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Generalized Riccati Equation Mapping Method: 27 Solutions for Josephson Junctions and Nonlinear Optics Models

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Abstract. The generalized Riccati equation mapping method, combined with Jumarie's Riemann–Liouville fractional derivative, is employed to obtain 27 distinct travelling wave solutions for two nonlinear space-time fractional (2+1)-dimensional models: the cubic Klein-Gordon equation, which is crucial for describing the propagation of fluxions in Josephson junctions and the Ablowitz-Kaup-Newell-Segur equation, a model with key applications in the field of nonlinear optics. Among the diverse solutions obtained, kink and periodic wave behaviors were identified and graphically represented through three-dimensional, two-dimensional, and contour plots to illustrate their dynamic characteristics. In comparison to other alternative methods, the generalized Riccati equation mapping method provides a broader array of solutions to these equations. This technique illustrates the solution's behavior as a wave in various shapes, as evidenced by the findings of this research.

2020 Mathematics Subject Classifications: 35C07, 35G20, 35R11

Key Words and Phrases: Fractional nonlinear partial differential equations, generalized Riccati equation mapping method, travelling wave solutions, cubic Klein-Gordon equation, Ablowitz-Kaup-Newell-Segur equation

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1. Introduction

Nonlinear evolution equations (NLEEs) emerge in various scientific and technical domains, such as particle physics, biophysics, beam propagation, ecology, nonlinear optics,

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fluid mechanics, marine engineering, elasticity theory, solid-state physics, and quantum field theory. They offer essential models for elucidating intricate real-world events. Exact solutions to nonlinear evolution equations are essential, as they provide crucial insights into the qualitative behavior of nonlinear systems and act as benchmarks for validating numerical approaches and simulations. Numerous strong and successful approaches have been developed to address the NLEEs, including the Kudyashov method [1], generalized Kudryashov method [2], first integral method [3], the Riccati-Bernoulli sub-ode method [4, 5], generalized Riccati equation mapping method [6], modified Khater method [7], generalized Khater method [8], Sardar sub-equation method [9], modified Sardar sub-equation method [10, 11], exponential rational function method [12, 13], G'/G-expansion method [14, 15], G'/G^2-expansion method [16, 17], functional variable method [18, 19], and simple equation method [20, 21]. In the process of finding an exact solution of the NLLEs, the researchers convert the nPDEs to ordinary differential equations (ODEs) using many approaches, including the conformable fractional derivative [12], Caputo [22, 23], Riemann-Liouville [24], modified Riemann-Liouville [25], and Jumarie's modified Riemann-Liouville [26], etc. One of the definitions that is used the most often in the field of study is the Jumarie's Riemann-Liouville fractional derivative [6, 21].

Travelling wave solutions are fundamental in the analysis of nonlinear partial differential equations (PDEs), as they reduce complex spatiotemporal dynamics into ordinary differential equations and simultaneously provide physically meaningful descriptions of wave phenomena. In solid-state physics, kink-type travelling waves model fluxon propagation in Josephson junctions and crystal dislocations [27], while in nonlinear optics, AKNS-type equations capture optical solitons and related interactions [28]. The importance of such solutions is amplified in the fractional-order setting, where memory and hereditary effects emerge and yield more realistic models of anomalous transport and dispersive processes. By adjusting the fractional order α, one can tune wave velocity and profile beyond what is possible in the classical case, thereby enriching the diversity of exact solutions. Although several analytical methods, including the $(1/G')$-expansion and functional variable techniques [28], have been employed to construct travelling waves, these approaches typically yield only limited classes of hyperbolic or trigonometric forms. In contrast, the generalized Riccati equation mapping method adopted in this work generates a broader spectrum of hyperbolic, trigonometric, and rational travelling wave solutions, underscoring both its mathematical versatility and physical relevance.

The Riemann-Liouville derivative of Jumarie was presented in the following manner in the year 2006 [29]:

Definition 1. *Riemann-Liouville's fractional derivative of Jumarie can be expressed in the following way:*

$$D_t^{\alpha}\Theta(t) = \begin{cases} \Theta(t) & ,\alpha = 0, \\ \dfrac{1}{\Gamma(1-\alpha)}\dfrac{d}{dt}\displaystyle\int_0^t (t-\beta)^{-\alpha}(\Theta(\beta)-\Theta(0))d\beta & ,0 < \alpha < 1, \\ \dfrac{d^n}{dt^n}D_t^{\alpha-n}\Theta(t) & ,n \le \alpha < n+1, n \ge 1, \end{cases} \tag{1}$$

where represents an order of the fractional derivative. The following is a list of features that were raised in 2009 regarding various attributes of the fractional derivative order of Jumarie's Riemann-Liouville [30]:

$$D_x^\alpha x^\Omega = \frac{\Gamma(\Omega+1)}{\Gamma(\Omega-\theta+1)} x^{\Omega-\alpha}, \quad \Omega \geq 0, \tag{2}$$

$$D_x^\alpha[\Theta(x)\Upsilon(x)] = \Theta(x) D_x^\alpha \Upsilon(x) + \Upsilon(x) D_x^\alpha \Theta(x), \tag{3}$$

$$\begin{aligned} D_x^\alpha \Theta[\Upsilon(x)] &= \Theta'_\Omega[\Upsilon(x)][D_x^\alpha \Upsilon(x)] && (4)\\ &= D_\Upsilon^\alpha \Theta[\Upsilon(x)][\Upsilon'(x)]^\alpha. && (5)\end{aligned}$$

These represent direct results that result from the equality $d^\alpha x(t) = \Gamma(1+\alpha)dx(t)$. This is true for functions that are not susceptible of being differentiated. In Equations (3) and (4), the function is not differentiable, however in Equation (5), it is differentiable.

It is possible to modify a variety of nonlinear phenomena by using the nonlinear (2+1)-dimensional cubic Klein-Gordon (cKG) equation. These phenomena [27], include the propagation of fluxions in Josephson junctions, the propagation of dislocation in crystals, and the behavior of elementary particles. The equation that represents the nonlinear (2+1)-dimensional cKG equation is as follows [20, 31]:

$$u_{xx} + u_{yy} - u_{tt} + \delta u + \lambda u^3 = 0, \tag{6}$$

where $u = u(x, y, t)$, x and y are considered to represent the distance of propagation and t represents the time, δ and λ are non-zero constants. Sanjun and Chankaew [20] have developed four distinct exact solutions to the cKG equation. These solutions were realized by the use of the simple equation method. The solutions have the form of hyperbolic and trigonometric form.

Ablowitz, Kaup, Newell, and Segur came up with the nonlinear (2+1)-dimensional Ablowitz-Kaup-Newell-Segur (AKNS) equations in 1970. These equations were inspired by the applications to nonlinear optics, which were the primary motivation for their creation. It is possible to convert the AKNS equation to a few nonlinear evolution equations, including the nonlinear Schrödinger equation, the sine-Gordon equations, the KdV equation, and a few more. The nonlinear (2+1)-dimensional AKNS equation of the fourth order, which is nonlinear, with the parameter ξ in this form [1, 28],

$$4u_{xt} + u_{xxxt} + 8u_x u_{xy} + 4u_{xx} u_y - \xi u_{xx} = 0, \tag{7}$$

where $u = u(x, y, t)$, x and y are considered to represent the distance of propagation and t represents the time, ξ is non-zero constants. The AKNS problem has two solutions, which were created by Durur and Yokus [28] in 2021 by using the $(1/G')$-expansion method. In terms of form, the solutions are hyperbolic type.

Several analytical methods have been applied to fractional nonlinear equations, yet they typically generate only a narrow class of hyperbolic or trigonometric solutions. Their effectiveness also decreases when extended to higher-dimensional or fractional-order problems. This limitation reveals a clear research gap: the need for a systematic approach that can provide a wider variety of exact solutions. In this work, we address this issue by employing the generalized Riccati equation mapping method with Jumarie's Riemann–Liouville fractional derivative to derive 27 analytic solutions of the cKG and AKNS equations, including kink-type and periodic wave behaviors.

The following is the outline of the paper: within Section 2, a comprehensive description of the algorithm for the generalized Riccati equation mapping method may be found. In Section 3, the method is used to solve the nonlinear evolution equations that were explored in the previous section. The discussion and results are presented in Section 4. To summarize, the conclusion is presented in Section 5.

2. Algorithm of Generalized Riccati Equation Mapping Method

For the purpose of solving fractional partial differential equations, we will be discussing the generalized Riccati equation mapping method in this section. An illustration of the general form of fractional PDEs is as follows:

$$\Psi(u, u_x, u_y, u_t, u_{xx}, u_{yx}, u_{tx}, \ldots) = 0, \tag{8}$$

where Ψ is a polynomial of $u(x, y, t)$ and its partial derivatives, devoting particular attention to the highest order derivatives and nonlinear terms in relation to the functions. Within the framework of the generalized Riccati equation mapping method, the following is an overview of the first step involved:

Step 1. Wave transforming

The traveling wave solution to fractional partial differential equations is a solution that meets the following conditions:

$$u(x, y, t) = U(\beta), \quad \beta = \frac{ax^{\alpha}}{\Gamma(\alpha+1)} + \frac{by^{\alpha}}{\Gamma(\alpha+1)} - \frac{ct^{\alpha}}{\Gamma(\alpha+1)}, \tag{9}$$

where β is a broad word for a wave that is moving, and c is a constant that represents the wave's velocity. We refer to a wave as stationary when c equals zero. When the value of c is more than zero, the wave goes in a positive direction, whereas when c is less than zero, the wave moves in a negative direction. Taking Equation (8) and decreasing it to an ODE

$$\Phi\left(U, \frac{dU}{d\beta}, \frac{d^2U}{d\beta^2}, \frac{d^3U}{d\beta^3}, \ldots\right) = 0, \tag{10}$$

where Φ is a polynomial in $U(\beta)$ and its derivatives.

Step 2. Assumption of solution

When written out in the form of a finite series, the solution to Equation (10) is expressed as

$$U(\beta) = \sum_{i=0}^{N} \rho_i H^i(\beta), \tag{11}$$

where ρ_i are real constants and $\rho_N \neq 0$, and $H(\beta)$ is dependent on the generalized Riccati equation mapping method [6], the following is what it says.

$$H'(\beta) = r + cH(\beta) + kH^2(\beta), \tag{12}$$

where r, s and k are the constants that are not always zero. For Equation (12), there are twenty-seven different solutions available.

Type I: when $\varphi = s^2 - 4rk > 0$ and $sk \neq 0$ or $rk \neq 0$, we obtained

$$H_1(\beta) = -\frac{1}{2k}\left(s + \sqrt{\varphi}\tanh\left(\frac{\sqrt{\varphi}}{2}\beta\right)\right), \tag{13}$$

$$H_2(\beta) = -\frac{1}{2k}\left(s + \sqrt{\varphi}\coth\left(\frac{\sqrt{\varphi}}{2}\beta\right)\right), \tag{14}$$

$$H_3(\beta) = -\frac{1}{2k}\left(s + \sqrt{\varphi}\left(\tanh\left(\sqrt{\varphi}\beta\right) \pm i\,\mathrm{sech}\left(\sqrt{\varphi}\beta\right)\right)\right), \tag{15}$$

$$H_4(\beta) = -\frac{1}{2k}\left(s + \sqrt{\varphi}\left(\coth\left(\sqrt{\varphi}\beta\right) \pm \mathrm{csch}\left(\sqrt{\varphi}\beta\right)\right)\right), \tag{16}$$

$$H_5(\beta) = -\frac{1}{4k}\left(2s + \sqrt{\varphi}\left(\tanh\left(\frac{\sqrt{\varphi}}{4}\beta\right) + \coth\left(\frac{\sqrt{\varphi}}{4}\beta\right)\right)\right), \tag{17}$$

$$H_6(\beta) = -\frac{1}{2k}\left(s - \frac{\sqrt{\varphi(\mu^2+\sigma^2)} - \mu\sqrt{\varphi}\cosh\left(\sqrt{\varphi}\beta\right)}{\mu\sinh\left(\sqrt{\varphi}\beta\right) + \sigma}\right), \tag{18}$$

$$H_7(\beta) = -\frac{1}{2k}\left(s - \frac{\sqrt{\varphi(\sigma^2-\mu^2)} + \mu\sqrt{\varphi}\sinh\left(\sqrt{\varphi}\beta\right)}{\mu\cosh\left(\sqrt{\varphi}\beta\right) + \sigma}\right), \tag{19}$$

where μ and σ are two real constants that are not zero, and they meet the condition that $\sigma^2 - \mu^2$.

$$H_8(\beta) = \frac{2r\cosh\left(\frac{\sqrt{\varphi}}{2}\beta\right)}{\sqrt{\varphi}\sinh\left(\frac{\sqrt{\varphi}}{2}\beta\right) - s\cosh\left(\frac{\sqrt{\varphi}}{2}\beta\right)}, \tag{20}$$

$$H_9(\beta) = \frac{-2r\sinh\left(\frac{\sqrt{\varphi}}{2}\beta\right)}{s\sinh\left(\frac{\sqrt{\varphi}}{2}\beta\right) - \sqrt{\varphi}\cosh\left(\frac{\sqrt{\varphi}}{2}\beta\right)}, \tag{21}$$

$$H_{10}(\beta) = \frac{2r\cosh\left(\sqrt{\varphi}\beta\right)}{\sqrt{\varphi}\sinh\left(\sqrt{\varphi}\beta\right) - s\cosh\left(\sqrt{\varphi}\beta\right) \pm i\sqrt{\varphi}}, \tag{22}$$

$$H_{11}(\beta) = \frac{2r\sinh\left(\sqrt{\varphi}\beta\right)}{-s\sinh\left(\sqrt{\varphi}\beta\right) + \sqrt{\varphi}\cosh\left(\sqrt{\varphi}\beta\right) \pm i\sqrt{\varphi}}, \tag{23}$$

$$H_{12}(\beta)=\frac{4r\sinh\left(\frac{\sqrt{\varphi}}{4}\beta\right)\cosh\left(\frac{\sqrt{\varphi}}{4}\beta\right)}{-2s\sinh\left(\frac{\sqrt{\varphi}}{4}\beta\right)\cosh\left(\frac{\sqrt{\varphi}}{4}\beta\right)+2\sqrt{\varphi}\cosh^2\left(\frac{\sqrt{\varphi}}{4}\beta\right)-\sqrt{\varphi}}, \tag{24}$$

Type II: when $\varphi=s^2-4rk<0$ and $sk\neq 0$ or $rk\neq 0$, we obtain

$$H_{13}(\beta)=-\frac{1}{2k}\left(s-\sqrt{-\varphi}\tan\left(\frac{\sqrt{-\varphi}}{2}\beta\right)\right), \tag{25}$$

$$H_{14}(\beta)=-\frac{1}{2k}\left(s+\sqrt{-\varphi}\cot\left(\frac{\sqrt{-\varphi}}{2}\beta\right)\right), \tag{26}$$

$$H_{15}(\beta)=-\frac{1}{2k}\left(s-\sqrt{-\varphi}\left(\tan\left(\sqrt{-\varphi}\beta\right)\pm\sec\left(\sqrt{-\varphi}\beta\right)\right)\right), \tag{27}$$

$$H_{16}(\beta)=-\frac{1}{2k}\left(s+\sqrt{-\varphi}\left(\cot\left(\sqrt{-\varphi}\beta\right)\pm\csc\left(\sqrt{-\varphi}\beta\right)\right)\right), \tag{28}$$

$$H_{17}(\beta)=-\frac{1}{4k}\left(2s-\sqrt{-\varphi}\left(\tan\left(\frac{\sqrt{-\varphi}}{4}\beta\right)-\cot\left(\frac{\sqrt{-\varphi}}{4}\beta\right)\right)\right), \tag{29}$$

$$H_{18}(\beta)=-\frac{1}{2k}\left(s-\frac{\pm\sqrt{-\varphi(\mu^2-\sigma^2)}-\mu\sqrt{-\varphi}\cos\left(\sqrt{-\varphi}\beta\right)}{\mu\sin\left(\sqrt{-\varphi}\beta\right)+\sigma}\right), \tag{30}$$

$$H_{19}(\beta)=-\frac{1}{2k}\left(s+\frac{\pm\sqrt{-\varphi(\mu^2-\sigma^2)}+\mu\sqrt{-\varphi}\sin\left(\sqrt{-\varphi}\beta\right)}{\mu\cos\left(\sqrt{-\varphi}\beta\right)+\sigma}\right), \tag{31}$$

where μ and σ are two real constants that are not zero, and they meet the condition that $\mu^2-\sigma^2>0$.

$$H_{20}(\beta)=\frac{-2r\cos\left(\frac{\sqrt{-\varphi}}{2}\beta\right)}{\sqrt{-\varphi}\sin\left(\frac{\sqrt{-\varphi}}{2}\beta\right)+s\cos\left(\frac{\sqrt{-\varphi}}{2}\beta\right)}, \tag{32}$$

$$H_{21}(\beta)=\frac{2r\sin\left(\frac{\sqrt{-\varphi}}{2}\beta\right)}{-s\sin\left(\frac{\sqrt{-\varphi}}{2}\beta\right)+\sqrt{-\varphi}\cos\left(\frac{\sqrt{-\varphi}}{2}\beta\right)}, \tag{33}$$

$$H_{22}(\beta)=\frac{-2r\cos\left(\sqrt{-\varphi}\beta\right)}{\sqrt{-\varphi}\sin\left(\sqrt{-\varphi}\beta\right)+s\cos\left(\sqrt{-\varphi}\beta\right)\pm\sqrt{-\varphi}}, \tag{34}$$

$$H_{23}(\beta)=\frac{2r\sin\left(\sqrt{-\varphi}\beta\right)}{-s\sin\left(\sqrt{-\varphi}\beta\right)+\sqrt{-\varphi}\cos\left(\sqrt{-\varphi}\beta\right)\pm\sqrt{-\varphi}}, \tag{35}$$

$$H_{24}(\beta) = \frac{4r\sin\left(\frac{\sqrt{-\varphi}}{4}\beta\right)\cos\left(\frac{\sqrt{-\varphi}}{4}\beta\right)}{-2s\sin\left(\frac{\sqrt{-\varphi}}{4}\beta\right)\cos\left(\frac{\sqrt{-\varphi}}{4}\beta\right)+2\sqrt{-\varphi}\cos^2\left(\frac{\sqrt{-\varphi}}{4}\beta\right)-\sqrt{-\varphi}}, \tag{36}$$

Type III: when $r = 0$ and $sk \neq 0$, we get

$$H_{25}(\beta) = \frac{-s\psi}{k(\psi + \cosh(s\beta) - \sinh(s\beta))}, \tag{37}$$

$$H_{26}(\beta) = \frac{-s(\cosh(s\beta) + \sinh(s\beta))}{k(\psi + \cosh(s\beta) - \sinh(s\beta))}, \tag{38}$$

for any constant ψ that is arbitrary.
Type IV: when $r = s = 0$ and $k \neq 0$, we get

$$H_{27}(\beta) = -\frac{1}{k\beta + \zeta}. \tag{39}$$

In this case, ζ is a constant that may be selected with no limitations.
Step 3. Determining the integer
For the purpose of obtaining the number N in Equation (11), it is necessary to achieve a balance between the highest-order derivative and the nonlinear parts.
Step 4. Obtaining the solutions
The parameters ρ_i, $(i = 0, 1, 2, 3, \ldots, N)$ and c can be determined by first gathering the coefficients of all terms that have the same order of H^i, $(j = 0, 1, 2, 3, \ldots)$ and then setting those coefficients to zero. Therefore, we are the analytical answers to Equation (10), which we have constructed.

List of Symbols

$u(x, y, t)$	Dependent variable (wave function or solution).
x, y, t	Independent spatial and temporal variables.
α	Fractional order parameter, $0 < \alpha \leq 1$.
$\Gamma(\cdot)$	Gamma function.
$D_x^\alpha, D_t^\alpha, D_{xx}^\alpha, D_{tt}^\alpha, \ldots$	Jumarie's modified Riemann–Liouville fractional derivatives with respect to x or t.
a, b, c	Constants in the traveling wave transformation; c denotes wave velocity.
β	Traveling wave variable defined in the transformation.
$U(\beta)$	Reduced ordinary differential equation (ODE) solution of $u(x, y, t)$.
ρ_i	Real coefficients in the finite series solution (Equation (11)).
r, s, k	Parameters in the generalized Riccati equation (Equation (12)).
ϕ	Discriminant parameter $\phi = s^2 - 4rk$ in Riccati equation cases.
μ, σ	Auxiliary constants for special Riccati solutions (Types I–II).
ψ, ζ	Arbitrary constants in Riccati solutions (Types III–IV).
δ, λ	Nonzero physical constants in the cKG equation.
ξ	Nonzero constant parameter in the AKNS equation.
N	Degree of finite series expansion in Equation (11).
$H(\beta)$	Riccati function satisfying Equation (12).
$u_i(x, y, t)$	Exact analytic solutions of the governing equations ($i = 1, \ldots, 27$).

3. Applications

In this paper, we show the traveling wave effects of the nonlinear space and time fractional (2+1)-dimensional cKG equation as well as the nonlinear space and time fractional (2+1)-dimensional AKNS equation.

3.1. The Space-Time Fractional Cubic Klein-Gordon Equation

The following statement provides a definition of the second-order nonlinear space-time fractional cKG equation:

$$D_{xx}^{2\alpha} u + D_{yy}^{2\alpha} u - D_{tt}^{2\alpha} u + \delta u + \lambda u^3 = 0, \quad t > 0, 0 < \alpha \leq 1, \tag{40}$$

where the value of δ, λ are constants and $u = u(x, y, t)$. Taking into consideration the fact that the solution $u(x, y, t) = U(\beta)$ and implementing the transformation

$$\beta = \frac{ax^{\alpha}}{\Gamma(\alpha+1)} + \frac{by^{\alpha}}{\Gamma(\alpha+1)} - \frac{ct^{\alpha}}{\Gamma(\alpha+1)}, \tag{41}$$

where a, b, and c are constants that are not equal to zero. There was a transformation of Equation (40) into an ODE.

$$a^2\frac{d^2U}{d\beta^2} + b^2\frac{d^2U}{d\beta^2} + c^2\frac{d^2U}{d\beta^2} + \delta U + \lambda U^3 = 0. \tag{42}$$

The solution is represented by Equation (11) and is constructed by the use of the generalized Riccati equation mapping method. After that, the nonlinear term in Equation (42) and the highest-order derivative are brought into balance. Now, $N = 1$. Equation (11), this is what we have:

$$U(\beta) = \rho_0 + \rho_1 H(\beta). \tag{43}$$

It is necessary to substitute Equation (43) for Equation (42). According to the following, we gathered all of the terms that were of the same power of $H(\beta)$ and set each coefficient to zero as shown below:

$$H^0(\beta) \quad : \quad a^2\rho_1 rs + b^2\rho_1 rs + c^2\rho_1 rs + \delta\rho_0 + \lambda\rho_0^3 = 0, \tag{44}$$

$$H^1(\beta) \quad : \quad a^2\rho_1 s^2 + 2a^2\rho_1 rk + b^2\rho_1 s^2 + 2b^2\rho_1 rk + c^2\rho_1 s^2 + 2c^2\rho_1 rk + \delta\rho_1 + 3\lambda\rho_0^2\rho_1 = 0, \tag{45}$$

$$H^2(\beta) \quad : \quad a^2\rho_1 sk + 2a^2\rho_1 sk + b^2\rho_1 sk + 2b^2\rho_1 sk + c^2\rho_1 sk + 2c^2\rho_1 sk + 3\lambda\rho_0\rho_1^2 = 0, \tag{46}$$

$$H^3(\beta) \quad : \quad 2a^2\rho_1 k^2 + 2b^2\rho_1 k^2 + 2c^2\rho_1 k^2 + \lambda\rho_1^3 = 0. \tag{47}$$

Attempting to solve the system of Equations (44) - (47), we obtain the following:
case I: we obtain

$$\rho_0 = \mp\frac{\sqrt{\delta}s}{\sqrt{-\lambda(s^2-4rk)}},\ \rho_1 = \mp\frac{2\sqrt{\delta}k}{\sqrt{-\lambda(s^2-4rk)}},\ \text{and } c = \frac{\sqrt{-2\delta + (a^2+b^2)(s^2-4rk)}}{\sqrt{-(s^2-4rk)}}, \tag{48}$$

case II: we get

$$\rho_0 = \mp\frac{\sqrt{\delta}s}{\sqrt{-\lambda(s^2-4rk)}},\ \rho_1 = \mp\frac{2\sqrt{\delta}k}{\sqrt{-\lambda(s^2-4rk)}},\ \text{and } c = -\frac{\sqrt{-2\delta + (a^2+b^2)(s^2-4rk)}}{\sqrt{-(s^2-4rk)}}. \tag{49}$$

In the nonlinear space and time fractional (2+1)-dimensional cKG equation, which is represented by Equations (13) - (39), Equations (41) and (48), there are 27 kinds of analytical solutions with an arbitrary constant. These solutions are described by the

equations. The following is a list of the several solutions:
Type I: when $\varphi = s^2 - 4rk > 0$ and $sk \neq 0$ or $rk \neq 0$, we obtained

$$u_1(x,y,t) = \mp \frac{\sqrt{\delta} s}{\sqrt{-\lambda(s^2-4rk)}} \mp \frac{2\sqrt{\delta} k}{\sqrt{-\lambda(s^2-4rk)}} \left(-\frac{1}{2k} \left(s + \sqrt{\varphi} \tanh\left(\frac{\sqrt{\varphi}}{2} \beta \right) \right) \right), \quad (50)$$

$$u_2(x,y,t) = \mp \frac{\sqrt{\delta} s}{\sqrt{-\lambda(s^2-4rk)}} \mp \frac{2\sqrt{\delta} k}{\sqrt{-\lambda(s^2-4rk)}} \left(-\frac{1}{2k} \left(s + \sqrt{\varphi} \coth\left(\frac{\sqrt{\varphi}}{2} \beta \right) \right) \right), \quad (51)$$

$$\begin{aligned} u_3(x,y,t) = &\mp \frac{\sqrt{\delta} s}{\sqrt{-\lambda(s^2-4rk)}} \\ &\mp \frac{2\sqrt{\delta} k}{\sqrt{-\lambda(s^2-4rk)}} \left(-\frac{1}{2k} \left(s + \sqrt{\varphi} \left(\tanh\left(\sqrt{\varphi} \beta \right) \pm i \,\mathrm{sech}\left(\sqrt{\varphi} \beta \right) \right) \right) \right), \end{aligned} \quad (52)$$

$$\begin{aligned} u_4(x,y,t) = &\mp \frac{\sqrt{\delta} s}{\sqrt{-\lambda(s^2-4rk)}} \\ &\mp \frac{2\sqrt{\delta} k}{\sqrt{-\lambda(s^2-4rk)}} \left(-\frac{1}{2k} \left(s + \sqrt{\varphi} \left(\coth\left(\sqrt{\varphi} \beta \right) \pm \mathrm{csch}\left(\sqrt{\varphi} \beta \right) \right) \right) \right), \end{aligned} \quad (53)$$

$$\begin{aligned} u_5(x,y,t) = &\mp \frac{\sqrt{\delta} s}{\sqrt{-\lambda(s^2-4rk)}} \\ &\mp \frac{2\sqrt{\delta} k}{\sqrt{-\lambda(s^2-4rk)}} \left(-\frac{1}{4k} \left(2s + \sqrt{\varphi} \left(\tanh\left(\frac{\sqrt{\varphi}}{4} \beta \right) + \coth\left(\frac{\sqrt{\varphi}}{4} \beta \right) \right) \right) \right), \end{aligned} \quad (54)$$

$$\begin{aligned} u_6(x,y,t) = &\mp \frac{\sqrt{\delta} s}{\sqrt{-\lambda(s^2-4rk)}} \\ &\mp \frac{2\sqrt{\delta} k}{\sqrt{-\lambda(s^2-4rk)}} \left(-\frac{1}{2k} \left(s - \frac{\sqrt{\varphi(\mu^2+\sigma^2)} - \mu\sqrt{\varphi} \cosh\left(\sqrt{\varphi} \beta \right)}{\mu \sinh\left(\sqrt{\varphi} \beta \right) + \sigma} \right) \right), \end{aligned} \quad (55)$$

$$\begin{aligned} u_7(x,y,t) = &\mp \frac{\sqrt{\delta} s}{\sqrt{-\lambda(s^2-4rk)}} \\ &\mp \frac{2\sqrt{\delta} k}{\sqrt{-\lambda(s^2-4rk)}} \left(-\frac{1}{2k} \left(s - \frac{\sqrt{\varphi(\sigma^2-\mu^2)} + \mu\sqrt{\varphi} \sinh\left(\sqrt{\varphi} \beta \right)}{\mu \cosh\left(\sqrt{\varphi} \beta \right) + \sigma} \right) \right), \end{aligned} \quad (56)$$

where μ and σ are two real constants that are not zero, and they meet the condition that $\sigma^2 - \mu^2$.

$$u_8(x,y,t) = \mp\frac{\sqrt{\delta}s}{\sqrt{-\lambda(s^2-4rk)}} \mp \frac{2\sqrt{\delta}k}{\sqrt{-\lambda(s^2-4rk)}}\left(\frac{2r\cosh\left(\frac{\sqrt{\varphi}}{2}\beta\right)}{\sqrt{\varphi}\sinh\left(\frac{\sqrt{\varphi}}{2}\beta\right) - s\cosh\left(\frac{\sqrt{\varphi}}{2}\beta\right)}\right), \tag{57}$$

$$u_9(x,y,t) = \mp\frac{\sqrt{\delta}s}{\sqrt{-\lambda(s^2-4rk)}} \mp \frac{2\sqrt{\delta}k}{\sqrt{-\lambda(s^2-4rk)}}\left(\frac{-2r\sinh\left(\frac{\sqrt{\varphi}}{2}\beta\right)}{s\sinh\left(\frac{\sqrt{\varphi}}{2}\beta\right) - \sqrt{\varphi}\cosh\left(\frac{\sqrt{\varphi}}{2}\beta\right)}\right), \tag{58}$$

$$u_{10}(x,y,t) = \mp\frac{\sqrt{\delta}s}{\sqrt{-\lambda(s^2-4rk)}} \mp \frac{2\sqrt{\delta}k}{\sqrt{-\lambda(s^2-4rk)}}\left(\frac{2r\cosh\left(\sqrt{\varphi}\beta\right)}{\sqrt{\varphi}\sinh\left(\sqrt{\varphi}\beta\right) - s\cosh\left(\sqrt{\varphi}\beta\right) \pm i\sqrt{\varphi}}\right), \tag{59}$$

$$u_{11}(x,y,t) = \mp\frac{\sqrt{\delta}s}{\sqrt{-\lambda(s^2-4rk)}} \mp \frac{2\sqrt{\delta}k}{\sqrt{-\lambda(s^2-4rk)}}\left(\frac{2r\sinh\left(\sqrt{\varphi}\beta\right)}{-s\sinh\left(\sqrt{\varphi}\beta\right) + \sqrt{\varphi}\cosh\left(\sqrt{\varphi}\beta\right) \pm i\sqrt{\varphi}}\right), \tag{60}$$

$$u_{12}(x,y,t) = \mp\frac{\sqrt{\delta}s}{\sqrt{-\lambda(s^2-4rk)}} \mp \frac{2\sqrt{\delta}k}{\sqrt{-\lambda(s^2-4rk)}}\left(\frac{4r\sinh\left(\frac{\sqrt{\varphi}}{4}\beta\right)\cosh\left(\frac{\sqrt{\varphi}}{4}\beta\right)}{-2s\sinh\left(\frac{\sqrt{\varphi}}{4}\beta\right)\cosh\left(\frac{\sqrt{\varphi}}{4}\beta\right) + 2\sqrt{\varphi}\cosh^2\left(\frac{\sqrt{\varphi}}{4}\beta\right) - \sqrt{\varphi}}\right), \tag{61}$$

Type II: when $\varphi = s^2 - 4rk < 0$ and $sk \neq 0$ or $rk \neq 0$, we obtain

$$u_{13}(x,y,t) = \mp\frac{\sqrt{\delta}s}{\sqrt{-\lambda(s^2-4rk)}} \mp \frac{2\sqrt{\delta}k}{\sqrt{-\lambda(s^2-4rk)}}\left(-\frac{1}{2k}\left(s - \sqrt{-\varphi}\tan\left(\frac{\sqrt{-\varphi}}{2}\beta\right)\right)\right), \tag{62}$$

$$\begin{aligned} u_{14}(x,y,t) = &\mp \frac{\sqrt{\delta} s}{\sqrt{-\lambda(s^2-4rk)}} \\ &\mp \frac{2\sqrt{\delta} k}{\sqrt{-\lambda(s^2-4rk)}} \left(-\frac{1}{2k}\left(s+\sqrt{-\varphi}\cot\left(\frac{\sqrt{-\varphi}}{2}\beta\right)\right)\right), \end{aligned} \tag{63}$$

$$\begin{aligned} u_{15}(x,y,t) = &\mp \frac{\sqrt{\delta} s}{\sqrt{-\lambda(s^2-4rk)}} \\ &\mp \frac{2\sqrt{\delta} k}{\sqrt{-\lambda(s^2-4rk)}} \left(-\frac{1}{2k}\left(s-\sqrt{-\varphi}\left(\tan\left(\sqrt{-\varphi}\beta\right) \pm \sec\left(\sqrt{-\varphi}\beta\right)\right)\right)\right), \end{aligned} \tag{64}$$

$$\begin{aligned} u_{16}(x,y,t) = &\mp \frac{\sqrt{\delta} s}{\sqrt{-\lambda(s^2-4rk)}} \\ &\mp \frac{2\sqrt{\delta} k}{\sqrt{-\lambda(s^2-4rk)}} \left(-\frac{1}{2k}\left(s+\sqrt{-\varphi}\left(\cot\left(\sqrt{-\varphi}\beta\right) \pm \csc\left(\sqrt{-\varphi}\beta\right)\right)\right)\right), \end{aligned} \tag{65}$$

$$\begin{aligned} u_{17}(x,y,t) = &\mp \frac{\sqrt{\delta} s}{\sqrt{-\lambda(s^2-4rk)}} \\ &\mp \frac{2\sqrt{\delta} k}{\sqrt{-\lambda(s^2-4rk)}} \left(-\frac{1}{4k}\left(2s-\sqrt{-\varphi}\left(\tan\left(\frac{\sqrt{-\varphi}}{4}\beta\right) - \cot\left(\frac{\sqrt{-\varphi}}{4}\beta\right)\right)\right)\right), \end{aligned} \tag{66}$$

$$\begin{aligned} u_{18}(x,y,t) = &\mp \frac{\sqrt{\delta} s}{\sqrt{-\lambda(s^2-4rk)}} \\ &\mp \frac{2\sqrt{\delta} k}{\sqrt{-\lambda(s^2-4rk)}} \left(-\frac{1}{2k}\left(s-\frac{\pm\sqrt{-\varphi(\mu^2-\sigma^2)} - \mu\sqrt{-\varphi}\cos\left(\sqrt{-\varphi}\beta\right)}{\mu\sin\left(\sqrt{-\varphi}\beta\right)+\sigma}\right)\right), \end{aligned} \tag{67}$$

$$\begin{aligned} u_{19}(x,y,t) = &\mp \frac{\sqrt{\delta} s}{\sqrt{-\lambda(s^2-4rk)}} \\ &\mp \frac{2\sqrt{\delta} k}{\sqrt{-\lambda(s^2-4rk)}} \left(-\frac{1}{2k}\left(s+\frac{\pm\sqrt{-\varphi(\mu^2-\sigma^2)} + \mu\sqrt{-\varphi}\sin\left(\sqrt{-\varphi}\beta\right)}{\mu\cos\left(\sqrt{-\varphi}\beta\right)+\sigma}\right)\right), \end{aligned} \tag{68}$$

where μ and σ are two real constants that are not zero, and they meet the condition that $\mu^2 - \sigma^2 > 0$.

$$u_{20}(x,y,t) = \mp\frac{\sqrt{\delta}s}{\sqrt{-\lambda(s^2-4rk)}} \mp \frac{2\sqrt{\delta}k}{\sqrt{-\lambda(s^2-4rk)}}\left(\frac{-2r\cos\left(\frac{\sqrt{-\varphi}}{2}\beta\right)}{\sqrt{-\varphi}\sin\left(\frac{\sqrt{-\varphi}}{2}\beta\right)+s\cos\left(\frac{\sqrt{-\varphi}}{2}\beta\right)}\right), \tag{69}$$

$$u_{21}(x,y,t) = \mp\frac{\sqrt{\delta}s}{\sqrt{-\lambda(s^2-4rk)}} \mp \frac{2\sqrt{\delta}k}{\sqrt{-\lambda(s^2-4rk)}}\left(\frac{2r\sin\left(\frac{\sqrt{-\varphi}}{2}\beta\right)}{-s\sin\left(\frac{\sqrt{-\varphi}}{2}\beta\right)+\sqrt{-\varphi}\cos\left(\frac{\sqrt{-\varphi}}{2}\beta\right)}\right), \tag{70}$$

$$u_{22}(x,y,t) = \mp\frac{\sqrt{\delta}s}{\sqrt{-\lambda(s^2-4rk)}} \mp \frac{2\sqrt{\delta}k}{\sqrt{-\lambda(s^2-4rk)}}\left(\frac{-2r\cos\left(\sqrt{-\varphi}\beta\right)}{\sqrt{-\varphi}\sin\left(\sqrt{-\varphi}\beta\right)+s\cos\left(\sqrt{-\varphi}\beta\right)\pm\sqrt{-\varphi}}\right), \tag{71}$$

$$u_{23}(x,y,t) = \mp\frac{\sqrt{\delta}s}{\sqrt{-\lambda(s^2-4rk)}} \mp \frac{2\sqrt{\delta}k}{\sqrt{-\lambda(s^2-4rk)}}\left(\frac{2r\sin\left(\sqrt{-\varphi}\beta\right)}{-s\sin\left(\sqrt{-\varphi}\beta\right)+\sqrt{-\varphi}\cos\left(\sqrt{-\varphi}\beta\right)\pm\sqrt{-\varphi}}\right), \tag{72}$$

$$u_{24}(x,y,t) = \mp\frac{\sqrt{\delta}s}{\sqrt{-\lambda(s^2-4rk)}} \mp \frac{2\sqrt{\delta}k}{\sqrt{-\lambda(s^2-4rk)}}\left(\frac{4r\sin\left(\frac{\sqrt{-\varphi}}{4}\beta\right)\cos\left(\frac{\sqrt{-\varphi}}{4}\beta\right)}{-2s\sin\left(\frac{\sqrt{-\varphi}}{4}\beta\right)\cos\left(\frac{\sqrt{-\varphi}}{4}\beta\right)+2\sqrt{-\varphi}\cos^2\left(\frac{\sqrt{-\varphi}}{4}\beta\right)-\sqrt{-\varphi}}\right), \tag{73}$$

Type III: when $r = 0$ and $sk \neq 0$, we get

$$u_{25}(x,y,t) = \mp\frac{\sqrt{\delta}s}{\sqrt{-\lambda(s^2-4rk)}} \mp \frac{2\sqrt{\delta}k}{\sqrt{-\lambda(s^2-4rk)}}\left(\frac{-s\psi}{k(\psi+\cosh(s\beta)-\sinh(s\beta))}\right), \tag{74}$$

$$u_{26}(x,y,t)=\mp\frac{\sqrt{\delta}s}{\sqrt{-\lambda(s^2-4rk)}}\mp\frac{2\sqrt{\delta}k}{\sqrt{-\lambda(s^2-4rk)}}\left(\frac{-s(\cosh(s\beta)+\sinh(s\beta))}{k(\psi+\cosh(s\beta)-\sinh(s\beta))}\right), \quad (75)$$

for any constant ψ that is arbitrary.

Type IV: when $r=s=0$ and $k\neq 0$, we get

$$u_{27}(x,y,t)=\mp\frac{\sqrt{\delta}s}{\sqrt{-\lambda(s^2-4rk)}}\mp\frac{2\sqrt{\delta}k}{\sqrt{-\lambda(s^2-4rk)}}\left(-\frac{1}{k\beta+\zeta}\right). \quad (76)$$

In this case, ζ is a constant that may be selected with no limitations, where $\beta=\frac{ax^{\alpha}}{\Gamma(\alpha+1)}+\frac{by^{\alpha}}{\Gamma(\alpha+1)}\mp\frac{\sqrt{-2\delta+(a^2+b^2)(s^2-4rk)}t^{\alpha}}{\sqrt{-(s^2-4rk)}\Gamma(\alpha+1)}$.

3.2. The Space-Time Fractional Ablowitz-Kaup-Newell-Segur Equation

According to the following definition, the nonlinear space and time fractional (2+1)-dimensional AKNS equation is determined as follows:

$$4D_t^{\alpha}D_x^{\alpha}u+D_t^{\alpha}D_{xxx}^{3\alpha}u+8D_x^{\alpha}uD_y^{\alpha}D_x^{\alpha}u+4D_{xx}^{2\alpha}D_y^{\alpha}u-\xi D_{xx}^{2\alpha}u=0,\ \ t>0, 0<\alpha\leq 1, \quad (77)$$

in which $u=u(x,y,t)$ and ξ is a constant. Comparable to the equation that came before it, we determine the solution and then become Equation (77) by Equation (41), which results in the following:

$$-4ac\frac{d^2U}{d\beta^2}-a^3c\frac{d^4U}{d\beta^4}+8a^2b\frac{dU}{d\beta}\frac{d^2U}{d\beta^2}+4a^2b\frac{dU}{d\beta}\frac{d^2U}{d\beta^2}-a^2\xi\frac{d^2U}{d\beta^2}=0, \quad (78)$$

$$-(4c+a\xi)\frac{d^2U}{d\beta^2}-a^2c\frac{d^4U}{d\beta^4}+12ab\frac{dU}{d\beta}\frac{d^2U}{d\beta^2}=0. \quad (79)$$

In order to integrate Equation (79), we must first set the constant to zero.

$$-(4c+a\xi)\frac{dU}{d\beta}-a^2c\frac{d^3U}{d\beta^3}+6ab\left(\frac{dU}{d\beta}\right)=0. \quad (80)$$

Equation (80) for balancing, we obtained the value of $N=1$. Consequently, Equation (11) turned out to be

$$U(\beta)=\rho_0+\rho_1H(\beta). \quad (81)$$

The Equation (80) is being replaced by the Equation (81). The process of gathering all of the terms that have the same power of $H(\beta)$. When we set each of their coefficients to zero, we get the following:

$$H^0(\beta)\ :\ -4c\rho_1r-a\xi\rho_1r-a^2c\rho_1rs^2-2a^2c\rho_1r^2k+6ab\rho_1^2r^2=0, \quad (82)$$

$$H^1(\beta)\ :\ -4c\rho_1s-a\xi\rho_1s-a^2c\rho_1s^3-8a^2c\rho_1rsk+12ab\rho_1^2r^2=0, \quad (83)$$

$$H^2(\beta) \quad : \quad -4c\rho_1 k - a\xi\rho_1 k - a^2c\rho_1 s^2 k - 6a^2 c\rho_1 s^2 k - 8a^2 c\rho_1 rk^2 + 12ab\rho_1^2 rk + 6ab\rho_1^2 s^2 = 0, \tag{84}$$

$$H^3(\beta) \quad : \quad -12a^2c\rho_1 sk^2 + 12ab\rho_1^2 sk = 0, \tag{85}$$

$$H^4(\beta) \quad : \quad -6a^2c\rho_1 k^3 + 6ab\rho_1^2 k^2 = 0. \tag{86}$$

By solving Equations (82)–(86), we are able to get

$$\rho_1 = \frac{a^2k\xi}{b(-4+4a^2kr-a^2s^2)} \quad \text{and} \quad c = \frac{a\xi}{-4+4a^2kr-a^2s^2}. \tag{87}$$

In 27 distinct scenarios, it is feasible to write the precise traveling wave solutions of the nonlinear space and time fractional (2+1)-dimensional AKNS equation. This is a possibility.

Type I: when $\varphi = s^2 - 4rk > 0$ and $sk \neq 0$ or $rk \neq 0$, we obtained

$$u_1(x,y,t) = \rho_0 + \frac{a^2k\xi}{b(-4+4a^2kr-a^2s^2)}\left(-\frac{1}{2k}\left(s+\sqrt{\varphi}\tanh\left(\frac{\sqrt{\varphi}}{2}\beta\right)\right)\right), \tag{88}$$

$$u_2(x,y,t) = \rho_0 + \frac{a^2k\xi}{b(-4+4a^2kr-a^2s^2)}\left(-\frac{1}{2k}\left(s+\sqrt{\varphi}\coth\left(\frac{\sqrt{\varphi}}{2}\beta\right)\right)\right), \tag{89}$$

$$u_3(x,y,t) = \rho_0 + \frac{a^2k\xi}{b(-4+4a^2kr-a^2s^2)}\left(-\frac{1}{2k}\left(s+\sqrt{\varphi}\left(\tanh\left(\sqrt{\varphi}\beta\right) \pm i\,\mathrm{sech}\left(\sqrt{\varphi}\beta\right)\right)\right)\right), \tag{90}$$

$$u_4(x,y,t) = \rho_0 + \frac{a^2k\xi}{b(-4+4a^2kr-a^2s^2)}\left(-\frac{1}{2k}\left(s+\sqrt{\varphi}\left(\coth\left(\sqrt{\varphi}\beta\right) \pm \mathrm{csch}\left(\sqrt{\varphi}\beta\right)\right)\right)\right), \tag{91}$$

$$u_5(x,y,t) = \rho_0 + \frac{a^2k\xi}{b(-4+4a^2kr-a^2s^2)}\left(-\frac{1}{4k}\left(2s+\sqrt{\varphi}\left(\tanh\left(\frac{\sqrt{\varphi}}{4}\beta\right)+\coth\left(\frac{\sqrt{\varphi}}{4}\beta\right)\right)\right)\right), \tag{92}$$

$$u_6(x,y,t) = \rho_0 + \frac{a^2k\xi}{b(-4+4a^2kr-a^2s^2)}\left(-\frac{1}{2k}\left(s-\frac{\sqrt{\varphi(\mu^2+\sigma^2)}-\mu\sqrt{\varphi}\cosh\left(\sqrt{\varphi}\beta\right)}{\mu\sinh\left(\sqrt{\varphi}\beta\right)+\sigma}\right)\right), \tag{93}$$

$$u_7(x,y,t) = \rho_0 + \frac{a^2k\xi}{b(-4+4a^2kr-a^2s^2)}\left(-\frac{1}{2k}\left(s-\frac{\sqrt{\varphi(\sigma^2-\mu^2)}+\mu\sqrt{\varphi}\sinh\left(\sqrt{\varphi}\beta\right)}{\mu\cosh\left(\sqrt{\varphi}\beta\right)+\sigma}\right)\right), \tag{94}$$

where μ and σ are two real constants that are not zero, and they meet the condition that $\sigma^2 - \mu^2$.

$$u_8(x,y,t) = \rho_0 + \frac{a^2k\xi}{b(-4+4a^2kr-a^2s^2)}\left(\frac{2r\cosh\left(\frac{\sqrt{\varphi}}{2}\beta\right)}{\sqrt{\varphi}\sinh\left(\frac{\sqrt{\varphi}}{2}\beta\right) - s\cosh\left(\frac{\sqrt{\varphi}}{2}\beta\right)}\right), \quad (95)$$

$$u_9(x,y,t) = \rho_0 + \frac{a^2k\xi}{b(-4+4a^2kr-a^2s^2)}\left(\frac{-2r\sinh\left(\frac{\sqrt{\varphi}}{2}\beta\right)}{s\sinh\left(\frac{\sqrt{\varphi}}{2}\beta\right) - \sqrt{\varphi}\cosh\left(\frac{\sqrt{\varphi}}{2}\beta\right)}\right), \quad (96)$$

$$u_{10}(x,y,t) = \rho_0 + \frac{a^2k\xi}{b(-4+4a^2kr-a^2s^2)}\left(\frac{2r\cosh\left(\sqrt{\varphi}\beta\right)}{\sqrt{\varphi}\sinh\left(\sqrt{\varphi}\beta\right) - s\cosh\left(\sqrt{\varphi}\beta\right) \pm i\sqrt{\varphi}}\right), \quad (97)$$

$$u_{11}(x,y,t) = \rho_0 + \frac{a^2k\xi}{b(-4+4a^2kr-a^2s^2)}\left(\frac{2r\sinh\left(\sqrt{\varphi}\beta\right)}{-s\sinh\left(\sqrt{\varphi}\beta\right) + \sqrt{\varphi}\cosh\left(\sqrt{\varphi}\beta\right) \pm i\sqrt{\varphi}}\right), \quad (98)$$

$$u_{12}(x,y,t) = \rho_0 + \frac{a^2k\xi}{b(-4+4a^2kr-a^2s^2)}\left(\frac{4r\sinh\left(\frac{\sqrt{\varphi}}{4}\beta\right)\cosh\left(\frac{\sqrt{\varphi}}{4}\beta\right)}{-2s\sinh\left(\frac{\sqrt{\varphi}}{4}\beta\right)\cosh\left(\frac{\sqrt{\varphi}}{4}\beta\right) + 2\sqrt{\varphi}\cosh^2\left(\frac{\sqrt{\varphi}}{4}\beta\right) - \sqrt{\varphi}}\right), \quad (99)$$

Type II: when $\varphi = s^2 - 4rk < 0$ and $sk \neq 0$ or $rk \neq 0$, we obtain

$$u_{13}(x,y,t) = \rho_0 + \frac{a^2k\xi}{b(-4+4a^2kr-a^2s^2)}\left(-\frac{1}{2k}\left(s - \sqrt{-\varphi}\tan\left(\frac{\sqrt{-\varphi}}{2}\beta\right)\right)\right), \quad (100)$$

$$u_{14}(x,y,t) = \rho_0 + \frac{a^2k\xi}{b(-4+4a^2kr-a^2s^2)}\left(-\frac{1}{2k}\left(s + \sqrt{-\varphi}\cot\left(\frac{\sqrt{-\varphi}}{2}\beta\right)\right)\right), \quad (101)$$

$$u_{15}(x,y,t) = \rho_0 + \frac{a^2k\xi}{b(-4+4a^2kr-a^2s^2)}\left(-\frac{1}{2k}\left(s - \sqrt{-\varphi}\left(\tan\left(\sqrt{-\varphi}\beta\right) \pm \sec\left(\sqrt{-\varphi}\beta\right)\right)\right)\right), \quad (102)$$

$$H_{16}(\beta) = -\frac{1}{2k}\left(s + \sqrt{-\varphi}\left(\cot\left(\sqrt{-\varphi}\beta\right) \pm \csc\left(\sqrt{-\varphi}\beta\right)\right)\right), \tag{103}$$

$$H_{17}(\beta) = -\frac{1}{4k}\left(2s - \sqrt{-\varphi}\left(\tan\left(\frac{\sqrt{-\varphi}}{4}\beta\right) - \cot\left(\frac{\sqrt{-\varphi}}{4}\beta\right)\right)\right), \tag{104}$$

$$u_{18}(x,y,t) = \rho_0 + \frac{a^2k\xi}{b(-4+4a^2kr-a^2s^2)}\left(-\frac{1}{2k}\left(s - \frac{\pm\sqrt{-\varphi(\mu^2-\sigma^2)} - \mu\sqrt{-\varphi}\cos\left(\sqrt{-\varphi}\beta\right)}{\mu\sin\left(\sqrt{-\varphi}\beta\right)+\sigma}\right)\right), \tag{105}$$

$$H_{19}(\beta) = -\frac{1}{2k}\left(s + \frac{\pm\sqrt{-\varphi(\mu^2-\sigma^2)} + \mu\sqrt{-\varphi}\sin\left(\sqrt{-\varphi}\beta\right)}{\mu\cos\left(\sqrt{-\varphi}\beta\right)+\sigma}\right), \tag{106}$$

where μ and σ are two real constants that are not zero, and they meet the condition that $\mu^2 - \sigma^2 > 0$.

$$u_{20}(x,y,t) = \rho_0 + \frac{a^2k\xi}{b(-4+4a^2kr-a^2s^2)}\left(\frac{-2r\cos\left(\frac{\sqrt{-\varphi}}{2}\beta\right)}{\sqrt{-\varphi}\sin\left(\frac{\sqrt{-\varphi}}{2}\beta\right) + s\cos\left(\frac{\sqrt{-\varphi}}{2}\beta\right)}\right), \tag{107}$$

$$u_{21}(x,y,t) = \rho_0 + \frac{a^2k\xi}{b(-4+4a^2kr-a^2s^2)}\left(\frac{2r\sin\left(\frac{\sqrt{-\varphi}}{2}\beta\right)}{-s\sin\left(\frac{\sqrt{-\varphi}}{2}\beta\right) + \sqrt{-\varphi}\cos\left(\frac{\sqrt{-\varphi}}{2}\beta\right)}\right), \tag{108}$$

$$u_{22}(x,y,t) = \rho_0 + \frac{a^2k\xi}{b(-4+4a^2kr-a^2s^2)}\left(\frac{-2r\cos\left(\sqrt{-\varphi}\beta\right)}{\sqrt{-\varphi}\sin\left(\sqrt{-\varphi}\beta\right) + s\cos\left(\sqrt{-\varphi}\beta\right) \pm \sqrt{-\varphi}}\right), \tag{109}$$

$$u_{23}(x,y,t) = \rho_0 + \frac{a^2k\xi}{b(-4+4a^2kr-a^2s^2)}\left(\frac{2r\sin\left(\sqrt{-\varphi}\beta\right)}{-s\sin\left(\sqrt{-\varphi}\beta\right) + \sqrt{-\varphi}\cos\left(\sqrt{-\varphi}\beta\right) \pm \sqrt{-\varphi}}\right), \tag{110}$$

$$u_{24}(x,y,t) = \rho_0 + \frac{a^2k\xi}{b(-4+4a^2kr-a^2s^2)}\left(\frac{4r\sin\left(\frac{\sqrt{-\varphi}}{4}\beta\right)\cos\left(\frac{\sqrt{-\varphi}}{4}\beta\right)}{-2s\sin\left(\frac{\sqrt{-\varphi}}{4}\beta\right)\cos\left(\frac{\sqrt{-\varphi}}{4}\beta\right) + 2\sqrt{-\varphi}\cos^2\left(\frac{\sqrt{-\varphi}}{4}\beta\right) - \sqrt{-\varphi}}\right), \tag{111}$$

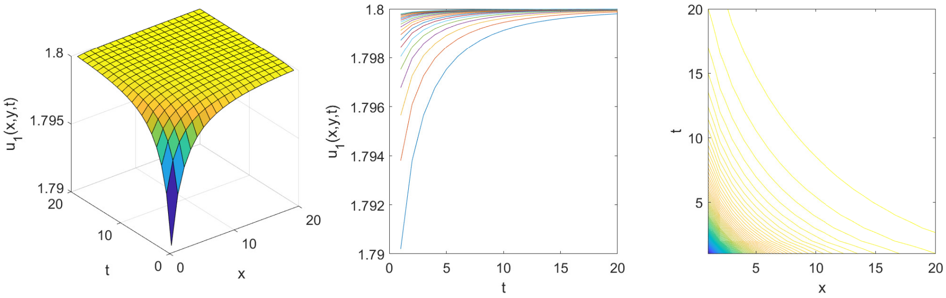

Figure 1: Three-dimensional, two-dimensional, and contour plots of the kink-type solution given by Equation (88) with parameters $(a, b, s, r, k, \xi, \alpha) = (1, 1, 3, 2, 1, 2, 0.5)$. The figure illustrates the formation of a localized traveling wave structure that remains stable along both x-direction and y-direction.

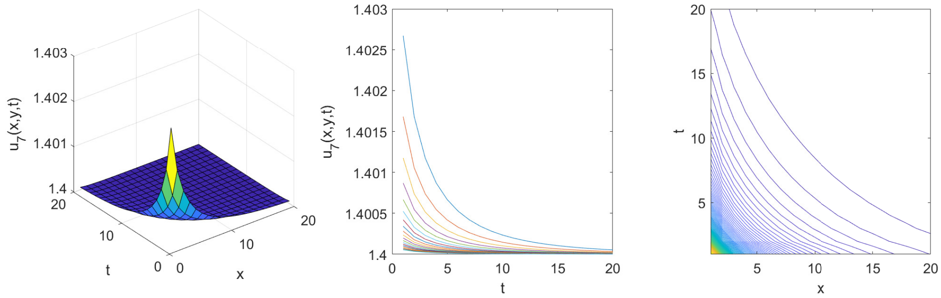

Figure 2: Three-dimensional, two-dimensional, and contour plots of the kink-type solution corresponding to Equation (94). The selected parameters highlight the sensitivity of the wave profile to changes in r and k, resulting in a steeper kink structure compared to Figure 1.

Type III: when $r = 0$ and $sk \neq 0$, we get

$$u_{25}(x, y, t) = \rho_0 + \frac{a^2 k\xi}{b(-4 + 4a^2kr - a^2s^2)} \left(\frac{-s\psi}{k(\psi + \cosh(s\beta) - \sinh(s\beta))} \right), \tag{112}$$

$$u_{26}(x, y, t) = \rho_0 + \frac{a^2 k\xi}{b(-4 + 4a^2kr - a^2s^2)} \left(\frac{-s(\cosh(s\beta) + \sinh(s\beta))}{k(\psi + \cosh(s\beta) - \sinh(s\beta))} \right), \tag{113}$$

for any constant ψ that is arbitrary.

Type IV: when $r = s = 0$ and $k \neq 0$, we get

$$u_{27}(x, y, t) = \rho_0 + \frac{a^2 k\xi}{b(-4 + 4a^2kr - a^2s^2)} \left(-\frac{1}{k\beta + \zeta} \right). \tag{114}$$

In this case, ζ is a constant that may be selected with no limitations,
where $\beta = \dfrac{ax^\alpha}{\Gamma(\alpha+1)} + \dfrac{by^\alpha}{\Gamma(\alpha+1)} - \dfrac{a\xi t^\alpha}{(-4 + 4a^2kr - a^2s^2)\Gamma(\alpha+1)}$.

4. Results and Discussions

This section examines the precise solutions derived for the space-time fractional cKG and AKNS equations by the application of the generalized Riccati equation mapping

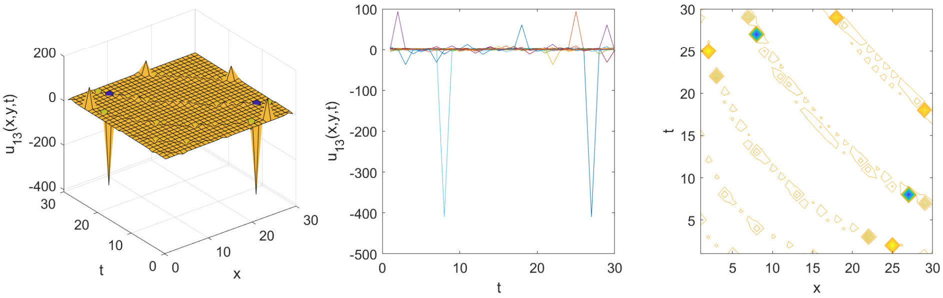

Figure 3: Three-dimensional, two-dimensional, and contour plots of the periodic solution corresponding to Equation (100) with parameters $s = 2$, $r = 2$ and $k = 2$. The oscillatory structure confirms the trigonometric nature of this class of exact solutions.

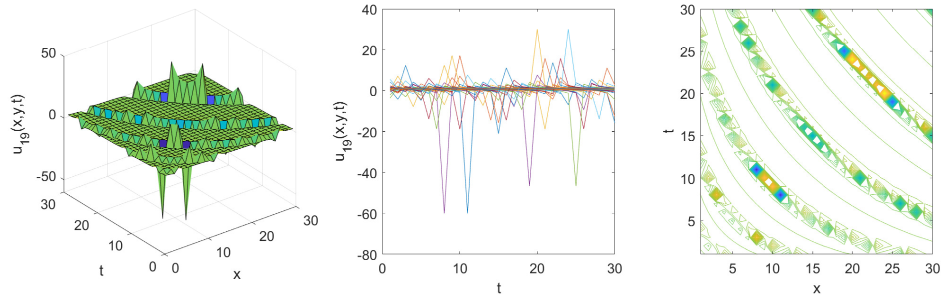

Figure 4: Three-dimensional, two-dimensional, and contour plots of the periodic solution derived from Equation (106). The parameter choice enhances the oscillatory pattern, demonstrating the flexibility of the generalized Riccati equation mapping method in capturing recurrent nonlinear wave dynamics.

method. A total of twenty-seven analytical solutions were obtained, comprising fourteen hyperbolic functions, twelve trigonometric functions, and one rational function. These solutions demonstrate kink-type and periodic behaviors contingent upon the parameter combinations. The parameter values used in the graphical simulations were carefully chosen to highlight different wave phenomena. Specifically, the constants $a = b = 1$ and $\alpha = 0.5$ were selected to simplify the traveling wave transformation while still reflecting the effect of fractional order. The Riccati parameters s, r, and k were tuned to control the balance between nonlinear and dispersive effects, thereby allowing a transition between kink-type and periodic solutions. Additional constants such as μ, σ, ψ, and ζ were assigned small integer values to generate nontrivial structures without introducing unnecessary complexity. The complete set of parameter values corresponding to each figure is summarized in Table 1, ensuring the reproducibility of the results.

Figures 1–5 depict representative three-dimensional, two-dimensional, and contour plots of the obtained solutions. Figure 1 pertains to Equation (88), wherein the selection $(a, b, s, r, k, \xi, \alpha) = (1, 1, 3, 2, 1, 2, 0.5)$ yields a confined kink configuration that maintains stability in both spatial dimensions. Figure 2, corresponding to Equation (94), illustrates an additional kink effect with a marginally modified steepness resulting from the impact of the Riccati constants r and k. Figure 3, aligned with Equation (100), distinctly illustrates periodic oscillations resulting from the parameters $s = 2$ and $k = 2$, so validating

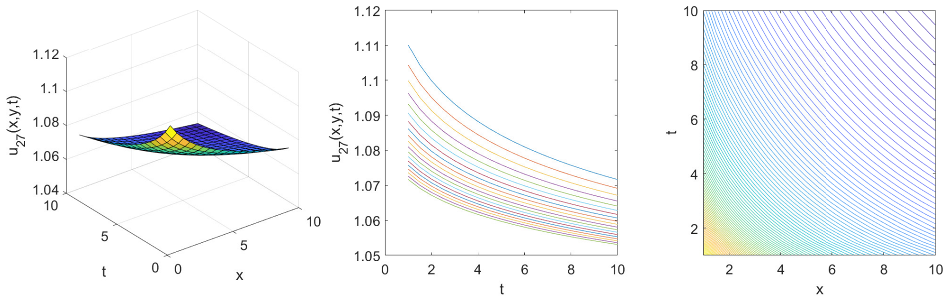

Figure 5: Three-dimensional, two-dimensional, and contour plots of the rational-type kink solution corresponding to Equation (114) under the conditions $s = r = 0$, $k = 2$, and $\zeta = 1$. The solution exhibits a decaying kink profile, showing the adaptability of the method to rationally structured waveforms.

Table 1: Configuration of parameters for Eqs. (88)-(114)

Equations	Parameters	Fig.	Wave behaviors
88	$\rho_0 = 1, a = 1, b = 1, s = 3, r = 2, k = 1, \xi = 2,$ $t = 10, \alpha = 0.5, 1 \leq x \leq 20, 1 \leq y \leq 20$	1	kink
89, 91-92, 95-96, 98-99	$\rho_0 = 1, a = 1, b = 1, s = 3, r = 2, k = 1, \xi = 2,$ $t = 10, \alpha = 0.5, 1 \leq x \leq 20, 1 \leq y \leq 20$	-	kink
93	$\rho_0 = 1, a = 1, b = 1, s = 3, r = 2, k = 1, \xi = 2, t = 10,$ $\alpha = 0.5, \mu = 1, \sigma = 2, 1 \leq x \leq 20, 1 \leq y \leq 20$	-	kink
94	$\rho_0 = 1, a = 1, b = 1, s = 3, r = 2, k = 1, \xi = 2, t = 10,$ $\alpha = 0.5, \mu = 1, \sigma = 2, 1 \leq x \leq 20, 1 \leq y \leq 20$	2	kink
100	$\rho_0 = 1, a = 1, b = 1, s = 2, r = 2, k = 2, \xi = 2,$ $t = 10, \alpha = 0.5, 1 \leq x \leq 30, 1 \leq y \leq 30$	3	periodic
101-104, 107-111	$\rho_0 = 1, a = 1, b = 1, s = 2, r = 2, k = 2, \xi = 2,$ $t = 10, \alpha = 0.5, 1 \leq x \leq 30, 1 \leq y \leq 30$	-	periodic
105	$\rho_0 = 1, a = 1, b = 1, s = 2, r = 2, k = 2, \xi = 2, t = 10,$ $\alpha = 0.5, \mu = 2, \sigma = 1, 1 \leq x \leq 30, 1 \leq y \leq 30$	-	periodic
106	$\rho_0 = 1, a = 1, b = 1, s = 2, r = 2, k = 2, \xi = 2, t = 10,$ $\alpha = 0.5, \mu = 2, \sigma = 1, 1 \leq x \leq 30, 1 \leq y \leq 30$	4	periodic
112-113	$\rho_0 = 1, a = 1, b = 1, s = 2, r = 0, k = 2, \xi = 2, t = 10,$ $\alpha = 0.5, \psi = 1, 1 \leq x \leq 30, 1 \leq y \leq 30$	-	kink
114	$\rho_0 = 1, a = 1, b = 1, s = 0, r = 0, k = 2, \xi = 2, t = 10,$ $\alpha = 0.5, \zeta = 1, 1 \leq x \leq 30, 1 \leq y \leq 30$	5	kink

the trigonometric characteristics of this solution family. Likewise, Figure 4 illustrates the periodic pattern of Equation (106), wherein the oscillatory profile is enhanced through a distinct parameter modification. Figure 5 depicts the rational-type kink solution derived from Equation (114), achieved under the parameters $s = r = 0$, $k = 2$, and $\zeta = 1$, demonstrating the versatility of the Riccati mapping approach for rationally decaying waves.

In comparison to alternative analytical methods, the generalized Riccati equation mapping method offers a wider array of exact solutions, encompassing kink, periodic, and rational families. The acquired solutions exhibit greater variability than those documented in earlier studies, broadening the application of analytical methods to higher-dimensional and fractional-order contexts [20, 28]. The solutions to the fractional cKG equation are complex-valued and not readily visualizable; yet, their algebraic structures yield significant insights into wave propagation in Josephson junctions. Conversely, the AKNS equation produces real-valued kink and periodic solutions, which hold physical significance in nonlinear optics. The results indicate that the figures serve as both mathematical representations and physically significant depictions of nonlinear wave phenomena.

5. Conclusions

In this study, we utilized the generalized Riccati equation mapping method alongside Jumarie's Riemann–Liouville fractional derivative to derive 27 unique analytic solutions for two essential nonlinear space-time fractional (2+1)-dimensional models: the cKG equation and the AKNS equation. The resulting solutions span a wide range of wave behaviors, including kink-type and periodic structures, thereby illustrating the adaptability and efficacy of the suggested method in comparison to traditional analytical techniques.

In addition to their mathematical originality, the results possess immediate physical significance. The cKG equation is esteemed for its application in modeling fluxion propagation in Josephson junctions, crystal dislocations, and particle dynamics, while the AKNS equation is significant in nonlinear optics, particularly in the study of soliton propagation and optical wave interactions. The solutions derived herein offer significant insights into nonlinear wave dynamics pertinent to both physics and engineering domains.

Nonetheless, the current study possesses specific limitations. The study is confined to precise analytic solutions, with no consideration given to stability, perturbation effects, or resilience against external influences. Subsequent study should concentrate on numerical validation, employing techniques such as spectrum simulations or finite-difference methods, to evaluate the precision and physical relevance of the results. Furthermore, the methodology can be used to additional categories of fractional nonlinear evolution equations, encompassing higher-order and coupled systems, which may exhibit more complex dynamics.

In conclusion, the extended Riccati equation mapping approach provides a systematic, adaptable, and robust framework for generating various solution families of fractional nonlinear equations. This paper integrates analytic theory with prospective physical applications, establishing a basis for further exploration of the dynamics of nonlinear fractional systems and facilitating future research in mathematical physics and engineering.

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Conflicts of Interest

The authors declare that there are no conflicts of interest to be disclosed with this publication.

References

[1] W Thadee, A Chankaew, and S Phoosree. Effects of wave solutions on shallow-water equation, optical fibre equation and electric-circuit equation. *Maejo International Journal of Science and Technology*, 16(3):262–274, 2022.

[2] H K Barman, M A Akbar, M S Osman, K S Nisar, M Zakarya, A. Abdel-Aty, and H Eleuch. Table of some basic fractional calculus formulae derived from a modified Riemann–Liouville derivative for non-differentiable functions. *Results in Physics*, 24:104092, 2021.

[3] F L Hasan. First integral method for constructing new exact solutions of the important nonlinear evolution equations in physics. In *Journal of Physics: Conference Series.*, page 012109, IRAQ, 2020. Imam Al-Kadhum International Conference for Modern Applications of Information and Communication Technology (MAICT).

[4] A R Alharbi and M B Almatrafi. Riccati-Bernoulli Sub-ODE approach on the partial differential equations and applications. *International Journal of Mathematics and Computer Science*, 15(1):367–388, 2020.

[5] M Sahoo and S Chakraverty. Riccati–Bernoulli sub-ode method-based exact solution of new coupled Konno–Oono equation. *International Journal of Modern Physics B*, 38(30):2440028, 2024.

[6] W Thadee, S Kirisri, A Kammanee, J Thepjinda, and S Phoosree. Novel solution behaviors for some communication equation and plasma physics equation via generalized Riccati equation mapping method. *Kuwait Journal of Science*, 52:100376, 2025.

[7] M M A Khater. Novel computational simulation of the propagation of pulses in optical fibers regarding the dispersion effect. *International Journal of Modern Physics B*, 37(9):2350083, 2023.

[8] M M A Khater. Prorogation of waves in shallow water through unidirectional Dullin–Gottwald–Holm model; computational simulations. *International Journal of Modern Physics B*, 37(8):2350071, 2023.

[9] H U Rehman, I Iqbal, S S Aiadi, N Mlaiki, and M S Saleem. Soliton solutions of Klein–Fock–Gordon equation using Sardar subequation method. *Mathematics*, 10(18):3377, 2022.

[10] M Nadeem, O A Arqub, A H Ali, and H A Neamah. Bifurcation, chaotic analysis and soliton solutions to the (3+ 1)-dimensional p-type model. *Alexandria Engineering Journal*, 107:245–253, 2024.

[11] M Nadeem, O A Arqub, and F M Alotaibi. Optical soliton solutions and modulation instability for unstable conformable Schrödinger model. *International Journal of Modern Physics C*, 36(01):2450175, 2025.

[12] G Bakıcıerler and E Mısırlı. On new analytical solutions of fractional systems in shallow water dynamics. *Revista mexicana de física*, 68(5):050701, 2022.

[13] M S Iqbal, A R Seadawy, M Z Baber, and M Qasim. Application of modified exponential rational function method to Jaulent–Miodek system leading to exact classical solutions. *Chaos, Solitons Fractals*, 164:112600, 2022.

[14] M Djilali and M Hakem. (G'/G)-expansion method to seek traveling wave solutions for some fractional nonlinear PDEs arising in natural sciences. *Advances in the Theory of Nonlinear Analysis and its Applications*, 7(2):303–318, 2023.

[15] S Phoosree and S Chinviriyasit. New analytic solutions of some fourth-order nonlinear space-time fractional partial differential equations by G'/G-expansion method. *Songklanakarin Journal of Science and Technology*, 43(3):795–801, 2021.

[16] S Behera and N H Aljahdaly. Nonlinear evolution equations and their traveling wave solutions in fluid media by modified analytical method. *Pramana*, 97:130, 2023.

[17] S Behera, N H Aljahdaly, and J P S Virdi. On the modified $\left(\frac{G'}{G^2}\right)$ expansion method for finding some analytical solutions of the traveling waves. *Journal of Ocean Engineering and Science*, 7(4):313–320, 2022.

[18] B Babajanov and F Abdikarimo. Nonlinear evolution equations and their traveling wave solutions in fluid media by modified analytical method. *Frontiers in Applied Mathematics and Statistics*, 8:912674, 2022.

[19] H Rezazadeh, J Vahidi, A Zafar, and A Bekir. The functional variable method to find new exact solutions of the nonlinear evolution equations with dual-power-law nonlinearity. *International Journal of Nonlinear Sciences and Numerical Simulation*, 21(3-4):249–257, 2020.

[20] J Sanjun and A Chankaew. Wave solutions of the DMBBM equation and the cKG equation using the simple equation method. *Frontiers in Applied Mathematics and Statistics*, 8:952668, 2022.

[21] W Thadee and S Phoosree. New wave behaviors generated by simple equation method with Riccati equation of some fourth-order fractional water wave equations. *Journal of the Physical Society of Japan*, 93:014002, 2024.

[22] H Ding and C Li. Numerical algorithms for the time-Caputo and space-Riesz fractional Bloch-Torrey equations. *Numerical Methods for Partial Differential Equations*, 36(41):772–799, 2020.

[23] A El-Kahlout and M Hakem. Exact solutions of partial differential equations of Caputo fractional order. *International Journal of Contemporary Mathematical Sciences*, 16(3):115–126, 2021.

[24] L F Seddek, A Ebaid, E R El-Zahar, and M D Aljoufi. Exact solution of non-homogeneous fractional differential system containing $2n$ periodic terms under physical conditions. *Mathematics*, 11(15):3308, 2023.

[25] O Guner and A Bekir. Exact solutions to the time-fractional differential equations via local fractional derivatives. *Waves in Random and Complex Media*, 28(1):139–149, 2017.

[26] H Zeng, Y Wang, M Xiao, and Y Wang. Fractional solitons: New phenomena and

exact solutions. *Frontiers in Physics*, 11:1177335, 2023.

[27] L Kaur and Kuldeep. Exact solutions of (2+1) dimensional cubic Klein-Gordon (cKG) equation. *International Journal of Mathematical, Engineering and Management Sciences*, 7(5):613–623, 2022.

[28] H Durur and A Yokuş. Exact solutions of (2 + 1)-Ablowitz-Kaup-Newell-Segur equation. *Applied Mathematics and Nonlinear Sciences*, 6(2):381–386, 2021.

[29] G Jumarie. Modified Riemann-Liouville derivative and fractional Taylor series of non-differentiable functions further results. *Computers Mathematics with Applications*, 51(9-10):1367–1376, 2006.

[30] G Jumarie. Table of some basic fractional calculus formulae derived from a modified Riemann–Liouville derivative for non-differentiable functions. *Applied Mathematics Letters*, 22(3):378–385, 2009.

[31] K M Abdul Al Woadud, M J Islam, D Kumar, and A R Khan. Analytical solutions to the (2+1)-dimensional cubic Klein–Gordon equation in the presence of fractional derivatives: A comparative study. *Partial Differential Equations in Applied Mathematics*, 12:101001, 2024.