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A New Numerical Solution for Prabhakar Fractional Differential Equations Using the Explicit Fractional Adams Method

Mohammad Abdel Aal[1,*], Mohammad S. Ghatasheh[2], Hussien Albadawi[1], Shaher Momani[3]

[1] *Department of Mathematics, Faculty of Arts and Sciences, The World Islamic Sciences & Education University (W.I.S.E), Amman, Jordan*
[2] *Department of Basic Sciences, Faculty of Arts and Educational Sciences, Middle East University, Amman, Jordan*
[3] *Department of Mathematics, Faculty of Sciences, The University of Jordan, Amman, Jordan*

Abstract. This paper introduces a new way to solve Prabhakar fractional differential equations using an Explicit Fractional Adams Method. These equations are difficult to work with because they have several parameters. The new method, which combines the Explicit Fractional Adams Method with Lagrange interpolation, effectively tackles these difficulties. The paper also includes an analysis to show how well the method works. It provides several examples to demonstrate the method's effectiveness and compares it with other existing methods. The results show that the proposed method is efficient and easy to use.

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Key Words and Phrases: Lagrange interpolation, Prabhakar fractional, fractional differential equations, Adams method

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1. Introduction

Fractional calculus is an extension of traditional calculus that broadens the scope of calculus operations by enabling the differentiation and integration of functions to arbitrary, non-integer orders [1]. This concept has gained significant traction in various scientific and engineering disciplines because it often provides a more accurate representation of real-world systems than classical integer-order calculus [2]. In particular, Fractional calculus has demonstrated its utility in mathematical modelling across diverse fields, including

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Email addresses: mohammad.abdelaal@wise.edu.jo (M. Abdel Aal), mghatasheh@meu.edu.jo (M. S. Ghatasheh), hussien.albadawi@wise.edu.jo (H. Albadawi), S.momani@ju.edu.jo (S. Momani)

science, physics [3], engineering [4], and finance [5]. Its ability to capture memory and hereditary properties of materials and processes makes it especially valuable for modelling complex systems that exhibit non-local behaviors. Recent studies, such as the work on new properties of differential transform via difference equations, further illustrate the expanding toolkit and applications of fractional calculus in discrete and hybrid systems [6].

The practical application of fractional derivatives and integrals usually relies on well-established definitions, such as the Riemann–Louville [7] and Caputo approaches [8, 9]. These classical methods have successfully addressed a range of challenges in dynamic systems, such as the well-studied Rössler system, and have been utilized in scenarios involving tempered fractional calculus [10]. However, these traditional techniques primarily emphasize fractional orders and may not adequately address the complexities that arise in certain equations [11].

Given the inherent complexity and non-local nature of fractional differential equations (FDEs), numerical methods are indispensable. Advanced approaches such as the hyperbolic, rational, fractional, and logarithmic non-polynomial spline methods have been developed to improve accuracy, efficiency, and convergence. Similarly, α-fractional and β-fractional finite difference methods extend the classical FDM by incorporating fractional-order operators to capture memory effects. Analytical–numerical approaches, such as the homotopy perturbation method, have been successfully applied to various functional equations, including fuzzy pantograph problems [13]. Moreover, fixed point theory provides a rigorous framework for proving the existence and uniqueness of solutions to fractional and multidimensional systems [12, 14], further strengthening the mathematical foundation of these techniques.

One area of increasing interest is the Prabhakar fractional differential equations, which incorporate multiple parameters that introduce additional layers of complexity. The challenges often arise from their non-local character and the intricate interplay between parameters, making them difficult to solve with standard techniques. Prabhakar derivatives were addressed in [15] using the fractional Fourier transform, demonstrating Mittag-Leffler-Hyers-Ulam stability. In Ref. [17], the invariant subspace method was employed to obtain exact solutions for time-fractional nonlinear differential equations involving the regularized Prabhakar derivative. For commensurate systems, [18] presented closed-form solutions using eigenvectors and eigenvalues, while [19] applied separation of variables to reduce the problem to a Cauchy-type fractional equation, expressible via a two-variable Mittag-Leffler function. The study in [16] explored the Explicit Euler Method for a nonlinear fractional oscillation equation using finite-difference schemes. In this paper, we propose a novel application of the Explicit Fractional Adams Method to solve Prabhakar fractional differential equations, known for their complexity due to multiple parameters. By integrating Lagrange interpolation, the proposed method achieves high accuracy and computational efficiency, outperforming existing techniques such as HPTM and FHPTM. A detailed convergence analysis and multiple numerical examples confirm the method's robustness, with potential extensions using shifted Legendre polynomials.

This outline is organized as follows: In Section 2, we discuss basic definitions and important properties related to Prabhakar's fractional integral. We focus on the basic definitions and key properties of Prabhakar's fractional integral. It also explain how it relates to other fractional integrals. Section 3 looks at the concept of the Explicit Fractional Adams Method for solving Prabhakar fractional differential equations. This method relates to the Volterra integral equation and addresses the initial value problem. Section 5 provides examples to show how the new explicit fractional Adams method for the Prabhakar derivative works and how accurate it is. We completed all calculations using the software Maple 18. Finally, Section 6 provides the conclusion.

2. Basic Definitions and Important Properties Related to Prabhakar's Fractional Integral

This section provides basic definitions and important properties related to Prabhakar's fractional integral. This will help readers understand its significance in studying complex problems.

2.1. Some Preliminaries Definition Relating to Prabhakar fractional integral

Definition 1. The one-parameter, three-parameter, and five-parameter Mittag-Leffler functions are defined as follows:

$$\begin{aligned} M_{\alpha}(\phi) &= \sum_{n=0}^{\infty} \frac{\phi^n}{\Gamma(\alpha n+1)}, \quad Re(\alpha) > 0, \\ M^{\gamma}_{\alpha,\beta}(\phi) &= \sum_{n=0}^{\infty} \frac{\Gamma(\gamma+n)\phi^n}{\Gamma(\gamma)\Gamma(\alpha n+\beta)n!}, \quad Re(\alpha) > 0, \\ M^{\gamma}_{\alpha_1,\alpha_2,\beta_1,\beta_2}(\phi) &= \sum_{n=0}^{\infty} \frac{\Gamma(\gamma+n)\phi^n}{\Gamma(\gamma)\Gamma(\alpha_1 n+\beta_1)\Gamma(\alpha_2 n+\beta_2)n!}, \quad Re(\alpha) > 0, \end{aligned} \tag{1}$$

Definition 2. For $f \in L^1(a,b)$, the Prabhakar fractional integral is defined using a three-parameter Mittag-Leffler function as follows:

$$I^{\gamma}_{\alpha,\beta,\tau,a} + p(x) = \int_a^x (x-w)^{\beta-1} M^{\gamma}_{\alpha,\beta}(\tau(x-w)^{\alpha}) p(x)dw \tag{2}$$

such that $Re(\beta) > 0$, $Re(\alpha) > 0$ and $\gamma, \alpha, \beta, \tau \in \mathbb{C}$ is integer part of α.

The definition in (1) can typically be expressed as

$$I^{\gamma}_{\alpha,\beta,\tau,a} + p(x) = \int_a^x m^{\gamma}_{\alpha,\beta}(x-w);\tau)p(x)dw, \tag{3}$$

such that

$$m^{\gamma}_{\alpha,\beta}(x;\tau) = x^{\beta-1}M^{\gamma}_{\alpha,\beta}(\tau x^{\alpha}). \tag{4}$$

Employing equation (1), we can explain the Prabhakar fractional derivative and the regularized Prabhakar derivative

Definition 3. For values of β between 0 and 1, $(0 < \beta < 1)$, if $f(x) \in M^1[a,b]$ is a function defined on the interval $[a,b]$, we define the Prabhakar fractional derivative using the Riemann-Liouville method in the following way:

$${}^{R}D^{\gamma}_{\alpha,\beta,\tau,a} + p(x) = \frac{d^{\nu}}{dx^{\nu}}\int_a^x m^{\gamma}_{\alpha,\beta}(x-w);\tau)p(x)dw, \tag{5}$$

where $Re(\alpha) > 0$, $Re(\beta) > 0$ and $\gamma,\alpha,\beta,\tau \in \mathbb{C}$ is integer part of α.

Definition 4. If $p(x)$ is a function that belongs to the class $AC^{\nu}(a,b)$, and if a and b are numbers such that $0 \leq a < x < b \leq \infty$, then the regularized Prabhakar derivative, using the Caputo definition, is defined as follows:

$${}^{R}C^{\gamma}_{\alpha,\beta,\tau,a} + p(x) = \int_a^x m^{\gamma}_{\alpha,\beta}(x-w);\tau)\frac{d^{\nu}p(x)}{dx^{\nu}}dw, \tag{6}$$

where $Re(\alpha) > 0$, $Re(\beta) > 0$ and $\gamma,\alpha,\beta,\tau \in \mathbb{C}$ is integer part of α,β.

Definition 5. This research paper defines Prabhakar fractional differential equations as follows:

$${}^{R}C^{\gamma}_{\alpha,\beta,\tau,a} + p(x) = \int_{a+}^x m^{\gamma}_{\alpha,\beta}(x-w);\tau)\frac{d^{\nu}p(x)}{dx^{\nu}}dw, \tag{7}$$

where $Re(\alpha) > 0$, $Re(\beta) > 0$ and $\gamma,\alpha,\beta,\tau \in \mathbb{C}$ is integer part of α,β.

2.2. Prabhakar fractional integral Key Properties and Relationships with Other Functions

The Prabhakar function is closely related to the standard Mittag-Leffler function and its two-parameter version, as well as other special functions. For values of $n \geq$, the following applies:

$$\varphi_n = \frac{\Gamma(\varphi+n)}{\Gamma(\varphi)} = \varphi(\varphi+1)....(\varphi+n-1) \tag{8}$$

When $\varphi = 1$, the equation

$$M^{0}_{\alpha,\beta} = \frac{1}{\Gamma(\beta)}. \tag{9}$$

Equation (9) holds true. When $\alpha = \beta = \varphi = 1$, we get the well-known two-parameter maximum likelihood function, written as;

$$M^{1}_{\alpha,\beta}(\phi) = M_{\alpha,\beta}(\phi) = \sum_{n=0}^{\infty}\frac{\phi^n}{\Gamma(\alpha n+\beta)}. \tag{10}$$

we then obtain the classic exponential function,

$$M_{1,1}^{1} = e^{\tau}. \tag{11}$$

The Prabhakar function is different from the two-parameter Mittag-Leffler function because it has a third parameter, ϕ. This difference means we need to examine the properties related to this additional parameter. An important formula from Prabhakar's original work in Ref. [23] allows for simplifying the third parameter. It is written as;

$$M_{\alpha,\beta}^{\varphi+1}(\phi) = \frac{M_{\alpha,\beta-1}(\phi) + (1-\beta+\alpha\varphi)M_{\alpha,\beta}(\phi)}{\alpha\varphi}, \tag{12}$$

Additionally, another simplification formula was derived later in [32] and appears as

$$M_{\alpha,\beta}^{\varphi+1}(\phi) = \frac{M_{\alpha,\beta-1}(\phi) + (1-\beta+\alpha\varphi)M_{\alpha,\beta}(\phi)}{\alpha\phi\varphi}, \tag{13}$$

where $\varphi \neq 0$.

We can then utilize both formulas to simplify the value of an additional parameter. These results become clearer when φ is an integer. For instance, let $\varphi = n \in \mathbb{N}$, where $\mathbb{N}$ is a natural number. In this scenario, we have

$$M_{\alpha,\beta}^{n+1}(\phi) = \frac{1}{\alpha^n N!}\sum_{i=0}^{n} a_i^{(n)} M_{\alpha,\beta-i}(\phi), \tag{14}$$

and

$$M_{\alpha,\beta}^{n+1}(\phi) = \frac{1}{\alpha^n \phi^n N!}\sum_{i=0}^{n} a_i^{(n)} M_{\alpha,\beta-\alpha n-i}(\phi), \tag{15}$$

where $\varphi \neq 0$. This implies that $M_{\alpha,\beta}^{n+1}$ from the right-side of (15) can be expressed as a combination of two-parameter ML functions. The coefficients $a_i^{(n)}$ in equations (15) and ((15) are derived from the recursive formula:

$$a_i^{(n)} = \begin{cases} (1+\alpha-\beta)a_0^{(n-1)} & for \quad i = 0 \\ a_i^{(n)} + (1+\alpha-\beta+i)a_i^{(n-1)} & for \quad i = 1, ..., n-1, \\ 1 & for \quad i = n. \end{cases} \tag{16}$$

3. The Concept of the Explicit Fractional Adams Method for solving Prabhakar fractional differential equations

In this section, we discuss the Explicit Fractional Adams Method. This method relates to the Volterra integral equation and addresses the initial value problem.

3.1. The Explicit Fractional Adams Method

The Volterra integral equation connects to the initial value problem as follows:

Definition 5. The explicit fractional Adams method that is related to the Volterra integral equation and addresses the initial value problem is define here as:

$$p(x) = p_0 + \int_0^x (x-w)^{\beta-1} M_{\alpha,\beta}^{\gamma}(\tau(x-w)^{\alpha})\mu(v;p(x))dw. \tag{17}$$

Then, further break down the integral from equation (17) as follows:

$$\begin{aligned} p(x_{k+1}) = p_{(x_k)} &+ \int_0^{x_{k+1}} (x_{k+1}-w)^{\beta-1} M_{\alpha,\beta}^{\gamma}(\tau(x_{k+1}-w)^{\alpha})\mu(v;p(x))dw \\ &- \int_0^{x_k} (x_k-w)^{\beta-1} M_{\alpha,\beta}^{\gamma}(\tau(x_k-w)^{\alpha})\mu(v;p(x))dw \end{aligned} \tag{18}$$

Using three-point Lagrange interpolation, equation (18) can be expressed in a different form.

$$\begin{aligned} p(x_{k+1}) = p_{(x_k)} &+ \int_0^{x_{k+1}} (x_{k+1}-w)^{\beta-1} M_{\alpha,\beta}^{\gamma}(\tau(x_{k+1}-w)^{\alpha}) \Bigg[\frac{(x-x_{k-1})(x-x_{k-2})}{(x_k-x_{k-1})(x_k-x_{k-2})} G_k \\ &+ \frac{(x-x_k)(x-x_{k-2})}{(x_{k-1}-x_k)(x_{k-1}-x_{k-2})} G_{k-1} + \frac{(x-x_{k-1})(x-x_{k-1})}{(x_{k-2}-x_k)(x_{k-2}-x_{k-1})} G_{k-2} \Bigg] dx \\ &- \int_0^{x_k} (x_k-w)^{\beta-1} M_{\alpha,\beta}^{\gamma}(\tau(x_k-w)^{\alpha}) \Bigg[\frac{(x-x_{k-1})(x-x_{k-2})}{(x_k-x_{k-1})(x_k-x_{k-2})} G_k \\ &+ \frac{(x-x_k)(x-x_{k-2})}{(x_{k-1}-x_k)(x_{k-1}-x_{k-2})} G_{k-1} + \frac{(x-x_{k-1})(x-x_{k-1})}{(x_{k-2}-x_k)(x_{k-2}-x_{k-1})} G_{k-2} \Bigg] dx. \end{aligned} \tag{19}$$

The first integral presented in equation (18) can be expressed in a more concise form as follows:

$$
\begin{aligned}
&\int_0^{x_{k+1}}(x_{k+1}-w)^{\beta-1}M^{\gamma}_{\alpha,\beta}(\tau(x_{k+1}-w)^{\alpha})\Bigg[\frac{(x-x_{k-1})(x-x_{k-2})}{(x_k-x_{k-1})(x_k-x_{k-2})}G_k\\
&+\frac{(x-x_k)(x-x_{k-2})}{(x_{k-1}-x_k)(x_{k-1}-x_{k-2})}G_{k-1}+\frac{(x-x_{k-1})(x-x_{k-1})}{(x_{k-2}-x_k)(x_{k-2}-x_{k-1})}G_{k-2}\Bigg]dx\\
&=\Bigg[\frac{G_k}{2h^2}\int_0^{x_{k+1}}(x_{k+1}-w)^{\beta-1}M^{\gamma}_{\alpha,\beta}(\tau(x_{k+1}-w)^{\alpha})(x-x_{k-1})(x-x_{k-2})dx\\
&-\frac{G_{k-1}}{h^2}\int_0^{x_{k+1}}(x_{k+1}-w)^{\beta-1}M^{\gamma}_{\alpha,\beta}(\tau(x_{k+1}-w)^{\alpha})(x-x_{k-1})(x-x_{k-2})dx\\
&+\frac{G_{k-1}}{h^2}\int_0^{x_{k+1}}(x_{k+1}-w)^{\beta-1}M^{\gamma}_{\alpha,\beta}(\tau(x_{k+1}-w)^{\alpha})(x-x_{k-1})(x-x_{k-1})\Bigg]dx\\
&=\frac{G_x}{2h^2}B_1-\frac{G_x}{h^2}B_2+\frac{G_{x-2}}{2h^2}B_3.
\end{aligned}
\tag{20}
$$

Setting $\int_0^x(x-w)^{\beta-1}M^{\gamma}_{\alpha,\beta}(\tau(x-w)^{\alpha})p^{u-1}dp=\Gamma(u)x^{\beta+u-1}M^{\gamma}_{\alpha,\beta}(\tau(x-w)^{\alpha})$, so that the specific formula is:

$$
\begin{aligned}
Z_1&=\int_0^{x_{k+1}}(x_{k+1}-w)^{\beta-1}M^{\gamma}_{\alpha,\beta}(\tau(x_{k+1}-w)^{\alpha})(x-x_{k-1})(x-x_{x-2})dx\\
&=\Gamma(3)(x_{k+1}-w)^{\beta+2}M^{\gamma}_{\alpha,\beta+3}(\tau(x_{k+1}-w)^{\alpha})-(x_{k-2}+x_{k-1})\Gamma(2)(x_{k+1})^{\beta+1}M^{\gamma}_{\alpha,\beta}(\tau(x_{k+1}-w)^{\alpha})\\
&+x_{k-1}x_{k-2}\Gamma(1)(x_{k+1})^{\beta}M^{\gamma}_{\alpha,\beta}(\tau(x_{k+1}-w)^{\alpha})\\
&=(k+1)^{\beta}h^{\beta+2}\Bigg[2(k+1)^2M^{\gamma}_{\alpha,\beta+3}(\tau(xh+h)^{\alpha})-(2k-3)(k+1)M^{\gamma}_{\alpha,\beta+2}(\tau(xh+h)^{\alpha})\\
&+(k-2)(k-1)M^{\gamma}_{\alpha,\beta+1}(\tau(xh+h)^{\alpha}\Bigg].
\end{aligned}
\tag{21}
$$

By employing the same established procedure, we successfully obtained.

$$
\begin{aligned}
Z_2=(k+1)^{\beta}h^{\beta+2}&\Bigg[2(k+1)^2M^{\gamma}_{\alpha,\beta+3}(\tau(xh+h)^{\alpha})-(2k-2)(k+1)M^{\gamma}_{\alpha,\beta+2}(\tau(xh+h)^{\alpha})\\
&+(k-2)(k-1)M^{\gamma}_{\alpha,\beta+1}(\tau(xh+h)^{\alpha}\Bigg].
\end{aligned}
\tag{22}
$$

and ditto to;

$$Z_3 = (k+1)^{\beta} h^{\beta+2} \Bigg[2(k+1)^2 M^{\gamma}_{\alpha,\beta+3}(\tau(xh+h)^{\alpha}) - (2k-1)(k+1) M^{\gamma}_{\alpha,\beta+2}(\tau(xh+h)^{\alpha}) + (k-2)(k-1) M^{\gamma}_{\alpha,\beta+1}(\tau(xh+h)^{\alpha} \Bigg]. \tag{23}$$

Using the established procedure, we successfully obtained the second integral in equation (18), which can be rewritten as:

$$\begin{aligned} &\int_0^{x_{k+1}} (x_{k+1}-w)^{\beta-1} M^{\gamma}_{\alpha,\beta}(\tau(x_{k+1}-w)^{\alpha}) \Bigg[\frac{(x-x_{k-1})(x-x_{k-2})}{(x_k-x_{k-1})(x_k-x_{k-2})} G_k \\ &+ \frac{(x-x_k)(x-x_{k-2})}{(x_{k-1}-x_k)(x_{k-1}-x_{k-2})} G_{k-1} + \frac{(x-x_k)(x-x_{k-1})}{(x_{k-2}-x_k)(x_{k-2}-x_{k-1})} G_{k-2} \Bigg] dx \\ &= \Bigg[\frac{G_k}{2h^2} \int_0^{x_{k+1}} (x_{k+1}-x)^{\beta-1} M^{\gamma}_{\alpha,\beta}(\tau(x_{k+1}-w)^{\alpha})(x-x_{k-1})(x-x_{k-2}) dx \\ &- \frac{G_{k-1}}{h^2} \int_0^{x_{k+1}} (x_{k+1}-x)^{\beta-1} M^{\gamma}_{\alpha,\beta}(\tau(x_{k+1}-w)^{\alpha})(x-x_{k-1})(x-x_{k-2}) dx \\ &+ \frac{G_{k-1}}{2h^2} \int_0^{x_{k+1}} (x_{k+1}-x)^{\beta-1} M^{\gamma}_{\alpha,\beta}(\tau(x_{k+1}-w)^{\alpha})(x-x_{k-1})(x-x_{k-1}) \Bigg] dx \\ &= \frac{G_x}{2h^2} B_1 - \frac{G_x}{h^2} B_2 + \frac{G_{x-2}}{2h^2} B_3. \end{aligned} \tag{24}$$

$$Z_4 = (k)^{\beta} h^{\beta+2} \Bigg[2k^2 M^{\gamma}_{\alpha,\beta+3}(\tau(xh)^{\alpha}) - (2k-3)k M^{\gamma}_{\alpha,\beta+2}(\tau(xh)^{\alpha}) + (k-2)(k-1) M^{\gamma}_{\alpha,\beta+1}(\tau(xh)^{\alpha} \Bigg]. \tag{25}$$

$$Z_5 = (k)^{\beta} h^{\beta+2} \Bigg[2k^2 M^{\gamma}_{\alpha,\beta+3}(\tau(xh)^{\alpha}) - (2k-2)k M^{\gamma}_{\alpha,\beta+2}(\tau(xh)^{\alpha}) + (k-2)(k-1) M^{\gamma}_{\alpha,\beta+1}(\tau(xh)^{\alpha} \Bigg]. \tag{26}$$

and ditto to;

$$Z_6 = (k)^{\beta} h^{\beta+2} \Bigg[2k^2 M^{\gamma}_{\alpha,\beta+3}(\tau(xh)^{\alpha}) - (2k-2)k M^{\gamma}_{\alpha,\beta+2}(\tau(xh)^{\alpha}) + (k-1)(k-1) M^{\gamma}_{\alpha,\beta+1}(\tau(xh)^{\alpha} \Bigg]. \tag{27}$$

A new method for solving the Prabhakar fractional differential equation is presented below:

$$p(x_{k+1}) = p(x_k) + \frac{G_x}{2h^2}B_1 - \frac{G_x}{h^2}B_2 + \frac{G_{x-2}}{2h^2}B_3 - \frac{G_{x-2}}{2h^2}B_4 + \frac{G_{x-2}}{h^2}B_5 - \frac{G_{x-2}}{2h^2}B_6. \tag{28}$$

In this paper, we seek to address the Prabhakar fractional differential equation, which is defined as follows:

$$D^{\gamma}_{\alpha,\beta,\tau}p(x) = f(x, p(x)), \tag{29}$$

subject to the initial condition $p(0) = p_0$, where $\tau \in R$ and $\alpha, \beta, \gamma \in (0, 1)$.

4. Convergence Analysis

In this subsection, we present the convergence analysis for the proposed numerical methods addressing the Prabhakar fractional differential equation. The following theorem summarizes our findings.

Theorem 1. *Let $p(x)$ represent the solution to the initial value problem outlined in equation (29). The error function associated with this solution is denoted as $\Re(t, \beta, k)$. Using the explicit fractional Adams method, we can calculate the numerical solution for the problem described in (29)). This approach involves approximating $p(x)$ based on previous values and using the error function to assess the accuracy of the solution.*

$$\begin{aligned} p(x_{k+1}) = p_{(x_k)} &+ \int_0^{x_{k+1}} (x_{k+1} - w)^{\beta-1} M^{\gamma}_{\alpha,\beta}(\tau(x_{k+1} - w)^{\alpha}) \left[\frac{(x - x_{k-1})(x - x_{k-2})}{(x_k - x_{k-1})(x_k - x_{k-2})} G_k \right. \\ &\left. + \frac{(x - x_k)(x - x_{k-2})}{(x_{k-1} - x_k)(x_{k-1} - x_{k-2})} G_{k-1} + \frac{(x - x_{k-1})(x - x_{k-1})}{(x_{k-2} - x_k)(x_{k-2} - x_{k-1})} G_{k-2} \right] dx \\ &- \int_0^{x_k} (x_k - w)^{\beta-1} M^{\gamma}_{\alpha,\beta}(\tau(x_k - w)^{\alpha}) \left[\frac{(x - x_{k-1})(x - x_{k-2})}{(x_k - x_{k-1})(x_k - x_{k-2})} G_k \right. \\ &\left. + \frac{(x - x_k)(x - x_{k-2})}{(x_{k-1} - x_k)(x_{k-1} - x_{k-2})} G_{k-1} + \frac{(x - x_{k-1})(x - x_{k-1})}{(x_{k-2} - x_k)(x_{k-2} - x_{k-1})} G_{k-2} \right] dx + \Re(t, \beta, k), \end{aligned} \tag{30}$$

such that $||\Re(t, \beta, k)||_{\infty} < \nu$.

Proof.

Based on the (18) and (19), we arrived at;

$$\begin{aligned}
p(x_{k+1}) &= p_{(x_k)} + \int_0^{x_{k+1}} (x_{k+1}-w)^{\beta-1} M^{\gamma}_{\alpha,\beta}(\tau(x_{k+1}-w)^{\alpha})\mu(v;p(x))dw \\
&- \int_0^{x_k} (x_k-w)^{\beta-1} M^{\gamma}_{\alpha,\beta}(\tau(x_k-w)^{\alpha})\mu(v;p(x))dw \\
&= p_{(x_k)} + \int_0^{x_{k+1}} (x_{k+1}-w)^{\beta-1} M^{\gamma}_{\alpha,\beta}(\tau(x_{k+1}-w)^{\alpha}) \Bigg[\frac{(x-x_{k-1})(x-x_{k-2})}{(x_k-x_{k-1})(x_k-x_{k-2})} G_k \\
&+ \frac{(x-x_k)(x-x_{k-2})}{(x_{k-1}-x_k)(x_{k-1}-x_{k-2})} G_{k-1} + \frac{(x-x_{k-1})(x-x_{k-1})}{(x_{k-2}-x_k)(x_{k-2}-x_{k-1})} G_{k-2} + \frac{G^{(k+1)}(x)}{(k+1)!} \prod_{i=1}^{k} (x-x_i) dx \Bigg] \\
&- \int_0^{x_k} (x_k-w)^{\beta-1} M^{\gamma}_{\alpha,\beta}(\tau(x_k-w)^{\alpha}) \Bigg[\frac{(x-x_{k-1})(x-x_{k-2})}{(x_k-x_{k-1})(x_k-x_{k-2})} G_k \\
&+ \frac{(x-x_k)(x-x_{k-2})}{(x_{k-1}-x_k)(x_{k-1}-x_{k-2})} G_{k-1} + \frac{(x-x_{k-1})(x-x_{k-1})}{(x_{k-2}-x_k)(x_{k-2}-x_{k-1})} G_{k-2} + \frac{G^{(k+1)}(x)}{(k+1)!} \prod_{i=1}^{k-1} (x-x_i) dx \Bigg] \\
&= x(x_k) + \mathbf{L}(x,\beta,x) + \Re(x,\beta,x).
\end{aligned} \tag{31}$$

The functions $\mathbf{L}(x,\beta,x)$ and $\Re(x,\beta,x)$ are defined as follows:

$$\begin{aligned}
\mathbf{L}(x,\beta,x) = \int_0^{x_{k+1}} & (x_{k+1}-w)^{\beta-1} M^{\gamma}_{\alpha,\beta}(\tau(x_{k+1}-w)^{\alpha}) \Bigg[\frac{(x-x_{k-1})(x-x_{k-2})}{(x_k-x_{k-1})(x_k-x_{k-2})} G_k \\
&+ \frac{(x-x_k)(x-x_{k-2})}{(x_{k-1}-x_k)(x_{k-1}-x_{k-2})} G_{k-1} + \frac{(x-x_{k-1})(x-x_{k-1})}{(x_{k-2}-x_k)(x_{k-2}-x_{k-1})} G_{k-2} \Bigg] dx \\
&- \int_0^{x_k} (x_k-w)^{\beta-1} M^{\gamma}_{\alpha,\beta}(\tau(x_k-w)^{\alpha}) \Bigg[\frac{(x-x_{k-1})(x-x_{k-2})}{(x_k-x_{k-1})(x_k-x_{k-2})} G_k \\
&+ \frac{(x-x_k)(x-x_{k-2})}{(x_{k-1}-x_k)(x_{k-1}-x_{k-2})} G_{k-1} + \frac{(x-x_{k-1})(x-x_{k-1})}{(x_{k-2}-x_k)(x_{k-2}-x_{k-1})} G_{k-2} \Bigg] dx,
\end{aligned} \tag{32}$$

and

$$\begin{aligned}
\Re(x,\beta,x) &= \int_0^{x_{k+1}} (x_{k+1}-w)^{\beta-1} M^{\gamma}_{\alpha,\beta}(\tau(x_{k+1}-w)^{\alpha}) \left[\frac{G^{(k+1)}(x)}{(k+1)!} \prod_{i=1}^{k} (x-x_i) \right] dx \\
&- \int_0^{x_{k+1}} (x_{k+1}-w)^{\beta-1} M^{\gamma}_{\alpha,\beta}(\tau(x_{k+1}-w)^{\alpha}) \left[\frac{G^{(k+1)}(x)}{(k+1)!} \prod_{i=1}^{k-1} (x-x_i) \right] dx.
\end{aligned} \tag{33}$$

It now remains for us to show that the proposed scheme is convergent. We do this by ensuring that $||\Re(t,\beta,k)||_{\infty} < \nu$.

$$
\begin{aligned}
||\Re(t,\beta,k)||_\infty &= \Bigg|\Bigg| \int_0^{x_{k+1}} (x_{k+1}-w)^{\beta-1} M^{\gamma}_{\alpha,\beta}(\tau(x_{k+1}-w)^{\alpha}) \left[\frac{G^{(k+1)}(x)}{(k+1)!}\prod_{i=1}^{k}(x-x_i)\right] dx \\
&- \int_0^{x_{k+1}} (x_{k+1}-w)^{\beta-1} M^{\gamma}_{\alpha,\beta}(\tau(x_{k+1}-w)^{\alpha}) \left[\frac{G^{(k+1)}(x)}{(k+1)!}\prod_{i=1}^{k-1}(x-x_i)\right] dx \Bigg|\Bigg|_\infty \\
&< \Bigg|\Bigg| \int_0^{x_{k+1}} (x_{k+1}-w)^{\beta-1} M^{\gamma}_{\alpha,\beta}(\tau(x_{k+1}-w)^{\alpha}) \left[\frac{G^{(k+1)}(x)}{(k+1)!}\prod_{i=1}^{k}(x-x_i)\right] dx \Bigg|\Bigg|_\infty \\
&+ \Bigg|\Bigg| \int_0^{x_{k}} (x_{k}-w)^{\beta-1} M^{\gamma}_{\alpha,\beta}(\tau(x_{k}-w)^{\alpha}) \left[\frac{G^{(k)}(x)}{(k)!}\prod_{i=1}^{k-1}(x-x_i)\right] dx \Bigg|\Bigg|_\infty \\
&< \max_{x\in[0,x_k]} \frac{|G^{(k+1)}(x)|}{(k+1)!} \Bigg|\Bigg| \prod_{i=1}^{k}(x-x_i) \Bigg|\Bigg|_\infty \int_0^{x_{k+1}} (x_{k+1}-w)^{\beta-1} M^{\gamma}_{\alpha,\beta}(\tau(x_{k+1}-w)^{\alpha}) dx \\
&+ \max_{x\in[0,x_k]} \frac{|G^{(k)}(x)|}{k!} \Bigg|\Bigg| \prod_{i=1}^{k}(x-x_i) \Bigg|\Bigg|_\infty \int_0^{x_{k}} (x_{k}-w)^{\beta-1} M^{\gamma}_{\alpha,\beta}(\tau(x_{k}-w)^{\alpha}) dx \\
&= \max_{x\in[0,x_k]} \frac{|G^{(k+1)}(x)|}{(k+1)!} \Bigg|\Bigg| \prod_{i=1}^{k}(x-x_i) \Bigg|\Bigg|_\infty x^{\beta}_{k+1} M^{\gamma}_{\alpha,\beta}(\tau(x^{k+1}_{\beta})) \\
&+ \max_{x\in[0,x_k]} \frac{|G^{(k)}(x)|}{(k)!} \Bigg|\Bigg| \prod_{i=1}^{k}(x-x_i) \Bigg|\Bigg|_\infty x^{\beta}_{k} M^{\gamma}_{\alpha,\beta}(\tau(x^{K}_{\beta})) \\
&< \sup_{x\in[0,x_{k+1}]} \left(\max_{x\in[0,x_k]} \frac{|G^{(k+1)}(x)|}{(k+1)!}, \max_{x\in[0,x_{k+1}]} \frac{|G^{(k)}(x)|}{(k)!} \right) \\
&\times \left[\frac{k!h^{k+1}}{4} x^{\beta}_{k+1} M^{\gamma}_{\alpha,\beta+1}(\tau(x^{\alpha}_{K+1})) + \frac{(k-1)!h^{k}}{4} x^{\beta}_{k} M^{\gamma}_{\alpha,\beta+1}(\tau(x^{\alpha}_{K})) \right] < \nu.
\end{aligned} \tag{34}
$$

5. Numerical Application of Tempered Fractional Differential Equations with Respect to Another Function

This section provides examples to show how the new explicit fractional Adams method for the Prabhakar derivative works and how accurate it is. We completed all calculations using the software Maple 18.

Example 1. We look at the simple nonlinear fractional differential equation defined using the Prabhakar derivative [22]:

$$ {}^{R}D^{\gamma}_{\alpha,\beta,\tau,a} + p(x) = g(x) - u^2(x). \tag{35} $$

Given the initial condition that

$$p(0) = 0, \tag{36}$$

so that

$$g(t) = 2\,t - 3\,t^2 - t^3 \\ + \left(2\,t\left(1 - \beta + \frac{\beta\,t^\beta}{\Gamma\left(\beta+2\right)}\right) - 3\,t^2\left(2\,t\left(1 - \beta + 2\,\frac{\beta\,t^\beta}{\Gamma\left(\beta+3\right)}\right) + t^3\left(1 - \beta + 6\,\frac{\beta\,t^\beta}{\Gamma\left(\beta+4\right)}\right)\right)\right)^2, \tag{37}$$

The exact solution is;

$$p(t) = \left(2\,t\left(1 - \beta + \frac{\beta\,t^\beta}{\Gamma\left(\beta+2\right)}\right) - 6\,t^3\left(1 - \beta + 2\,\frac{\beta\,t^\beta}{\Gamma\left(\beta+3\right)}\right) + t^3\left(1 - \beta + 6\,\frac{\beta\,t^\beta}{\Gamma\left(\beta+4\right)}\right)\right)^2. \tag{38}$$

Solution: To check how accurate the explicit fractional Adams method from Theorem 1 is, we use it to estimate the Prabhakar derivative derivative for Example 1. We applied the five-parameter Mittag-Leffler functions as defined:

For

$${}^{R}D^{\gamma}_{\alpha,\beta,\tau,a} + p(t) = \sum_{k=0}^{\infty} \frac{\Gamma\left(-r+k\right) w^k \Gamma\left(p+1\right) t^{\alpha\,k-\beta+p}}{\Gamma\left(\alpha\,k-\beta+p+1\right)\Gamma\left(-r\right)\Gamma\left(k\right)k} \tag{39}$$

where $\beta = 0.99, \alpha = 1, w = 1, r = 1$

So that,

$${}^{R}D^{\gamma}_{\alpha,\beta,\tau,a} + p(x) = -\frac{32\,t^{\frac{7}{4}}}{315\,\Gamma\left(\frac{3}{4}\right)\pi}\left(7\,\sqrt{t}\sqrt{2}\left(\Gamma\left(\frac{3}{4}\right)\right)^2 {}_2\mathrm{F}_2(\frac{1}{4},\frac{3}{4};\frac{3}{2},\frac{13}{4};t) - 15\,{}_2\mathrm{F}_2(-\frac{1}{4},\frac{1}{4};\frac{1}{2},\frac{11}{4};t)\pi\right) \tag{40}$$

Table 1: The comparison between the Homotopy Perturbation Transform Method (HPTM) and proposed solutions at when $\beta = 0.99$

t	Exact	HPTM solutions	Absolute Error	
0.01	0.0001	0.0003	0.0000	0.00000E+00
0.02	0.0004	0.0008	0.0005	1.69000E-01
0.03	0.0009	0.0015	0.0012	2.72000E-01
0.04	0.0015	0.0023	0.0021	3.03000E-01
0.05	0.0024	0.0034	0.0031	2.56000E-01
0.06	0.0034	0.0046	0.0044	1.25000E-01
0.07	0.0046	0.0059	0.0058	9.60000E-0
0.08	0.0059	0.0075	0.0073	4.13000E-01
0.09	0.0074	0.0091	0.0090	8.32000E-01

We used the proposed method to get the results shown in Table 1. This Table compares our results with the iteration method (HPTM) in Ref. [22], using a step size of $h = 0.01$. Our numerical results show that even with just a few terms (N = 2), we achieved good results compared to the method in Ref. [22].

Figure 1 shows a comparison between the exact equation and the approximate solution. This comparison demonstrates that the proposed method is accurate, as the approximate solution closely matches the exact values. Also, Figure 1 shows the results for different values of the parameter β: 0.95, 0.75, and 0.55, as explained in Example 1. Changing the value of β has a significant effect on the process. This demonstrates how adjustments in this parameter can influence the results and confirms that the proposed method is strong enough to work in various situations.

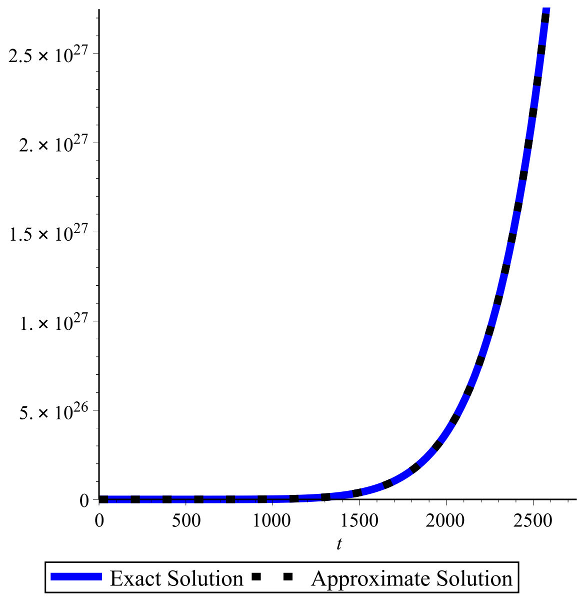

Figure 1: The results of the comparison between the approximate and exact solutions for Example 1

Table 2 compares our results with those obtained from the Fractional Homotopy Perturbation Transform Method (FHPTM) when $\beta = 0.95$. Our findings indicate that we achieved favourable results compared to the method described in Ref. [22]. The numerical results demonstrate that our proposed method is an efficient algorithm when compared to the HPTM and FHPTM methods discussed in Ref. [22].

Example 2.

We look at nonlinear fractional differential equation defined using the Prabhakar derivative;

$$^{R}D^{\gamma}_{\alpha,\beta,\tau,a} + p(x) = g(x) - u^2(x). \tag{41}$$

Given the initial condition that

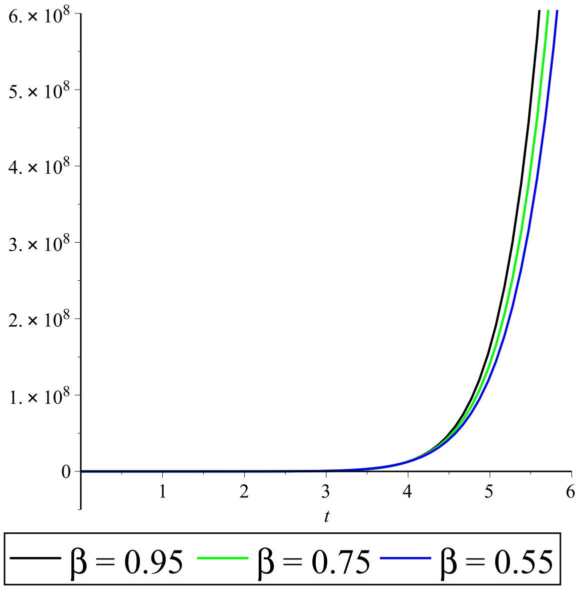

Figure 2: The results of the comparison between the approximate and exact solutions for Example 1

Table 2: The comparison between the Homotopy Perturbation Transform Method (HPTM), Fractional Homotopy Perturbation Transform Method (FHPTM) and proposed solutions at when $\beta = 0.95$

t	Exact	HPTM	FHPTM	Approximate solution	Absolute Error
0.01	0.0001	0.0011	0.0008	0.000001	1.33339E+00
0.02	0.0004	0.0024	0.0017	0.000006	9.18072E-01
0.03	0.0009	0.0039	0.0027	0.000016	5.18988E-01
0.04	0.0015	0.0056	0.0039	0.000034	1.57268E-01
0.05	0.0024	0.0074	0.0052	0.000062	1.59033E-01
0.06	0.0034	0.0093	0.0065	0.000101	4.25645E-01
0.07	0.0046	0.0115	0.0080	0.000155	6.39892E-01
0.08	0.0059	0.0137	0.0096	0.000225	7.99914E-01
0.09	0.0074	0.0161	0.0113	0.000316	9.04328E-01

$$p(0) = 0 \tag{42}$$

So that

$$g(x) = (2t - 3t^2 + t^3)^2 + \frac{2t^{1-\beta}}{\Gamma(2-\beta)} - \frac{6t^{2-\beta}}{\Gamma(3-\beta)} + \frac{6t^{3-\beta}}{\Gamma(4-\beta)}. \tag{43}$$

The exact solution is;

$$p(t) = (-t^3 - 3t^2 + 2t)^2. \tag{44}$$

Solution:

To check how accurate the explicit fractional Adams method from Theorem 1 is, we use it to estimate the Prabhakar derivative derivative for Example 2. We applied the five-parameter Mittag-Leffler functions as defined, where $\beta = 0.95, \alpha = 1, w = 1, r = 1$

So that,

$$^{R}D^{\gamma}_{\alpha,\beta,\tau,a}+p(x) = \left(-t^3 - 3\,t^2 + 2\,t\right)^2+2.054433730\,t^{0.05}-5.869810658\,t^{1.05}+2.863322272\,t^{2.05} \tag{45}$$

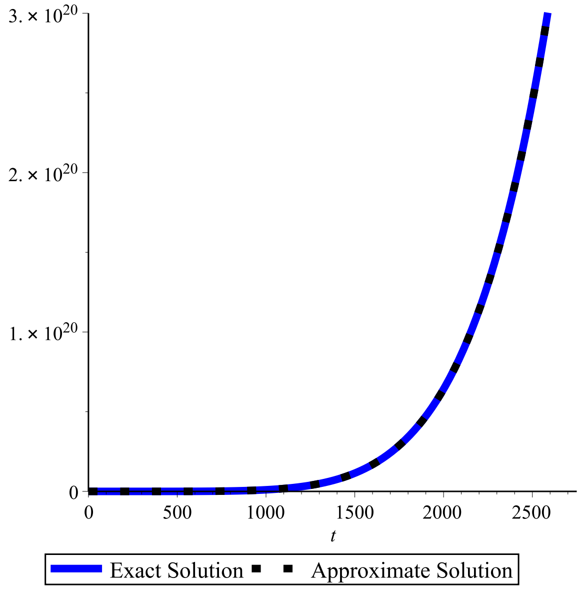

Figure 3: The results of the comparison between the approximate and exact solutions for Example 2

Figure 3 presents a comparison between the exact equation and the approximate solution. This analysis demonstrates the accuracy of the proposed method, as the approximate solution shows a close alignment with the exact values.

Table 4 presents a comprehensive comparison between the approximate and exact solutions, with particular emphasis on the associated error rates. A thorough analysis, alongside established methodologies in the field, reveals that the proposed approach not only markedly reduces computational costs but also delivers improved efficiency when compared to solutions within both the approximate and exact frameworks. This reduction in computational expense is vital for applications where resources and time are constrained.

Example 3. We look at nonlinear fractional differential equation defined using the

Table 3: The comparison between the exact solutions and the absolute error for Example 2

t	Exact solutions	Absolute error	Absolute error
0.1	1.3619489510	0.0285610000	1.33339E+00
0.2	0.9920555733	0.0739840000	9.18072E-01
0.3	0.6107966552	0.0918090000	5.18988E-01
0.4	0.2228042575	0.0655360000	1.57268E-01
0.5	-0.1434077531	0.0156250000	1.59033E-01
0.6	-0.4164291000	0.0092160000	4.25645E-01
0.7	-0.4693227850	0.1705690000	6.39892E-01
0.8	-0.1076897890	0.6922240000	7.99914E-01
0.9	0.9425527730	1.8468810000	9.04328E-01
1.0	3.0479453440	4.0000000000	9.52055E-01

Prabhakar derivative in Ref. [25];

$$^{R}D^{\gamma}_{\alpha,\beta,\tau,a} + p(x) = g(x) - u^2(x). \tag{46}$$

Given the initial condition that

$$p(0) = 0 \tag{47}$$

So that

$$g(x) = \frac{2t^{2-\beta}}{\Gamma(3-\beta)} + t^2. \tag{48}$$

The exact solution is;

$$p(t) = t^2. \tag{49}$$

Solution:
(48) is solved to be;

$$1.956603553t^{1.05} + t^2. \tag{50}$$

Figure 4 illustrates a comparative analysis between the exact equation and the approximate solution derived through the proposed method. This analysis demonstrates the accuracy of the method, as there is a strikingly close alignment between the approximate solutions and the exact values, indicating that the approach effectively captures the underlying mathematical behaviour. In addition, Figure 5 showcases the results obtained for varying values of the parameter β, specifically 0.95, 0.75, and 0.55, as detailed in Example 3. The variations in β significantly influence the overall process, revealing how changes in this parameter can impact the outcome and further validating the robustness of the proposed method in accommodating different scenarios.

6. Conclusion

In our study, we present an approach to solve Prabhakar fractional differential equations through the Explicit Fractional Adams Method. The inherent complexity of these equations, due to the multitude of parameters involved, poses significant challenges. However, our innovative numerical scheme effectively addresses these issues by utilizing Lagrange interpolation within the Explicit Fractional Adams Method. We provide a convergence analysis of our newly developed scheme, supported by several illustrative examples that clearly demonstrate its superior effectiveness compared to existing solutions. Our findings confirm that this method is not only highly efficient but also remarkably straightforward to implement, revealing its considerable potential for practical applications. Furthermore, we strongly advocate for further research to explore additional enhancements to

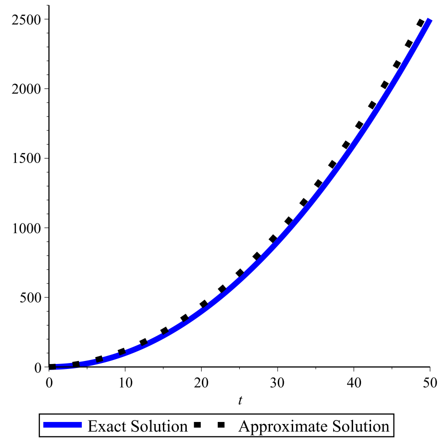

Figure 4: The results of the comparison between the approximate and exact solutions for Example 3

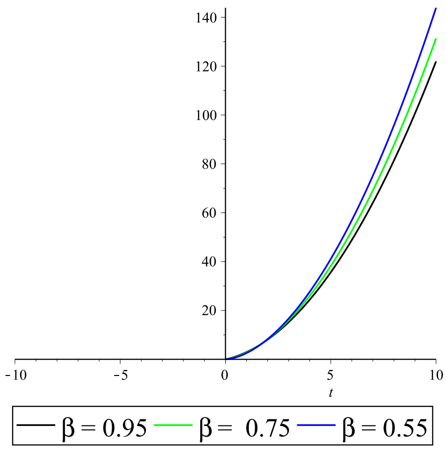

Figure 5: The results of the comparison for different value of β for Example 3

Table 4: The comparison between the exact solutions and the absolute error for Example 3

t	Exact solutions	Absolute error	Absolute Error
0.1	0.210	0.010	2.000E-01
0.2	0.440	0.040	4.000E-01
0.3	0.690	0.090	6.000E-01
0.4	0.960	0.160	8.000E-01
0.5	1.250	0.250	1.000E+00
0.6	1.560	0.360	1.200E+00
0.7	1.890	0.490	1.400E+00
0.8	2.240	0.640	1.600E+00
0.9	2.610	0.810	1.800E+00
1.0	3.000	1.000	2.000E+00

this method, particularly in conjunction with the shifted Legendre polynomials technique. Such advancements will undoubtedly lead to a more robust and comprehensive solution for the tempered fractional problem, with far-reaching implications across diverse fields such as physics, engineering, and finance.

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