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A Comparative Analysis of the Non-linear Time Fractional Whitham-Broer-Kaup Equations under Aboodh Decomposition Transform

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Abstract. This article offers a comprehensive analysis of nonlinear time fractional Whitham-Broer-Kaup equations under Aboodh Decomposition Transform (ADT). The study model examines the effect of different fractional derivative operators on the solution behavior of these equations by comparing them with accuracy, calculation efficiency, and physical characteristics. By employing the Aboodh Transform, a mathematical tool that is powerful in solving fractional differential equations by implementing the Adomian decomposition method, we derive an approximate solutions for models assessed for specific values of the fractional order; the solutions obtained are shown in 2D and 3D. In addition, comparative analyses are performed to clarify the effect of various fractional derivative operators on the solutions achieved, which shows the accuracy and efficiency of ADT in the handling of these complex non-linear fractional partial differential equations. Furthermore, the exact and approximate solutions are compared to the constructed problem to identify the absolute errors.

2020 Mathematics Subject Classifications: 26A33, 35R11, 65H10, 65R10

Key Words and Phrases: System of Time Fractional PDEs, Fractional Whitham-Broer-Kaup Equations, Aboodh Transform, Adomian Decomposition Method, Caputo Operator, Atangana-Baleanu-Caputo Operator, Caputo-Fabrizio Operator

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1. Introduction

Calculus simulates diffusion, control, and viscoelasticity, making applied mathematics more popular in recent decades [1–3]. Researchers in physics and engineering use non-linear differential equations [4–6]. There are many methods to solve differential equations [7–9]. The field of non-linear partial differential equations (PDEs) [3, 7] has gained substantial interest over recent decades because of their extensive applications across multiple scientific domains such as fluid mechanics, plasma physics, and non-linear optics.

The Whitham-Broer-Kaup (WBK) system [5], which characterizes shallow water wave propagation, has attracted extensive scholarly research [13, 14, 16, 17, 22]. As an extension of traditional calculus approaches, fractional calculus [20, 21] demonstrates powerful capabilities in representing complex systems with memory-dependent behavior [27, 29]. The application of fractional derivatives [1, 6] to the WBK system enables the modeling of long-range interactions and anomalous diffusion found in many real-world systems [18, 24].

The well-known Whitham-Broer-Kaup equations are given as follows:

$$\begin{aligned} \frac{\partial g(t,x)}{\partial t} + g\frac{\partial g(t,x)}{\partial x} + \frac{\partial h(t,x)}{\partial x} + d_1\frac{\partial^2 g(t,x)}{\partial x^2} = 0, \\ \frac{\partial h(t,x)}{\partial t} + g(t,x)\frac{\partial h(t,x)}{\partial x} + h(t,x)\frac{\partial g(t,x)}{\partial x} + d_2\frac{\partial^3 g(t,x)}{\partial x^3} - d_1\frac{\partial^2 g(t,x)}{\partial x^2} = 0, \end{aligned} \tag{1.1}$$

where, the function $g(t,x)$ measures the velocity deviation from fluid balance, while function $h(t,x)$ measures the height deviation from fluid balance. External forces or internal instability generates disruptions in both the motion and positioning of fluids as described by these quantities. The function $g(t,x)$ indicates the deviation from straight-line velocity while $h(t,x)$ shows the deviation in height, and both functions depend on parameters t and x that represent system-controlling physical and geometric properties. The constants d_1 and d_2 are associated with various dissemination forces acting on fluid systems. These constants characterize the effects of various disruptive mechanisms, such as viscous waste, thermal spread, and other spreading processes. They lead the scaling and behavior of these forces, influencing the rate at which the perturbations in velocity and height disorders or propagate throughout the fluid.

The Whitham-Broer-Kaup (WBK) system consists of a pair of partial differential equations that model non-linear wave phenomena, especially when it comes to shallow water waves. This system can be regarded as a higher-order generalization of the classic KdV equation, which extends the ability to describe more complex wave interactions. This provides an effective structure to study the behavior of solitary waves, shock waves, and other types of non-linear waves, accounting for both nonlinear effects and spread [23]. As mentioned above, the WBK system includes two equations: one controls the development of the height of the fluid and the other describes the velocity of the fluid. These equations can be achieved from the basic Euler equation or shallow water equations under appropriate assumptions about the depth of the fluid and the nature of the waves. The WBK system captures the interplay of nonlinear advection and dispersion, which is an improvement for such complex types of wave behavior over simpler models like the KdV equation. To study the fractional WBK equations [12], we apply Aboodh transform with Adomian decomposition method [11], which is an integrated transformation method that is relatively new and it has a great potential for solving fractional ordinary and partial differential equations [15, 19]. This is a relatively new and powerful hybrid analytical method that combines the Aboodh integral transform with the Adomian Decomposition Method (ADM). The Aboodh transform, similar to the Laplace transform, helps in converting differential equations into algebraic equations, simplifying the problem. The Adomian Decomposition Method, on the other hand, is particularly effective at handling non-linear terms by decomposing the solution into an infinite series of components, with each component recursively determined. By this method, it has notable benefits such as the ability to find an accurate solution after a finite number of iterations and resistance with the discretion or disruption problems [25, 26]. The Aboodh transform [9, 10] provides several benefits over traditional integral transforms, such as Laplace and Fourier transforms [8], which include the ability to handle the initial conditions more efficiently and its suitability for solving an extended range of fractional differential equations [2, 4, 28].

The purpose of our article is to provide a comprehensive comparative analysis of the fractional WBK equations under the Aboodh decomposition transform. We will get an approximate solution when using different analytical techniques and detect the properties of these solutions under different fractional operators. In addition, we will perform numerical simulations to visualize the behavior of the system and validate our analytical results. Moreover, the proposed method is investigated for convergence and error. The insight gained from these findings provides new perspectives to the existing literature. By studying fractional WBK equations, we expect to achieve a greater intensive understanding of the effect of fractional derivatives on the dynamics of nonlinear wave phenomena. Our findings may have important implications for applications in fields such as coastal engineering, oceanography, and atmospheric sciences.

The paper is structured as follows: Section 2 provides an overview of some fundamental concepts, which is useful for the successive sections. Section 3 the procedure for the Aboodh decomposition transformation scheme for non-linear time fractional Whitham-Broer-Kaup equations in the Caputo, Caputo-Fabrizio, and Atangana-Baleanu-Caputo senses. In section 4, we present numerical results for the solutions obtained to the time-fractional WBK equations to describe the efficiency of the method. Section 5 presents the necessary conditions for convergence and presents an error analysis for the proposed method using graphs and tables compared to exact solutions, and Section 6 provides the conclusion.

2. Fundamental Concepts

This section provides an overview of the fundamental definitions and properties of fractional calculus, which are employed to describe the proposed method.

Definition 1: [29] The Caputo derivative of the fractional order γ, $0 < \gamma < 1$, is defined as

$$ {}^{C}D_t^{\gamma} g(t) = \frac{1}{\Gamma(1-\gamma)} \int_0^t (t-\eta)^{-\gamma} g'(\eta) d\eta, \tag{2.1} $$

where, the gamma function $\Gamma(.)$, has the integral representation shown below.

$$ \Gamma(x) = \int_0^{\infty} e^{-t} t^{x-1} dt. $$

Definition 2: [29] The Mittag-Leffler function used in fractional calculus, is defined for complex t and $\gamma > 0$ as

$$ E_{\gamma}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(1+\gamma k)}. \tag{2.2} $$

Definition 3: [15] Let $g \in H^1(a,b),\ a < b,\ a \in (-\infty, t),\ 0 < \gamma < 1$, the Caputo-Fabrizio fractional derivative in the Caputo sense is given as

$$ {}^{CF}D_t^{\gamma} g(t) = \frac{M(\gamma)}{1-\gamma} \int_0^t e^{\frac{-\gamma(t-\eta)}{1-\gamma}} g'(\eta) d\eta, $$

where, $M(\gamma) > 0$ is a normalization function that satisfies $M(0) = M(1) = 1$.

Definition 4: [10] Let $g \in H^1(a,b),\ a < b,\ a \in (-\infty, t),\ 0 < \gamma < 1$, the Atangana-Baleanu-Caputo fractional derivative is expressed as

$$ {}^{ABC}D_t^{\gamma} g(t) = \frac{M(\gamma)}{1-\gamma} \int_a^t g'(\eta) E_{\gamma} \left[\frac{-\gamma(t-\eta)^{\gamma}}{1-\gamma} \right] d\eta, $$

where, $M(\gamma) > 0$ is a normalization function satisfying $M(0) = M(1) = 1$ and E_{α} is known as the generalized Mittag-Leffler function.

Definition 5: [11] The Aboodh transform of $g(t)$ is defined in the following manner:

$$ \mathcal{A}\left[g(t)\right] = \frac{1}{s} \int_0^{\infty} g(t) e^{-st} dt = A(s), \quad t \geq 0, \tag{2.3} $$

easily one can see that, the Aboodh transform is linear.

Definition 6: [11] If $\mathcal{A}[g(t)] = A(s)$, the inverse Aboodh transform of $g(t)$ is defined as :

$$ g(t) = \mathcal{A}^{-1}\left[A(s)\right]. \tag{2.4} $$

Definition 7: [19] The Aboodh transform for the Caputo operator of order γ is defined as follows:

$$ \mathcal{A}\left[{}^{C}D_t^{\gamma} g(t)\right] = s^{\gamma} \mathcal{A}\left[g(t)\right] - \sum_{k=0}^{m-1} \frac{g^{(k)}(0)}{s^{2-\gamma+k}},\ m \in \mathbb{N}, \quad m-1 < \gamma \leq m. \tag{2.5} $$

Definition 8: [13] The Aboodh transform of Atangana-Baleanu fractional operator by Caputo's meaning is given as :

$$ \mathcal{A}\left[{}^{ABC}D_t^{\gamma} g(t)\right] = \frac{M(\gamma)}{1-\gamma+\gamma s^{-\gamma}} \left(\mathcal{A}\left[g(t)\right] - s^{-2} g(0)\right). \tag{2.6} $$

Definition 9: [18] The Aboodh transform of the Caputo-Fabrizio operator is expressed as:

$$\mathcal{A}\left[{}^{CF}D_t^{\gamma} g(t)\right] = \frac{s^{1+\gamma}}{s^2(1-\gamma)+\gamma s}\mathcal{A}\left[g(t)\right] - \sum_{k=0}^{m-1}\frac{g^{(k)}(0)}{s^{2-\gamma+k}},\; m\in\mathbb{N},\quad m-1<\gamma\leq m. \tag{2.7}$$

3. The Aboodh Decomposition Transform for Time-Fractional WBK Equations

The fundamental concept behind the Aboodh decomposition transform to the time fractional Whitham-Broer-Kaup equations is demonstrated in this part.

Consider the time fractional Whitham-Broer-Kaup equations expressed in the following way:

$$\begin{aligned}
&\frac{\partial^{\gamma} g}{\partial t^{\gamma}} + g\frac{\partial^{\beta} g}{\partial x^{\beta}} + \frac{\partial^{\beta} h}{\partial x^{\beta}} + d_1\frac{\partial^{2\beta} g}{\partial x^{2\beta}} = 0,\\
&\frac{\partial^{\gamma} h}{\partial t^{\gamma}} + g\frac{\partial^{\beta} h}{\partial x^{\beta}} + h\frac{\partial^{\beta} g}{\partial x^{\beta}} + d_2\frac{\partial^{3\beta} g}{\partial x^{3\beta}} - d_1\frac{\partial^{2\beta} g}{\partial x^{2\beta}} = 0,
\end{aligned} \tag{3.1}$$

where, $t,\ x\geq 0$, $0<\gamma,\ \beta\leq 1$, with initial conditions given as :

$$\begin{aligned}
g(0,x) &= v(x),\\
h(0,x) &= w(x).
\end{aligned} \tag{3.2}$$

3.1. Aboodh Decomposition Transform under Caputo operator.

Now, we describe the steps of ADT method, which is employed to address our overall model.

Step 1 : The Aboodh transform is implemented on both sides of Eqs. (3.1) to achieve the desired outcome:

$$\begin{aligned}
&\mathcal{A}\left[\frac{\partial^{\gamma} g}{\partial t^{\gamma}} + g\frac{\partial^{\beta} g}{\partial x^{\beta}} + \frac{\partial^{\beta} h}{\partial x^{\beta}} + d_1\frac{\partial^{2\beta} g}{\partial x^{2\beta}} = 0\right],\\
&\mathcal{A}\left[\frac{\partial^{\gamma} h}{\partial t^{\gamma}} + g\frac{\partial^{\beta} h}{\partial x^{\beta}} + h\frac{\partial^{\beta} g}{\partial x^{\beta}} + d_2\frac{\partial^{3\beta} g}{\partial x^{3\beta}} - d_1\frac{\partial^{2\beta} g}{\partial x^{2\beta}} = 0\right],
\end{aligned} \tag{3.3}$$

using the **Aboodh transform in Caputo's form** given by Eq. (2.5), we get :

$$\begin{aligned}
s^{\gamma}\mathcal{A}\left[g(t,x)\right] &= \frac{1}{s^{2-\gamma}}g(0,x) - \mathcal{A}\left[g\frac{\partial^{\beta} g}{\partial x^{\beta}} + \frac{\partial^{\beta} h}{\partial x^{\beta}} + d_1\frac{\partial^{2\beta} g}{\partial x^{2\beta}}\right],\\
s^{\gamma}\mathcal{A}\left[h(t,x)\right] &= \frac{1}{s^{2-\gamma}}h(0,x) - \mathcal{A}\left[g\frac{\partial^{\beta} h}{\partial x^{\beta}} + h\frac{\partial^{\beta} g}{\partial x^{\beta}} + d_2\frac{\partial^{3\beta} g}{\partial x^{3\beta}} - d_1\frac{\partial^{2\beta} g}{\partial x^{2\beta}}\right].
\end{aligned} \tag{3.4}$$

Now, using I.C given in Eqs. (3.2), yields

$$\begin{aligned} \mathcal{A}\left[g(t,x)\right] &= \frac{1}{s^2}v(x) - s^{-\gamma}\mathcal{A}\left[g\frac{\partial^\beta g}{\partial x^\beta} + \frac{\partial^\beta h}{\partial x^\beta} + d_1\frac{\partial^{2\beta} g}{\partial x^{2\beta}}\right], \\ \mathcal{A}\left[h(t,x)\right] &= \frac{1}{s^2}w(x) - s^{-\gamma}\mathcal{A}\left[g\frac{\partial^\beta h}{\partial x^\beta} + h\frac{\partial^\beta g}{\partial x^\beta} + d_2\frac{\partial^{3\beta} g}{\partial x^{3\beta}} - d_1\frac{\partial^{2\beta} g}{\partial x^{2\beta}}\right]. \end{aligned} \tag{3.5}$$

Step 2 : Applying the inverse Aboodh transformation $\mathcal{A}^{-1}$ to Eqs. (3.5), we obtain

$$\begin{aligned} g(t,x) &= v(x) - \mathcal{A}^{-1}\left[s^{-\gamma}\mathcal{A}\left[g\frac{\partial^\beta g}{\partial x^\beta} + \frac{\partial^\beta h}{\partial x^\beta} + d_1\frac{\partial^{2\beta} g}{\partial x^{2\beta}}\right]\right], \\ h(t,x) &= w(x) - \mathcal{A}^{-1}\left[s^{-\gamma}\mathcal{A}\left[g\frac{\partial^\beta h}{\partial x^\beta} + h\frac{\partial^\beta g}{\partial x^\beta} + d_2\frac{\partial^{3\beta} g}{\partial x^{3\beta}} - d_1\frac{\partial^{2\beta} g}{\partial x^{2\beta}}\right]\right], \end{aligned} \tag{3.6}$$

where, the nonlinear terms can be decomposed as follows :

$$g\frac{\partial^\beta g}{\partial x^\beta} = \sum_{j=0}^{\infty} L_j,\ g\frac{\partial^\beta h}{\partial x^\beta} = \sum_{j=0}^{\infty} B_j,\ h\frac{\partial^\beta g}{\partial x^\beta} = \sum_{j=0}^{\infty} C_j, \tag{3.7}$$

where, $L_j,\ B_j,\ C_j$ are Adomian polynomials.

Step 3 : Substituting Eqs. (3.6) and (3.7) into Eq. (3.5), produces :

$$\begin{aligned} \sum_{j=0}^{\infty} g_j(t,x) &= v(x) - \mathcal{A}^{-1}\left[s^{-\gamma}\mathcal{A}\left[\sum_{j=0}^{\infty} L_j + \frac{\partial^\beta h}{\partial x^\beta} + d_1\frac{\partial^{2\beta} g}{\partial x^{2\beta}}\right]\right], \\ \sum_{j=0}^{\infty} h_j(t,x) &= w(x) - \mathcal{A}^{-1}\left[s^{-\gamma}\mathcal{A}\left[\sum_{j=0}^{\infty} B_j + \sum_{j=0}^{\infty} C_j + d_2\frac{\partial^{3\beta} g}{\partial x^{3\beta}} - d_1\frac{\partial^{2\beta} g}{\partial x^{2\beta}}\right]\right]. \end{aligned} \tag{3.8}$$

Consequently, the recurrence relations shown below are found :

$$\begin{cases} g_0(t,x) = v(x), \\ h_0(t,x) = w(x), \end{cases} \tag{3.9}$$

$$\begin{cases} g_1(t,x) = -\mathcal{A}^{-1}\left[s^{-\gamma}\mathcal{A}\left[L_0 + \dfrac{\partial^\beta h_0}{\partial x^\beta} + d_1\dfrac{\partial^{2\beta} g_0}{\partial x^{2\beta}}\right]\right], \\ h_1(t,x) = -\mathcal{A}^{-1}\left[s^{-\gamma}\mathcal{A}\left[B_0 + C_0 + d_2\dfrac{\partial^{3\beta} g_0}{\partial x^{3\beta}} - d_1\dfrac{\partial^{2\beta} g_0}{\partial x^{2\beta}}\right]\right], \end{cases} \tag{3.10}$$

$$\begin{cases} g_2(t,x) = -\mathcal{A}^{-1}\left[s^{-\gamma}\mathcal{A}\left[L_1 + \dfrac{\partial^\beta h_1}{\partial x^\beta} + d_1\dfrac{\partial^{2\beta} g_1}{\partial x^{2\beta}}\right]\right], \\ h_2(t,x) = -\mathcal{A}^{-1}\left[s^{-\gamma}\mathcal{A}\left[B_1 + C_1 + d_2\dfrac{\partial^{3\beta} g_1}{\partial x^{3\beta}} - d_1\dfrac{\partial^{2\beta} g_1}{\partial x^{2\beta}}\right]\right], \end{cases} \tag{3.11}$$

$$\begin{cases} g_3(t,x) = -\mathcal{A}^{-1}\left[s^{-\gamma}\mathcal{A}\left[L_2 + \dfrac{\partial^\beta h_2}{\partial x^\beta} + d_1\dfrac{\partial^{2\beta} g_2}{\partial x^{2\beta}}\right]\right], \\ h_3(t,x) = -\mathcal{A}^{-1}\left[s^{-\gamma}\mathcal{A}\left[B_2 + C_2 + d_2\dfrac{\partial^{3\beta} g_2}{\partial x^{3\beta}} - d_1\dfrac{\partial^{2\beta} g_2}{\partial x^{2\beta}}\right]\right], \end{cases} \tag{3.12}$$

and so on. Therefore, The series solution of Eqs. ((3.1)-(3.2)) is expressed as :

$$\begin{aligned} g(t,x) &= \sum_{j=0}^{\infty} g_j(t,x) = g_0(t,x) + g_1(t,x) + g_2(t,x) + g_3(t,x) + ..., \\ h(t,x) &= \sum_{j=0}^{\infty} h_j(t,x) = h_0(t,x) + h_1(t,x) + h_2(t,x) + h_3(t,x) + ... \end{aligned} \tag{3.13}$$

3.2. Aboodh Decomposition Transform under Atangana-Baleanu-Caputo operator.

Employing the **Aboodh transform in the Atangana-Baleanu-Caputo sense** given by Eq. (2.6) on Eqs. (3.2), we get :

$$\begin{aligned} \frac{M(\gamma)}{1-\gamma+\gamma s^{-\gamma}}\mathcal{A}\left[g(t,x)\right] &= \frac{1}{s^2}g(0,x) - \mathcal{A}\left[g\frac{\partial^\beta g}{\partial x^\beta} + \frac{\partial^\beta h}{\partial x^\beta} + d_1\frac{\partial^{2\beta} g}{\partial x^{2\beta}}\right], \\ \frac{M(\gamma)}{1-\gamma+\gamma s^{-\gamma}}\mathcal{A}\left[h(t,x)\right] &= \frac{1}{s^2}h(0,x) - \mathcal{A}\left[g\frac{\partial^\beta h}{\partial x^\beta} + h\frac{\partial^\beta g}{\partial x^\beta} + d_2\frac{\partial^{3\beta} g}{\partial x^{3\beta}} - d_1\frac{\partial^{2\beta} g}{\partial x^{2\beta}}\right]. \end{aligned} \tag{3.14}$$

Therefore,

$$\begin{aligned} \mathcal{A}\left[g(t,x)\right] &= \frac{1}{s^2}g(0,x) - \frac{1-\gamma+\gamma s^{-\gamma}}{M(\gamma)}\mathcal{A}\left[g\frac{\partial^\beta g}{\partial x^\beta} + \frac{\partial^\beta h}{\partial x^\beta} + d_1\frac{\partial^{2\beta} g}{\partial x^{2\beta}}\right], \\ \mathcal{A}\left[h(t,x)\right] &= \frac{1}{s^2}h(0,x) - \frac{1-\gamma+\gamma s^{-\gamma}}{M(\gamma)}\mathcal{A}\left[g\frac{\partial^\beta h}{\partial x^\beta} + h\frac{\partial^\beta g}{\partial x^\beta} + d_2\frac{\partial^{3\beta} g}{\partial x^{3\beta}} - d_1\frac{\partial^{2\beta} g}{\partial x^{2\beta}}\right]. \end{aligned} \tag{3.15}$$

By following the same process as before, we obtain :

$$\begin{aligned} g(t,x) &= g(0,x) - \mathcal{A}^{-1}\left[\frac{1-\gamma+\gamma s^{-\gamma}}{M(\gamma)}\mathcal{A}\left[g\frac{\partial^\beta g}{\partial x^\beta} + \frac{\partial^\beta h}{\partial x^\beta} + d_1\frac{\partial^{2\beta} g}{\partial x^{2\beta}}\right]\right], \\ h(t,x) &= h(0,x) - \mathcal{A}^{-1}\left[\frac{1-\gamma+\gamma s^{-\gamma}}{M(\gamma)}\mathcal{A}\left[g\frac{\partial^\beta h}{\partial x^\beta} + h\frac{\partial^\beta g}{\partial x^\beta} + d_2\frac{\partial^{3\beta} g}{\partial x^{3\beta}} - d_1\frac{\partial^{2\beta} g}{\partial x^{2\beta}}\right]\right. \end{aligned} \tag{3.16}$$

Hence, the recurrence relations are given as follows :

$$\begin{cases} g_0(t,x) = v(x), \\ h_0(t,x) = w(x), \end{cases} \tag{3.17}$$

$$\begin{cases} g_1(t,x) = -\mathcal{A}^{-1}\left[\dfrac{1-\gamma+\gamma s^{-\gamma}}{M(\gamma)}\mathcal{A}\left[L_0 + \dfrac{\partial^\beta h_0}{\partial x^\beta} + d_1\dfrac{\partial^{2\beta} g_0}{\partial x^{2\beta}}\right]\right], \\ h_1(t,x) = -\mathcal{A}^{-1}\left[\dfrac{1-\gamma+\gamma s^{-\gamma}}{M(\gamma\gamma)}\mathcal{A}\left[B_0 + C_0 + d_2\dfrac{\partial^{3\beta} g_0}{\partial x^{3\beta}} - d_1\dfrac{\partial^{2\beta} g_0}{\partial x^{2\beta}}\right]\right], \end{cases} \tag{3.18}$$

$$\begin{cases} g_2(t,x) = -\mathcal{A}^{-1}\left[\dfrac{1-\gamma+\gamma s^{-\gamma}}{M(\gamma)}\mathcal{A}\left[L_1 + \dfrac{\partial^\beta h_1}{\partial x^\beta} + d_1\dfrac{\partial^{2\beta} g_1}{\partial x^{2\beta}}\right]\right], \\ h_2(t,x) = -\mathcal{A}^{-1}\left[\dfrac{1-\gamma+\gamma s^{-\gamma}}{M(\gamma)}\mathcal{A}\left[B_1 + C_1 + d_2\dfrac{\partial^{3\beta} g_1}{\partial x^{3\beta}} - d_1\dfrac{\partial^{2\beta} g_1}{\partial x^{2\beta}}\right]\right], \end{cases} \tag{3.19}$$

$$\begin{cases} g_3(t,x) = -\mathcal{A}^{-1}\left[\dfrac{1-\gamma+\gamma s^{-\gamma}}{M(\gamma)}\mathcal{A}\left[L_2 + \dfrac{\partial^\beta h_2}{\partial x^\beta} + d_1\dfrac{\partial^{2\beta} g_2}{\partial x^{2\beta}}\right]\right], \\ h_3(t,x) = -\mathcal{A}^{-1}\left[\dfrac{1-\gamma+\gamma s^{-\gamma}}{M(\gamma)}\mathcal{A}\left[B_2 + C_2 + d_2\dfrac{\partial^{3\beta} g_2}{\partial x^{3\beta}} - d_1\dfrac{\partial^{2\beta} g_2}{\partial x^{2\beta}}\right]\right], \end{cases} \tag{3.20}$$

Thus, The series solution is given as follows :

$$\begin{aligned} g(t,x) &= g_0(t,x) + g_1(t,x) + g_2(t,x) + g_3(t,x) + ..., \\ h(t,x) &= h_0(t,x) + h_1(t,x) + h_2(t,x) + h_3(t,x) + ... \end{aligned} \tag{3.21}$$

3.3. Aboodh Decomposition Transform under Caputo-Fabrizio operator.

Employing the **Aboodh transform in the sense of Caputo-Fabrizio** from Eq. (2.7) on Eqs. (3.1), we get:

$$\begin{aligned} \frac{s^{1+\gamma}}{s^2(1-\gamma)+\gamma s}\mathcal{A}\left[g(t,x)\right] &= \frac{1}{s^{2-\gamma}}g(0,x) - \mathcal{A}\left[g\frac{\partial^\beta g}{\partial x^\beta} + \frac{\partial^\beta h}{\partial x^\beta} + d_1\frac{\partial^{2\beta} g}{\partial x^{2\beta}}\right], \\ \frac{s^{1+\gamma}}{s^2(1-\gamma)+\gamma s}\mathcal{A}\left[h(t,x)\right] &= \frac{1}{s^{2-\gamma}}h(0,x) - \mathcal{A}\left[g\frac{\partial^\beta h}{\partial x^\beta} + h\frac{\partial^\beta g}{\partial x^\beta} + d_2\frac{\partial^{3\beta} g}{\partial x^{3\beta}} - d_1\frac{\partial^{2\beta} g}{\partial x^{2\beta}}\right]. \end{aligned} \tag{3.22}$$

Thus,

$$\begin{aligned} \mathcal{A}\left[g(t,x)\right] &= \frac{1}{s^{2-\gamma}}g(0,x) - \frac{s^2(1-\gamma)+\gamma s}{s^3}\mathcal{A}\left[g\frac{\partial^\beta g}{\partial x^\beta} + \frac{\partial^\beta h}{\partial x^\beta} + d_1\frac{\partial^{2\beta} g}{\partial x^{2\beta}}\right], \\ \mathcal{A}\left[h(t,x)\right] &= \frac{1}{s^{2-\gamma}}h(0,x) - \frac{s^2(1-\gamma)+\gamma s}{s^3}\mathcal{A}\left[g\frac{\partial^\beta h}{\partial x^\beta} + h\frac{\partial^\beta g}{\partial x^\beta} + d_2\frac{\partial^{3\beta} g}{\partial x^{3\beta}} - d_1\frac{\partial^{2\beta} g}{\partial x^{2\beta}}\right]. \end{aligned} \tag{3.23}$$

By following the same process as before, we obtain :

$$\begin{aligned} g(t,x) &= g(0,x) - \mathcal{A}^{-1}\left[\frac{s^2(1-\gamma)+\gamma s}{s^3}\mathcal{A}\left[g\frac{\partial^\beta g}{\partial x^\beta} + \frac{\partial^\beta h}{\partial x^\beta} + d_1\frac{\partial^{2\beta} g}{\partial x^{2\beta}}\right]\right], \\ h(t,x) &= h(0,x) - \mathcal{A}^{-1}\left[\frac{s^2(1-\gamma)+\gamma s}{s^3}\mathcal{A}\left[g\frac{\partial^\beta h}{\partial x^\beta} + h\frac{\partial^\beta g}{\partial x^\beta} + d_2\frac{\partial^{3\beta} g}{\partial x^{3\beta}} - d_1\frac{\partial^{2\beta} g}{\partial x^{2\beta}}\right]\right. \end{aligned} \tag{3.24}$$

Hence, the recurrence relations are given as follows :

$$\begin{cases} g_0(t,x) = v(x), \\ h_0(t,x) = w(x), \end{cases} \tag{3.25}$$

$$\begin{cases} g_1(t,x) = -\mathcal{A}^{-1}\left[\dfrac{s^2(1-\gamma)+\gamma s}{s^3}\mathcal{A}\left[L_0 + \dfrac{\partial^\beta h_0}{\partial x^\beta} + d_1\dfrac{\partial^{2\beta} g_0}{\partial x^{2\beta}}\right]\right], \\ h_1(t,x) = -\mathcal{A}^{-1}\left[\dfrac{s^2(1-\gamma)+\gamma s}{s^3}\mathcal{A}\left[B_0 + C_0 + d_2\dfrac{\partial^{3\beta} g_0}{\partial x^{3\beta}} - d_1\dfrac{\partial^{2\beta} g_0}{\partial x^{2\beta}}\right]\right], \end{cases} \tag{3.26}$$

$$\begin{cases} g_2(t,x) = -\mathcal{A}^{-1}\left[\dfrac{s^2(1-\gamma)+\gamma s}{s^3}\mathcal{A}\left[L_1 + \dfrac{\partial^\beta h_1}{\partial x^\beta} + d_1\dfrac{\partial^{2\beta} f_1}{\partial x^{2\beta}}\right]\right], \\ h_2(t,x) = -\mathcal{A}^{-1}\left[\dfrac{s^2(1-\gamma)+\gamma s}{s^3}\mathcal{A}\left[B_1 + C_1 + d_2\dfrac{\partial^{3\beta} g_1}{\partial x^{3\beta}} - d_1\dfrac{\partial^{2\beta} g_1}{\partial x^{2\beta}}\right]\right], \end{cases} \tag{3.27}$$

$$\begin{cases} g_3(t,x) = -\mathcal{A}^{-1}\left[\dfrac{s^2(1-\gamma)+\gamma s}{s^3}\mathcal{A}\left[L_2 + \dfrac{\partial^\beta h_2}{\partial x^\beta} + d_1\dfrac{\partial^{2\beta} g_2}{\partial x^{2\beta}}\right]\right], \\ h_3(t,x) = -\mathcal{A}^{-1}\left[\dfrac{s^2(1-\gamma)+\gamma s}{s^3}\mathcal{A}\left[B_2 + C_2 + d_2\dfrac{\partial^{3\beta} g_2}{\partial x^{3\beta}} - d_1\dfrac{\partial^{2\beta} g_2}{\partial x^{2\beta}}\right]\right], \end{cases} \tag{3.28}$$

Finally, The series solution is given as follows :

$$\begin{aligned} g(t,x) &= g_0(t,x) + g_1(t,x) + g_2(t,x) + g_3(t,x) + ..., \\ h(t,x) &= h_0(t,x) + h_1(t,x) + h_2(t,x) + h_3(t,x) + ... \end{aligned} \tag{3.29}$$

4. Approximate solutions to the time fractional WBK equations

This section is dedicated to the numerical findings of solutions obtained for the time-fractional WBK equations. Examine the coupled system of the Whitham-Broer-Kaup equations where, $d_1 = 1$ and $d_2 = 3$. We achieve the suggested fractional equations as given by.

4.1. The time fractional WBK equations in Caputo framework.

We consider the following Caputo-type time fractional WBK equations

$$\begin{aligned} \frac{\partial^\gamma g}{\partial t^\gamma} + g\frac{\partial^\beta g}{\partial x^\beta} + \frac{\partial^\beta h}{\partial x^\beta} + \frac{\partial^{2\beta} g}{\partial x^{2\beta}} = 0, \\ \frac{\partial^\gamma h}{\partial t^\gamma} + g\frac{\partial^\beta h}{\partial x^\beta} + h\frac{\partial^\beta g}{\partial x^\beta} + 3\frac{\partial^{3\beta} g}{\partial x^{3\beta}} - \frac{\partial^{2\beta} g}{\partial x^{2\beta}} = 0, \end{aligned} \tag{4.1}$$

subjected to the initial conditions

$$\begin{aligned} g(0,x) &= \frac{1}{2} - 8tanh(-2\frac{x^{\beta}}{\Gamma(\beta+1)}), \\ h(0,x) &= 16 - 16tanh^{2}(-2\frac{x^{\beta}}{\Gamma(\beta+1)}) \end{aligned} \tag{4.2}$$

where $t,\ x \geq 0$ and $0 < \gamma,\ \beta \leq 1$.

Following the procedure defined in **section 3.1** provides us with the components that follows:

$$\begin{cases} g_0(t,x) = \frac{1}{2} - 8tanh(-2\frac{x^{\beta}}{\Gamma(\beta+1)}), \\ h_0(t,x) = 16 - 16tanh^{2}(-2\frac{x^{\beta}}{\Gamma(\beta+1)}), \end{cases} \tag{4.3}$$

$$\begin{cases} g_1(t,x) = -8sech^{2}(-2\frac{x^{\beta}}{\Gamma(\beta+1)})\frac{t^{\alpha}}{\Gamma(\gamma+1)}, \\ h_1(t,x) = -32sech^{2}(-2\frac{x^{\beta}}{\Gamma(\beta+1)})tanh(-2\frac{x^{\beta}}{\Gamma(\beta+1)})\frac{t^{\gamma}}{\Gamma(\gamma+1)}, \end{cases} \tag{4.4}$$

$$\begin{cases} g_2(t,x) = 16sech^{2}(-2\frac{x^{\beta}}{\Gamma(\beta+1)})[4sech^{2}(-2\frac{x^{\beta}}{\Gamma(\beta+1)}) - 8tanh^{2}(-2\frac{x^{\beta}}{\Gamma(\beta+1)}) \\ \qquad +3tanh(-2\frac{x^{\beta}}{\Gamma(1+\beta)})]\frac{t^{2\gamma}}{\Gamma(1+2\gamma)}, \\ h_2(t,x) = -32sech^{2}(-2\frac{x^{\beta}}{\Gamma(\beta+1)})[-25sech^{2}(-2\frac{x^{\beta}}{\Gamma(\beta+1)}) + 40sech^{2}(-2\frac{x^{\beta}}{\Gamma(\beta+1)})tanh(-2\frac{x^{\beta}}{\Gamma(\beta+1)}) \\ \qquad -2tanh^{2}(-2\frac{x^{\beta}}{\Gamma(\beta+1)}) - 32tanh^{3}(-2\frac{x^{\beta}}{\Gamma(\beta+1)}) + 96tanh(-2\frac{x^{\beta}}{\Gamma(\beta+1)})]\frac{t^{2\gamma}}{\Gamma(2\gamma+1)}. \end{cases} \tag{4.5}$$

The other iterative terms can be employed in precisely the same manner. The solutions are defined by

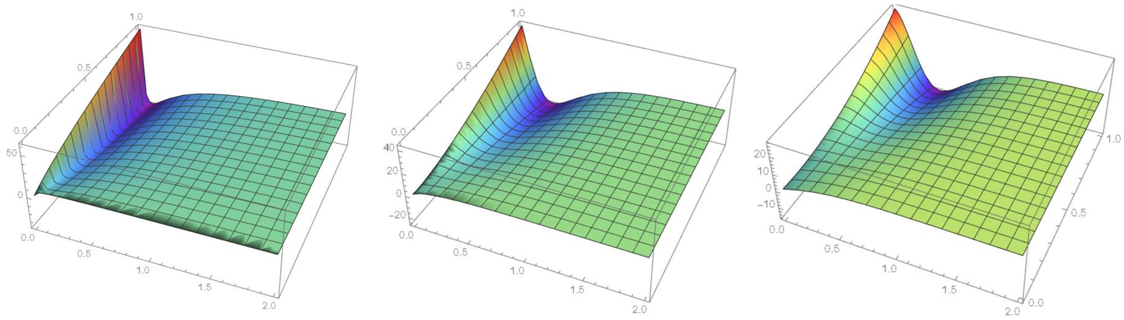
$$\begin{aligned} g(t,x) &= g_0(t,x) + g_1(t,x) + g_2(t,x) + g_3(t,x) + ..., \\ h(t,x) &= h_0(t,x) + h_1(t,x) + h_2(t,x) + h_3(t,x) + ... \end{aligned} \tag{4.6}$$

We achieve the following desired solutions:

$$\begin{aligned} g(t,x) &= \frac{1}{2} - 8tanh(-2\frac{x^{\beta}}{\Gamma(1+\beta)}) - 8sech^{2}(-2\frac{x^{\beta}}{\Gamma(1+\beta)})\frac{t^{\gamma}}{\Gamma(\gamma+1)} \\ &+16sech^{2}(-2\frac{x^{\beta}}{\Gamma(1+\beta)})[4sech^{2}(-2\frac{x^{\beta}}{\Gamma(1+\beta)})-8tanh^{2}(-2\frac{x^{\beta}}{\Gamma(\beta+1)})+3tanh(-2\frac{x^{\beta}}{\Gamma(\beta+1)})]\frac{t^{2\gamma}}{\Gamma(2\gamma+1)}+\dots \end{aligned} \tag{4.7}$$

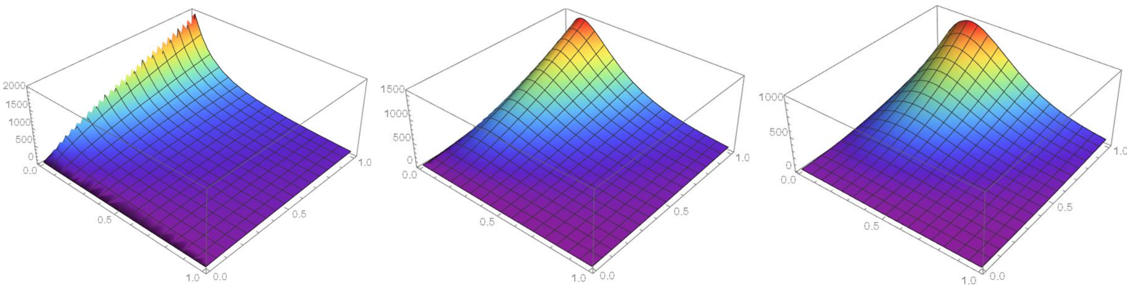
$$
\begin{aligned}
h(t,x) &= 16 - 16tanh^2(-2\frac{x^\beta}{\Gamma(1+\beta)}) - 32sech^2(-2\frac{x^\beta}{\Gamma(1+\beta)})tanh(-2\frac{x^\beta}{\Gamma(1+\beta)})\frac{t^\gamma}{\Gamma(1+\gamma)} \\
&-32sech^2(-2\frac{x^\beta}{\Gamma(1+\beta)})[-25sech^2(-2\frac{x^\beta}{\Gamma(\beta+1)})+40sech^2(-2\frac{x^\beta}{\Gamma(\beta+1)})tanh(-2\frac{x^\beta}{\Gamma(\beta+1)}) \\
&-2tanh^2(-2\frac{x^\beta}{\Gamma(\beta+1)}) - 32tanh^3(-2\frac{x^\beta}{\Gamma(\beta+1)}) + 96tanh(-2\frac{x^\beta}{\Gamma(1+\beta)})]\frac{t^{2\gamma}}{\Gamma(1+2\gamma)} + \dots
\end{aligned} \tag{4.8}
$$

After obtaining the approximate solutions of the considered system, they are represented in 3D for various values of the fractional derivatives γ and β. The surface plots can be seen in Fig. (1)-(2).



(A) (B) (C)

Figure 1: The approximate solution g of the problem (4.1-4.2) for $\gamma = \beta = 0.35, 0.7$, and 0.95



(A) (B) (D)

Figure 2: The approximate solution h of the problem (4.1-4.2) for $\gamma = \beta = 0.35, 0.7$, and 0.95

Furthermore, the behavior of the solutions g and h to the problem (4.1-4.2) for different values of fractional derivatives are also represented in 2D as follows.

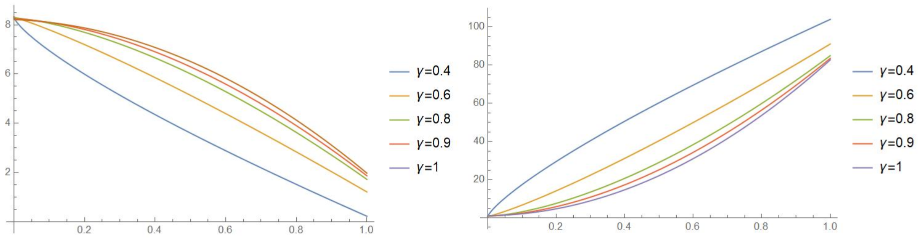


Figure 3: The approximate solution g and h of the problem (4.1-4.2) in 2D

4.2. The time fractional WBK equations in Atangana-Baleanu-Caputo framework.

Taking into account the following Atangana-Baleanu-Caputo-type time fractional WBK equations

$$\begin{aligned} &\frac{\partial^{\gamma} g}{\partial t^{\gamma}} + g\frac{\partial^{\beta} g}{\partial x^{\beta}} + \frac{\partial^{\beta} h}{\partial x^{\beta}} + \frac{\partial^{2\beta} g}{\partial x^{2\beta}} = 0, \\ &\frac{\partial^{\gamma} h}{\partial t^{\gamma}} + g\frac{\partial^{\beta} h}{\partial x^{\beta}} + h\frac{\partial^{\beta} g}{\partial x^{\beta}} + 3\frac{\partial^{3\beta} g}{\partial x^{3\beta}} - \frac{\partial^{2\beta} g}{\partial x^{2\beta}} = 0, \end{aligned} \tag{4.9}$$

subjected to the initial conditions

$$\begin{aligned} g(0,x) &= \frac{1}{2} - 8tanh(-2\frac{x^{\beta}}{\Gamma(\beta+1)}), \\ h(0,x) &= 16 - 16tanh^2(-2\frac{x^{\beta}}{\Gamma(\beta+1)}) \end{aligned} \tag{4.10}$$

where $t,\ x \geq 0$ and $0 < \gamma,\ \beta \leq 1$.

Following the procedure defined in **section 3.2** provides us with the following components:

$$\begin{cases} g_0(t,x) = \frac{1}{2} - 8tanh(-2\frac{x^{\beta}}{\Gamma(\beta+1)}), \\ h_0(t,x) = 16 - 16tanh^2(-2\frac{x^{\beta}}{\Gamma(\beta+1)}), \end{cases} \tag{4.11}$$

$$\begin{cases} g_1(t,x) = -8sech^2(-2\frac{x^{\beta}}{\Gamma(\beta+1)})\frac{1}{M(\gamma)}((1-\gamma) + \frac{\gamma t^{\gamma}}{\Gamma(\gamma+1)}), \\ h_1(t,x) = -32sech^2(-2\frac{x^{\beta}}{\Gamma(\beta+1)})tanh(-2\frac{x^{\beta}}{\Gamma(\beta+1)})\frac{1}{M(\gamma)}((1-\gamma) + \frac{\gamma t^{\gamma}}{\Gamma(\gamma+1)}), \end{cases} \tag{4.12}$$

$$\begin{cases} g_2(t,x) = 16sech^2(-2\frac{x^\beta}{\Gamma(\beta+1)})[4sech^2(-2\frac{x^\beta}{\Gamma(\beta+1)}) - 8tanh^2(-2\frac{x^\beta}{\Gamma(\beta+1)}) \\ \quad +3tanh(-2\frac{x^\beta}{\Gamma(\beta+1)})]\frac{1}{M^2(\gamma)}((1-\gamma)^2 + \frac{2\gamma(1-\gamma)t^\gamma}{\Gamma(\gamma+1)} + \frac{\gamma^2 t^\gamma}{\Gamma(2\gamma+1)}), \\ h_2(t,x) = -32sech^2(-2\frac{x^\beta}{\Gamma(\beta+1)})[-25sech^2(-2\frac{x^\beta}{\Gamma(\beta+1)}) + 40sech^2(-2\frac{x^\beta}{\Gamma(\beta+1)})tanh(-2\frac{x^\beta}{\Gamma(\beta+1)}) \\ -2tanh^2(-2\frac{x^\beta}{\Gamma(\beta+1)}) - 32tanh^3(-2\frac{x^\beta}{\Gamma(\beta+1)}) + 96tanh(-2\frac{x^\beta}{\Gamma(\beta+1)})]\frac{1}{M^2(\gamma)}((1-\gamma)^2 + \frac{2\gamma(1-\gamma)t^\gamma}{\Gamma(\gamma+1)} + \frac{\gamma^2 t^\gamma}{\Gamma(2\gamma+1)}). \end{cases} \tag{4.13}$$

The other iterative terms can be employed in precisely the same manner. As the solutions are defined by

$$\begin{aligned} g(t,x) &= g_0(t,x) + g_1(t,x) + g_2(t,x) + g_3(t,x) + ..., \\ h(t,x) &= h_0(t,x) + h_1(t,x) + h_2(t,x) + h_3(t,x) + ... \end{aligned} \tag{4.14}$$

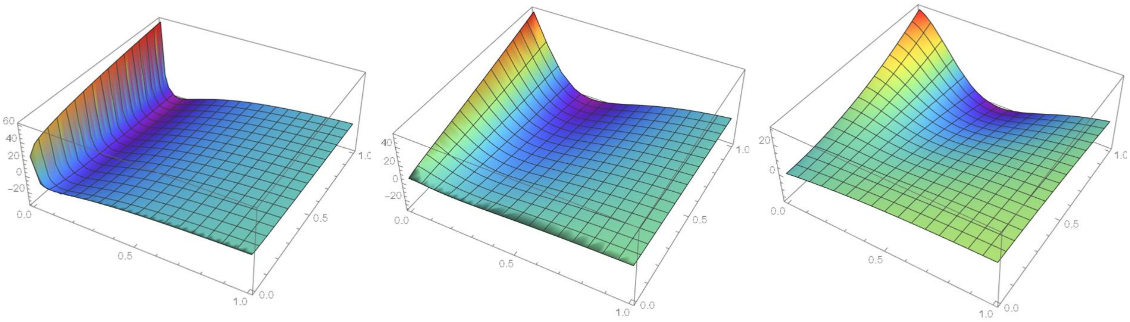
We acquire the requisite solutions:

$$\begin{aligned} g(t,x) = \frac{1}{2} - 8tanh(-2\frac{x^\beta}{\Gamma(\beta+1)}) - 8sech^2(-2\frac{x^\beta}{\Gamma(\beta+1)})\frac{1}{M(\gamma)}((1-\gamma) + \frac{\gamma t^\gamma}{\Gamma(\gamma+1)}) \\ +16sech^2(-2\frac{x^\beta}{\Gamma(1+\beta)})[4sech^2(-2\frac{x^\beta}{\Gamma(1+\beta)}) - 8tanh^2(-2\frac{x^\beta}{\Gamma(1+\beta)}) \\ +3tanh(-2\frac{x^\beta}{\Gamma(1+\beta)})]\frac{1}{M^2(\gamma)}((-\gamma+1)^2 + \frac{2\gamma(-\gamma+1)t^\gamma}{\Gamma(1+\gamma)} + \frac{\gamma^2 t^\gamma}{\Gamma(1+2\gamma)}) + \dots \end{aligned} \tag{4.15}$$

$$\begin{aligned} h(t,x) = 16-16tanh^2(-2\frac{x^\beta}{\Gamma(1+\beta)}) - 32sech^2(-2\frac{x^\beta}{\Gamma(1+\beta)})tanh(-2\frac{x^\beta}{\Gamma(1+\beta)})\frac{1}{M(\gamma)}((-\gamma+1) + \frac{\gamma t^\gamma}{\Gamma(1+\gamma)}) \\ -32sech^2(-2\frac{x^\beta}{\Gamma(1+\beta)})[-25sech^2(-2\frac{x^\beta}{\Gamma(1+\beta)}) + 40sech^2(-2\frac{x^\beta}{\Gamma(1+\beta)})tanh(-2\frac{x^\beta}{\Gamma(1+\beta)}) \\ -2tanh^2(-2\frac{x^\beta}{\Gamma(1+\beta+)}) - 32tanh^3(-2\frac{x^\beta}{\Gamma(\beta+1)}) + 96tanh(-2\frac{x^\beta}{\Gamma(1+\beta)})]\frac{1}{M^2(\gamma)}((-\gamma+1)^2 + \frac{2\gamma(-\gamma+1)t^\gamma}{\Gamma(\gamma+1)} \\ + \frac{\gamma^2 t^\gamma}{\Gamma(2\gamma+1)}) + \dots \end{aligned} \tag{4.16}$$

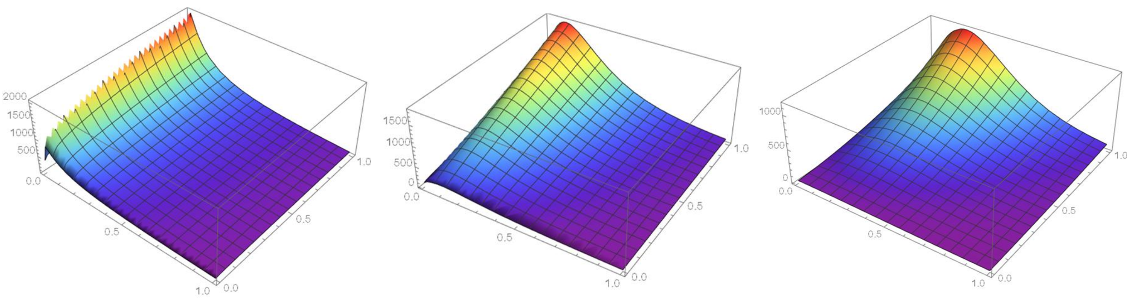
After obtaining the estimated solutions of the considered system, they are represented in 3D with different fractional derivative values γ and β. The surface plots can be seen in Fig. (4)-(5). In addition to the 3D representation, the behavior of the solutions g and h

of the problem (4.9-4.10) for different values of fractional derivatives are shown in 2D as follows.



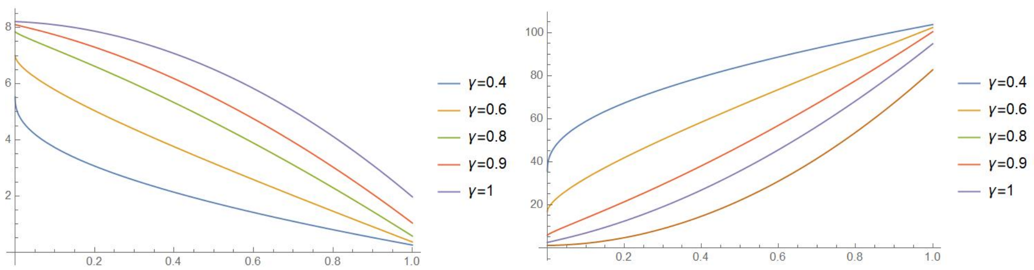
(A) (B) (C)

Figure 4: The approximate solution g of the problem (4.9-4.10) for $\gamma = \beta = 0.35, 0.7$, and 0.95



(A) (B) (C)

Figure 5: The approximate solution h of the problem (4.9-4.10) for $\gamma = \beta = 0.35, 0.7$, and 0.95



(A) (B)

Figure 6: The approximate solution g and h of the problem (4.9-4.10) in 2D

4.3. The time-fractional WBK equations in Caputo-Fabrizio sense.

Consider the following Caputo-Fabrizio-type time fractional WBK equations

$$\begin{aligned} \frac{\partial^\gamma g}{\partial t^\gamma} + g\frac{\partial^\beta g}{\partial x^\beta} + \frac{\partial^\beta h}{\partial x^\beta} + \frac{\partial^{2\beta} g}{\partial x^{2\beta}} = 0, \\ \frac{\partial^\gamma h}{\partial t^\gamma} + g\frac{\partial^\beta h}{\partial x^\beta} + h\frac{\partial^\beta g}{\partial x^\beta} + 3\frac{\partial^{3\beta} g}{\partial x^{3\beta}} - \frac{\partial^{2\beta} g}{\partial x^{2\beta}} = 0, \end{aligned} \tag{4.17}$$

subjected to the initial conditions

$$\begin{aligned} g(0,x) = \frac{1}{2} - 8tanh(-2\frac{x^\beta}{\Gamma(\beta+1)}), \\ h(0,x) = 16 - 16tanh^2(-2\frac{x^\beta}{\Gamma(\beta+1)}) \end{aligned} \tag{4.18}$$

where $t,\ x \geq 0$ and $0 < \gamma,\ \beta \leq 1$.

Following the procedure defined in **section 3.3** provides us with the following components:

$$\begin{cases} g_0(t,x) = \frac{1}{2} - 8tanh(-2\frac{x^\beta}{\Gamma(\beta+1)}), \\ h_0(t,x) = 16 - 16tanh^2(-2\frac{x^\beta}{\Gamma(\beta+1)}), \end{cases} \tag{4.19}$$

$$\begin{cases} g_1(t,x) = -8sech^2(-2\frac{x^\beta}{\Gamma(\beta+1)})(1-\gamma+\gamma t), \\ h_1(t,x) = -32sech^2(-2\frac{x^\beta}{\Gamma(\beta+1)})tanh(-2\frac{x^\beta}{\Gamma(\beta+1)})(1-\gamma+\gamma t), \end{cases} \tag{4.20}$$

$$\begin{cases} g_2(t,x) = 16sech^2(-2\frac{x^\beta}{\Gamma(\beta+1)})[4sech^2(-2\frac{x^\beta}{\Gamma(\beta+1)}) - 8tanh^2(-2\frac{x^\beta}{\Gamma(\beta+1)}) \\ \qquad +3tanh(-2\frac{x^\beta}{\Gamma(\beta+1)})]((1-\gamma)^2 + 2\gamma(1-\gamma)t + \gamma^2 t^2), \\ h_2(t,x) = -32sech^2(-2\frac{x^\beta}{\Gamma(\beta+1)})[-25sech^2(-2\frac{x^\beta}{\Gamma(\beta+1)}) + 40sech^2(-2\frac{x^\beta}{\Gamma(\beta+1)})tanh(-2\frac{x^\beta}{\Gamma(\beta+1)}) \\ \qquad -2tanh^2(-2\frac{x^\beta}{\Gamma(\beta+1)}) - 32tanh^3(-2\frac{x^\beta}{\Gamma(\beta+1)}) + 96tanh(-2\frac{x^\beta}{\Gamma(\beta+1)})]((1-\gamma)^2 + 2\gamma(1-\delta)t + \gamma^2 t^2). \end{cases} \tag{4.21}$$

The other iterative terms can be employed in precisely the same manner. As the solutions are defined by

$$
\begin{aligned}
g(t,x) &= g_0(t,x) + g_1(t,x) + g_2(t,x) + g_3(t,x) + ...,\\
h(t,x) &= h_0(t,x) + h_1(t,x) + h_2(t,x) + h_3(t,x) + ...
\end{aligned}
\tag{4.22}
$$

we acquire the desired following solutions:

$$
\begin{aligned}
g(t,x) = \frac{1}{2} - 8tanh(-2\frac{x^\beta}{\Gamma(\beta+1)}) - 8sech^2(-2\frac{x^\beta}{\Gamma(\beta+1)})(1-\gamma+\gamma t)\\
+16sech^2(-2\frac{x^\beta}{\Gamma(1+\beta)})[4sech^2(-2\frac{x^\beta}{\Gamma(1+\beta)}) - 8tanh^2(-2\frac{x^\beta}{\Gamma(1+\beta)})\\
+3tanh(-2\frac{x^\beta}{\Gamma(1+\beta)})]((1-\gamma)^2 + 2\gamma(1-\gamma)t + \gamma^2t^2) + \dots
\end{aligned}
\tag{4.23}
$$

$$
\begin{aligned}
h(t,x) = 16 - 16tanh^2(-2\frac{x^\beta}{\Gamma(\beta+1)}) - 32sech^2(-2\frac{x^\beta}{\Gamma(\beta+1)})tanh(-2\frac{x^\beta}{\Gamma(\beta+1)})(1-\gamma+\gamma t)\\
-32sech^2(-2\frac{x^\beta}{\Gamma(\beta+1)})[-25sech^2(-2\frac{x^\beta}{\Gamma(\beta+1)}) + 40sech^2(-2\frac{x^\beta}{\Gamma(\beta+1)})tanh(-2\frac{x^\beta}{\Gamma(\beta+1)})\\
-2tanh^2(-2\frac{x^\beta}{\Gamma(\beta+1)}) - 32tanh^3(-2\frac{x^\beta}{\Gamma(\beta+1)}) + 96tanh(-2\frac{x^\beta}{\Gamma(\beta+1)})]((1-\gamma)^2 + 2\gamma(1-\gamma)t + \gamma^2t^2) + \dots
\end{aligned}
\tag{4.24}
$$

After obtaining the approximate solutions for the considered system, they are represented in 3D with different fractional derivatives values γ and β. The surface plots can be seen in Fig. (7)-(8). Moreover, the behaviour of the solutions g and h to the problem (4.17-4.18) for different values of fractional derivatives are also drawn in 2D as follows.

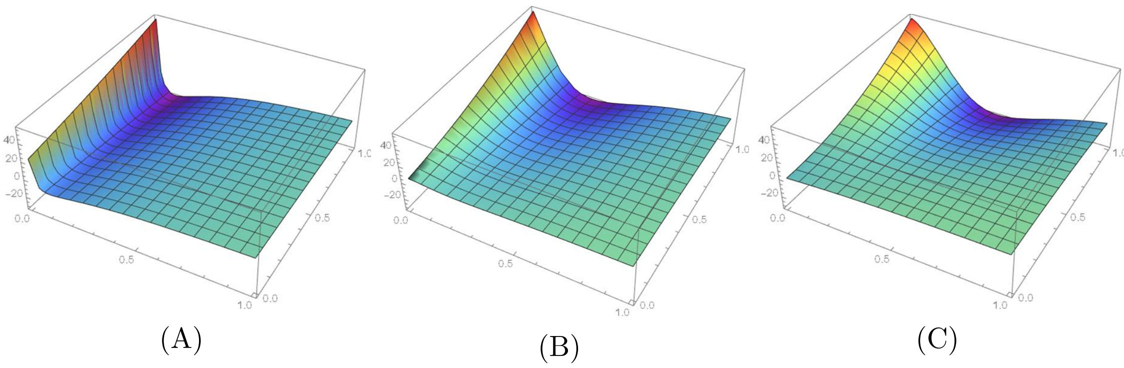
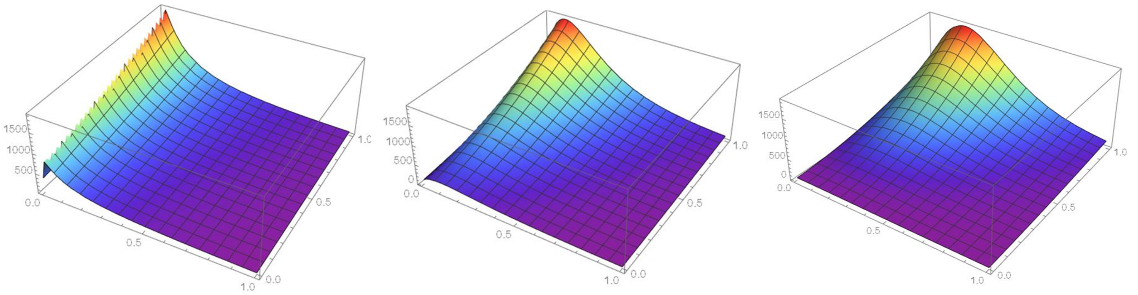
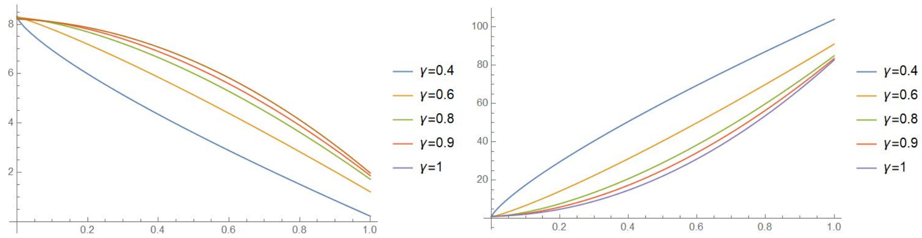


Figure 7: The approximate solution g of the problem (4.17-4.18) for $\gamma = \beta = 0.35, 0.7,$ and 0.95



(A) (B) (C)

Figure 8: The approximate solution h of the problem (4.17-4.18) for $\gamma = \beta = 0.35, 0.7,$ and 0.95



(A) (B)

Figure 9: The approximate solution g and h of the problem (4.17-4.18) in 2D

5. Convergence-Error Analysis and Numerical Findings

In this part, we provide sufficient conditions for convergence and provide an error analysis for the method described above. It is proposed that the solution is presented in the form that follows:

$$f(t,x) = \sum_{j=0}^{\infty} f_j(t,x). \tag{5.1}$$

Theorem 1. *Let $f(t,x)$ and $\sum_{j=0}^{\infty} f_j(t,x)$ be an exact and approximate solutions, respectively. then the solution (5.1) is convergent to the exact solution provided that there exists $0 < K < 1$ s.t*

$$||f_{j+1}(t,x)|| \le K||f_j(t,x)||, \tag{5.2}$$

for some j_0 s.t $\forall j \ge j_0$.

This convergence is also determined in another way.

Definition 1. *For all $j \in N$, K_j is defined as* $K_j = \begin{cases} \frac{||f_{j+1}(t,x)||}{||f_j(t,x)||} & ||f_j(t,x)|| \neq 0, \\ 0 & ||f_j(t,x)|| = 0. \end{cases}$

Theorem 2. *If* $0 \leq K_j < 1$ *for* $j = 0, 1, 2, ...,$ *then the approximate solution* $\sum_{j=0}^{\infty} f_j(t,x)$ *converges to the exact solution* $f(t,x)$.

The estimate for maximum absolute truncated error is calculated according to the theorem presented below.

Theorem 3. *Assume that* $\sum_{j=0}^{m} f_j(t,x)$ *is an approximate solution. Then the maximum absolute error between the exact and approximate solutions is defined as*

$$||f(t,x) - \sum_{j=0}^{m} f_j(t,x)|| \leq \frac{K^{m+1}}{1-K} ||f_0(t,x)||. \tag{5.3}$$

Now, the absolute error between the exact and approximate solutions to the problems (4.1), (4.9), and (4.17) are calculated and shown in the following Tab. (1)-(6), separately.

$\|g_{Ex.} - g_{Appr.}\|$				
t	$\gamma = \beta = 0.35$	$\gamma = \beta = 0.5$	$\gamma = \beta = 0.75$	$\gamma = \beta = 0.95$
0.1	1.50265×10^{-3}	1.95801×10^{-5}	6.14509×10^{-10}	5.32907×10^{-15}
0.2	2.04476×10^{-3}	2.92565×10^{-5}	9.93525×10^{-10}	7.10543×10^{-15}
0.4	3.69422×10^{-3}	6.02222×10^{-5}	2.27974×10^{-10}	1.42109×10^{-14}
0.6	6.21062×10^{-3}	108025×10^{-4}	4.29171×10^{-9}	2.84217×10^{-14}
0.8	9.63061×10^{-3}	173139×10^{-4}	7.03805×10^{-9}	4.44089×10^{-14}
1.0	1.39675×10^{-2}	2.55749×10^{-4}	1.05225×10^{-8}	6.75016×10^{-14}

Table 1: Absolute error between the exact and approximate solution g of the problem (4.1).

$\|h_{Ex.} - h_{Appr.}\|$				
t	$\gamma = \beta = 0.35$	$\gamma = \beta = 0.5$	$\gamma = \beta = 0.75$	$\gamma = \beta = 0.95$
0.1	5.91396×10^{-3}	1.34564×10^{-5}	9.90875×10^{-10}	5.72004×10^{-15}
0.2	2.00379×10^{-3}	7.71889×10^{-5}	4.07759×10^{-9}	2.87398×10^{-14}
0.4	1.56917×10^{-2}	3.57241×10^{-4}	1.65687×10^{-8}	1.05689×10^{-13}
0.6	4.05531×10^{-2}	8.41049×10^{-4}	3.74073×10^{-8}	2.38151×10^{-13}
0.8	7.65353×10^{-2}	1.52828×10^{-3}	6.64439×10^{-8}	4.19059×10^{-13}
1.0	1.23619×10^{-1}	2.41829×10^{-3}	1.03475×10^{-7}	6.48432×10^{-13}

Table 2: Absolute error between the exact and approximate solution h of the problem (4.1)

Tab. 1 and 2 show the absolute error between the exact and approximate solutions for the WBK system (4.1). As it is seen in the given tables, the absolute error decreases gradually as γ and β increase for both g and h. It reaches around 10^{-15} as $\gamma, \beta \to 1$ while the absolute error is around 10^{-3} as $\gamma, \beta \to 0.35$ for smaller t. Tab. 3 and 4 indicate that the absolute error also reaches around 10^{-14} and 10^{-13} as $\gamma, \beta \to 1$ while it is around 10^{-2} and 10^{-1} as $\gamma, \beta \to 0.35$ for g and h, respectively. If it is checked for the values of t, one can see easily that it increases as t goes from 0 to 1. For g, it gets at 1.3×10^{-2} for $t = 1.0$ while it is 1.5×10^{-3} for $t = 0.1$. For h, it attains to 1.2×10^{-1} while it is 5.9×10^{-3} for $t = 0.1$. Furthermore, it takes the values $\{5.3 \times 10^{-15}, 6.7 \times 10^{-14}\}$ for $t = \{0.1, 1.0\}$ as $\gamma, \beta \to 0.95$.

$\lvert g_{Ex.} - g_{Appr.} \rvert$				
t	$\gamma = \beta = 0.35$	$\gamma = \beta = 0.5$	$\gamma = \beta = 0.75$	$\gamma = \beta = 0.95$
0.1	1.88522×10^{-3}	2.06715×10^{-5}	5.53346×10^{-10}	5.50671×10^{-14}
0.2	2.48246×10^{-3}	2.77811×10^{-5}	7.20409×10^{-10}	8.88178×10^{-15}
0.4	3.52542×10^{-3}	4.21993×10^{-5}	1.12275×10^{-9}	1.42109×10^{-14}
0.6	4.51883×10^{-3}	5.8159×10^{-5}	1.65663×10^{-9}	2.66454×10^{-14}
0.8	5.51245×10^{-3}	7.62188×10^{-5}	2.36798×10^{-9}	3.90799×10^{-14}
1.0	6.52445×10^{-3}	9.67237×10^{-5}	3.31146×10^{-9}	5.50671×10^{-14}

Table 3: Absolute error between the exact and approximate solution g of the problem (4.9).

$\lvert h_{Ex.} - h_{Appr.} \rvert$				
t	$\gamma = \beta = 0.35$	$\gamma = \beta = 0.5$	$\gamma = \beta = 0.75$	$\gamma = \beta = 0.95$
0.1	2.47306×10^{-1}	2.44281×10^{-3}	3.06249×10^{-8}	3.33153×10^{-14}
0.2	2.74199×10^{-1}	2.94872×10^{-3}	4.49372×10^{-8}	7.2239×10^{-14}
0.4	3.09498×10^{-1}	3.72426×10^{-3}	7.39622×10^{-8}	1.75933×10^{-13}
0.6	3.34918×10^{-1}	4.36154×10^{-3}	1.0436×10^{-7}	3.27597×10^{-13}
0.8	3.55348×10^{-1}	4.92211×10^{-3}	1.36085×10^{-7}	5.18844×10^{-13}
1.0	3.72604×10^{-1}	5.42929×10^{-3}	1.68860×10^{-7}	7.48604×10^{-13}

Table 4: Absolute error between the exact and approximate solution h of the problem (4.9).

Tab. 3 and 4, the absolute errors between the exact and approximate solutions for the WBK system (4.9) are shown. As it is seen in these tables, the absolute error decreases gradually as γ and β increase for both g and h, same as in the system (4.1). Here, it also increases as t goes from 0 to 1. Moreover, Tab. 3 and 4 demonstrate that the absolute error also reaches around 10^{-14} and 10^{-13} as $\gamma, \beta \to 1$ while it is around 10^{-3} and 10^{-1} as $\gamma, \beta \to 0.35$ for f and g, respectively. For g, it reaches 6.5×10^{-3} for $t = 1.0$ while it is 1.8×10^{-3} for $t = 0.1$. For h, it gets at 3.7×10^{-1} for $t = 1$, whereas it is 2.4×10^{-1} for $t = 0.1$. Besides, it takes the values $\{3.3 \times 10^{-14}, 7.4 \times 10^{-13}\}$ for $t = \{0.1, 1.0\}$ as $\alpha, \beta \to 0.95$.

$\lvert g_{Ex.} - g_{Appr.} \rvert$				
t	$\gamma = \beta = 0.35$	$\gamma = \beta = 0.5$	$\gamma = \beta = 0.75$	$\gamma = \beta = 0.95$
0.1	1.81254×10^{-2}	1.78661×10^{-4}	2.28515×10^{-9}	3.55271×10^{-15}
0.2	2.00367×10^{-2}	2.15021×10^{-4}	3.4817×10^{-9}	8.88178×10^{-15}
0.4	2.25946×10^{-2}	2.7168×10^{-4}	6.03487×10^{-9}	2.13163×10^{-14}
0.6	2.4485×10^{-2}	3.19148×10^{-4}	8.8086×10^{-9}	4.26326×10^{-14}
0.8	2.60448×10^{-2}	3.6173×10^{-4}	1.17706×10^{-8}	6.75016×10^{-14}
1.0	2.73989×10^{-2}	4.01099×10^{-4}	1.48857×10^{-8}	1.03029×10^{-13}

Table 5: Absolute error between the exact and approximate solution g of the problem (4.17).

$\lvert h_{Ex.} - h_{Appr.} \rvert$				
t	$\gamma = \beta = 0.35$	$\gamma = \beta = 0.5$	$\gamma = \beta = 0.75$	$\gamma = \beta = 0.95$
0.1	2.37233×10^{-1}	2.33207×10^{-3}	3.0211×10^{-8}	3.88519×10^{-14}
0.2	2.61046×10^{-1}	2.79242×10^{-3}	4.52939×10^{-8}	9.60104×10^{-14}
0.4	2.92223×10^{-1}	3.50361×10^{-3}	7.74618×10^{-8}	2.68959×10^{-13}
0.6	3.14592×10^{-1}	4.09156×10^{-3}	1.12513×10^{-7}	5.31062×10^{-13}
0.8	3.32501×10^{-1}	4.61058×10^{-3}	1.50073×10^{-7}	8.72187×10^{-13}
1.0	3.47563×10^{-1}	5.08115×10^{-3}	1.89684×10^{-7}	1.29011×10^{-12}

Table 6: Absolute error between the exact and approximate solution h of the problem (4.17).

From Tab. 5 and 6, it is seen that the absolute errors between the exact and approximate solutions for the WBK system (4.17) also decrease rapidly as γ and β increase for both f and g. Note that it increases as t goes from 0 to 1, as well. Furthermore, Tab. 5 and 6 show that the absolute error also reaches around 10^{-13} and 10^{-12} as $\gamma, \beta \to 1$ while it is around 10^{-2} and 10^{-1} as $\gamma, \beta \to 0.35$ for f and g, respectively. For g, it reaches 6.5×10^{-3} for $t = 1.0$ while it is 1.8×10^{-3} for $t = 0.1$. For h, it gets at 3.7×10^{-1} for $t = 1$, whereas it is 2.4×10^{-1} for $t = 0.1$. Besides, it takes the values $\{3.3 \times 10^{-14}, 7.4 \times 10^{-13}\}$ for $t = \{0.1, 1.0\}$ as $\gamma, \beta \to 0.95$.

6. Conclusion

This paper presented a new approach to solving and analyzing the non-linear time fractional Whitham-Broer-Kaup (WBK) equations, employing the powerful Aboodh Decomposition Transform (ADT). This method successfully harnessed the power of the Aboodh transform with the Adomian decomposition method, which can efficiently handle non-linearity and fractional derivatives arising in the equations. By decomposing the solution into an infinite series of components, ADT effectively addressed the non-linear terms and facilitated the calculation of analytical and approximate solutions under different fractional derivative operators. The solutions are compared to different values of γ and β in both 3D and 2D, and the plots suggest that the approximate solutions converge to the exact ones as soon as 1 is reached. Numerical results demonstrate that used method is very effective and reliable to achieve an approximate solution for non-linear fractional partial differential equations. The comparative analysis of these solutions has provided valuable insights under the influence of fractional calculus regarding the system's behavior. The results demonstrate the versatility and effectiveness of the Aboodh Decomposition Transform in addressing fractional differential equations and offering a valuable contribution to the field of mathematical physics and non-linear dynamics. In the future, we intend solve some new fractional models, such as in [39–43] and make comparisons with other numerical methods [44–48].

Conflicts of Interest: The authors declare that they have no conflict of interest.

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