^Th(\varsigma,\varepsilon,{}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\varepsilon(\varsigma))\Bigg]d\varsigma,\end{aligned}$$

- Set of constraints for (WD-1):

$$\begin{aligned}&\phi^L_\varepsilon(\varsigma,\varepsilon,{}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\varepsilon)+\phi^U_\varepsilon(\varsigma,\varepsilon,{}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\varepsilon)+(\bar{\Theta})^T(\varsigma)h_\varepsilon(\varsigma,\varepsilon,{}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\varepsilon(\varsigma))\\&=-{}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_2-}\Big\{\phi^L_{{}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\varepsilon(\varsigma)}(\varsigma,\varepsilon,{}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\varepsilon)+\phi^U_{{}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\varepsilon(\varsigma)}(\varsigma,\varepsilon,{}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\varepsilon)\\&\qquad+(\bar{\Theta})^T(\varsigma)h_{{}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\varepsilon(\varsigma)}(\varsigma,\varepsilon,{}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\varepsilon(\varsigma))\Big\},\\&\int_{a_1}^{a_2}(\bar{\Theta})^T(\varsigma)h(\varsigma,\varepsilon(\varsigma),{}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\varepsilon(\varsigma))d\varsigma\leq 0,\\&\bar{\Theta}(\xi)\geq 0,\varsigma\in\Im,\text{where } a_1=0,a_2=1.\end{aligned}$$

- Set of feasible point for (WD-1):

$$\mathbb{W}_3 = \{(\bar{\Theta}, \bar{\varepsilon}) : \bar{\Theta} \in \mathbb{R}^m, \varepsilon \in X : \text{satisfying constraints of (WD-1), forall } \varsigma \in \Im\}$$

- Set of self data:

where,
$\phi^L \varsigma, \varkappa(\varsigma), {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\varkappa(\varsigma)),$
$\phi^U(\varsigma, \varkappa(\varsigma), {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\varkappa(\varsigma)),$
$h(\varsigma, \varkappa(\varsigma), {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\varkappa(\varsigma))$
and
$\varkappa(\varsigma)$are continuously differentiable functions and are invexity for. $\varsigma \in \Phi_3$.

output:

- Verification of invexity:

$$\begin{aligned}
&\phi^L(\varsigma, \varkappa(\varsigma), {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\varkappa(\varsigma))\\
&\phi^U(\varsigma, \varkappa(\varsigma), {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\varkappa(\varsigma)\\
&\phi^L(\varsigma, \varepsilon(\varsigma), {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\varepsilon(\varsigma)) + (\bar{\Theta})^T(\varsigma)h(\varsigma, \varepsilon(\varsigma),\\
&\phi^U(\varsigma, \varepsilon(\varsigma), {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\varepsilon(\varsigma)) + (\bar{\Theta})^T(\varsigma)h(\varsigma, \varepsilon(\varsigma),\\
&h(\varsigma, \varkappa(\varsigma), {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\varkappa(\varsigma)), \varsigma \in \Phi_3, \mathbb{W}_3.
\end{aligned}$$

the point $\bar{\varkappa}$, the point $\bar{\varepsilon}$ is satisfying the Definition12.
Set of self data.

Begin:
Main Optimization Loop:
Primary problem solution :

- while not converged:
- Compute Caputo-Fabrizio fractional derivative:
  ${}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\varkappa(\varsigma)$
- Evaluate objective functions

$$J^L = \int_{a_1}^{a_2} \phi^L(\varsigma, \varkappa(\varsigma), {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\varkappa(\varsigma))d\varsigma,$$
$$J^U = \int_{a_1}^{a_2} \phi^U(\varsigma, \varkappa(\varsigma), {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\varkappa(\varsigma))d\varsigma$$

- Check constraints

$h(\varsigma, \varkappa(\varsigma), {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\varkappa(\varsigma)) \leq 0,$
$\forall\ \varsigma \in [a_1,\ a_2]$
and

- Boundary conditions are satisfied:
- Solution is feasible
- Store current solution
- else:
- Apply constraint projection
- Update solution using gradient descent or other optimization

$\bar{\varkappa}_{k+1} = \text{Update solution}(\bar{\varkappa}_k, J^L, J^U, Constraints)$

- Check convergence criteria

Wolfe-Type Dual problem (WD-1) solution:

- while not converged:
- Compute adjoint system Solve the Euler-Lagrange type equation:

$$\phi^L_\varepsilon(\varsigma, \varepsilon, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\varepsilon) + \phi^U_\varepsilon(\varsigma, \varepsilon, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\varepsilon) + (\bar{\Theta})^T(\varsigma) h_\varepsilon(\varsigma, \varepsilon, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\varepsilon(\varsigma))$$
$$+ {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_2-}\Big\{\phi^L_{{}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\varepsilon(\varsigma)}(\varsigma, \varepsilon, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\varepsilon) + \phi^U_{{}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\varepsilon(\varsigma)}(\varsigma, \varepsilon, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\varepsilon)$$
$$+(\bar{\Theta})^T(\varsigma) h_{{}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\varepsilon(\varsigma)}(\varsigma, \varepsilon, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\varepsilon(\varsigma))\Big\} = 0.$$

- Check dual constraints

$$\int_{a_1}^{a_2} (\bar{\Theta})^T(\varsigma) h(\varsigma, \varepsilon(\varsigma), {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\varepsilon(\varsigma))d\varsigma \leq 0,$$

- and boundary conditions

$\bar{\Theta}(\xi) \geq 0, \varsigma \in \Im = [a_1, a_2].$

Solution is feasible

Store current dual solution

else;

- Apply constraint projection
- Update solution using gradient descent or other optimization

$(\bar{\Theta}_{k+1}, \bar{\varepsilon}_{k+1})$ = Update Dual variables$(\bar{\Theta}_k, \bar{\varepsilon}_k)$

- Check convergence criteria

Verification and Duality Check:

- Duality Verification:

For feasible solutions $(\bar{\Theta}_{k+1},\ \bar{\varsigma}_{k+1})$,

for the primal problem (P-4) and$(\bar{\Theta}_{k+1},\ \bar{\varepsilon}_{k+1})$for dual problem (WD-1)

Verify that the interval objective values are not comparable under the $\preceq_{LU}$

else stop;

end;

We provide an example for the weak duality theorem in the following way:

Example 6. *Let us consider the following interval-valued variational programming problem under Caputo-Fabrizio fractional derivative:*

$$(P\text{-}4):\ \min\left[\int_{a_1}^{a_2}\phi^L(\varsigma,\varkappa(\varsigma),{}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\varkappa(\varsigma))d\varsigma,\int_{a_1}^{a_2}\phi^U(\varsigma,\varkappa(\varsigma),{}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\varkappa(\varsigma))d\varsigma\right]$$

subject to,

$$-2\varkappa^2(\varsigma)-4\varkappa(\varsigma)+3\frac{\sqrt{(\pi)}+1}{\sqrt{(\pi)}}(1-e^{-\varsigma})\leq 0,$$

$$\varkappa(0)=1,\quad \varkappa(1)=1,\ \varsigma\in[0,1],$$

where,

$$\phi^L(\varsigma,\varkappa(\varsigma),{}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\varkappa(\varsigma))=\varkappa(\varsigma)+\frac{1}{2}\frac{\sqrt{(\pi)}+1}{\sqrt{(\pi)}}(1-e^{-\varsigma}),$$

$$\phi^U(\varsigma,\varkappa(\varsigma),{}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\varkappa(\varsigma))=\frac{1}{3}\varkappa^2(\varsigma)+3\frac{\sqrt{(\pi)}+1}{\sqrt{(\pi)}}(1-e^{-\varsigma}),$$

and

$$\varkappa(\varsigma)=\frac{7}{2}\varsigma^2-\frac{5.5636}{1.5896}\varsigma+1.$$

The feasible region of (P-4) is $\Phi_3 = \{\varkappa \in X : -2\varkappa^2(\varsigma) - 4\varkappa(\varsigma) + 3\frac{\sqrt{(\pi)}+1}{\sqrt{(\pi)}}(1-e^{-\varsigma}) \leq 0, \varkappa(0) = 1, \varkappa(1) = 1\}$.
For $\varkappa \in \Phi_3$, *the Wolfe-type dual problem for the primary problem (P-4) is given by*

$$(WD\text{-}1): \quad \max \int_{a_1}^{a_2} \Big[\{\phi^L(\varsigma, \varepsilon(\varsigma), {}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\varepsilon(\varsigma))d\varsigma, \quad \phi^U(\varsigma, \varepsilon(\varsigma), {}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\varepsilon(\varsigma))\} + (\bar{\Theta})^T h(\varsigma, \varepsilon, {}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\varepsilon(\varsigma)) \Big] d\varsigma,$$

subject to,

$$\begin{aligned}
&\varepsilon(0) = 1, \varepsilon(1) = 1,\\
&(1 + 0.7825e^{-\varsigma}) + (\frac{2}{3}\varepsilon + 4.7688e^{-\varsigma}) + (\bar{\Theta})^T(-4\varepsilon - 4 + 4.7688e^{-\varsigma})\\
&\quad + 0.9292\frac{d}{d\varsigma}\{\int_{\varsigma}^{1} \{(1 + \frac{2}{3}\varepsilon + 5.5513e^{-\varsigma})\\
&\qquad + (\bar{\Theta})^T(-2\varepsilon - 4 + 1.5641e^{-\varsigma})\}d\varsigma\} = 0\\
&\int_{\varsigma}^{1} \{(\bar{\Theta})^T(-4\varepsilon - 4 + 4.7688e^{-\varsigma})\}d\varsigma \leq 0.
\end{aligned}$$

Let $\mathbb{W}_3$ *be the collection of all feasible solutions of the problem (WD-1), that is* $\mathbb{W}_3 = \{(\bar{\Theta}, \bar{\varepsilon}) : \bar{\Theta} \in \mathbb{R}^m, \varepsilon \in X$*: satisfying constraints of (WD-1), forall* $\varsigma \in \Im\}$. *For the feasible solutions* $(\bar{\Theta} = 0, \bar{\varkappa} = 1)$ *of* $(P-4)$ *and* $(\bar{\Theta} = 0, \bar{\varepsilon} = 1)$ *of (WD-1), one can easily verify that*

$$\int_{a_1}^{a_2} \Big[\{\phi^L(\varsigma, \varepsilon(\varsigma), {}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\varepsilon(\varsigma)),\ \phi^U(\varsigma, \varepsilon(\varsigma), {}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\varepsilon(\varsigma))\} + (\bar{\Theta})^T h(\varsigma, \varepsilon, {}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\varepsilon(\varsigma)) \Big] d\varsigma,$$

is invex at $\bar{\varepsilon}$ *on* $\Phi_3 \cup \mathbb{W}_3$, *we observe that*

$$\left[\int_{a_1}^{a_2} \phi^L(\varsigma, \bar{\varkappa}, {}^{CF}D^{\theta^\bullet}_{a_1+}\bar{\varkappa})d\varsigma, \int_{a_1}^{a_2} \phi^U(\varsigma, \bar{\varkappa}, {}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})d\varsigma \right]$$

$$\not\prec_{LU} \left[\int_{a_1}^{a_2} (\phi^L + (\bar{\Theta})^T h)(\varsigma, \bar{\varepsilon}, {}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varepsilon})d\varsigma, \int_{a_1}^{a_2} (\phi^U + (\bar{\Theta})^T h)(\varsigma, \bar{\varepsilon}, {}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varepsilon})d\varsigma \right].$$

Theorem 5 (Strong Duality). *Let $\bar{\varkappa}$ be an LU optimal point for (P), and the Slater's constraint qualification is satisfied at $\bar{\varkappa}$. Then there are piecewise smooth functions $\bar{\Theta} : \Im \to \mathbb{R}^m$, $\bar{\Theta} \geq 0$, such that $(\bar{\varkappa}, \bar{\Theta})$ is a feasible point for (WD) and the two objective functions are equivalent at $\bar{\varkappa}$ and $(\bar{\varkappa}, \bar{\Theta})$ for (P) and (WD), respectively. Furthermore, if weak duality Theorem 4 holds between (P) and (WD), then $(\bar{\varkappa}, \bar{\Theta})$ is LU-optimality for (WD).*

Proof. According to the hypothesis of the theorem, $\bar{\varkappa}$ is a LU optimum point for (P), hence according to Theorem 1, there exist piecewise smooth functions $\bar{\Theta} : \Im \to \mathbb{R}^m$ such that

$$\phi^L_{\bar{\varkappa}}(\varsigma, \bar{\varkappa}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa}) + \phi^U_{\bar{\varkappa}}(\varsigma, \bar{\varkappa}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa}) + (\bar{\Theta})^T(\varsigma) h_{\bar{\varkappa}}(\varsigma, \bar{\varkappa}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})$$

$$= -{}^{\mathrm{CFR}}\mathbb{D}^{\theta^\bullet}_{a_2-} \Bigg[\phi^L_{{}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a+}\bar{\varkappa}(\varsigma)}(\varsigma, \bar{\varkappa}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa}) + \phi^U_{{}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa}(\varsigma)}(\varsigma, \bar{\varkappa}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})$$

$$+ (\bar{\Theta})^T(\varsigma) h_{{}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa}(\varsigma)}(\varsigma, \bar{\varkappa}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a+}\bar{\varkappa}) \Bigg],$$

$$(\bar{\Theta})^T(\varsigma) h(\varsigma, \bar{\varkappa}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa}) = 0, (\bar{\Theta})(\varsigma) \geq 0.$$

It follows that $(\bar{\varkappa}, \bar{\Theta})$ is a feasible point for (WD) and the objective values of (P) and (WD) are equal. According to Theorem 4, $(\bar{\varkappa}, \bar{\Theta})$ is the optimal point for (WD).

Theorem 6 (Strict Converse Duality). *Let $\bar{\varkappa}$ and $(\bar{\Theta}, \bar{\varepsilon})$ be the feasibile points for (P) and (WD) respectively, such that*

$$\left[\int_{a_1}^{a_2} \phi^L(\varsigma, \bar{\varkappa}, {}^{CF}D^{\theta^\bullet}_{a_1+}\bar{\varkappa}) d\varsigma, \int_{a_1}^{a_2} \phi^U(\varsigma, \bar{\varkappa}, {}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa}) d\varsigma \right]$$

$$+ \int_{a_1}^{a_2} (\bar{\Theta})^T(\varsigma) h(\xi, \bar{\varkappa}, {}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa}) d\varsigma$$

$$= \left[\int_{a_1}^{a_2} \phi^L(\varsigma, \bar{\varepsilon}, {}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varepsilon}) d\varsigma, \int_{a_1}^{a_2} \phi^U(\varsigma, \bar{\varepsilon}, {}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varepsilon}) d\varsigma \right]$$

$$+ \int_{a_1}^{a_2} (\bar{\Theta})^T(\varsigma) h(\varsigma, \bar{\varepsilon}, {}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varepsilon}) d\varsigma. \tag{20}$$

Further, suppose that $\varepsilon(\varsigma) \in X$, $\varepsilon(\varsigma) \geq 0$ and the functional $\int_{a_1}^{a_2} [\phi^L + \phi^U + (\bar{\Theta})^T h](\varsigma, \bar{\varepsilon}(\varsigma),$ ${}^{CF}\mathbb{D}^{\theta^\bullet}_{a_1+}\ \bar{\varepsilon}(\varsigma))\ d\varsigma$ is strictly-invex at $\bar{\varepsilon}$ on X. Then, $\bar{\varkappa} = \bar{\varepsilon}$ and $\bar{\varepsilon}$ is an LU optimal point for (P).

Proof. Assume, contrary to the outcome, $\bar{\varkappa} \neq \bar{\varepsilon}$. By (20), we have

$$\left[\int_{a_1}^{a_2} \phi^L(\varsigma, \bar{\varkappa}, {}^{\mathrm{CF}}D^{\theta^\bullet}_{a+}\bar{\varkappa})d\varsigma, \int_{a_1}^{a_2} \phi^U(\varsigma, \bar{\varkappa}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})d\varsigma\right]$$

$$+\int_{a_1}^{a_2} (\bar{\Theta})^T(\varsigma) h(\varsigma, \bar{\varkappa}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})d\varsigma$$

$$= \left[\int_{a_1}^{a_2} \phi^L(\varsigma, \bar{\varepsilon}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varepsilon})d\varsigma, \int_{a_1}^{a_2} \phi^U(\varsigma, \bar{\varepsilon}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varepsilon})d\varsigma\right]$$

$$+\int_{a_1}^{a_2} (\bar{\Theta})^T(\varsigma) h(\varsigma, \bar{\varepsilon}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varepsilon})d\varsigma.$$

That is,

$$\int_{a_1}^{a_2} (\phi^L + (\bar{\Theta})^T h)(\varsigma, \bar{\varkappa}, {}^{\mathrm{CF}}D^{\theta^\bullet}_{a_1+}\bar{\varkappa})d\varsigma = \int_{a_1}^{a_2} (\phi^L + (\bar{\Theta})^T h)(\varsigma, \bar{\varepsilon}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varepsilon})d\varsigma, \tag{21}$$

$$\int_{a_1}^{a_2} (\phi^U + (\bar{\Theta})^T h)(\varsigma, \bar{\varkappa}, {}^{\mathrm{CF}}D^{\theta^\bullet}_{a_1+}\bar{\varkappa})d\varsigma = \int_{a_1}^{a_2} (\phi^U + (\bar{\Theta})^T h)(\varsigma, \bar{\varepsilon}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varepsilon})d\varsigma. \tag{22}$$

On adding (21) and (22), we get

$$\int_{a_1}^{a_2} \left[\phi^L + \phi^U + (\bar{\Theta})^T h\right](\varsigma, \bar{\varkappa}, {}^{\mathrm{CF}}D^{\theta^\bullet}_{a_1+}\bar{\varkappa})d\varsigma$$

$$= \int_{a_1}^{a_2} \left[\phi^L + \phi^U + (\bar{\Theta})^T h\right](\varsigma, \bar{\varepsilon}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varepsilon})d\varsigma. \tag{23}$$

On the other hand, by using strictly-invex of $(\phi^L + \phi^U + (\bar{\Theta})^T h)(\varsigma, \bar{\varkappa}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varkappa})$ at $\bar{\varepsilon}$ on $\varkappa$, we have

$$\int_{a_1}^{a_2} \left[\phi^L + \phi^U + (\bar{\Theta})^T h\right](\varsigma, \bar{\varkappa}, {}^{\mathrm{CF}}D^{\theta^\bullet}_{a_1+}\bar{\varkappa})d\varsigma$$

$$-\int_{a_1}^{a_2} \left[\phi^L + \phi^U + (\bar{\Theta})^T h\right](\varsigma, \bar{\varepsilon}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varepsilon})d\varsigma$$

$$> \int_{a_1}^{a_2} \aleph(\varsigma, \bar{\varkappa}, \bar{\varepsilon})^T \left[\phi^L_{\bar{\varepsilon}} + \phi^U_{\bar{\varepsilon}} + (\bar{\Theta})^T h_{\bar{\varepsilon}}\right](\varsigma, \bar{\varepsilon}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varepsilon})$$

$$+{}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\aleph(\varsigma, \bar{\varkappa}, \bar{\varepsilon})^T \Big[\phi^L_{{}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varepsilon}} + \phi^U_{{}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varepsilon}}$$

$$+(\bar{\Theta})^T h_{{}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varepsilon}}\Big](\varsigma,\bar{\varepsilon},{}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varepsilon})d\varsigma. \quad (24)$$

By using dual constraints (13), (14) with Proposition 2.1, in view of the Weak-duality Theorem 4, we get

$$\int_{a_1}^{a_2} \aleph(\varsigma,\bar{\varkappa},\bar{\varepsilon})^T\left[\phi^L_{\bar{\varepsilon}}+\phi^U_{\bar{\varepsilon}}+(\bar{\Theta})^T h_{\bar{\varepsilon}}\right](\varsigma,\bar{\varepsilon},{}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varepsilon})+{}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\aleph(\varsigma,\bar{\varkappa},\bar{\varepsilon})^T$$

$$\left[\phi^L_{{}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varepsilon}}+\phi^U_{{}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varepsilon}}+(\bar{\Theta})^T h_{{}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varepsilon}}\right](\varsigma,\bar{\varepsilon},{}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varepsilon})d\varsigma=0. \quad (25)$$

(24) and (25), give us

$$\int_{a_1}^{a_2}\left[\phi^L+\phi^U+(\bar{\Theta})^T h\right](\varsigma,\bar{\varkappa},{}^{\mathrm{CF}}D^{\theta^\bullet}_{a_1+}\bar{\varkappa})d\varsigma$$

$$-\int_{a_1}^{a_2}\left[\phi^L+\phi^U+(\bar{\Theta})^T h\right](\varsigma,\bar{\varepsilon},{}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varepsilon})d\varsigma>0$$

which contradicts (23). Hence $\bar{\varkappa}=\bar{\varepsilon}$.
Further, if $\bar{\varkappa}$ is not an LU optimal point for (P), then there is another feasible point $\varkappa$ for (P) such that

$$\left[\int_{a_1}^{a_2}\phi^L(\varsigma,\varkappa,{}^{\mathrm{CF}}D^{\gamma}_{a+}\varkappa)d\varsigma,\int_{a_1}^{a_2}\phi^U(\varsigma,\varkappa,{}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\varkappa)d\aleph\right]$$

$$\prec_{LU}\left[\int_{a_1}^{a_2}\phi^L(\varsigma,\bar{\varepsilon},{}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varepsilon})d\varsigma,\int_{a_1}^{a_2}\phi^U(\varsigma,\bar{\varepsilon},{}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\nu})d\varsigma\right].$$

Since $\varkappa$ and $(\bar{\Theta},\bar{\nu})$ represent the respective feasibility points for (P) and (WD), then from the Theorem 4, we have

$$\left[\int_{a_1}^{a_2}\phi^L(\varsigma,\varkappa,{}^{\mathrm{CF}}D^{\theta^\bullet}_{a_1+}\varkappa)d\varsigma,\int_{a_1}^{a_2}\phi^U(\varsigma,\varkappa,{}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\varkappa)d\varsigma\right]$$

$$\not\prec_{LU}\left[\int_{a_1}^{a_2}\phi^L(\varsigma,\bar{\varepsilon},{}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varepsilon})d\varsigma,\int_{a_1}^{a_2}\phi^U(\varsigma,\bar{\varepsilon},{}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varepsilon})d\varsigma\right]$$

$$+\int_{a_1}^{a_2}(\bar{\Theta})^T(\varsigma)h(\varsigma,\bar{\varepsilon},{}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varepsilon})d\varsigma.$$

Considering the feasibility of $\varkappa$ for (P), it follows that

$$\left[\int_{a_1}^{a_2} \phi^L(\varsigma, \varkappa, {}^{\mathrm{CF}}D^{\gamma}_{a_1+}\varkappa)d\varsigma, \int_{a_1}^{a_2} \phi^U(\varsigma, \varkappa, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\varkappa)d\varsigma\right]$$

$$+ \int_{a_1}^{a_2} (\bar{\Theta})^T(\varsigma) h(\varsigma, \varkappa, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\varkappa)d\varsigma$$

$$\not<_{LU} \left[\int_{a_1}^{a_2} \phi^L(\varsigma, \bar{\varepsilon}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varepsilon})d\varsigma, \int_{a_1}^{a_2} \phi^U(\varsigma, \bar{\varepsilon}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varepsilon})d\varsigma\right]$$

$$+ \int_{a_1}^{a_2} (\bar{\Theta})^T(\varsigma) h(\varsigma, \bar{\varepsilon}, {}^{\mathrm{CF}}\mathbb{D}^{\theta^\bullet}_{a_1+}\bar{\varepsilon})d\varsigma,$$

it is in contradiction with (20). This implies that $\bar{\varepsilon}$ is a LU optimum point for (P) and hence the proof.

5. Conclusion

LU optimality and generalized-invexity are employed to establish optimality conditions for a broader class of interval-valued variational programming problems involving objective functions with Caputo-Fabrizio fractional derivatives. For the associated Wolfe-type dual problems, we have derived results pertaining to weak, strong, and strict converse duality. These theoretical developments are further substantiated through illustrative examples.

The utilization of the Caputo-Fabrizio fractional derivative introduces memory effects without involving singular kernels, thereby enabling more accurate modeling of physical and engineering systems. Meanwhile, incorporating interval-valued objective functions strengthens the robustness of the model in handling uncertainties that frequently arise in real-world scenarios. The introduction of generalized-invexity significantly extends the framework of optimality and duality, surpassing traditional convexity assumptions and offering greater flexibility and generality.

The proposed framework relies on the assumption that the interval-valued objective and constraint functions are both well-defined and bounded. However, in real-world scenarios characterized by high volatility or deep uncertainty, these assumptions may not always hold, potentially affecting the model's accuracy and reliability. Additionally, the use of Caputo-Fabrizio fractional derivatives, though beneficial for capturing memory effects without singularities, may not be appropriate for systems where long-range memory behavior is best modeled using singular kernels (e.g., power-law decay), thereby limiting its applicability in certain contexts.

This model is well-suited for addressing resource allocation problems in uncertain market environments, particularly when costs or returns exhibit memory-dependent behavior and are best represented using interval values. It also has relevance in control system design and structural optimization, where degradation of material properties over time and underlying uncertainty play a critical role. The framework's flexibility in handling both interval uncertainty and memory effects makes it applicable across various engineering, economic, and decision-making problems involving complex dynamic systems.

Future research may extend this work by exploring fuzzy interval level sets, closed and bounded intervals of real numbers that act as a bridge between fuzzy set theory and classical mathematical analysis. Based on existing literature, such as [34] and [17], which describe applications of fuzzy set theory in system analysis, the concept of fuzzy interval level sets can be used to extend results from interval spaces to fuzzy intervals. Consequently, this line of inquiry could pave the way for developing optimality conditions for fuzzy interval-valued variational programming problems with Caputo-Fabrizio fractional derivatives, marking a novel direction in this field of study.

Conflicts of interest or competing interests

The authors declare that they have no conflicts of interest.

Data and code Availability

No data were used to support this study

Supplementary information

Not Applicable

Ethical Approval

This article does not contain any studies with human participants or animals performed by any of the authors

Informed Consent

The authors are fully aware and satisfied with the contents of the article.

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