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A Boundary-Value Problem with Caputo-Hadamard Fractional Derivative: Analysis and Numerical Solution

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Abstract. We investigate a boundary-value problem governed by a fractional differential equation, which is non-linear. The fractional derivative is the combined Caputo-Hadamard fractional derivative. We establish the required conditions for the existence and uniqueness of solutions, utilising the two standard fixed-point theorems, Banach fixed-point and Sadovskii fixed-point. Furthermore, we address and analyse the problem's stability through demanding conditions using the Ulam-Hyers and Ulam-Hyers-Rassias stability methods. To illustrate the theoretical results, we present an example that validates the existence, uniqueness, and stability criteria. The analytical solution obtained from the problem is discretised with fractional rectangular, $L_{\ln,1}$ interpolation on a non-uniform mesh. The resulting system of non-linear equations is solved using the Newton-Raphson method, incorporating the Jacobian matrix to couple $\eta(t)$ and $\theta(t, \eta(t))$ non-linearly. The stability and reliability of the suggested numerical approach are examined through illustrative examples.

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Key Words and Phrases: Lane-Emden fractional differential equation, existance, uniqueness, stability, numerical solution

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1. Introduction

It is an indisputable fact that fractional differential equations (FDEs) are crucial for modeling complex systems where traditional integer-order equations fall short, see [1, 2]. They offer a more flexible and accurate framework for governing processes with non-local behaviors and irregular dynamics. This makes FDEs valuable across various fields, including physics, biology, engineering, and finance, where systems exhibit non-linear and complex characteristics. The ability of FDEs to capture a wider range of phenomena makes them an essential tool for advancing our understanding of intricate real-world problems. See [3–8] and references therein.

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The classical Lane-Emden equation, rooted in astrophysics, was pioneered by [9] and [10] to model the equilibrium of self-gravitating polytropic gas spheres. Its dimensionless form,

$$\eta''(t) + \frac{\mu}{t}\eta'(t) = -\eta^n, \qquad 0 < t \leq 1, \mu > 0, \tag{1}$$

governs density profiles in stars, where $\eta(t)$ represents scaled density and n is the polytropic index [11]. Chandrasekhar's seminal work formalized its role in stellar structure theory, linking solutions (polytropes) to configurations of stars and gaseous planets. Prialnik in his book [12] further contextualized its applications in stellar evolution, particularly in modeling pre-main-sequence stars and degenerate cores. Beyond astrophysics, the equation's singular term t^{-1} and nonlinearity η^n inspired adaptations in plasma physics and radiative cooling. The Lane-Emden fractional differential equation (LEFDE) emerged as a modern extension, replacing integer derivatives with fractional operators to incorporate memory effects and anomalous transport. A generalized form,

$$D^{\beta+\alpha}\eta(t) + \frac{\mu}{t}D^{\alpha}\eta(t) = \theta(t, \eta(t)), \qquad 0 < \alpha, \beta < 1, \tag{2}$$

addresses non-local dynamics in systems like viscoelastic collapsing clouds or turbulent plasmas. $D^{\beta+\alpha}$ and D^{α} are the fractional derivatives of order $\beta + \alpha$ and α, respectively. While [11], [13] and [12] focus on classical theory, recent studies leverage fractional calculus to resolve discrepancies in observational data, such as non-isothermal collapse in molecular clouds. This fractional framework retains the singular coefficient t^{-1} but introduces flexibility in modeling multi-scale phenomena, bridging gaps between classical polytropic assumptions and complex astrophysical systems.

Lately, the study of initial and boundary-value problems that are governed by LEFDE has garnered substantial attention from many researchers. Ibrahim in [14] studied the existence of the non-linear LEFDE

$$D^{\beta}\left(D^{\alpha} + \frac{\mu}{t}\right)\eta(t) = \theta(t, \eta(t)), \tag{3}$$

for $0 < \alpha, \beta \leq 1, 0 < t \leq 1$, subject to the boundary conditions $\eta(0) = \eta(t_1) = \eta(1) = 0$ for some $t_1 \in (0, 1)$. The same author in [15] studied the stability of the linear LEFDE

$$D^{\beta}\left(D^{\alpha} + \frac{\mu}{t}\right)\eta(t) = \theta(t), \tag{4}$$

for $0 < \alpha, \beta \leq 1, 0 < t \leq 1$, with boundary conditions $\eta(0) = l_1$ and $\eta(1) = l_2$, where l_1 and l_2 are constants. In [16] the authors used a combination of Chebyshev wavelets and a finite difference approaches to numerically solve the LEFDE

$$D^{\alpha}\eta(t) + \frac{\mu}{t^{\alpha-\beta}}D^{\beta}\eta(t) = \theta(t, \eta(t)), \tag{5}$$

for $1 < \alpha \leq 2, 0 < \beta \leq 1, 0 < t \leq 1$, subject to the initial or boundary conditions. In a study by [17], they considered the LEFDE in an n-dimensional system where each equation consists of two arbitrary differential orders in terms of Caputo fractional derivative. They proved the existence and uniqueness using Krasnoselskii and Banach's fixed point theorems and they showed that the system is both Ulam–Hyers and Ulam-Hyers-Rassias stable according to the proposed conditions. The existence and stability of LEFDE has been studied by [18] using advanced monotonicity, concentration-compactness, and Sobolev-type inequalities techniques. In [19], the author presents a modern analytical method for solving nonlinear singular type of LEFDE with Liouville–Caputo derivatives, focusing on the conditions that are proving the existence and uniqueness of solutions. The author combines techniques in fractional calculus alongside with advanced analytical methods to establish the conditions under which these solutions exist and are unique. The stability of solutions for two classes of LEFDE has been studied by [20] thought Lyapunov's direct method while the existence and uniqueness of solutions are demonstrated using Banach's fixed-point theory. The solution of LEFDE analytically studied by [21] using the techniques of power series approaches. Most recently, [22] explored the LEFDE with Caputo derivatives. The author managed to study the existence and uniqueness of mild solutions undergoing the Bielecki-type norm. In addition, the author manifested that, the mild solution is Ulam-Hyres type stable.

Motivated by the above studies, we examine the existence, uniqueness, and stability of the fractional Lane-Emden Boundary-Value Problem. We consider the nonlinear Lane-Emden equation with multiple fractional derivatives:

$$ {}_{\mathrm{CH}}D^{\beta}\left({}_{\mathrm{CH}}D^{\alpha}+\frac{\mu}{t}\right)\eta(t)=\theta(t,\eta(t)), \tag{6}$$

supplemented with boundary conditions:

$$\eta(a)=\xi_1, \tag{7}$$

$$\eta(1)=\xi_2, \tag{8}$$

where $\mu,\xi_1,\xi_2\in R$, $t\in[a,1]$ and $0<a<1$. In equation (6), α and β are positive non-integer numbers less than one; we obtain the classical Lane-Emden equation for $\alpha=\beta=1$. The two functions $\eta(t)$ and $\theta(t,\eta(t))$ belong to the set of all continuous function on $[a,1]$, i.e. $\eta(t),\theta(t,\eta(t))\in \mathrm{C}\left([a,1],R\right)$. Here ${}_{\mathrm{CH}}D^{\beta}$ represents the Caputo-Hadamard fractional derivative of order β as stated in Definition 3.

We study the existence, uniqueness, and stability of the problem (6-8) on the interval $[a,1]$ such that $1/a$ is finite. In addition, we assemble a numerical scheme, by applying the fractional rectangular, $L_{\ln,1}$ interpolation on a logarithmic grid to approximate the derived analytical solution (18). The resulting system of nonlinear equation solved by the Newton-Raphson method using the Jacobian matrix to handle the non-linear coupling between $\eta(t)$ and $\theta(t,\eta(t))$.

Numerical approaches to solve Caputo-Hadamard fractional differential equations are complicated due to the logarithmic kernel, which is weakly singular. Recently, some researchers are investigating the numerical schemes that can be considered. In a study by [23], they examined finite difference methods for Caputo-Hadamard derivative fractional differential equations. In their study, the analogous Volterra integral equations were approximated via fractional rectangle, $L_{log,1}$ interpolation when they used the modified predictor–corrector for Caputo-Hadamard fractional differential equations. [24] used the local discontinuous Galerkin method to solve boundary-value problems with the Caputo-Hadamard fractional derivative. Furthermore, [25] numerically solved Caputo–Hadamard fractional differential equations with the graded meshes.

The organization of this paper is as follows: Section 2 presents the essential definitions, theorems, remarks, and lemmas that support our current investigation. Section 3 addresses the existence and uniqueness of the solution for the problem (6-8). Section 4 examines the Ulam-Hyers and Ulam-Hyers-Rassias stability results, whilst Section 5 presents the numerical scheme using Newton-Raphson method to solve the Caputo-Hadamard fractional problem. Examples along with relevant graphs are presented in Sections 4-5.

2. Preliminaries

This section presents key definitions, lemmas, and remarks that form the foundational concepts for this study and will be referenced and served as the groundwork for the analysis and discussions in this study.

Definition 1. *[7] For a given function $\eta(t)$, the left-sided fractional integral of order $\beta > 0$ of Hadamard type is defined as*

$$ {}_{H}I_{a}^{\beta}\eta(t) = \frac{1}{\Gamma(\beta)} \int_{a}^{t} \left(\ln \left(\frac{t}{\tau} \right) \right)^{\beta-1} \frac{\eta(\tau)}{\tau} \, \mathrm{d}\tau, \tag{9} $$

where $\Gamma(\cdot)$ refers to the Euler Gamma function.

Definition 2. *[7] For a given function $\eta(t)$, the left-sided fractional derivative of order $\beta > 0$ $(n-1 < \beta < n \in \mathbb{N})$ of Hadamard type is defined as*

$$ {}_{H}D_{a}^{\beta}\eta(t) = \frac{1}{\Gamma(n-\beta)} \delta^{n} \int_{a}^{t} \left(\ln \left(\frac{t}{\tau} \right) \right)^{n-\beta-1} \frac{\eta(\tau)}{\tau} \, \mathrm{d}\tau, \tag{10} $$

where $\delta^{n} = \left(t\frac{\mathrm{d}}{\mathrm{d}t}\right)^{n}$.

With changing the order of integration and derivative in (10) we arrive at the following.

Definition 3. *[26] If $\eta(t)$ belongs to $\mathrm{C}([a,1],\mathbb{R})$, then, the left-sided fractional derivative of order $\beta > 0$ of Caputo-Hadamard type is defined as*

$$ {}_{\mathcal{CH}}D_{a}^{\beta}\eta(t) = \frac{1}{\Gamma(n-\beta)} \int_{a}^{t} \left(\ln \left(\frac{t}{\tau} \right) \right)^{n-\beta-1} \delta^{n}\eta(\tau) \frac{\mathrm{d}\tau}{\tau}, \tag{11} $$

where $\delta^{n} = \left(\tau\frac{\mathrm{d}}{\mathrm{d}\tau}\right)^{n}$ and $n-1 < \beta < n$.

Lemma 1. *[27] For a given function $\eta(t)$, if $\beta > 0$ $(n-1 < \beta < n)$ where n is an integer greater than 0, then, the Hadamard integral operates on the Caputo-Hadamard derivatives as*

$$
{}_{H}I^{\beta}_{a^+}\ {}_{\mathcal{CH}}D^{\beta}_{a^+}\ \eta(t) = \eta(t) + c_0 + c_1 \ln\left(\frac{t}{a}\right) + c_2\left(\ln\left(\frac{t}{a}\right)\right)^2 + \cdots + c_{n-1}\left(\ln\left(\frac{t}{a}\right)\right)^{n-1}, \tag{12}
$$

where $c_i = \frac{\delta^i \eta(a)}{i!}, i = 0, 1, \ldots, n-1,$ are constants.

Lemma 2. *[27] For $\beta > 0$ and $\alpha > 0$, we have*

$$
{}_{H}I^{\beta}_{a^+}\left(\ln\left(\frac{t}{a}\right)\right)^{\alpha} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\beta+1)}\left(\ln\left(\frac{t}{a}\right)\right)^{\alpha+\beta}. \tag{13}
$$

Theorem 1. *[3] Let ϕ_1 and ϕ_2 be two opertors such that their combined mapping $\phi_1 + \phi_2$ forms a k^*set-contraction for $0 \le k^* < 1$. This map is also condensing if the following conditions are satisfied:*

I- $\phi_1, \phi_2 : \mathbb{B} \subseteq \mathbb{Q} \to \mathbb{Q}$ are operators on the Banach space $\mathbb{Q}$ where $\mathbb{Q} = \mathrm{C}([a,1],\mathbb{R})$.

II- ϕ_1 is k^*-contractive, that is, $\|\phi_1(x) - \phi_1(y)\| \le k^*\|x-y\|$ for all x and y in the domain and fixed $k^* \in [0,1)$.

III- ϕ_2 is compact.

Theorem 2. *[3] (Sadovskii Fixed Point Theorem) For a subset $\mathbb{B}$ of a Banach space $\mathbb{Q}$ which is convex, bounded, and closed the condensing map $\phi : \mathbb{B} \to \mathbb{B}$ has a fixed point.*

Theorem 3. *[28] (Banach Fixed Point Theorem). For a continuous operator $\phi : \mathbb{B} \to \mathbb{B}$ which is k^*-contractive there is a unique fixed point.*

Definition 4. *[29] The boundary-value problem (6-8) is Ulam-Hyers stable if there exists a real constant $c_h > 0$ such that for any $\epsilon > 0$, and for every solution $\eta(t) \in \mathbb{Q}$ of the inequality*

$$
\left| {}_{\mathcal{CH}}D^{\beta}\left({}_{\mathcal{CH}}D^{\alpha} + \frac{\mu}{t}\right)\eta(t) - \theta(t,\eta(t))\right| \le \epsilon, \tag{14}
$$

there exists a solution $Z(t) \in \mathbb{Q}$ of the problem (6-8) with

$$
|\eta(t) - Z(t)| \le c_h \epsilon, \tag{15}
$$

for all $t \in [a,1]$.

Remark 1. *[30] A function $\eta(t) \in \mathbb{Q}$ is a solution of the inequality (14) if and only if there exists a function $g(t) \in \mathbb{Q}$ (which is dependent on η), such that:*

(i) $|g(t)| \le \epsilon$ for all $t \in [a,1]$,

(ii) ${}_{\mathcal{H}}D^{\beta}\left({}_{\mathcal{H}}D^{\alpha}+\frac{\mu}{t}\right)\eta(t)=\theta(t,\eta(t))+g(t)$.

Definition 5. *[29] The boundary-value problem (6-8) has the stability of the type Ulam-Hyers-Rassiass if there exists a real constant $c_\lambda > 0$ such that for any $\epsilon > 0$, and for every solution $\eta(t) \in \mathbb{Q}$ of the inequality*

$$\left|{}_{\mathcal{H}}D^{\beta}\left({}_{\mathcal{H}}D^{\alpha}+\frac{\mu}{t}\right)\eta(t)-\theta(t,\eta(t))\right| \leq \epsilon\psi(t), \tag{16}$$

where $\psi(t) \in \mathbb{Q}$ is the control function, positive and non-decreasing, there exists a solution $Z(t) \in \mathbb{Q}$ of the problem (6-8) with

$$|\eta(t) - Z(t)| \leq c_\lambda \epsilon \psi(t), \tag{17}$$

for all $t \in [a, 1]$.

Remark 2. *[30] A function $\eta(t) \in \mathbb{Q}$ is a solution of the inequality (16) if and only if there exists a function $g(t) \in \mathbb{Q}$ (which is dependent on η), such that:*

(i) $|g(t)| \leq \epsilon\psi(t)$ *for all* $t \in [a, 1]$,

(ii) ${}_{\mathcal{H}}D^{\beta}\left({}_{\mathcal{H}}D^{\alpha}+\frac{\mu}{t}\right)\eta(t)=\theta(t,\eta(t))+g(t)$.

Lemma 3. *The solution of the boundary-value problem (6-8) is a function $\eta(t)$ that belongs to $\mathrm{C}([a,1],\mathbb{R})$ which takes the following form:*

$$\begin{aligned}\eta(t) = \frac{1}{\Gamma(\beta+\alpha)}\int_a^t \left(\ln\left(\frac{t}{\tau}\right)\right)^{\beta+\alpha-1}\frac{\theta(\tau,\eta(\tau))}{\tau}\mathrm{d}\tau - \frac{\mu}{\Gamma(\alpha)}\int_a^t\left(\ln\left(\frac{t}{\tau}\right)\right)^{\alpha-1}\frac{\eta(\tau)}{\tau^2}\mathrm{d}\tau \\ -\frac{\left(\ln\left(\frac{t}{a}\right)\right)^{\alpha}}{b\Gamma(\beta+\alpha)}\int_a^1\left(\ln\left(\frac{1}{\tau}\right)\right)^{\beta+\alpha-1}\frac{\theta(\tau,\eta(\tau))}{\tau}\mathrm{d}\tau \\ +\frac{\mu\left(\ln\left(\frac{t}{a}\right)\right)^{\alpha}}{b\Gamma(\alpha)}\int_a^1\left(\ln\left(\frac{1}{\tau}\right)\right)^{\alpha-1}\frac{\eta(\tau)}{\tau^2}\mathrm{d}\tau + \Phi(t),\end{aligned} \tag{18}$$

where $\Phi(t) = \xi_1 + \frac{1}{b}(\xi_2-\xi_1)\left(\ln\left(\frac{t}{a}\right)\right)^{\alpha}$ and $b = \left(\ln\left(\frac{1}{a}\right)\right)^{\alpha}$.

Proof By implementing the fractional Hadamard integral operator of order β, ${}^{H}_{a}I^{\beta}$, given by Definition 1 on both sides of (6) and following Lemma 1 and Definition 1 we arrive at

$$D^{\alpha}\eta(t) = \frac{1}{\Gamma(\beta)}\int_a^t\left(\ln\left(\frac{t}{\tau}\right)\right)^{\beta-1}\frac{\theta(\tau,\eta(\tau))}{\tau}\mathrm{d}\tau - \frac{\mu}{t}\eta(t) + \bar{c}, \tag{19}$$

where $\bar{c}$ is a constant to be determined. Next, we operate both sides of (19) by the fractional Hadamard integral operator of order α, ${}^{H}_{a}I^{\alpha}$, through Lemma 1, to provide the following general solution

$$\eta(t)+\underline{c}=\frac{1}{\Gamma(\beta+\alpha)}\int_a^t\left(\ln\left(\frac{t}{\tau}\right)\right)^{\beta+\alpha-1}\frac{\theta(\tau,\eta(\tau))}{\tau}\mathrm{d}\tau$$
$$-\frac{\mu}{\Gamma(\alpha)}\int_a^t\left(\ln\left(\frac{t}{\tau}\right)\right)^{\alpha-1}\frac{\eta(\tau)}{\tau^2}\mathrm{d}\tau+\frac{\bar{c}}{\Gamma(\alpha+1)}\left(\ln\left(\frac{t}{a}\right)\right)^{\alpha}. \tag{20}$$

With applying the two boundary conditions (7-8) on equation (20) we obtain $\underline{c}=-\xi_1$ and

$$\bar{c}=\frac{\xi_2\Gamma(\alpha+1)}{b}-\frac{\xi_1\Gamma(\alpha+1)}{b}-\frac{\Gamma(\alpha+1)}{b\Gamma(\beta+\alpha)}\int_a^1\left(\ln\left(\frac{1}{\tau}\right)\right)^{\beta+\alpha-1}\frac{\theta(\tau,\eta(\tau))}{\tau}\mathrm{d}\tau$$
$$+\frac{\alpha\mu}{b}\int_a^1\left(\ln\left(\frac{1}{\tau}\right)\right)^{\alpha-1}\frac{\eta(\tau)}{\tau^2}\mathrm{d}\tau. \tag{21}$$

After plugging in the known values of $\bar{c}$ and $\underline{c}$ into (20), it turns out that the solution to the boundary-value problem (6-8) is given by (18). Through direct computation, the converse is obtained. The proof for Lemma 3 has been finalised.

3. Existence and Uniqueness of Solution

In this section, first we apply the Banach fixed point theorem to show the existence and uniqueness of the boundary-value problem (6-8), and then, the Sadovoskii fixed point theorem for the existence of the solution to our problem. Let $\mathbb{B}_r=\{\eta(t)\in\mathbb{Q}:\|\eta\|\le r\}$ be a closed, bounded, and convex subset of $\mathbb{Q}=\mathrm{C}([a,1],\mathbb{R})$ where r is a positive constant such that

$$r\ge\frac{\frac{2b\Omega}{\Gamma(\beta+\alpha+1)}\left(\ln\left(\frac{1}{a}\right)\right)^{\beta}+\|\Phi\|}{1-\frac{2b\mu}{a\Gamma(\alpha+1)}},\qquad 2b\mu<a\Gamma(\alpha+1). \tag{22}$$

Here $\mathbb{Q}$ is a Banach space of all continuous function from $[a,1]$ to $\mathbb{R}$ with the norm

$$\|\eta\|=\sup_t\{|\eta(t)|,t\in[a,1]\}\ \text{ for all } \eta\in\mathbb{Q}. \tag{23}$$

We proceed with Section 3 under the following two assumptions, namely $\mathbb{H}_1$ and $\mathbb{H}_2$, which state:

$\mathbb{H}_1$: There exists a constant $\Omega>0$, such that $|\theta(t,\eta(t))|\le\Omega$ for all $t\in[a,1]$ and $\eta\in\mathbb{Q}$.

$\mathbb{H}_2$: There exists a constant $k>0$, such that $\|\theta(t,\eta_1)-\theta(t,\eta_2)\|\le k\|\eta_1-\eta_2\|$ for all $t\in[a,1]$ and η_1, $\eta_2\in\mathbb{Q}$ if:

$$0<\frac{k}{\Gamma(\beta+\alpha+1)}\left(\ln\left(\frac{1}{a}\right)\right)^{\beta}+\frac{\mu}{a\Gamma(\alpha+1)}<\frac{1}{2b}.$$

We initiate by calling the operator $\phi:\mathbb{Q}\longrightarrow\mathbb{Q}$ to be set as:

$$\phi\left(\eta(t)\right)=\frac{1}{\Gamma(\beta+\alpha)}\int_a^t\left(\ln\left(\frac{t}{\tau}\right)\right)^{\beta+\alpha-1}\frac{\theta(\tau,\eta(\tau))}{\tau}\mathrm{d}\tau$$

$$
\begin{aligned}
&- \frac{\mu}{\Gamma(\alpha)} \int_a^t \left(\ln\left(\frac{t}{\tau} \right) \right)^{\alpha-1} \frac{\eta(\tau)}{\tau^2} \mathrm{d}\tau \\
&- \frac{\left(\ln\left(\frac{t}{a}\right)\right)^{\alpha}}{b\Gamma(\beta+\alpha)} \int_a^1 \left(\ln\left(\frac{1}{\tau} \right) \right)^{\beta+\alpha-1} \frac{\theta(\tau,\eta(\tau))}{\tau} \mathrm{d}\tau \\
&+ \frac{\mu\left(\ln\left(\frac{t}{a}\right)\right)^{\alpha}}{b\Gamma(\alpha)} \int_a^1 \left(\ln\left(\frac{1}{\tau} \right) \right)^{\alpha-1} \frac{\eta(\tau)}{\tau^2} \mathrm{d}\tau + \Phi(t), \qquad t \in [a,1]. \qquad (24)
\end{aligned}
$$

It has to be shown that ϕ has a fixed point. The fixed point is a solution to the boundary-value problem (6-8). We have to manifest that $\phi\left(\mathbb{B}_r\right) \subset \mathbb{B}_r$. For any $\eta \in \mathbb{B}_r$:

$$
\begin{aligned}
\|\phi(\eta)\| = \operatorname*{Sup}_t \Bigg| & \frac{1}{\Gamma(\beta+\alpha)} \int_a^t \left(\ln\left(\frac{t}{\tau} \right) \right)^{\beta+\alpha-1} \frac{\theta(\tau,\eta(\tau))}{\tau} \mathrm{d}\tau \\
&- \frac{\mu}{\Gamma(\alpha)} \int_a^t \left(\ln\left(\frac{t}{\tau} \right) \right)^{\alpha-1} \frac{\eta(\tau)}{\tau^2} \mathrm{d}\tau \\
&- \frac{\left(\ln\left(\frac{t}{a}\right)\right)^{\alpha}}{b\Gamma(\beta+\alpha)} \int_a^1 \left(\ln\left(\frac{1}{\tau} \right) \right)^{\beta+\alpha-1} \frac{\theta(\tau,\eta(\tau))}{\tau} \mathrm{d}\tau \\
&+ \frac{\mu\left(\ln\left(\frac{t}{a}\right)\right)^{\alpha}}{b\Gamma(\alpha)} \int_a^1 \left(\ln\left(\frac{1}{\tau} \right) \right)^{\alpha-1} \frac{\eta(\tau)}{\tau^2} \mathrm{d}\tau + \Phi(t) \Bigg|, \qquad t \in [a,1]. \qquad (25)
\end{aligned}
$$

Utilizing the assumption $\mathbb{H}_1$ and the condition (22), we arrive at:

$$
\|\phi(\eta)\| \leq \Omega\varphi_1 + r\varphi_2 + \|\Phi\| \leq r, \tag{26}
$$

where $\varphi_1 = \frac{2b}{\Gamma(\beta+\alpha+1)} \left(\ln\left(\frac{1}{a}\right)\right)^{\beta}$ and $\varphi_2 = \frac{2b\mu}{a\Gamma(\alpha+1)}$, and inequality (26) reveal that $\phi\left(\mathbb{B}_r\right) \subset \mathbb{B}_r$. Progressing from the prior, we now prove the contraction requirement on the mapping. For any two functions η_1 and η_2 in $\mathbb{B}_r$, the norm of their difference follows:

$$
\begin{aligned}
\|\phi(\eta_2) - \phi(\eta_1)\| = \Bigg\| & \frac{1}{\Gamma(\beta+\alpha)} \int_a^t \left(\ln\left(\frac{t}{\tau} \right) \right)^{\beta+\alpha-1} \left[\theta(\tau,\eta_2(\tau)) - \theta(\tau,\eta_1(\tau))\right] \frac{\mathrm{d}\tau}{\tau} \\
&- \frac{\mu}{\Gamma(\alpha)} \int_a^t \left(\ln\left(\frac{t}{\tau} \right) \right)^{\alpha-1} \left[\eta_2(\tau) - \eta_1(\tau)\right] \frac{\mathrm{d}\tau}{\tau^2} \\
&- \frac{\left(\ln\left(\frac{t}{a}\right)\right)^{\alpha}}{b\Gamma(\beta+\alpha)} \int_a^1 \left(\ln\left(\frac{1}{\tau} \right) \right)^{\beta+\alpha-1} \left[\theta(\tau,\eta_2(\tau)) - \theta(\tau,\eta_1(\tau))\right] \frac{\mathrm{d}\tau}{\tau} \\
&+ \frac{\mu\left(\ln\left(\frac{t}{a}\right)\right)^{\alpha}}{b\Gamma(\alpha)} \int_a^1 \left(\ln\left(\frac{1}{\tau} \right) \right)^{\alpha-1} \left[\eta_2(\tau) - \eta_1(\tau)\right] \frac{\mathrm{d}\tau}{\tau^2} \Bigg\|, \qquad t \in [a,1].
\end{aligned}
\tag{27}
$$

Equation (27) follows:

$$
\|\phi(\eta_2) - \phi(\eta_1)\| \leq \frac{\left(\ln\left(\frac{t}{a}\right)\right)^{\alpha+\beta}}{\Gamma(\beta+\alpha+1)} \|\theta_2 - \theta_1\| + \frac{\mu\left(\ln\left(\frac{t}{a}\right)\right)^{\alpha}}{a\Gamma(\alpha+1)} \|\eta_2 - \eta_1\|
$$

$$
\begin{aligned}
&+ \frac{\left(\ln\left(\frac{t}{a}\right)\right)^{\alpha}\left(\ln\left(\frac{1}{a}\right)\right)^{\alpha+\beta}}{b\Gamma(\beta+\alpha+1)}\left\|\theta_2-\theta_1\right\| + \frac{\mu\left(\ln\left(\frac{t}{a}\right)\right)^{\alpha}\left(\ln\left(\frac{1}{a}\right)\right)^{\alpha}}{ab\Gamma(\alpha+1)}\left\|\eta_2-\eta_1\right\|. \\
&= \varphi_1\left\|\theta_2-\theta_1\right\| + \varphi_2\left\|\eta_2-\eta_1\right\|, \qquad (28)
\end{aligned}
$$

Using assumption (H_2), (28) arrives at:

$$
\left\|\phi\left(\eta_2\right)-\phi\left(\eta_1\right)\right\| \leq \Upsilon\left\|\eta_2-\eta_1\right\|, \qquad (29)
$$

where $0 < \Upsilon = k\varphi_1 + \varphi_2 < 1$. Using Theorem 3, we can show that the solution to the boundary-value problem (6–8) is unique. For the existence of a solution for the boundary-value problem (6-8), we apply the Sadovoskii fixed point theorem. We consider the operator to be ϕ, as it is defined in 24. At this point, we introduce two operators called ϕ_1 and ϕ_2 that map $\mathbb{Q}$ to itself as follows:

$$
\begin{aligned}
\phi_1\left(\eta(t)\right) = &-\frac{\mu}{\Gamma(\alpha)}\int_a^t\left(\ln\left(\frac{t}{\tau}\right)\right)^{\alpha-1}\frac{\eta(\tau)}{\tau^2}\mathrm{d}\tau \\
&+\frac{\mu\left(\ln\left(\frac{t}{a}\right)\right)^{\alpha}}{b\Gamma(\alpha)}\int_a^1\left(\ln\left(\frac{1}{\tau}\right)\right)^{\alpha-1}\frac{\eta(\tau)}{\tau^2}\mathrm{d}\tau + \Phi(t), \qquad t\in[a,1], \qquad (30)
\end{aligned}
$$

and

$$
\begin{aligned}
\phi_2\left(\eta(t)\right) = &\frac{1}{\Gamma(\beta+\alpha)}\int_a^t\left(\ln\left(\frac{t}{\tau}\right)\right)^{\beta+\alpha-1}\frac{\theta(\tau,\eta(\tau))}{\tau}\mathrm{d}\tau \\
&-\frac{\left(\ln\left(\frac{t}{a}\right)\right)^{\alpha}}{b\Gamma(\beta+\alpha)}\int_a^1\left(\ln\left(\frac{1}{\tau}\right)\right)^{\beta+\alpha-1}\frac{\theta(\tau,\eta(\tau))}{\tau}\mathrm{d}\tau, \qquad t\in[a,1], \qquad (31)
\end{aligned}
$$

respectively. Instead of seeking the existence of a fixed point for the operator $\phi\left(\eta(t)\right)$, we shall do the same through the sum of the two previously defined operators in (30-31). We will use Theorem 2 and the contraction condition $0 \leq \mu(b+1)/(a\Gamma(\alpha+1)) < 1$ to show that $\phi_1+\phi_2$ has a fixed point using the steps below.

Step 1: It has previously manifested that $\phi\left(\mathbb{B}_r\right) \subset \mathbb{B}_r$.

Step 2: We are compelled to exhibit that ϕ_2 is compact. For any $t_1, t_2 \in [a,1]$ with $t_1 < t_2$ and for any $\eta(t) \in B_r$ we have:

$$
\begin{aligned}
\left|\phi_2(\eta(t_2)) - \phi_2(\eta(t_1))\right| = &\Bigg|\frac{1}{\Gamma(\beta+\alpha)}\int_a^{t_2}\left(\ln\left(\frac{t_2}{\tau}\right)\right)^{\beta+\alpha-1}\frac{\theta(\tau,\eta(\tau))}{\tau}\mathrm{d}\tau \\
&-\frac{\left(\ln\left(\frac{t_2}{a}\right)\right)^{\alpha}}{b\Gamma(\beta+\alpha)}\int_a^1\left(\ln\left(\frac{1}{\tau}\right)\right)^{\beta+\alpha-1}\frac{\theta(\tau,\eta(\tau))}{\tau}\mathrm{d}\tau \\
&-\frac{1}{\Gamma(\beta+\alpha)}\int_a^{t_1}\left(\ln\left(\frac{t_1}{\tau}\right)\right)^{\beta+\alpha-1}\frac{\theta(\tau,\eta(\tau))}{\tau}\mathrm{d}\tau \\
&+\frac{\left(\ln\left(\frac{t_1}{a}\right)\right)^{\alpha}}{b\Gamma(\beta+\alpha)}\int_a^1\left(\ln\left(\frac{1}{\tau}\right)\right)^{\beta+\alpha-1}\frac{\theta(\tau,\eta(\tau))}{\tau}\mathrm{d}\tau\Bigg|. \qquad (32)
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{1}{\Gamma(\beta+\alpha)} \int_a^{t_1} \left[\left(\ln\left(\frac{t_2}{\tau}\right)\right)^{\beta+\alpha-1} - \left(\ln\left(\frac{t_1}{\tau}\right)\right)^{\beta+\alpha-1} \right] \frac{|\theta(\tau,\eta(\tau))|}{\tau} \mathrm{d}\tau \\
&\quad + \frac{1}{\Gamma(\beta+\alpha)} \int_{t_1}^{t_2} \left(\ln\left(\frac{t_2}{\tau}\right)\right)^{\beta+\alpha-1} \frac{|\theta(\tau,\eta(\tau))|}{\tau} \mathrm{d}\tau \\
&\quad + \frac{1}{b\Gamma(\beta+\alpha)} \left| \left(\ln\left(\frac{t_1}{a}\right)\right)^{\alpha} - \left(\ln\left(\frac{t_2}{a}\right)\right)^{\alpha} \right| \int_a^1 \left(\ln\left(\frac{1}{\tau}\right)\right)^{\beta+\alpha-1} \frac{|\theta(\tau,\eta(\tau))|}{\tau} \mathrm{d}\tau. \qquad (33)
\end{aligned}
$$

It is shown in (33) that as $|t_2 - t_1| \longrightarrow 0$ we have $|\phi_2(\eta(t_2)) - \phi_1(\eta(t_1))| \longrightarrow 0$. Hence ϕ is equicontinuous. Since $\phi(\mathbb{B}_r) \subset \mathbb{B}_r$, then, ϕ_2 is uniformly bounded. As a consequence of the Arzelà-Ascoli theorem, $\phi_2(\mathbb{B}_r)$ is compact.

Step 3: For this part, we have to demonstrate that ϕ_1 is k*-contractive. For any η_1 and η_2 in $\mathbb{B}_r$, we have:

$$
\begin{aligned}
\|\phi_1(\eta_1) - \phi_1(\eta_2)\| &\leq \operatorname*{Sup}_t \left\{ \frac{\mu}{\Gamma(\alpha)} \int_a^t \left(\ln\left(\frac{t}{\tau}\right)\right)^{\alpha-1} \frac{|\eta_1(\tau) - \eta_2(\tau)|}{\tau^2} \mathrm{d}\tau \right. \\
&\quad \left. + \frac{\mu\left(\ln\left(\frac{t}{a}\right)\right)^{\alpha}}{b\Gamma(\alpha)} \int_a^1 \left(\ln\left(\frac{1}{\tau}\right)\right)^{\alpha-1} \frac{|\eta_1(\tau) - \eta_2(\tau)|}{\tau^2} \mathrm{d}\tau \right\}, \qquad t \in [a,1]. \\
&\leq \frac{\mu b}{a\Gamma(\alpha+1)} \|\eta_1 - \eta_2\| + \frac{\mu}{a\Gamma(\alpha+1)} \|\eta_1 - \eta_2\| \\
&= \mathrm{k}^* \|\eta_1 - \eta_2\|. \qquad (34)
\end{aligned}
$$

Where $\mathrm{k}^* = \frac{\mu(b+1)}{a\Gamma(\alpha+1)}$.

Step 4: In the last part, we need to show that ϕ is condensing. Since ϕ_1 is continuous and k*-contractive, and we found that ϕ_2 is compact, therefore, by Theorem 1, we observe that $\phi = \phi_1 + \phi_2$ is a condensing map on $\mathbb{B}_r$.

From the above four steps and by the Sadovski Theorem 2, we arrive at the conclusion that the map ϕ has a fixed point.

4. Stability

This section examines the stability of the boundary-value problem (6-8) utilising two established definitions of stability: Ulam-Heyrs stability and Ulam-Heyrs-Rassias stability.

Theorem 4. *Assume that $\theta : [a,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function and the assumption $\mathbb{H}_2$ holds. Then, the solution of the boundary-value problem (6-8) is Ulam-Hyers stable.*

Proof: Let $\eta(t) \in \mathrm{C}([a,1],\mathbb{R})$ be a solution of the inequality (14) which satisfies boundary conditions (7-8). Through Remark 1, we have

$$
{}_{\mathfrak{H}}D^{\beta}\left(D^{\alpha} + \frac{\mu}{t}\right)\eta(t) = \theta(t,\eta(t)) + g(t), \qquad (35)
$$

where $g(t)$ possesses the same property mentioned in Remark 1. The solution of the perturbed problem (35) supplemented with (7-8) can be found to be:

$$\begin{aligned}\eta(t) = &\frac{1}{\Gamma(\beta+\alpha)}\int_a^t \left(\ln\left(\frac{t}{\tau}\right)\right)^{\beta+\alpha-1}\frac{\theta(\tau,\eta(\tau))}{\tau}\mathrm{d}\tau\\ &-\frac{\mu}{\Gamma(\alpha)}\int_a^t \left(\ln\left(\frac{t}{\tau}\right)\right)^{\alpha-1}\frac{\eta(\tau)}{\tau^2}\mathrm{d}\tau\\ &+\frac{1}{\Gamma(\beta+\alpha)}\int_a^t \left(\ln\left(\frac{t}{\tau}\right)\right)^{\beta+\alpha-1}\frac{g(\tau)}{\tau}\mathrm{d}\tau\\ &-\frac{\left(\ln\left(\frac{t}{a}\right)\right)^{\alpha}}{b\Gamma(\beta+\alpha)}\int_a^1 \left(\ln\left(\frac{1}{\tau}\right)\right)^{\beta+\alpha-1}\frac{\theta(\tau,\eta(\tau))}{\tau}\mathrm{d}\tau\\ &+\frac{\mu\left(\ln\left(\frac{t}{a}\right)\right)^{\alpha}}{b\Gamma(\alpha)}\int_a^1 \left(\ln\left(\frac{1}{\tau}\right)\right)^{\alpha-1}\frac{\eta(\tau)}{\tau^2}\mathrm{d}\tau\\ &-\frac{\left(\ln\left(\frac{t}{a}\right)\right)^{\alpha}}{b\Gamma(\beta+\alpha)}\int_a^1 \left(\ln\left(\frac{1}{\tau}\right)\right)^{\beta+\alpha-1}\frac{g(\tau)}{\tau}\mathrm{d}\tau+\Phi(t),\end{aligned} \tag{36}$$

where (36) satisfies the following inequality:

$$\begin{aligned}&\Bigg|\eta(t) - \frac{1}{\Gamma(\beta+\alpha)}\int_a^t \left(\ln\left(\frac{t}{\tau}\right)\right)^{\beta+\alpha-1}\frac{\theta(\tau,\eta(\tau))}{\tau}\mathrm{d}\tau\\ &+\frac{\mu}{\Gamma(\alpha)}\int_a^t \left(\ln\left(\frac{t}{\tau}\right)\right)^{\alpha-1}\frac{\eta(\tau)}{\tau^2}\mathrm{d}\tau\\ &+\frac{\left(\ln\left(\frac{t}{a}\right)\right)^{\alpha}}{b\Gamma(\beta+\alpha)}\int_a^1 \left(\ln\left(\frac{1}{\tau}\right)\right)^{\beta+\alpha-1}\frac{\theta(\tau,\eta(\tau))}{\tau}\mathrm{d}\tau\\ &-\frac{\mu\left(\ln\left(\frac{t}{a}\right)\right)^{\alpha}}{b\Gamma(\alpha)}\int_a^1 \left(\ln\left(\frac{1}{\tau}\right)\right)^{\alpha-1}\frac{\eta(\tau)}{\tau^2}\mathrm{d}\tau-\Phi(t)\Bigg| \leq \frac{2\ \epsilon\ b\left(\ln\left(\frac{1}{a}\right)\right)^{\beta}}{\Gamma(\beta+\alpha+1)},\end{aligned} \tag{37}$$

for all $t \in [a,1]$. Let $Z(t)$ be the unique solution for our boundary-value problem (6-8). The left side of inequality (15) conforms to

$$\begin{aligned}|\eta(t)-Z(t)| \leq &\frac{2\ \epsilon\left(\ln\left(\frac{1}{a}\right)\right)^{\beta+\alpha}}{\Gamma(\beta+\alpha+1)} + \frac{1}{\Gamma(\beta+\alpha)}\int_a^t \left(\ln\left(\frac{t}{\tau}\right)\right)^{\beta+\alpha-1}|\theta(\tau,\eta(\tau))-\theta(\tau,Z(\tau))|\frac{\mathrm{d}\tau}{\tau}\\ &+\frac{\mu}{\Gamma(\alpha)}\int_a^t \left(\ln\left(\frac{t}{\tau}\right)\right)^{\alpha-1}|\eta(\tau)-Z(\tau)|\frac{\mathrm{d}\tau}{\tau^2}\\ &+\frac{\left(\ln\left(\frac{t}{a}\right)\right)^{\alpha}}{b\Gamma(\beta+\alpha)}\int_a^1 \left(\ln\left(\frac{1}{\tau}\right)\right)^{\beta+\alpha-1}|\theta(\tau,\eta(\tau))-\theta(\tau,Z(\tau))|\frac{\mathrm{d}\tau}{\tau}\\ &+\frac{\mu\left(\ln\left(\frac{t}{a}\right)\right)^{\alpha}}{b\Gamma(\alpha)}\int_a^1 \left(\ln\left(\frac{1}{\tau}\right)\right)^{\alpha-1}|\eta(\tau)-Z(\tau)|\frac{\mathrm{d}\tau}{\tau^2}.\end{aligned} \tag{38}$$

Considering the Lipchitz condition $\mathbb{H}_2$, the inequality (38) reads

$$|\eta(t)-Z(t)| \leq \frac{2b\epsilon\left(\ln\left(\frac{1}{a}\right)\right)^{\beta}}{\Gamma(\beta+\alpha+1)} + (k\varphi_1+\varphi_2)\,|\eta(t)-Z(t)|\,, \tag{39}$$

and by rewriting inequality (39), we obtain

$$|\eta(t)-Z(t)| \leq c_h\epsilon, \tag{40}$$

where $c_h = \frac{\varphi_1}{1-\Upsilon}$. Inequality (40) via Definition 4 ensures that the boundary-value problem (6-8) is stable of type Ulam-Hayrs.

To investigate the Ulam-Hayrs-Rassias type stability, we imposed a new assumption on the problem (6-8).

$\mathbb{H}_3$: The control function $\psi(t)$ in Definition 5 is a positive non-decreasing function such that:

$${}_{a}^{H}I^{\alpha+\beta}\psi(t) \leq \lambda\psi(t), \tag{41}$$

where $\lambda > 0$.

Theorem 5. *Assume that $\theta : [a,1]\times\mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function and both hypotheses $\mathbb{H}_2$ and $\mathbb{H}_3$ hold. Then, the solution of the boundary-value problem (6-8) is Ulam-Hyers-Rassias type stable.*

Proof: Let $\eta(t) \in \mathrm{C}([a,1],\mathbb{R})$ be a solution of the inequality (16) which satisfies boundary conditions (7-8). Through Remark 2, we obtain the perturbed problem (35), where $|g(t)|$ is bounded by $\epsilon\psi(t)$ as stated in Remark 2, and $\psi(t)$ is the control function that fulfills assumption $\mathbb{H}_3$. Thus we have

$$\begin{aligned}
&\Bigg|\eta(t) - \frac{1}{\Gamma(\beta+\alpha)}\int_a^t \left(\ln\left(\frac{t}{\tau}\right)\right)^{\beta+\alpha-1}\frac{\theta(\tau,\eta(\tau))}{\tau}\mathrm{d}\tau \\
&+\frac{\mu}{\Gamma(\alpha)}\int_a^t \left(\ln\left(\frac{t}{\tau}\right)\right)^{\alpha-1}\frac{\eta(\tau)}{\tau^2}\mathrm{d}\tau \\
&+\frac{\left(\ln\left(\frac{t}{a}\right)\right)^{\alpha}}{b\Gamma(\beta+\alpha)}\int_a^1 \left(\ln\left(\frac{1}{\tau}\right)\right)^{\beta+\alpha-1}\frac{\theta(\tau,\eta(\tau))}{\tau}\mathrm{d}\tau \\
&-\frac{\mu\left(\ln\left(\frac{t}{a}\right)\right)^{\alpha}}{b\Gamma(\alpha)}\int_a^1 \left(\ln\left(\frac{1}{\tau}\right)\right)^{\alpha-1}\frac{\eta(\tau)}{\tau^2}\mathrm{d}\tau - \Phi(t)\Bigg| \leq 2\epsilon\lambda\psi(t),
\end{aligned} \tag{42}$$

for all $t \in [a,1]$. If we call $Z(t)$ the unique solution for our boundary-value problem (6-8), then, the left side of inequality (17) conforms to

$$\begin{aligned}
|\eta(t)-Z(t)| \leq 2\epsilon\lambda\psi(t) &+ \frac{1}{\Gamma(\beta+\alpha)}\int_a^t \left(\ln\left(\frac{t}{\tau}\right)\right)^{\beta+\alpha-1}|\theta(\tau,\eta(\tau))-\theta(\tau,Z(\tau))|\frac{\mathrm{d}\tau}{\tau} \\
&+\frac{\mu}{\Gamma(\alpha)}\int_a^t \left(\ln\left(\frac{t}{\tau}\right)\right)^{\alpha-1}|\eta(\tau)-Z(\tau)|\frac{\mathrm{d}\tau}{\tau^2}
\end{aligned}$$

$$+\frac{\left(\ln\left(\frac{t}{a}\right)\right)^{\alpha}}{b\Gamma(\beta+\alpha)}\int_a^1\left(\ln\left(\frac{1}{\tau}\right)\right)^{\beta+\alpha-1}|\theta(\tau,\eta(\tau))-\theta(\tau,Z(\tau))|\frac{\mathrm{d}\tau}{\tau}$$
$$+\frac{\mu\left(\ln\left(\frac{t}{a}\right)\right)^{\alpha}}{b\Gamma(\alpha)}\int_a^1\left(\ln\left(\frac{1}{\tau}\right)\right)^{\alpha-1}|\eta(\tau)-Z(\tau)|\frac{\mathrm{d}\tau}{\tau^2}. \tag{43}$$

Considering the second assumption $\mathbb{H}_2$, the inequality (43) reads

$$|\eta(t)-Z(t)|\leq 2\epsilon\lambda\psi(t)+(k\varphi_1+\varphi_2)\,|\eta(t)-Z(t)|\,, \tag{44}$$

and by rewriting inequality (44), we obtain

$$|\eta(t)-Z(t)|\leq \epsilon c_\lambda\psi(t), \tag{45}$$

where $c_\lambda=\frac{2\lambda}{(1-\Upsilon)}$. Inequality (45) guarantees the stability of type Ulam-Hyers-Rassias according to Definition 5.

Example 1: Consider the following multifractional boundary-value problem:

$$_{\mathrm{H}}D^{\beta}\left(_{\mathrm{H}}D^{\alpha}+\frac{\mu}{t}\right)\eta(t)=\frac{|\eta(t)|\,e^{-2t}}{(10+t^2)\,(1+\eta(t)^2)}. \tag{46}$$
$$\eta(a)=0, \tag{47}$$
$$\eta(1)=0.1. \tag{48}$$

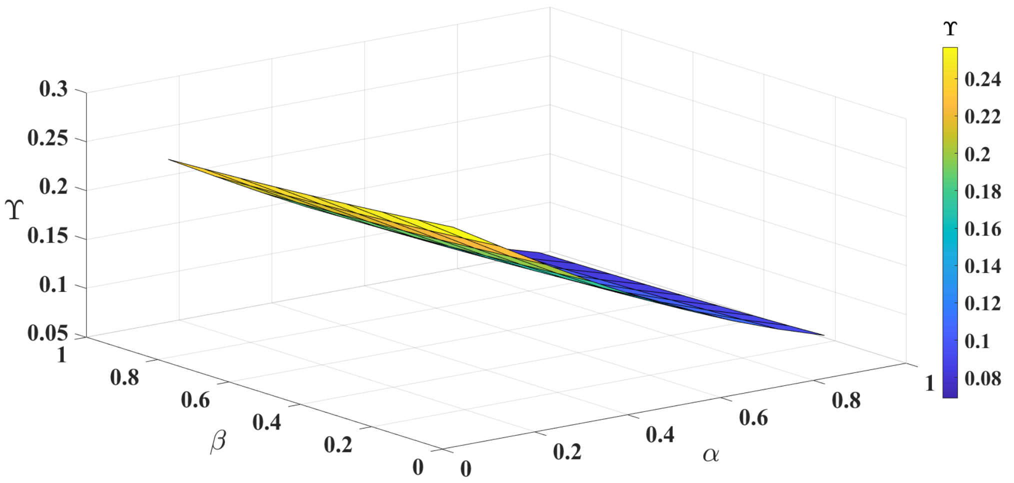
In this problem, (46-48), which is defined on $[a,1]$ where $0<a<1$, we have $\theta(t,\eta(t))=\frac{|\eta(t)|e^{-2t}}{(10+t^2)(1+\eta^2)}$ which satisfies both hypotheses $\mathbb{H}_1$ and $\mathbb{H}_2$ with $\Omega=k=\frac{e^{-2a}}{10+a^2}$, i.e.:

$$|\theta(t,\eta(t))|\leq\Omega=\frac{e^{-2a}}{10+a^2}, \tag{49}$$

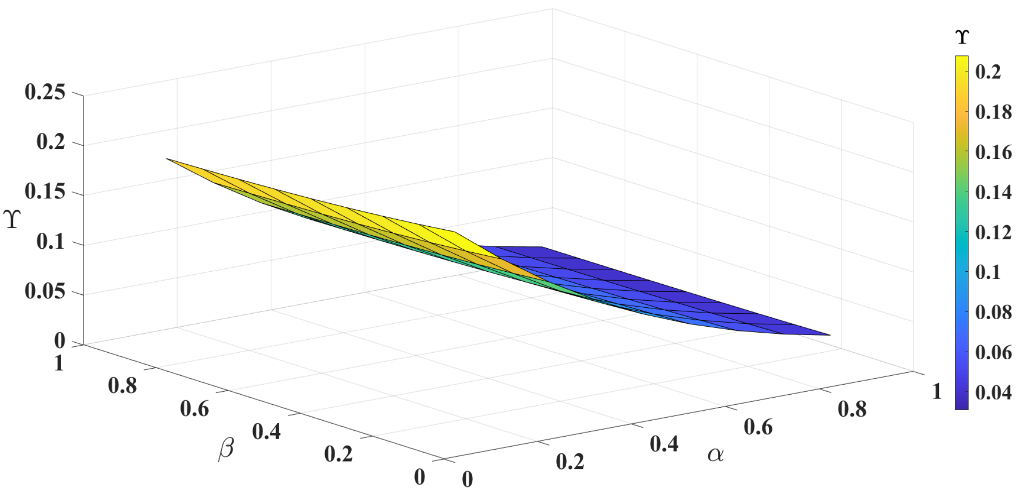
and

$$\|\theta(t,\eta_1)-\theta(t,\eta_2)\|\leq\frac{e^{-2a}}{10+a^2}\|\eta_1-\eta_2\|. \tag{50}$$

Hence by Banach's Fixed Point Theorem there exists a unique solution to the problem (46-48). To analyze the problem (46-48), we set the interval $[a,1]$ with $a=0.8$, and $a=0.9$, which read $k\approx 0.01898$, and $k\approx 0.01529$, respectively.Also, we set $\mu=0.1$. Next, we take the values of α and β to run on $[0.1,0.9]$. We demonstrate the proposed contraction condition in inequality (29), $\Upsilon=k\varphi_1+\varphi_2$, which depends on α, β, μ, and a.



(a)



(b)

Figure 1: The range of contraction parameter Υ, with $\mu = 0.1$ as α and β move from 0.1 to 0.9 for the values of a: (a) $a = 0.8$, and (b) $a = 0.9$.

For stability, we assume that $\eta(t)$ and $Z(t)$ are the solutions for the perturbed (for some $\epsilon > 0$) and unperturbed problem (46-48), respectively. Hence, $\eta(t)$ is a solution for the inequality (14). Since both hypotheses $\mathbb{H}_1$ and $\mathbb{H}_2$ are satisfied through (49-50), then, by Theorem 4, both $\eta(t)$ and $Z(t)$ satisfy inequality (15), and consequently the problem (46-48) is Ulam-Hyers stable with $c_h = \frac{\varphi_1}{1-\Upsilon}$.

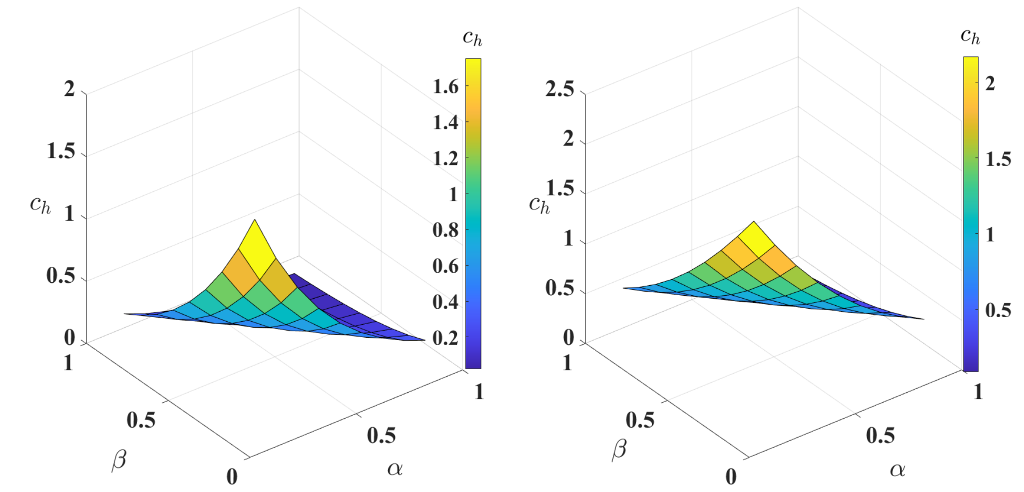
The last part is to establish how the problem is stable by the means of Ulam-Hyers-Rassiass as stated in Definition 5. If $\eta(t)$ is a solution to the inequality (16) for some constant $\epsilon > 0$ and the control function is $\psi(t) = \ln\left(\frac{t}{a}\right)$, for $t \in [a, 1]$, then, $\psi(t)$ satisfies the hypothesis $\mathbb{H}_3$. In other words,

$$
\begin{aligned}
{}_a^H I^{\alpha+\beta}\psi(t) &= {}_a^H I^{\alpha+\beta} \ln\left(\frac{t}{a}\right) \\
&= \frac{\Gamma(2)}{\Gamma(\beta+\alpha+2)}\left(\ln\left(\frac{t}{a}\right)\right)^{\alpha+\beta+1} \\
&= \frac{1}{\Gamma(\beta+\alpha+2)}\left(\ln\left(\frac{t}{a}\right)\right)^{\alpha+\beta} \ln\left(\frac{t}{a}\right) \\
&\leq \lambda\psi(t),
\end{aligned}
\tag{51}
$$

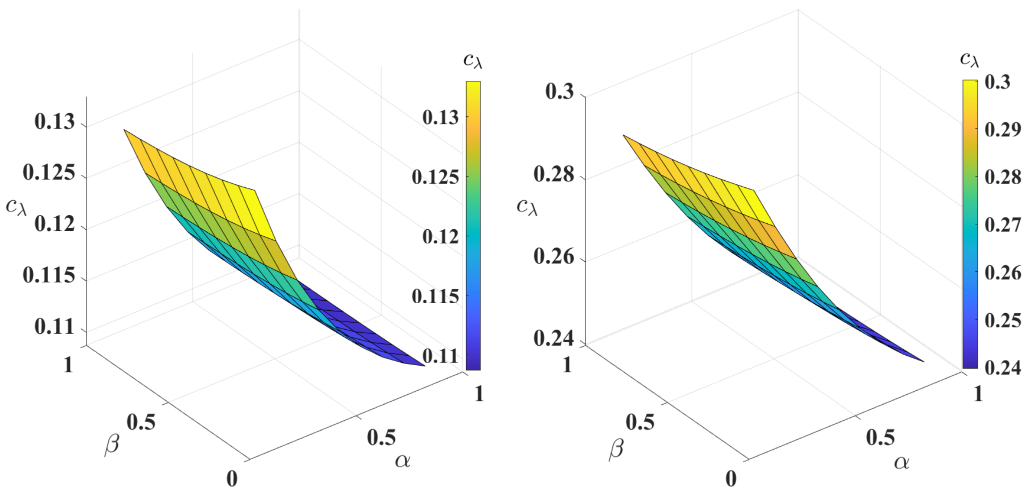
where $\lambda = \frac{\gamma}{\Gamma(\beta+\alpha+2)}$ and γ represents the maximum value of $\left(\ln\left(\frac{t}{a}\right)\right)^{\alpha+\beta}$ on $[a, 1]$. The value of λ depends on three parameters, a, α, and β. For example, for $a = 0.9$, $\alpha = \beta = 0.5$, we have $\lambda = 0.05268$. Consequently, by Theorem 5, the problem (46-48) is Ulam-Hyers-Rassiass stable as the following inequality holds,

$$
|\eta(t) - Z(t)| \leq \epsilon c_\lambda \psi(t), \tag{52}
$$

where $c_\lambda = \frac{2\lambda}{(1-k\varphi_1-\varphi_2)}$. Both parameters, c_h and c_λ are depicted in Figure 2 for both $a = 0.9$ and $a = 0.8$.



(a)



(b)

Figure 2: The range of values of c_h and c_λ as both orders, α and β, grow from 0.1 up to 0.9 for $a = 0.9$ (left) and $a = 0.8$ (right), (a) c_h, and (b) c_λ.

5. Numerical Scheme

In this section we introduce a numerical scheme to approximately solve the problem (6-8) through the Newton-Raphson method. We apply the fractional rectangular, $L_{\ln,1}$, interpolation on a logarithmic grid to approximate the derived analytical solution (18).

This results in a system of non-linear equations which will be solved by the Newton-Raphson method using the Jacobian matrix to handle the non-linear coupling between $\eta(t)$ and $\theta(t, \eta(t))$.

We subdivide the domain using a non-uniform mesh on $[a, 1]$, $\mathcal{B} = \{t_0, t_1, \cdots, t_N\}$, for some positive integer N where $t_0 = a$ and $t_N = 1$. The following formula is used to generate the graded mesh:

$$\ln(t_i) = \ln(t_0) + \Delta \left(\frac{i}{N}\right)^s, \tag{53}$$

where $\Delta = \ln(t_N) - \ln(t_0)$, for the considered interval we have $\Delta = -\ln(t_0)$, and s is the mesh graded parameter which is sensitive. For $s = 1$ we obtain the uniform mesh (in the logarithmic scale), while for optimized $s > 1$, both the resolution of the non-local effects inherent to fractional derivatives and the logarithmic kernel close to $t = a$ and the convergence rate are improved. For $i = 0$, formula (53) generates $\ln(t_0)$ and for $i = N$, it generates $\ln(t_N)$.

The numerical solution of (18) is equivalent to solving the two-point fractional boundary-value problem (6-8). At $t = t_q, (0 \leq q \leq N)$, $t_q \in \mathcal{B}$, denote $\eta_q \approx \eta(t_q)$, we have:

$$\begin{aligned}\eta_q = \frac{1}{\Gamma(\beta+\alpha)} \int_a^{t_q} \left(\ln\left(\frac{t_q}{\tau}\right)\right)^{\beta+\alpha-1} \frac{\theta(\tau, \eta(\tau))}{\tau} \mathrm{d}\tau - \frac{\mu}{\Gamma(\alpha)} \int_a^{t_q} \left(\ln\left(\frac{t_q}{\tau}\right)\right)^{\alpha-1} \frac{\eta(\tau)}{\tau^2} \mathrm{d}\tau \\ - \frac{\left(\ln\left(\frac{t_q}{a}\right)\right)^{\alpha}}{b\Gamma(\beta+\alpha)} \int_a^1 \left(\ln\left(\frac{1}{\tau}\right)\right)^{\beta+\alpha-1} \frac{\theta(\tau, \eta(\tau))}{\tau} \mathrm{d}\tau \\ + \frac{\mu\left(\ln\left(\frac{t_q}{a}\right)\right)^{\alpha}}{b\Gamma(\alpha)} \int_a^1 \left(\ln\left(\frac{1}{\tau}\right)\right)^{\alpha-1} \frac{\eta(\tau)}{\tau^2} \mathrm{d}\tau + \Phi_q,\end{aligned} \tag{54}$$

where $\Phi_q = \Phi(t_q) = \xi_1 + \frac{1}{b}(\xi_2 - \xi_1)\left(\ln\left(\frac{t_q}{a}\right)\right)^{\alpha}$. Furthermore,

$$\begin{aligned}\eta_q = \frac{1}{\Gamma(\beta+\alpha)} \sum_{i=1}^{q} \int_{t_{i-1}}^{t_i} \left(\ln\left(\frac{t_q}{\tau}\right)\right)^{\beta+\alpha-1} \theta(\tau, \eta(\tau)) \frac{\mathrm{d}\tau}{\tau} \\ - \frac{\mu}{\Gamma(\alpha)} \sum_{i=1}^{q} \int_{t_{i-1}}^{t_i} \left(\ln\left(\frac{t_q}{\tau}\right)\right)^{\alpha-1} \frac{\eta(\tau)}{\tau} \frac{\mathrm{d}\tau}{\tau} + f(t_q),\end{aligned} \tag{55}$$

where

$$\begin{aligned}f(t_q) = -\frac{\left(\ln\left(\frac{t_q}{a}\right)\right)^{\alpha}}{b\Gamma(\beta+\alpha)} \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} \left(\ln\left(\frac{1}{\tau}\right)\right)^{\beta+\alpha-1} \theta(\tau, \eta(\tau)) \frac{\mathrm{d}\tau}{\tau} \\ + \frac{\mu\left(\ln\left(\frac{t_q}{a}\right)\right)^{\alpha}}{b\Gamma(\alpha)} \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} \left(\ln\left(\frac{1}{\tau}\right)\right)^{\alpha-1} \frac{\eta(\tau)}{\tau} \frac{\mathrm{d}\tau}{\tau} + \Phi_q.\end{aligned} \tag{56}$$

Note that, the first sum of (56) is the same as the first sum of (55), for $q = N$. The same is true for the second sums of (55) and (56). Applying the fractional rectangular, $L_{\ln,1}$ interpolation to approximate the nonlinear terms $\theta(\tau, \eta(\tau))$ and $\frac{\eta(\tau)}{\tau}$, that is

$$\theta(\tau, \eta(\tau)) \approx \frac{\ln\left(\frac{\tau}{t_i}\right)}{\ln\left(\frac{t_{i-1}}{t_i}\right)}\theta_{i-1} + \frac{\ln\left(\frac{\tau}{t_{i-1}}\right)}{\ln\left(\frac{t_i}{t_{i-1}}\right)}\theta_i, \tag{57}$$

$$\eta(\tau) \approx \frac{\ln\left(\frac{\tau}{t_i}\right)}{\ln\left(\frac{t_{i-1}}{t_i}\right)}\eta_{i-1} + \frac{\ln\left(\frac{\tau}{t_{i-1}}\right)}{\ln\left(\frac{t_i}{t_{i-1}}\right)}\eta_i, \tag{58}$$

where $\theta_i = \theta(t_i, \eta(t_i))$ and $\eta_i = \eta(t_i)$. Now, in conjunction with (57-58) and η_q in (55) we arrive at:

$$\begin{aligned}
\eta_q &= \frac{1}{\Gamma(\beta+\alpha)} \sum_{i=1}^{q} \int_{t_{i-1}}^{t_i} \left(\ln\left(\frac{t_q}{\tau}\right)\right)^{\beta+\alpha-1} \left[\frac{\ln\left(\frac{\tau}{t_i}\right)}{\ln\left(\frac{t_{i-1}}{t_i}\right)}\theta_{i-1} + \frac{\ln\left(\frac{\tau}{t_{i-1}}\right)}{\ln\left(\frac{t_i}{t_{i-1}}\right)}\theta_i\right] \frac{\mathrm{d}\tau}{\tau} \\
&\quad - \frac{\mu}{\Gamma(\alpha)} \sum_{i=1}^{q} \int_{t_{i-1}}^{t_i} \left(\ln\left(\frac{t_q}{\tau}\right)\right)^{\alpha-1} \left[\frac{\ln\left(\frac{\tau}{t_i}\right)}{\ln\left(\frac{t_{i-1}}{t_i}\right)}\eta_{i-1} + \frac{\ln\left(\frac{\tau}{t_{i-1}}\right)}{\ln\left(\frac{t_i}{t_{i-1}}\right)}\eta_i\right] \frac{\mathrm{d}\tau}{\tau^2} + f(t_q) \\
&= \frac{1}{\Gamma(\beta+\alpha)} \sum_{i=1}^{q} \left[\frac{\theta_{i-1}}{\ln\left(\frac{t_{i-1}}{t_i}\right)} \int_{t_{i-1}}^{t_i} \left(\ln\left(\frac{t_q}{\tau}\right)\right)^{\beta+\alpha-1} \ln\left(\frac{\tau}{t_i}\right) \frac{\mathrm{d}\tau}{\tau}\right. \\
&\qquad \left. + \frac{\theta_i}{\ln\left(\frac{t_i}{t_{i-1}}\right)} \int_{t_{i-1}}^{t_i} \left(\ln\left(\frac{t_q}{\tau}\right)\right)^{\beta+\alpha-1} \ln\left(\frac{\tau}{t_{i-1}}\right) \frac{\mathrm{d}\tau}{\tau}\right] \\
&\quad - \frac{\mu}{\Gamma(\alpha)} \sum_{i=1}^{q} \left[\frac{\eta_{i-1}}{\ln\left(\frac{t_{i-1}}{t_i}\right)} \int_{t_{i-1}}^{t_i} \left(\ln\left(\frac{t_q}{\tau}\right)\right)^{\alpha-1} \ln\left(\frac{\tau}{t_i}\right) \frac{\mathrm{d}\tau}{\tau^2}\right. \\
&\qquad \left. + \frac{\eta_i}{\ln\left(\frac{t_i}{t_{i-1}}\right)} \int_{t_{i-1}}^{t_i} \left(\ln\left(\frac{t_q}{\tau}\right)\right)^{\alpha-1} \ln\left(\frac{\tau}{t_{i-1}}\right) \frac{\mathrm{d}\tau}{\tau^2}\right] + f(t_q),
\end{aligned} \tag{59}$$

proceeding the integration in (59) using the change of variables $t_q = \tau e^u$ we obtain:

$$\begin{aligned}
\eta_q = \frac{1}{\Gamma(\beta+\alpha)} \sum_{i=1}^{q} &\left[\frac{\theta_{i-1}}{\ln\left(\frac{t_{i-1}}{t_i}\right)} \ln\left(\frac{t_q}{t_i}\right) \frac{\left(\ln\frac{t_q}{t_{i-1}}\right)^{\beta+\alpha} - \left(\ln\frac{t_q}{t_i}\right)^{\beta+\alpha}}{\beta+\alpha}\right. \\
&- \frac{\theta_{i-1}}{\ln\left(\frac{t_{i-1}}{t_i}\right)} \frac{\left(\ln\frac{t_q}{t_{i-1}}\right)^{\beta+\alpha+1} - \left(\ln\frac{t_q}{t_i}\right)^{\beta+\alpha+1}}{\beta+\alpha+1}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\theta_i}{\ln\left(\frac{t_i}{t_{i-1}}\right)}\ln\left(\tfrac{t_q}{t_{i-1}}\right)\frac{\left(\ln\frac{t_q}{t_{i-1}}\right)^{\beta+\alpha}-\left(\ln\frac{t_q}{t_i}\right)^{\beta+\alpha}}{\beta+\alpha}\\
&-\frac{\theta_i}{\ln\left(\frac{t_i}{t_{i-1}}\right)}\frac{\left(\ln\frac{t_q}{t_{i-1}}\right)^{\beta+\alpha+1}-\left(\ln\frac{t_q}{t_i}\right)^{\beta+\alpha+1}}{\beta+\alpha+1}\Bigg]\\
&-\frac{\mu}{\Gamma(\alpha)}\sum_{i=1}^{q}\left[\frac{(-1)^{\alpha}\eta_{i-1}}{t_q\ln\left(\frac{t_{i-1}}{t_i}\right)}\left[\Gamma\left(\alpha+1,-\ln\left(\frac{t_q}{t_{i-1}}\right)\right)-\Gamma\left(\alpha+1,-\ln\left(\frac{t_q}{t_i}\right)\right)\right]\right.\\
&-\frac{(-1)^{\alpha-1}\eta_{i-1}}{t_q\ln\left(\frac{t_{i-1}}{t_i}\right)}\ln\left(\frac{t_q}{t_{i-1}}\right)\left[\Gamma\left(\alpha,-\ln\left(\frac{t_q}{t_{i-1}}\right)\right)-\Gamma\left(\alpha,-\ln\left(\frac{t_q}{t_i}\right)\right)\right]\\
&+\frac{(-1)^{\alpha-1}\eta_i}{t_q\ln\left(\frac{t_i}{t_{i-1}}\right)}\ln\left(\frac{t_q}{t_i}\right)\left[\Gamma\left(\alpha,-\ln\left(\frac{t_q}{t_{i-1}}\right)\right)-\Gamma\left(\alpha,-\ln\left(\frac{t_q}{t_i}\right)\right)\right]\\
&\left.-\frac{(-1)^{\alpha}\eta_i}{t_q\ln\left(\frac{t_i}{t_{i-1}}\right)}\left[\Gamma\left(\alpha+1,-\ln\left(\frac{t_q}{t_{i-1}}\right)\right)-\Gamma\left(\alpha+1,-\ln\left(\frac{t_q}{t_i}\right)\right)\right]\right],
\end{aligned}
\tag{60}
$$

where $\Gamma(.,.)$ is the incomplete gamma function. Rearranging (60)

$$
\eta_q+\frac{\mu}{\Gamma(\alpha)}\sum_{i=1}^{q}\left[\nu_{qi}\eta_{i-1}+\bar{\nu}_{qi}\eta_i\right]=\frac{1}{\Gamma(\beta+\alpha)}\sum_{i=1}^{q}\left[\omega_{qi}\theta_{i-1}+\bar{\omega}_{qi}\theta_i\right]+f(t_q),
\tag{61}
$$

where

$$
\begin{aligned}
\nu_{qi}&=\frac{(-1)^{\alpha}}{t_q\ln\left(\frac{t_{i-1}}{t_i}\right)}\left[\Gamma\left(\alpha+1,-\ln\left(\frac{t_q}{t_{i-1}}\right)\right)-\Gamma\left(\alpha+1,-\ln\left(\frac{t_q}{t_i}\right)\right)\right]\\
&-\frac{(-1)^{\alpha-1}}{t_q\ln\left(\frac{t_{i-1}}{t_i}\right)}\ln\left(\frac{t_q}{t_{i-1}}\right)\left[\Gamma\left(\alpha,-\ln\left(\frac{t_q}{t_{i-1}}\right)\right)-\Gamma\left(\alpha,-\ln\left(\frac{t_q}{t_i}\right)\right)\right],\\
\bar{\nu}_{qi}&=\frac{(-1)^{\alpha-1}}{t_q\ln\left(\frac{t_i}{t_{i-1}}\right)}\ln\left(\frac{t_q}{t_i}\right)\left[\Gamma\left(\alpha,-\ln\left(\frac{t_q}{t_{i-1}}\right)\right)-\Gamma\left(\alpha,-\ln\left(\frac{t_q}{t_i}\right)\right)\right]\\
&-\frac{(-1)^{\alpha}}{t_q\ln\left(\frac{t_i}{t_{i-1}}\right)}\left[\Gamma\left(\alpha+1,-\ln\left(\frac{t_q}{t_{i-1}}\right)\right)-\Gamma\left(\alpha+1,-\ln\left(\frac{t_q}{t_i}\right)\right)\right],\\
\omega_{qi}&=\ln\left(\tfrac{t_q}{t_i}\right)\frac{\left(\ln\frac{t_q}{t_{i-1}}\right)^{\beta+\alpha}-\left(\ln\frac{t_q}{t_i}\right)^{\beta+\alpha}}{(\beta+\alpha)\ln\left(\frac{t_{i-1}}{t_i}\right)}-\frac{\left(\ln\frac{t_q}{t_{i-1}}\right)^{\beta+\alpha+1}-\left(\ln\frac{t_q}{t_i}\right)^{\beta+\alpha+1}}{(\beta+\alpha+1)\ln\left(\frac{t_{i-1}}{t_i}\right)},\\
\bar{\omega}_{qi}&=\ln\left(\tfrac{t_q}{t_{i-1}}\right)\frac{\left(\ln\frac{t_q}{t_{i-1}}\right)^{\beta+\alpha}-\left(\ln\frac{t_q}{t_i}\right)^{\beta+\alpha}}{(\beta+\alpha)\ln\left(\frac{t_i}{t_{i-1}}\right)}-\frac{\left(\ln\frac{t_q}{t_{i-1}}\right)^{\beta+\alpha+1}-\left(\ln\frac{t_q}{t_i}\right)^{\beta+\alpha+1}}{(\beta+\alpha+1)\ln\left(\frac{t_i}{t_{i-1}}\right)}.
\end{aligned}
\tag{62}
$$

Based on the right-hand side of (61), we introduce the vector $\mathtt{H} = [\eta_0, \eta_1, \dots, \eta_N]^T$. We further introduce an $(N+1) \times (N+1)$ matrix $\mathtt{A}$. The first and last rows of the matrix $\mathtt{A}$ are $[1, 0, 0, \dots, 0]$ and $[0, 0, \dots, 0, 1]$, respectively, as a result of enforcing the boundary conditions (7) and (8) on η_0 and η_N, respectively. The elements of the interior rows are represented by $(\mathtt{A}_{qi})_{i=1}^{N}$ with $q = 2, \dots, N-1$, so that:

$$\mathtt{A}_{qi} = \frac{\mu}{\Gamma(\alpha)} \begin{cases} \nu_{qi} & ; i = 1, \\ \nu_{qi} + \bar{\nu}_{qi} & ; i = 2, 3, \dots, q-1, \\ \Gamma(\alpha)\mu^{-1} + \bar{\nu}_{qi} & ; i = q, \\ 0 & ; q < i \leq N. \end{cases} \tag{63}$$

For the left-hand side of (61), we assemble the vector $\mathtt{B} = [\mathtt{B}_0, \mathtt{B}_1, \dots, \mathtt{B}_N]^T$, where

$$\mathtt{B}q = \begin{cases} \xi_1 & ; q = 0, \\ \frac{1}{\Gamma(\beta+\alpha)} \sum_{i=1}^{q} [\omega_{qi}\theta_{i-1} + \bar{\omega}_{qi}\theta_i] + f(t_q) & ; q = 1, 2, \dots, N-1, \\ \xi_2 & ; q = N. \end{cases} \tag{64}$$

Therefore, the two-point boundary-value problem (6-8) numerically will be solved by implementing the Newton-Raphson iteration on the non-linear system:

$$\mathtt{AH} = \mathtt{B}. \tag{65}$$

We rearrange the nonlinear system in (65) as:

$$\mathtt{F}(\mathtt{H}) = \mathtt{AH} - \mathtt{B} = 0, \tag{66}$$

and we use a Matlab code to solve iteratively the following:

$$\mathtt{H}^{(k+1)} = \mathtt{H}^{(k)} - \mathtt{J}(\mathtt{H}^{(\mathtt{k})})^{-1}\mathtt{F}(\mathtt{H}^{(\mathtt{k})}). \tag{67}$$

Here, $\mathtt{J}(\mathtt{H})$ is the Jacobian matrix with entries $\mathtt{J}_{pi} = \mathtt{A}_{pi} - \frac{\partial \mathtt{B}_q}{\partial \eta_i}$. Initially, we start with $\mathtt{H}^{(0)}$, which is generated by applying the linear interpolation $\eta_p = \xi_1 + (t_p - a)/(1-a)(\xi_2 - \xi_1)$ using the boundary conditions (7) and (8). Then, from (67) we obtain $\mathtt{H}^{(1)}$ and so on. The iteration stops when we achieve the required tolerance. In the following examples we set s to $\frac{2-\beta}{\beta}$ (see [31]).

Example 2: Consider the following multifractional boundary-value problem:

$$_{\mathfrak{H}}D^{\beta}\left({}_{\mathfrak{H}}D^{\alpha} + \frac{\mu}{t}\right)\eta(t) = \frac{\eta(t)}{t}\left(2 + \mu \operatorname{erf}\left(\sqrt{\ln\left(\frac{\eta(t)}{at}\right)}\right)\right), \tag{68}$$

$$\eta(a) = a^2, \tag{69}$$

$$\eta(1) = 1. \tag{70}$$

Here, the non-linear function $\theta(t, \eta(t)) = \frac{\eta(t)}{t}\left(2 + \mu \text{ erf}\left(\sqrt{\ln \frac{\eta(t)}{at}}\right)\right)$, where erf$(\cdot)$ is the error function. For $\alpha = \beta = 0.5$ and $\mu = 0.1$, $\eta(t) = t^2$ is the exact solution for the problem (68-70). The exact solution of the problem (68-70) versus the numerical solution is depicted in Figures (3)a-b. Both graphs presented with $N = 512$, and the convergence was achieved with tolerance 1e-06 at the 4th iteration.

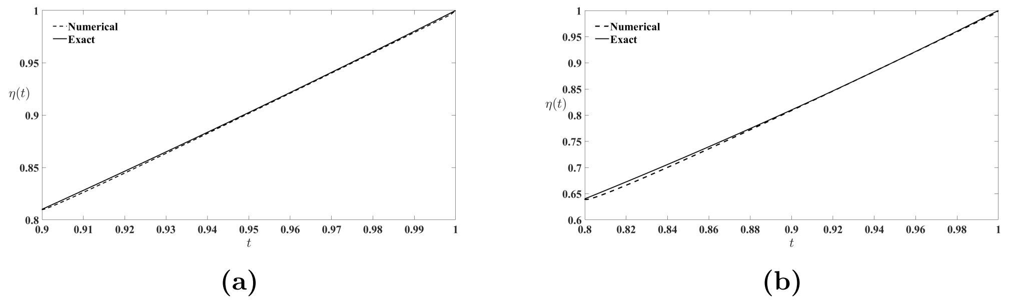


Figure 3: Exact solution, $\eta(t) = t^2$, (solid line) versus numerical solution (dashed line) for the problem (68-70), with $\mu = 0.1$, $\alpha = \beta = 0.5$, $N = 512$, (a) $a = 0.9$, and (b) $a = 0.8$.

The reduction of the absolute error is depicted in Figures (4)a-b, which is of order O(1e-03), as the mesh increases.

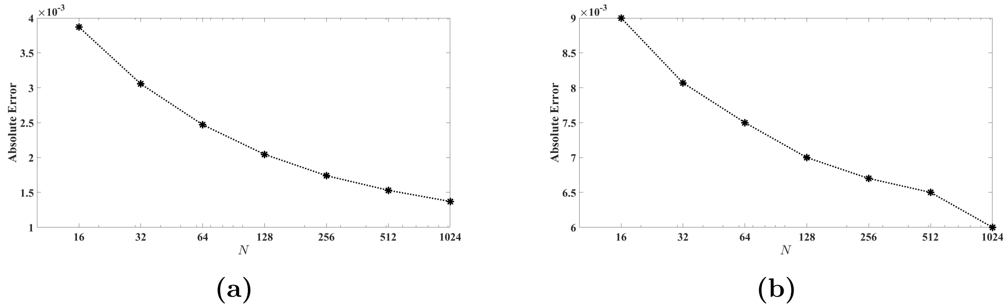


Figure 4: Absolute error for the problem (68-70) as the mesh number (N) increases: (a) for $a = 0.9$ at $t = 0.92$, and (b) for $a = 0.8$ at $t = 8.2$.

Example 3: Consider the following multifractional boundary-value problem:

$$_{\mathrm{CH}}D^{\beta}\left(_{\mathrm{CH}}D^{\alpha} + \frac{\mu}{t}\right)\eta(t) = \frac{\eta(t)}{t} + \frac{2\mu}{\sqrt{\pi t}}\sqrt{\eta(t)} + 1. \tag{71}$$

$$\eta(a) = 0, \tag{72}$$

$$\eta(1) = \ln\left(\frac{1}{a}\right). \tag{73}$$

Here, the non-linear function $\theta(t, \eta(t)) = \frac{\eta(t)}{t} + \frac{2\mu}{\sqrt{\pi t}}\sqrt{\eta(t)} + 1$. For $\alpha = \beta = 0.5$ and $\mu = 0.1$, the exact solution for the problem (71-73) is $\eta(t) = t\ln\left(\frac{t}{a}\right)$. The exact solution of the problem (71-73) versus the numerical solution is depicted in Figures (6)a-b. Both graphs presented with $N = 128$, and the convergence achieved with tolerance 1e-06 at the 3rd iteration.

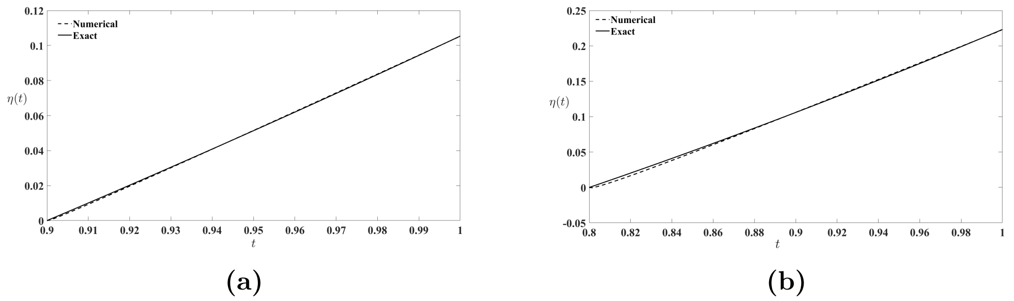


Figure 5: Absolute error for the problem (68-70) as the mesh number (N) increases, (a) for $a = 0.9$ at $t = 0.92$, and (b) for $a = 0.8$ at $t = 8.2$.

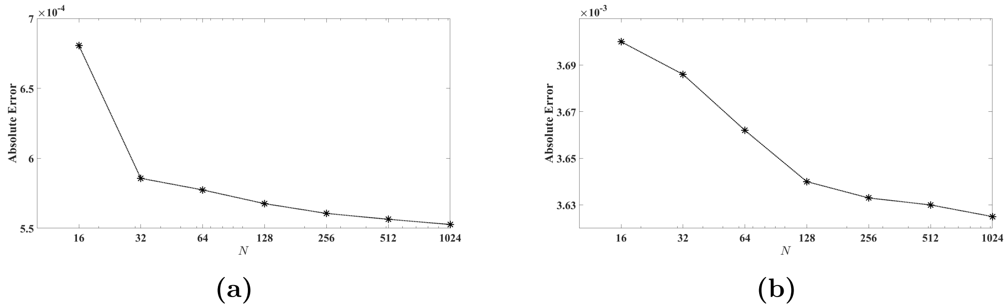


Figure 6: Exact solution (solid line) versus numerical solution (dashed line) for the problem (71-73), with $\mu = 0.1$, $\alpha = \beta = 0.5$, $N = 128$, (a) $a = 0.9$, and (b) $a = 0.8$.

The reduction of absolute error is depicted in Figures (5)a-b as the mesh increases. For $a = 0.9$, the absolute error is of order $O(1e - 04)$ and it is of order $O(1e - 03)$ for $a = 0.8$. A similar behavior for the absolute error was observed for the previous example.

6. Conclusion

This study has provided a detailed examination of the boundary-value problem (6-8), which is governed by the Lane-Emden fractional differential equation in terms of existence, uniqueness, and stability. The fractional derivative is of the Caputo-Hadamard type. The Banach and Sodavoski's fixed-point theorems proved the existence and uniqueness of the solution. The stability of the problem of type Ulam-Hyers and Ulam-Hyers-Rassiass has been investigated. The existence, uniqueness, and stability of the solutions are demonstrated with an example in Section 4. The derived analytical solution to the problem is discretised on a graded mesh using the fractional rectangular, $L_{\ln,1}$ interpolation. The resulting set of non-linear equations is solved using the Newton-Raphson method, which involves the Jacobian matrix to connect $\eta(t)$ and $\theta(t, \eta(t))$ in a non-linear way. The stability and reliability of the proposed numerical scheme were studied through providing examples.

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