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Secure Pointwise Non-Domination and Secure Hop Domination in Graphs

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Abstract. In this paper, we revisit the concept of secure hop domination in graphs and define a new concept called secure pointwise non-domination. A pointwise non-dominating set S is a *secure pointwise non-dominating set* if for every $u \in V(G) \setminus S$, there exists $v \in S \setminus N_G(u)$ such that $(S \setminus \{v\}) \cup \{u\}$ is a pointwise non-dominating set. The *secure pointwise non-domination number* $spnd(G)$ of G is the smallest cardinality of a secure pointwise non-dominating set in G. In this paper, we give bounds on the secure pointwise non-domination number and characterize those graphs which attain these bounds. We also determine the secure pointwise non-domination number of some classes of graphs. Necessary and sufficient conditions for a subset in the join of graphs to be a secure hop dominating set is given. Moreover, we show that given positive integers a and b with $2 \leq a \leq b$, there exists a connected graph such that $\gamma_h(G) = a$ and $\gamma_{sh}(G) = b$, where $\gamma_h(G)$ and $\gamma_{sh}(G)$ are the hop domination number and secure hop domination number of G, respectively.

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Key Words and Phrases: Pointwise non-domination number, secure pointwise non-domination number, hop domination, secure hop domination number, join of graphs

1. Introduction

The concept of security in graphs could provide a framework for modeling a security system [1] that could guarantee that no area is left unmonitored. Recently, Alfeche et al. [2] studied the secure hop dominating sets in graphs, where they gave bounds on the secure hop domination number and determine the secure hop domination numbers of the shadow graph and complementary prism. In this paper, we introduce and study the concept of secure pointwise non-dominating set in a graph. We give bounds on the parameter called secure pointwise non-domination number and give necessary and sufficient conditions for

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those graphs that attain these bounds. Furthermore, we use this newly defined concept to characterize the secure hop dominating sets in the join of graphs.

Other related studies could be found in [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], and [18].

2. Terminology and Notation

Let $G = V(G), E(G))$ be an undirected graph. For any two vertices u and v of G, the distance $d_G(u, v)$ is the length of a shortest path joining u and v. Any u-v path of length $d_G(u, v)$ is called a *u-v geodesic*. The interval $I_G[u, v]$ consists of u, v, and all vertices lying on a u-v geodesic. The interval $I_G(u, v) = I_G[u, v] \setminus \{u, v\}$. Vertices u and v are adjacent (or neighbors) if $uv \in E(G)$. The set of neighbors of a vertex u in G, denoted by $N_G(u)$, is called the *open neighborhood* of u. The *closed neighborhood* of u is the set $N_G[u] = N_G(u) \cup \{u\}$. If $X \subseteq V(G)$, the *open neighborhood* of X is the set $N_G(X) = \bigcup_{u \in X} N_G(u)$. The *closed neighborhood* of X is the set $N_G[X] = N_G(X) \cup X$.

A set $D \subseteq V(G)$ is a *dominating set* in G if for every $v \in V(G) \setminus D$, there exists $u \in D$ such that $uv \in E(G)$, that is, $N_G[D] = V(G)$. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set in G. Any dominating set in G with cardinality $\gamma(G)$, is called a γ-set in G. If $\gamma(G) = 1$ and $\{v\}$ is a dominating set in G, then we call v a *dominating vertex* in G. A dominating set $D \subseteq V(G)$ is *secure dominating* in G if for every $v \in V(G) \setminus D$, there exists $w \in D \cap N_G(v)$ such that $(D \setminus \{w\}) \cup \{v\}$ is a dominating set in G.

A vertex v in G is a *hop neighbor* of vertex u in G if $d_G(u, v) = 2$. The set $N_G^2(u) = \{v \in V(G) : d_G(v, u) = 2\}$ is called the *open hop neighborhood* of u. The *closed hop neighborhood* of u is given by $N_G^2[u] = N_G^2(u) \cup \{u\}$. The *open hop neighborhood* of $X \subseteq V(G)$ is the set $N_G^2(X) = \bigcup_{u \in X} N_G^2(u)$. The *closed hop neighborhood* of X is the set $N_G^2[X] = N_G^2(X) \cup X$. If $S \subseteq V(G)$ and $v \in S$, then a vertex $w \in V(G) \setminus S$ is an *external private hop neighbor* of v if $N_G^2(w) \cap S = \{v\}$. The set containing all the external private hop neighbors of v with respect to S is denoted by $ephn(v; S)$.

A set $S \subseteq V(G)$ is a *hop dominating set* in G if $N_G^2[S] = V(G)$, that is, for every $v \in V(G) \setminus S$, there exists $u \in S$ such that $d_G(u, v) = 2$. The minimum cardinality among all hop dominating sets in G, denoted by $\gamma_h(G)$, is called the *hop domination number* of G. Any hop dominating set with cardinality equal to $\gamma_h(G)$ is called a γ_h-set.

A set $S \subseteq V(G)$ is a *pointwise non-dominating set* of G if for each $v \in V(G) \setminus S$, there exists $u \in S$ such that $v \notin N_G(u)$. The smallest cardinality of a pointwise non-dominating set of G, denoted $pnd(G)$, is called the *pointwise non-domination number* of G.

A pointwise non-dominating set S is a *secure pointwise non-dominating set* if for every $u \in V(G) \setminus S$, there exists $v \in S \setminus N_G(u)$ such that $(S \setminus \{v\}) \cup \{u\}$ is a pointwise non-dominating set. The *secure pointwise non-domination number* $spnd(G)$ of G is the smallest cardinality of a secure pointwise non-dominating set in G. A pointwise (secure pointwise non-dominating) set in G having cardinality equal to $pnd(G)$ (resp. $spnd(G)$) is called a *pnd*-set (resp. *spnd*-set) in G.

A hop dominating set S is *secure hop dominating* if for each $v \in V(G) \setminus S$, there exists $w \in S \cap N_G^2(v)$ such that $(S \setminus \{w\}) \cup \{v\}$ is a hop dominating set in G. The minimum cardinality among all secure hop dominating sets of G, denoted by $\gamma_{sh}(G)$, is called the *secure hop domination number* of G. Any secure hop dominating set with cardinality equal to $\gamma_{sh}(G)$ is called a γ_{sh}-set.

The *join* of graphs G_1 and G_2, denoted $G_1 + G_2$, is the graph with $V(G_1 + G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1) \text{ and } v \in V(G_2)\}$. For other graph theoretic terms not mentioned here, readers may refer to [19] and [20].

3. Results

We shall need the following results.

Theorem 1. [2] $\gamma_{sh}(K_n) = \gamma_{sh}(\overline{K}_n) = n$ *for every positive integer* n.

Theorem 2. [21] *Let* G *be a graph of order* n. *Then each of the following holds:*

(i) $pnd(G) = 1$ *if and only if* G *has an isolated vertex.*

(ii) $pnd(G) = n$ *if and only if* $G = K_n$.

It should be noted that every graph admits a secure pointwise non-dominating sets.

Theorem 3. *Let* G *be a graph of order* n. *Then* $1 \leq pnd(G) \leq spnd(G) \leq n$. *Moreover, each of the following statements holds.*

(i) $spnd(G) = 1$ *if and only if* $G = \overline{K}_n$.

(ii) $spnd(G) = 2$ *if and only if the following conditions hold:*

- (j_1) $G \neq \overline{K}_n$;
- (j_2) *there exist distinct vertices* p, q *such that* $N_G(p) \cap N_G(q) = \varnothing$; *and*
- (j_3) *for each* $x \in V(G) \setminus \{p, q\}$, *either* $xp \notin E(G)$ *and* $N_G(x) \cap N_G(q) = \varnothing$ *or* $xq \notin E(G)$ *and* $N_G(x) \cap N_G(p) = \varnothing$.

(iii) $spnd(G) = n$ *if and only if* $G = K_n$.

Proof. Since every secure pointwise non-dominating set is pointwise non-dominating and $1 \leq pnd(G)$, it follows that $1 \leq pnd(G) \leq spnd(G) \leq n$.
(i) Suppose $spnd(G) = 1$. Then $pnd(G) = 1$. By Theorem 2, G has an isolated vertex. Let $S = \{v\}$ be an *spnd*-set in G. Then v is an isolated vertex of G (otherwise S is not a pointwise non-dominating set). Let $x \in V(G) \setminus S$. Since S is a secure pointwise non-dominating set, $(S \setminus \{v\}) \cup \{x\} = \{x\}$ is a pointwise non-dominating set. This implies that x is an isolated vertex. Therefore, $G = \overline{K}_n$.

Conversely, suppose $G = \overline{K}_n$. Choose any vertex u of G. Then $\{u\}$ is a secure pointwise non-dominating set in G. Hence, $spnd(G) = 1$.

(ii) Suppose $spnd(G) = 2$. Then $G \neq \overline{K}_n$. Let $D = \{p, q\}$ be an *spnd*-set in G. Since $D = \{p, q\}$ is a pointwise non-dominating set, $N_G(p) \cap N_G(q) = \varnothing$. Let $x \in V(G) \setminus D$. Since D is secure pointwise non-dominating, $xp \notin E(G)$ and $D_x = \{x, q\}$ is pointwise non-dominating or $xq \notin E(G)$ and $D_x = \{x, p\}$ is pointwise non-dominating. Thus, $xp \notin E(G)$ and $N_G(x) \cap N_G(q) = \varnothing$ or $xq \notin E(G)$ and $N_G(x) \cap N_G(p) = \varnothing$. Hence, conditions (j_1), (j_2), and (j_3) hold.

Conversely, suppose conditions (j_1), (j_2), and (j_3) hold. By (j_1) and part *(i)*, it follows that $spnd(G) \geq 2$. Let $S = \{p, q\}$. By (j_2), S is a pointwise non-dominating set in G. Next, let $x \in V(G) \setminus S$. Suppose $xp \notin E(G)$. Set $S_x = (S \setminus \{p\}) \cup \{x\} = \{x, q\}$. By (j_3), $N_G(x) \cap N_G(q) = \varnothing$. This implies that S_x is pointwise non-dominating. If $xp \in E(G)$, then $xq \notin E(G)$ by condition (j_2) and $N_G(x) \cap N_G(p) = \varnothing$ by (j_3). It follows that $S'_x = (S \setminus \{q\}) \cup \{x\} = \{x, p\}$ is pointwise non-dominating in G. Therefore, S is a secure pointwise non-dominating set in G and $spnd(G) = |S| = 2$.

(iii) Suppose $spnd(G) = n$. Suppose further that $G \neq K_n$. Then there exist non-adjacent vertices x and y of G. Let $S = V(G) \setminus \{x\}$. Since $y \in S$ and $x \notin N_G(y)$, it follows that S is a pointwise non-dominating set in G. Moreover, because $V(G) \setminus \{y\}$ is also pointwise non-dominating, S is a secure pointwise non-dominating set in G. Thus, $spnd(G) \leq |S| = n-1$, a contradiction. Therefore, $G = K_n$.

The converse is clear. □

Corollary 1. *Let n be a positive integer. Then*

$$spnd(P_n) = \begin{cases} 1 & \text{if } n = 1 \\ 2 & \text{if } n \geq 2. \end{cases}$$

Proof. Let $P_n = [v_1, v_2, \cdots, v_n]$. By Theorem 3*(iii)*, $spnd(P_1) = 1$ and $spnd(P_2) = 2$. Suppose $n \geq 3$. Then $spnd(P_n) \geq 2$ by Theorem 3*(i)*. Let $p = v_1$ and $q = v_2$. Then $N_{P_n}(p) \cap N_{P_n}(q) = \varnothing$. Let $x \in V(P_n) \setminus \{p, q\}$. If $x = v_3$, then $xp \notin E(P_n)$ and $N_{P_n}(q) \cap N_{P_n}(x) = \varnothing$. If $x \neq v_3$, then $xq \notin E(P_n)$ and $N_{P_n}(p) \cap N_{P_n}(x) = \varnothing$. Therefore, $spnd(P_n) = 2$ by Theorem 3*(ii)*. □

Corollary 2. *Let n be a positive integer and $n \geq 3$. Then*

$$spnd(C_n) = \begin{cases} 3 & \text{if } n = 3, 5 \\ 2 & \text{if } n \notin \{3, 5\}. \end{cases}$$

Proof. Let $C_n = [v_1, v_2, \cdots, v_n, v_1]$. By Theorem 3*(iii)*, $spnd(C_3) = 3$. Suppose $n = 5$. It is easy to show that any set $S \subset V(C_n)$ with $|S| = 2$ is not secure pointwise non-dominating. Since $\{v_1, v_2, v_3\}$ is secure pointwise non-dominating, it follows that $spnd(C_5) = 3$. Next, suppose $n \notin \{3, 5\}$. Then $\{v_1, v_2\}$ is a secure pointwise non-dominating set. Therefore, $spnd(C_n) = 2$. □

Theorem 4. *Let $G_1, G_2, \cdots, G_k$ be the components of G where $k \geq 2$ and at least one component is non-trivial. Then each of the following statements holds:*

(*i*) *If* $spnd(G_j) = 2$ *for some* $j \in [k] = \{1, 2, \cdots, k\}$ *or* G *has two trivial components, then* $spnd(G) = 2$.

(*ii*) *If* $k \geq 3$, G *has at most one trivial component, and* $pnd(G_j) \neq 2$ *for all* $j \in [k]$, *then* $spnd(G) = 3$.

(*iii*) *If* $k = 2$ *and* G_j *is complete for each* $j \in \{1, 2\}$, *then*

$$spnd(G) = \begin{cases} 2 & \textit{if } G = K_2 \cup K_m \textit{ where } m \geq 1 \\ 3 & \textit{if } G = K_3 \cup K_m \textit{ where } m \geq 3 \\ 4 & \textit{if } G = K_r \cup K_m \textit{ where } r, m \geq 4 \\ m & \textit{if } G = K_1 \cup K_m \textit{ where } m \geq 1. \end{cases}$$

Proof. (*i*) Suppose $spnd(G_j) = 2$ for some $j \in [k]$. Let $S = \{p, q\}$ be an *spnd*-set in G_j. Then S is a pointwise non-dominating set in G. Let $x \in V(G) \setminus S$. If $x \in V(G_j)$, then there exists $v \in S \setminus N_G(x)$, say $v = p$, such that $(S \setminus \{p\}) \cup \{x\} = \{x, q\}$ is pointwise non-dominating in G_j because S is secure pointwise non-dominating in G_j. Hence, $\{x, q\}$ is pointwise non-dominating in G. Suppose $x \in V(G_i)$ for $i \neq j$. Then, $(S \setminus \{q\}) \cup \{x\} = \{x, p\}$ is pointwise non-dominating in G. Therefore, S is secure pointwise non-dominating in G. This implies that $spnd(G) = 2$. Next, suppose G has two trivial components, say G_1 and G_2. Then $D = V(G_1) \cup V(G_2)$ is cleary an *spnd*-set in G. Hence, $spnd(G) = 2$.

(*ii*) Suppose $k \geq 3$, G has at most one trivial component, and $pnd(G_j) \neq 2$ for all $j \in [k]$. Then $spnd(G) \geq 2$. Suppose $spnd(G) = 2$, say $S = \{x, y\}$ is an *spnd*-set in G. Since $pnd(G_j) \neq 2$ for all $j \in [k]$, it follows that $x \in V(G_i)$ and $y \in V(G_j)$ for distinct indices $i, j \in [k]$. Since G has at most one trivial component, we may assume that G_i is a non-trivial graph. Let $z \in N_G(x)$. Since S is secure pointwise non-dominating, $(S \setminus \{y\}) \cup \{z\} = \{x, z\}$ is pointwise non-dominating in G. Hence, $\{x, z\}$ is pointwise non-dominating in G_i, a contradiction. Thus, $spnd(G) \geq 3$. Pick any $a_j \in V(G_j)$ for $j \in \{1, 2, 3\}$ and set $D = \{a_1, a_2, a_3\}$. Then clearly, D is a secure pointwise non-dominating set in G. Therefore, $spnd(G) = |D| = 3$.

(*iii*) Suppose now that $k = 2$ and G_j is complete for each $j \in \{1, 2\}$. Since $spnd(K_2) = 2$, it follows from (*i*) that $spnd(G) = 2$ whenever $G = K_2 \cup K_m$ for $m \geq 1$. Next, suppose $G = K_3 \cup K_m$ where $m \geq 3$. Clearly, $spnd(G) \geq 3$. Since $S = V(K_3)$ is a secure pointwise non-dominating set in G, it follows that $spnd(G) = |S| = 3$.

Next, suppose $G = K_r \cup K_m$ where $r, m \geq 4$. Clearly, $spnd(G) \geq 3$. Suppose $spnd(G) = 3$, say $Q = \{x, y, z\}$ is an *spnd*-set in G. Since $pnd(K_r) = r \geq 4$ and $pnd(K_m) = m \geq 4$, we may assume that $x, y \in V(K_r)$ and $z \in V(K_m)$. Let $p \in V(K_r) \setminus \{x, y\}$. Since Q is secure pointwise non-dominating, it follows that $(Q \setminus \{z\}) \cup \{p\} = \{x, y, p\}$ is pointwise non-dominating in G (and in K_r), contrary to the fact that $pnd(K_r) \geq 4$. Thus, $spnd(G) \geq 4$. Choose any $a, b \in V(K_r)$ and $c, d \in V(K_m)$. Then $R = \{a, b, c, d\}$ is a secure pointwise non-dominating set in G. Thus, $spnd(G) = 4$.

Finally, suppose $G = K_1 \cup K_m$ where $m \geq 1$. Clearly, $spnd(G) = m$ if $m \in \{1, 2\}$. Suppose $m \geq 3$. Let S be an *spnd*-set in G. Suppose $|S| < m$. Since $pnd(K_m) = m$, it follows that $V(K_1) \subseteq S$. This implies that $|V(K_m) \cap S| \leq m - 2$. Let $w \in V(K_m) \setminus S$ and let $V(K_1) = \{q\}$. Since S is secure pointwise non-dominating in G, $S_w = (S \setminus \{q\}) \cup \{w\}$ is pointwise non-dominating in G (and in K_m), a contradiction because $pnd(K_m) = m > m-1 \geq |S_w|$. Therefore, $spnd(G) \geq m$. Since $V(K_m)$ is a secure pointwise non-dominating set in G, it follows that $spnd(G) = m$. □

Given a graph G, the vertex set $V(G)$ is a secure hop dominating set of G. Thus, every graph admits a secure hop dominating set.

Theorem 5. *[22] Let G and H be any two graphs. A set $S \subseteq V(G+H)$ is hop dominating set in $G+H$ if and only if $S = S_G \cup S_H$, where S_G and S_H are pointwise non-dominating sets in G and H, respectively.*

Corollary 3. *[22] Let G and H be any two graphs. Then*

$$\gamma_h(G + H) = pnd(G) + pnd(H).$$

Theorem 6. *Let G and H be any graphs. Then $S \subseteq V(G+H)$ is a secure hop dominating set if and only if $S = S_G \cup S_H$ and S_G and S_H are secure pointwise non-dominating sets in G and H, respectively.*

Proof. Suppose S is a secure hop dominating set in $G+H$. Since S is a hop dominating set, $S = S_G \cup S_H$ where S_G and S_H are pointwise non-dominating sets in G and H, respectively, by Theorem 5. Let $v \in V(G) \setminus S_G$. Then $v \in V(G + H) \setminus S$. Since S is a secure hop dominating set, there exists $w \in S \setminus N_{G+H}(v)$ such that $S_v = (S \setminus \{w\}) \cup \{v\}$ is a hop dominating set. Note that since $w \in S \setminus N_{G+H}(v)$, $w \in S_G \setminus N_G(v)$. Hence, $S_v = [S_G \setminus \{w\}) \cup \{v\}] \cup S_H$. By Theorem 5, $(S_G \setminus \{w\}) \cup \{v\}$ is a pointwise non-dominating set because S_v is a hop dominating set. This shows that S_G is a secure pointwise non-dominating set in G. Similarly, S_H is a secure pointwise non-dominating set in H.

For the converse, suppose that $S = S_G \cup S_H$ and S_G and S_H are secure pointwise non-dominating sets in G and H, respectively. Then S_G and S_H are pointwise non-dominating sets, S is a hop dominating set in $G + H$ by Theorem 5. Let $x \in V(G + H) \setminus S$. We may assume that $x \in V(G)$. Then $x \notin S_G$. Since S_G is a secure pointwise non-dominating set in G, there exists $y \in S \setminus N_G(x)$ such that $(S_G \setminus \{y\}) \cup \{x\}$ is a pointwise non-dominating set in G. It follows from Theorem 5 that

$$(S \setminus \{y\}) \cup \{x\} = [(S_G \setminus \{y\}) \cup \{x\}] \cup S_H$$

is a hop dominating set in $G + H$. Hence, S is a secure hop dominating set in $G + H$. □

Corollary 4. *Let G and H be any graphs. Then*

$$\gamma_{sh}(G + H) = spnd(G) + spnd(H).$$

Proof. Suppose D_G and D_H are *spnd*-sets in G and H, respectively. Then $D = D_G \cup D_H$ is a secure hop dominating set in $G + H$ by Theorem 6. Hence, $\gamma_{sh}(G + H) \leq |D| = |D_G| + |D_H| = spnd(G) + spnd(H)$.

Next, let S be an *spnd*-set in $G + H$. Then $S_G = S \cap V(G)$ and $S_H = S \cap V(H)$ are pointwise non-dominating sets in G and H, respectively, by Theorem 6. It follows that $\gamma_{sh}(G + H) = |S| = |S_G| + |S_H| \geq spnd(G) + spnd(H)$. This establishes the desired equality. □

Corollary 5. *Each of the following statements holds.*

(i) $\gamma_{sh}(K_{1,n}) = spnd(K_1) + spnd(\overline{K}_n) = 2$ *for all* $n \geq 1$.

(ii) If n *is a positive integer, then*

$$\gamma_{sh}(F_n) = \gamma_{sh}(K_1 + P_n) = \begin{cases} 2 & \text{if } n = 1 \\ 3 & \text{if } n \geq 2. \end{cases}$$

(iii) If n *is a positive integer and* $n \geq 3$*, then*

$$\gamma_{sh}(W_n) = \gamma_{sh}(K_1 + C_n) = \begin{cases} 4 & \text{if } n = 3, 5 \\ 3 & \text{if } n \notin \{3, 5\}. \end{cases}$$

(iv) If m *and* n *are positive integers with* $m \leq n$*, then*

$$\gamma_{sh}(P_m + P_n) = spnd(P_m) + spnd(P_n) = \begin{cases} 2 & \text{if } n = 1 \\ 3 & \text{if } m = 1 \text{ and } n = 2 \\ 4 & \text{if } m \geq 2. \end{cases}$$

(v) If m *and* n *are positive integers with* $3 \leq m \leq n$*, then*

$$\gamma_{sh}(C_m + C_n) = spnd(C_m) + spnd(C_n) = \begin{cases} 6 & \text{if } n = 3, 5 \\ 5 & \text{if } m = 3, 5 \text{ and } n \notin \{3, 5\} \\ 4 & \text{if } m, n \notin \{3, 5\}. \end{cases}$$

(v) If $1 \leq m_1 \leq m_2 \leq \cdots \leq m_k$*, where* $k \geq 2$*, then* $\gamma_{sh}(K_{m_1, m_2, \cdots, m_k}) = k$.
In particular, $\gamma_{sh}(K_{m,n}) = 2$ *for all* $m, n \geq 2$.

Proof. Note that by Corollary 4, Theorem 3(i), Corollary 1, and Corollary 2, statements (i), (ii), (iii), (iv), and (v) follow. By repetitive application of Corollary 4 and by Theorem 4(i), we have

$$\gamma_{sh}(K_{m_1, m_2, \cdots, m_k}) = \sum_{j \in [k]} spnd(\overline{K}_{m_j}) = k.$$

Therefore, the assertions hold. □

Theorem 7. *Let a and b be positive integers such that $2 \leq a \leq b$. Then there exists a connected graph G such that $\gamma_h(G) = a$ and $\gamma_{sh}(G) = b$.*

Proof. Suppose $a = b$. Consider $G = K_a$. Then $\gamma_h(G) = a$ because $V(K_a)$ is the only hop dominating set in K_a. With the same reason, $\gamma_{sh}(G) = a$ (also by Theorem 3(iii)). Next, suppose $a < b$. Let $m = b - a$ and let $G = (K_1 \cup K_{m+1}) + K_{a-1}$. By Corollary 3 and Theorem 2,

$$\gamma_h(G) = pnd(K_1 \cup K_{m+1}) + pnd(K_{a-1}) = 1 + (a - 1) = a.$$

By Corollary 4, and Theorem 3(iii), we have

$$\gamma_{sh}(G) = spnd(K_1 \cup K_{m+1}) + spnd(K_{a-1}) = 1 + (a - 1) = (m + 1) + (a - 1) = b.$$

This proves the assertion. □

The next result is found in [22].

Theorem 8. *Let G and H be any two graphs. A set $C \subseteq V(G \circ H)$ is a hop dominating set of $G \circ H$ if and only if $C = A \cup (\cup_{v \in V(G) \cap N_G(A)} S_v)) \cup (\cup_{w \in V(G) \setminus N_G(A)} E_w)$; where*

- *(i) $A \subseteq V(G)$ such that for each $w \in V(G) \setminus A$, there exists $x \in A$ with $d_G(w, x) = 2$ or there exists $y \in N_G(w)$ with $V(H^y) \cap C \neq \varnothing$,*
- *(ii) $S_v \subseteq V(H^v)$ for each $v \in N_G(A)$, and*
- *(iii) E_w is a pointwise non-dominating set in H^w for each $w \in V(G) \setminus N_G(A)$.*

Theorem 9. *Let G and H be any two non-trivial graphs. If $C = A \cup (\cup_{v \in V(G)} S_v))$, where A is a secure hop dominating set in G and S_v is a secure pointwise non-dominating set in H^v for each $v \in V(G)$, then C is a secure hop dominating set in $G \circ H$.*

Proof. By Theorem 8, C is a hop dominating set in $G \circ H$. Let $x \in V(G \circ H) \setminus C$ and let $v \in V(G)$ such that $x \in V(v + H^v)$. Consider the following cases:

Case 1. $x = v$.
Then $x \in V(G) \setminus A$. Since A is secure hop dominating in G, $(A \setminus \{y\}) \cup \{x\}$ is hop dominating for some $y \in A \cap N_G^2(x)$. Hence,

$$(C \setminus \{y\}) \cup \{x\} = [(A \setminus \{y\}) \cup \{x\}] \cup (\cup_{w \in V(G)} S_w)$$

is hop dominating in $G \circ H$ by Theorem 8.

Case 2. $x \in V(H^v)$.

Then $x \in V(H^v) \setminus S_v$. Since S_v is secure pointwise non-dominating in H^v, there exists $p \in S_v \setminus N_{H^v}(x)$ such that $(S_v \setminus \{p\}) \cup \{x\}$ is pointwise non-dominating in H^v. Therefore, by Theorem 8,

$$(C \setminus \{p\}) \cup \{x\} = A \cup [(\cup_{w \in V(G)\setminus\{v\}} S_w)] \cup ((S_v \setminus \{p\}) \cup \{x\})$$

is hop dominating in $G \circ H$.

Therefore, C is a secure hop dominating set in $G \circ H$. □

Corollary 6. *Let G and H be any two non-trivial graphs. Then*

$$\gamma_{sh}(G \circ H) \leq \gamma_{sh}(G) + |V(G)|spnd(H).$$

Proof. Let A be a γ_{sh}-set in G and let S_v be an *spnd*-set in H^v for each $v \in V(G)$. Then $C = A \cup (\cup_{v \in V(G)} S_v))$ is a secure hop dominating set in $G \circ H$ by Theorem 9. Thus,

$$\begin{aligned}
\gamma_{sh}(G \circ H) &\leq |C| \\
&= |A| + \sum_{v \in V(G)} |S_v| \\
&= \gamma_{sh}(G) + \sum_{v \in V(G)} spnd(H) \\
&= \gamma_{sh}(G) + |V(G)|spnd(H).
\end{aligned}$$

This proves the assertion. □

Remark 1. *The bound given in Corollary 6 is sharp. Strict inequality is also attainable.*

To see this, consider $G_1 = \overline{K}_2$, $H_1 = K_2$, $G_2 = K_2$, and $H_2 = \overline{K}_2$. Then

$$\gamma_{sh}(G_1 \circ H_1) = \gamma_{sh}(K_3 \cup K_3) = 6 = \gamma_{sh}(G_1) + 2spnd(H_1)$$

and

$$\gamma_{sh}(G_2 \circ H_2) = 2 < 4 = \gamma_{sh}(G_2) + 2spnd(H_2).$$

Theorem 10. *Let G be a non-trivial connected graph with $\delta(G) \geq 2$ and let H be any graph. Then $\gamma_{sh}(G \circ H) \leq |V(G)|$.*

Proof. By Theorem 8, $C = V(G)$ is a hop dominating set in $G \circ H$. Next, let $x \in V(G \circ H) \setminus C$ and let $v \in V(G)$ such that $x \in V(v + H^v)$. Since $C = V(G)$, it follows that $x \in V(H^v) \setminus S_v$. Pick any $w \in N_G(v)$ and let $C_x = [V(G) \setminus \{w\}] \cup \{x\}$. Let $p \in V(G \circ H) \setminus C_x$. Suppose $p = w$. Then $x \in C_x \cap N^2_{G \circ H}(p)$. Suppose $p \in V(H^z)$ for some $z \in V(G)$. If $z = w$, then $v \in C_x \cup N_{G \circ H}(p)$ and $d_{G \circ H}(u, p) = 2$. Suppose $z \neq w$. Since $\delta(G) \geq 2$, we may choose $u \in N_G(z) \setminus \{w\}$. This implies that $u \in C_x$ and $d_{G \circ H}(u, p) = 2$. Thus, C_x is hop dominating in $G \circ H$. Since x was arbitrarily chosen in $V(G \circ H) \setminus C$, it follows that C is a secure hop dominating set in $G \circ H$. Therefore, $\gamma_{sh}(G \circ H) \leq |V(G)|$. □

4. Conclusion

Secure pointwise non-domination was introduced and investigated in this study. Bounds on the secure pointwise non-domination number were established, and graphs attaining these bounds were characterized. Necessary and sufficient conditions for a subset in the join of graphs to be a secure pointwise non-dominating set was obtained. Moreover, it was shown that given positive integers a and b with $2 \leq a \leq b$, there exists a connected graph such that $\gamma_h(G) = a$ and $\gamma_{sh}(G) = b$, where $\gamma_h(G)$ and $\gamma_{sh}(G)$ are the hop domination number and secure hop domination number of G, respectively.

Secure pointwise non-domination may be used to characterize the secure hop dominating sets in the corona and lexicographic product of graphs.

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