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Exactness of the Functors $\mathbf{Hom}_{\mathscr{A}}(X,-)$, $\mathbf{Hom}_{\mathscr{A}}(-,X)$, $\mathbf{Hom}_{\mathbf{Comp}(\mathscr{A})}(X,-)$, $\mathbf{Hom}_{\mathbf{Comp}(\mathscr{A})}(-,X)$ and the Homological Functors $\tilde{H}_n(X,-)$ and $\tilde{H}_n(-,X)$ in a Balanced Abelian Category

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Abstract. This article presents several results concerning the exactness of covariant and contravariant Hom functors and their derived functors in a balanced abelian category $\mathscr{A}$. In particular:

(i) The functors $\mathrm{Hom}_{\mathscr{A}}(X,-)$ and $\mathrm{Hom}_{\mathscr{A}}(-,X)$ are left exact, and become exact if and only if X is projective (resp. injective).

(ii) The functors $\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X,-)$ and $\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(-,X)$ on the category of complexes $\mathrm{Comp}(\mathscr{A})$ preserve this behavior.

(iii) The homological functors $\tilde{H}_n(X,-)$ and $\tilde{H}_n(-,X)$ are constructed for all $n \in \mathbb{Z}$.

(iv) For projective X, the connecting morphism $\lambda_n : \tilde{H}_n(X,-)((T,\gamma)) \to \tilde{H}_{n+1}(X,-)((Y,\alpha))$ allows $\tilde{H}_n(X,-)$ to send short exact sequences in $\mathrm{Comp}(\mathscr{A})$ into long exact sequences in Ab.

(v) Similarly, for injective X, the morphism $\delta_n : \tilde{H}_n(-,X)((Y,\alpha)) \to \tilde{H}_{n+1}(-,X)((T,\gamma))$ shows that $\tilde{H}_n(-,X)$ also preserves long exact sequences.

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Key Words and Phrases: Abelian category, balanced category, homological functors, projective object, injective object, category of abelian groups

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Introduction

The main objective of this article is to study the exactness of the functors $\mathrm{Hom}_{\mathscr{A}}(X,-)$, $\mathrm{Hom}_{\mathscr{A}}(-,X) : \mathscr{A} \longrightarrow \mathrm{Ab}, \mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X,-), \mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(-,X) : \mathrm{Comp}(\mathscr{A}) \longrightarrow \mathrm{Comp}(\mathrm{Ab})$

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and the homological functors $\tilde{H}_n(X,-), \tilde{H}_n(-,X) : \mathrm{Comp}(\mathscr{A}) \longrightarrow \mathrm{Ab}$ where $\mathscr{A}$ is a balanced abelian category, Ab the category of abelian groups, and n an integer in $\mathbb{Z}$. This study is motivated by extending the fundamental concepts of homological algebra from the category of modules to an arbitrary balanced abelian category. This generalization is not straightforward, as evidenced by the proofs of the various results mentioned in the abstract. Furthermore, this work is inspired by the exactness of the functors $\mathrm{Hom}_{\mathrm{A}}(X,-)$, $\mathrm{Hom}_{\mathrm{A}}(-,X)$ and the homological functor $H_n : \mathrm{Comp(A\text{-}Mod)} \longrightarrow \mathrm{Ab}$ in the category of left A-modules A-Mod (resp. right A-modules Mod-A) and complexes of left A-modules (Comp(A-Mod)) (resp. complexes of right A-modules Comp(Mod-A)): "The functor and its relationship with homological functor" [1], "Localization, Isomorphisms and Adjoint Isomorphism in the Category Comp(A- Mod"[2],"Localization of hopfian and cohopfian objects in the categories of a-mod, agr (a-mod) and comp(agr(a-mod))" [3], "Adjunction and localization in the category a-alg of a-algebras" [4], "Modules and rings" [5], "Notes on homological algebras" [6], "Abelian categories" [7], "Des catégories abéliennes" [8], "An Introduction to Homological Algebra" [9],"An introduction to homological algebra" [10] and other important results on abelian category and homological functors by the autors: ElHadjOusseynou [11], Joseph .J Rotman [12], [13], Bassirou DEMBELE [14], Ahmed OULD CHBIH [15], Charles A weibel [16], Ahmed OULD CHBIH et al [17],[18] and Moussa Thiaw [19]. Thus, the article is structured as follows:

In **Section 1** titled preliminary results, we provided the following definitions: abelian category, balanced category, $\mathrm{Comp}(\mathscr{A})$, exact sequence in $\mathrm{Comp}(\mathscr{A})$. And we presented some preliminary results.

In **Section 2**, titled the exactness of the functors $\mathrm{Hom}_{\mathscr{A}}(X,-)$ and $\mathrm{Hom}_{\mathscr{A}}(-,X)$, where $\mathscr{A}$ is a balanced abelian category and X is an object in $\mathscr{A}$. The following results have been shown:

(i) Let $\mathscr{A}$ be a balanced abelian category and X an object in $\mathscr{A}$. Then the covariant functor denoted $Hom_{\mathscr{A}}(X,-) : \mathscr{A} \longrightarrow \mathrm{Ab}$, defined by:

(a) $\forall Y \in \mathrm{Ob}(\mathscr{A})$, $Hom_{\mathscr{A}}(X,-)(Y) = Hom_{\mathscr{A}}(X,Y) \in \mathrm{Ob(Ab)}$;

(b) $\forall f \in \mathrm{Hom}_{\mathscr{A}}(Y,Z)$,

$$Hom_{\mathscr{A}}(X,-)(f) = Hom_{\mathscr{A}}(X,f) = f^* : \begin{array}{ccc} Hom_{\mathscr{A}}(X,Y) & \longrightarrow & Hom_{\mathscr{A}}(X,Z) \\ \phi & \longmapsto & f \circ \phi \end{array}$$

; is additive, left-exact functor and it is exact if and only if X is a projective object in $\mathscr{A}$.

(ii) Let $\mathscr{A}$ be a balanced abelian category and X an object in $\mathscr{A}$. Then the contravariant functor denoted $Hom_{\mathscr{A}}(-,X) : \mathscr{A} \longrightarrow \mathrm{Ab}$, defined by:

(a) $\forall Y \in \mathrm{Ob}(\mathscr{A})$, $Hom_{\mathscr{A}}(-,X)(Y) = Hom_{\mathscr{A}}(Y,X) \in \mathrm{Ob(Ab)}$

(b) $\forall f \in \mathrm{Hom}_{\mathscr{A}}(Y,Z)$,

$$Hom_{\mathscr{A}}(f,X) = f^* : \begin{array}{ccc} Hom_{\mathscr{A}}(Z,X) & \longrightarrow & Hom_{\mathscr{A}}(Y,X) \\ \phi & \longmapsto & \phi \circ f \end{array}$$

is additive, left-exact functor and it is exact if and only if X is an injective object in $\mathscr{A}$.

In Section 3, titled the exactness of the functors $\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X,-)$ and $\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(-,X)$ where $\mathscr{A}$ is a balanced abelian category and X is an object in $\mathscr{A}$. We proved the following results:

(i) Let $\mathscr{A}$ be a balanced abelian category and X an object of $\mathscr{A}$. Then:

- (a) the functor $\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X,-) : \mathrm{Comp}(\mathscr{A}) \to \mathrm{Comp}(Ab)$ is a covariant, additive, and left-exact functor;
- (b) the functor $\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X,-) : \mathrm{Comp}(\mathscr{A}) \to \mathrm{Comp}(Ab)$ is exact if and only if X is a projective object in $\mathscr{A}$.

(ii) Let $\mathscr{A}$ be a balanced abelian category and X an object of $\mathscr{A}$. Then:

- (a) the functor $\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(-,X) : \mathrm{Comp}(\mathscr{A}) \to \mathrm{Comp}(Ab)$ is a contravariant, additive, and left-exact functor;
- (b) the functor $\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(-,X) : \mathrm{Comp}(\mathscr{A}) \to \mathrm{Comp}(Ab)$ is exact if and only if X is an injective object in $\mathscr{A}$.

In Section 4, we studied the homological functors of degree n: $\tilde{H}_n(X,-), \tilde{H}_n(-,X) : \mathrm{Comp}(\mathscr{A}) \to Ab$ where $\mathscr{A}$ is a balanced abelian category and n is an integer in $\mathbb{Z}$. We proved the following results:

(i) The functor $\tilde{H}_n(X,-)$ is a covariant additive functor.

(ii) Let $(0) \longrightarrow (Y,\alpha) \xrightarrow{f} (Z,\beta) \xrightarrow{g} (T,\gamma) \longrightarrow (0)$ be a short exact sequence of morphisms in $\mathrm{Comp}(\mathscr{A})$, where X is a projective object in $\mathscr{A}$ and $\mathscr{A}$ is a balanced abelian category. Then:

- (a) We call the morphism of connection associated to the functor $\tilde{H}_n(X,-)$, denoted by λ_n, the morphism defined by:

$$\begin{array}{rccc} \lambda_n : & \tilde{H}_n(X,-)((T,\gamma)) & \longrightarrow & \dfrac{\tilde{H}_{n+1}(X,-)((Y,\alpha))}{f^{*-1}_{n+2}(\beta^*_{n+1}(g^{*-1}_{n+1}(k_{n+1})))} \quad \forall n \in \mathbb{Z} \\ & \overline{k_{n+1}} & \longmapsto & \end{array}$$

- (b) The sequence

$$\cdots \longrightarrow \tilde{H}_n(X,-)((Y,\alpha)) \xrightarrow{\tilde{H}_n(X,-)(f)} \tilde{H}_n(X,-)((Z,\beta)) \xrightarrow{\tilde{H}_n(X,-)(g)} \tilde{H}_n(X,-)((T,\gamma)) \xrightarrow{\lambda_n}$$

$$\tilde{H}_{n+1}(X,-)((Y,\alpha)) \xrightarrow{\tilde{H}_{n+1}(X,-)(f)} \tilde{H}_{n+1}(X,-)((Z,\beta)) \xrightarrow{\tilde{H}_{n+1}(X,-)(g)} \tilde{H}_{n+1}(X,-)((T,\gamma)) \xrightarrow{\lambda_{n+1}} \cdots$$

is a long exact sequence of abelian group morphisms. That is, for all $n \in \mathbb{Z}$:

$$\begin{cases} \mathrm{Im}(\tilde{H}_n(X,-)(f)) = \mathrm{Ker}(\tilde{H}_n(X,-)(g)) \\ \mathrm{Im}(\tilde{H}_n(X,-)(g)) = \mathrm{Ker}(\lambda_n) \\ \mathrm{Im}(\lambda_n) = \mathrm{Ker}(\tilde{H}_{n+1}(X,-)(f)) \end{cases}$$

(iii) The functor $\tilde{H}_n(-, X)$ is a contravariant additive functor.

(iv) Let $(0) \longrightarrow (Y, \alpha) \xrightarrow{f} (Z, \beta) \xrightarrow{g} (T, \gamma) \longrightarrow (0)$ be a short exact sequence in Comp($\mathscr{A}$), where $\mathscr{A}$ is a balanced abelian category and X is an injective object in $\mathscr{A}$. Then:

(a) We call the morphism of connection associated to the functor $\tilde{H}_n(-, X)$, denoted by λ_n, the morphism defined by:

$$\delta_n : \begin{array}{ccc} \tilde{H}_n(-,X)((Y,\alpha)) & \longrightarrow & \dfrac{\tilde{H}_{n+1}(-,X)((T,\gamma))}{} \\ \overline{k_{n+1}} & \longmapsto & \overline{g^{*-1}_{n+2}(\beta^*_{n+1}(f^{*-1}_{n+1}(k_{n+1})))} \end{array}, \quad \forall n \in \mathbb{Z}$$

(b) The sequence

$$\cdots \longrightarrow \tilde{H}_n(-,X)((T,\gamma)) \xrightarrow{\tilde{H}_n(-,X)(g)} \tilde{H}_n(-,X)((Z,\beta)) \xrightarrow{\tilde{H}_n(-,X)(f)} \tilde{H}_n(-,X)((Y,\alpha)) \xrightarrow{\delta_n}$$

$$\tilde{H}_{n+1}(-,X)((T,\gamma)) \xrightarrow{\tilde{H}_{n+1}(-,X)(g)} \tilde{H}_{n+1}(-,X)((Z,\beta)) \xrightarrow{\tilde{H}_{n+1}(-,X)(f)} \tilde{H}_{n+1}(-,X)((Y,\alpha)) \xrightarrow{\delta_{n+1}} \cdots$$

is a long exact sequence in Ab. That is (for all $n \in \mathbb{Z}$):

$$\begin{cases} \mathrm{Im}(\tilde{H}_n(-,X)(g)) = \mathrm{Ker}(\tilde{H}_n(-,X)(f)) \\ \mathrm{Im}(\tilde{H}_n(-,X)(f)) = \mathrm{Ker}(\delta_n) \\ \mathrm{Im}(\delta_n) = \mathrm{Ker}(\tilde{H}_{n+1}(-,X)(g)). \end{cases}$$

1. Preliminary Results

[**Abelian Category**]
An abelian category is a category $\mathscr{A}$ that satisfies the following conditions:

(i) the category $\mathscr{A}$ has a zero object;

(ii) for all objects X and Y in $\mathscr{A}$, the set $\mathrm{Hom}_{\mathscr{A}}(X, Y)$ is endowed with an abelian group structure whose composition law is denoted additively ;

(iii) the composition in $\mathscr{A}$ is bilinear with respect to the additions;

(iv) every finite family of objects in $\mathscr{A}$ has a coproduct;

(v) every morphism in $\mathscr{A}$ has a kernel and a cokernel;

(vi) every monomorphism in $\mathscr{A}$ is the kernel of its cokernel;

(vii) every epimorphism in $\mathscr{A}$ is the cokernel of its kernel.

[**Balanced Category**]
A category $\mathscr{C}$ is balanced if:

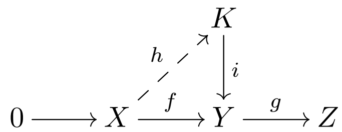
(i) every monomorphism of $\mathscr{C}$ is retractable;

(ii) and every epimorphism of $\mathscr{C}$ is splittable.

[**Left Exact Sequence in** $\mathscr{A}$]

Let (S): $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z$ be a short sequence of morphisms in $\mathscr{A}$. Then (S) is called left exact if:

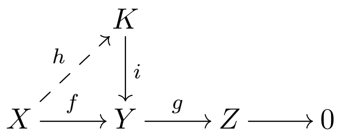
$$\begin{cases} g \circ f = e_{X,Z}, \text{ where } e_{X,Z} \text{ is the zero morphism of the abelian group } \mathrm{Hom}_{\mathscr{A}}(X,Z) \\ N(f) = (0, e_{0,X}) \text{ where } e_{0,X} : 0 \to X \text{ is the zero morphism of the abelian group } \mathrm{Hom}_{\mathscr{A}}(0,X) \\ \mathrm{coN}(h) = (0, e_{K,0}) \text{ where } e_{K,0} : K \to 0 \text{ is the zero morphism of the abelian group } \mathrm{Hom}_{\mathscr{A}}(K,0) \\ \text{with} \quad h \quad \text{the morphism satisfying} \quad i \circ h = f \quad \text{and} \quad \mathrm{Ker}\, g = (K,i). \end{cases}$$



[**Right Exact Sequence in** $\mathscr{A}$]

Let (S): $X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ be a short sequence of morphisms in $\mathscr{A}$. Then (S) is called right exact if:

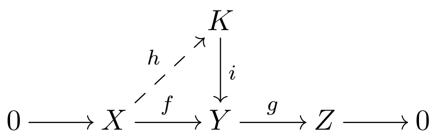
$$\begin{cases} g \circ f = e_{X,Z}, \text{ where } e_{X,Z} \text{ is the zero morphism of the abelian group } \mathrm{Hom}_{\mathscr{A}}(X,Z) \\ \mathrm{CoN}\ (g) = (0, e_{Z,0}) \text{ where } e_{Z,0} : Z \to 0 \text{ is the zero morphism of the abelian group } \mathrm{Hom}_{\mathscr{A}}(Z,0) \\ \mathrm{coN}(h) = (0, e_{K,0}) \text{ where } e_{K,0} : K \to 0 \text{ is the zero morphism of the abelian group } \mathrm{Hom}_{\mathscr{A}}(K,0) \\ \text{with} \quad h \quad \text{the morphism satisfying} \quad i \circ h = f \quad \text{and} \quad \mathrm{Ker}\, g = (K,i). \end{cases}$$



[**Exact Sequence in** $\mathscr{A}$]

Let (S): $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ be a short sequence of morphisms in $\mathscr{A}$. Then (S) is called exact if it is both left exact and right exact. That is:

$$\begin{cases} g \circ f = e_{X,Z}, \text{ where } e_{X,Z} \text{ is the zero morphism of the abelian group } \mathrm{Hom}_{\mathscr{A}}(X,Z) \\ N(f) = (0, e_{0,X}) \text{ where } e_{0,X} : 0 \to X \text{ is the zero morphism of the abelian group } \mathrm{Hom}_{\mathscr{A}}(0,X) \\ \mathrm{coN}\ (g) = (0, e_{Z,0}) \text{ where } e_{Z,0} : Z \to 0 \text{ is the zero morphism of the abelian group } \mathrm{Hom}_{\mathscr{A}}(Z,0) \\ \mathrm{coN}(h) = (0, e_{K,0}) \text{ where } e_{K,0} : K \to 0 \text{ is the zero morphism of the abelian group } \mathrm{Hom}_{\mathscr{A}}(K,0) \\ \text{with} \quad h \quad \text{the morphism satisfying} \quad i \circ h = f \quad \text{and} \quad \mathrm{Ker}\, g = (K,i). \end{cases}$$



Remark 1.
0 *denotes the zero object of the category* $\mathscr{A}$.

[**Comp**($\mathscr{A}$)]
The category of complexes of an abelian category $\mathscr{A}$, denoted Comp($\mathscr{A}$), is defined by:

(i) The objects are complex sequence in $\mathscr{A}$.
A complex sequence in $\mathscr{A}$ is a sequence of morphisms in $\mathscr{A}$ $(\alpha_n : X_n \to X_{n+1})_{n\in\mathbb{Z}}$, denoted (X, α), such that $\alpha_{n+1} \circ \alpha_n = e_{X_n,X_{n+2}} \quad \forall n \in \mathbb{Z}$, where $e_{X_n,X_{n+2}}$ is the zero morphism of the abelian group $\mathrm{Hom}_{\mathscr{A}}(X_n, X_{n+2})$.

(ii) The morphisms (arrows) are complex chains in $\mathscr{A}$.
Let $(X, \alpha) = (\alpha_n : X_n \to X_{n+1})_{n\in\mathbb{Z}}$ and $(Y, \beta) = (\beta_n : Y_n \to Y_{n+1})_{n\in\mathbb{Z}}$ be two complex sequences in $\mathscr{A}$. A complex chain $(f_n : X_n \longrightarrow Y_n)_{n\in\mathbb{Z}}$, denoted $f : (X, \alpha) \to (Y, \beta)$, is a sequence of morphisms in $\mathscr{A}$ such that: $f_{n+1} \circ \alpha_n = \beta_n \circ f_n \quad \forall n \in \mathbb{Z}$.

[**Right Exact Sequence in Comp**($\mathscr{A}$)]
Let $(S) : \ (Y, \alpha) \xrightarrow{\ f\ } (Z, \beta) \xrightarrow{\ g\ } (T, \gamma) \longrightarrow (0)$ be a short sequence of morphisms in Comp($\mathscr{A}$) where $\mathscr{A}$ is an abelian category. Then we say that (S) is right exact if for every integer n in $\mathbb{Z}$ the sequence $Y_n \xrightarrow{\ f_n\ } Z_n \xrightarrow{\ g_n\ } T_n \longrightarrow 0$ is a right exact short sequence of morphisms in $\mathscr{A}$. [**Left Exact Sequence in Comp**($\mathscr{A}$)]
Let $(S) : \ (0) \longrightarrow (Y, \alpha) \xrightarrow{\ f\ } (Z, \beta) \xrightarrow{\ g\ } (T, \gamma)$ be a short sequence of morphisms in Comp($\mathscr{A}$) where $\mathscr{A}$ is an abelian category. Then we say that (S) is left exact if for every integer n in $\mathbb{Z}$ the sequence $0 \longrightarrow Y_n \xrightarrow{\ f_n\ } Z_n \xrightarrow{\ g_n\ } T_n$ is a left exact short sequence of morphisms in $\mathscr{A}$. [**Exact Sequence in Comp**($\mathscr{A}$)]
Let $(S) : \ (0) \longrightarrow (Y, \alpha) \xrightarrow{\ f\ } (Z, \beta) \xrightarrow{\ g\ } (T, \gamma) \longrightarrow (0)$ be a short sequence of morphisms in Comp($\mathscr{A}$) where $\mathscr{A}$ is an abelian category. Then we say that (S) is exact if for every integer n in $\mathbb{Z}$ the sequence $0 \longrightarrow Y_n \xrightarrow{\ f_n\ } Z_n \xrightarrow{\ g_n\ } T_n \longrightarrow 0$ is a short exact sequence of morphisms in $\mathscr{A}$. Let $\mathscr{A}$ be an abelian category. Then Comp($\mathscr{A}$) is an abelian category.
Proof. See Page 319 [9].

Let $\mathscr{A}$ be a balanced abelian category. Then Comp($\mathscr{A}$) is a balanced abelian category.
Proof.
Let $\mathscr{A}$ be a balanced abelian category, which means: every monomorphism in $\mathscr{A}$ is a retraction and every epimorphism in $\mathscr{A}$ is a section.

- Let $f : (X, \alpha) \to (Y, \beta)$ be a monomorphism in Comp($\mathscr{A}$).
  Since f is a monomorphism, it follows that for every $n \in \mathbb{Z}, f_n : X_n \longrightarrow Y_n$ is a monomorphism in $\mathscr{A}$. And since $\mathscr{A}$ is balanced, for every $n \in \mathbb{Z}, f_n : X_n \longrightarrow Y_n$ is a retraction. Therefore, f is a retraction.

- Let $f : (X, \alpha) \to (Y, \beta)$ be an epimorphism in Comp($\mathscr{A}$). Since f is an epimorphism, it follows that for every $n \in \mathbb{Z}$, $f_n : X_n \longrightarrow Y_n$ is an epimorphism in $\mathscr{A}$. And since $\mathscr{A}$ is balanced, for every $n \in \mathbb{Z}$, $f_n : X_n \longrightarrow Y_n$ is a section. Therefore, f is a section. Hence, every monomorphism in Comp($\mathscr{A}$) is a retraction and every epimorphism of Comp($\mathscr{A}$) is a section.

Thus, Comp($\mathscr{A}$) is balanced. According to the proposition 1, Comp($\mathscr{A}$) is balanced abelian category.

2. Exactness of the Functors Hom$_{\mathscr{A}}(X, -)$ and Hom$_{\mathscr{A}}(-, X)$

Let $\mathscr{A}$ be an abelian category, $X, Y \in$ Ob($\mathscr{A}$), and $f : X \to Y$ a morphism of $\mathscr{A}$. Then:

(i) The kernel of f, $N(f)$ is zero if and only if f is a monomorphism.

(ii) The cokernel of f, $coN(f)$ is zero if and only if f is an epimorphism.

Proof.

[label=)](⇒) Suppose that $N(f)$ is zero and let us show that f is a monomorphism. Let $u, v : Z \to X$ be two morphisms in $\mathscr{A}$ such that $f \circ u = f \circ v$.
We have:

$$f \circ u = f \circ v \Rightarrow f \circ (u - v) = e_{Z,Y}.$$

By the definition of the kernel of f, we have: $N(f) = (K, i)$ implies that $f \circ i = e_{K,Y}$. If $N(f) = (K, i)$ is zero, then $i = e_{K,X}$ and thus $f \circ i = e_{K,Y} \Rightarrow f \circ e_{K,X} = e_{K,Y}$. There exists a unique $h : Z \to K$ such that $e_{K,X} \circ h = u - v$. Hence,
$\forall i \in Hom_{\mathscr{A}}(K, X)$, with $e_{K,X}$ being the neutral element of the abelian group $Hom_{\mathscr{A}}(K, X)$, we have:

$$\begin{aligned}
(i + e_{K,X}) \circ h = (e_{K,X} + i) \circ h = i \circ h &\Rightarrow i \circ h + e_{K,X} \circ h = e_{K,X} \circ h + i \circ h = i \circ h \\
&\Rightarrow \begin{cases} i \circ h + e_{K,X} \circ h = i \circ h + e_{Z,X} = i \circ h \\ e_{K,X} \circ h + i \circ h = e_{Z,X} + i \circ h = i \circ h \end{cases} \\
&\Rightarrow e_{K,X} \circ h = e_{Z,X}
\end{aligned}$$

Thus, we have:

$$\begin{aligned}
e_{K,X} \circ h = u - v = e_{Z,X} &\Rightarrow u - v = e_{Z,X} \\
&\Rightarrow u - v + v = e_{Z,X} + v \\
&\Rightarrow u + e_{Z,X} = v \\
&\Rightarrow u = v.
\end{aligned}$$

Thus, f is a monomorphism.
(⇐) Suppose that f is a monomorphism and let us show that $N(f)$ is zero.

Let $i : K \to X$ be a morphism such that $f \circ i = e_{K,X}$.
But $e_{K,Y} = f \circ e_{K,X}$. Indeed: $\forall i \in Hom_{\mathscr{A}}(K, X)$, with $e_{K,X}$ being the neutral element of the abelian group $Hom_{\mathscr{A}}(K, X)$, we have:

$$\begin{aligned} f \circ (i + e_{K,X}) = f \circ (e_{K,X} + i) = f \circ i &\Rightarrow f \circ i + f \circ e_{K,X} = f \circ e_{K,X} + f \circ i = f \circ i \\ &\Rightarrow f \circ e_{K,X} = e_{K,Y} \end{aligned}$$

Since f is a monomorphism, $i = e_{K,X}$. Thus, the kernel of f, $N(f)$, is zero. $(\Rightarrow)$ Suppose that $coN(f)$ is zero and let us show that f is an epimorphism.
Let $g, h : Y \to Z$ be two morphisms such that $g \circ f = h \circ f$. We have:

$$g \circ f = h \circ f \Rightarrow (g - h) \circ f = e_{X,Z}.$$

Since $coN(f) = (Y, p)$ implies $p \circ f = e_{X,T}$, and since $coN(f)$ is zero, we have $p = e_{Y,T}$, and thus:

$$k \circ e_{Y,T} = g - h = e_{Y,Z}$$

Thus, we have:

$$\begin{aligned} k \circ e_{Y,T} = g - h = e_{Y,Z} &\Rightarrow g - h = e_{Z,X} \\ &\Rightarrow g - h + h = e_{Y,Z} + h \\ &\Rightarrow g + e_{Y,Z} = h \\ &\Rightarrow g = h. \end{aligned}$$

Thus, f is an epimorphism.
$(\Leftarrow)$ Suppose that f is an epimorphism and let us show that $coN(f)$ is zero.
Let $p : Y \to T$ be a morphism such that $p \circ f = e_{X,T}$.
Now, $e_{X,T} = e_{X,T} \circ f$. Indeed:
$\forall p \in \mathrm{Hom}_{\mathscr{A}}(Y, T)$, where $e_{Y,T}$ is the neutral element of the abelian group $\mathrm{Hom}_{\mathscr{A}}(Y, T)$, we have:

$$\begin{aligned} (p + e_{Y,T}) \circ f = (e_{Y,T} + p) \circ f = p \circ f &\Rightarrow p \circ f + e_{Y,T} \circ f = e_{Y,T} \circ f + p \circ f = p \circ f \\ &\Rightarrow \begin{cases} p \circ f + e_{Y,T} \circ f = p \circ f + e_{X,T} = p \circ f \\ e_{Y,T} \circ f + p \circ f = e_{X,T} + p \circ f = p \circ f \end{cases} \\ &\Rightarrow e_{Y,T} \circ f = e_{X,T} \end{aligned}$$

We have:

$$p \circ f = e_{Y,T} \circ f = e_{X,T}.$$

Since f is an epimorphism, $p = e_{Y,T}$. Therefore, the cokernel of f, $coN(f)$, is zero.

Let $\mathscr{A}$ be a balanced abelian category and X an object in $\mathscr{A}$. Then the functor denoted by $Hom_{\mathscr{A}}(X, -) : \mathscr{A} \longrightarrow \mathrm{Ab}$ defined by:

(i) $\forall Y \in \mathrm{Ob}(\mathscr{A})$, $Hom_{\mathscr{A}}(X,-)(Y) = Hom_{\mathscr{A}}(X,Y) \in \mathrm{Ob}(\mathrm{Ab})$;

(ii) $\forall f \in \mathrm{Hom}_{\mathscr{A}}(Y,Z)$,

$$Hom_{\mathscr{A}}(X,-)(f) = Hom_{\mathscr{A}}(X,f) = f^* : \begin{array}{ccc} Hom_{\mathscr{A}}(X,Y) & \longrightarrow & Hom_{\mathscr{A}}(X,Z) \\ \phi & \longmapsto & f \circ \phi \end{array}$$

is covariant, additive, left exact functor, and it is exact if and only if X is a projective object in $\mathscr{A}$.

Proof.

- It's evident that $\mathrm{Hom}_{\mathscr{A}}(X,-) : \mathscr{A} \longrightarrow \mathrm{Ab}$ is an additive covariant functor.
- Let us show that $\mathrm{Hom}_{\mathscr{A}}(X,-) : \mathscr{A} \longrightarrow \mathrm{Ab}$ is a left-exact functor.
Consider the following short left-exact sequence of morphisms in $\mathscr{A}$:

$$0 \longrightarrow Y \xrightarrow{f} Z \xrightarrow{g} T$$

We will show that

$$\{e_{X,0}\} \longrightarrow \mathrm{Hom}_{\mathscr{A}}(X,Y) \xrightarrow{\mathrm{Hom}_{\mathscr{A}}(X,f)=f^*} \mathrm{Hom}_{\mathscr{A}}(X,Z) \xrightarrow{\mathrm{Hom}_{\mathscr{A}}(X,g)=g^*} \mathrm{Hom}_{\mathscr{A}}(X,T)$$

is a left-exact sequence. It suffices to show that $\mathrm{Ker}(f^*) = \{e_{X,Y}\}$, where $e_{X,Y}$ is the neutral element (the zero morphism) of $\mathrm{Hom}_{\mathscr{A}}(X,Y)$, and that $\mathrm{Im}(f^*) = \mathrm{Ker}(g^*)$.
• Show that $\mathrm{Ker}(f^*) = \{e_{X,Y}\}$. Since f^* is a morphism of abelian groups, the kernel of f^* is:

$$\mathrm{Ker}(f^*) = \{\phi \in \mathrm{Hom}_{\mathscr{A}}(X,Y) : f^*(\phi) = f \circ \phi = e_{X,Z}\},$$

where $e_{X,Z}$ is the neutral element (the zero morphism) of the abelian group $\mathrm{Hom}_{\mathscr{A}}(X,Z)$.
Let $\phi \in \mathrm{Ker}(f^*)$.
We have:

$$\phi \in \mathrm{Ker}(f^*) \Rightarrow f^*(\phi) = f \circ \phi = e_{X,Z}.$$

Now, $f \circ e_{X,Y} = e_{X,Z}$. Indeed, for all $\phi \in \mathrm{Hom}_{\mathscr{A}}(X,Y)$, where $e_{X,Y}$ is the neutral element of the abelian group $\mathrm{Hom}_{\mathscr{A}}(X,Y)$, we have:

$$\begin{aligned} f \circ (\phi + e_{X,Y}) = f \circ (e_{X,Y} + \phi) = f \circ \phi &\Rightarrow f \circ \phi + f \circ e_{X,Y} = f \circ e_{X,Y} + f \circ \phi = f \circ \phi \\ &\Rightarrow \begin{cases} f \circ \phi + f \circ e_{X,Y} = f \circ \phi + e_{X,Z} = f \circ \phi \\ f \circ e_{X,Y} + f \circ \phi = e_{X,Z} + f \circ \phi = f \circ \phi \end{cases} \\ &\Rightarrow f \circ e_{X,Y} = e_{X,Z}. \end{aligned}$$

Thus, we have:

$$f \circ \phi = f \circ e_{X,Y} = e_{X,Z}.$$

Or by hypothesis, the kernel of f is zero $N(f) = (0, e_{X,0})$. Bylemma 2, f is a monomorphism. Hence,

$$f \circ \phi = f \circ e_{X,Y} \Rightarrow \phi = e_{X,Y}.$$

Thus, $\mathrm{Ker}(f^*) = \{e_{X,Y}\}$.

• Show that $\mathrm{Im}(f^*) = \mathrm{Ker}(g^*)$.
- First, show that $\mathrm{Im}(f^*) \subset \mathrm{Ker}(g^*)$. It suffices to show that $g^* \circ f^* = e_{\mathrm{Hom}_{\mathscr{A}}(X,Y),\mathrm{Hom}_{\mathscr{A}}(X,T)}$, where $e_{\mathrm{Hom}_{\mathscr{A}}(X,Y),\mathrm{Hom}_{\mathscr{A}}(X,T)}$ is the zero morphism of the abelian group $\mathrm{Hom}_{\mathrm{Ab}}(\mathrm{Hom}_{\mathscr{A}}(X,Y), \mathrm{Hom}_{\mathscr{A}}(X,T))$. We have:

$$g^* \circ f^* : \mathrm{Hom}_{\mathscr{A}}(X,Y) \to \mathrm{Hom}_{\mathscr{A}}(X,T).$$

Let $\phi \in \mathrm{Hom}_{\mathscr{A}}(X,Y)$. We have:

$$\begin{aligned} g^* \circ f^*(\phi) &= g^*(f^*(\phi)) = g^*(f \circ \phi) = g \circ (f \circ \phi) = (g \circ f) \circ \phi \\ &= e_{Y,T} \circ \phi \quad \text{(since by hypothesis, } g \circ f = e_{Y,T}) \\ &= e_{X,T} \quad \text{because} \end{aligned}$$

for all $\psi \in \mathrm{Hom}_{\mathscr{A}}(Y,T)$, where $e_{Y,T}$ is the neutral element of the abelian group $\mathrm{Hom}_{\mathscr{A}}(Y,T)$, we have:

$$\begin{aligned} (\psi + e_{Y,T}) \circ \phi = (e_{Y,T} + \psi) \circ \phi = \psi \circ \phi &\Rightarrow \psi \circ \phi + e_{Y,T} \circ \phi = e_{Y,T} \circ \phi + \psi \circ \phi = \psi \circ \phi \\ &\Rightarrow \begin{cases} \psi \circ \phi + e_{Y,T} \circ \phi = \psi \circ \phi + e_{X,T} = \psi \circ \phi \\ e_{Y,T} \circ \phi + \psi \circ \phi = e_{X,T} + \psi \circ \phi = \psi \circ \phi \end{cases} \\ &\Rightarrow e_{Y,T} \circ \phi = e_{X,T}. \end{aligned}$$

Thus, $g^* \circ f^* = e_{\mathrm{Hom}_{\mathscr{A}}(X,Y),\mathrm{Hom}_{\mathscr{A}}(X,T)}$, and therefore $\mathrm{Im}(f^*) \subset \mathrm{Ker}(g^*)$.
- Now, show that $\mathrm{Ker}(g^*) \subset \mathrm{Im}(f^*)$.
We have:

$$\mathrm{Ker}(g^*) = \{\phi \in \mathrm{Hom}_{\mathscr{A}}(X,Z) : g^*(\phi) = g \circ \phi = e_{X,T}\}.$$

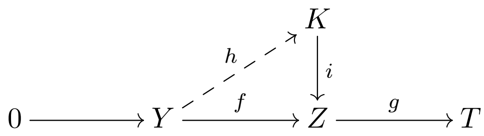
Let $\psi \in \mathrm{Ker}(g^*) = \mathrm{Ker}(\mathrm{Hom}_{\mathscr{A}}(X,g))$. Show that $\psi \in \mathrm{Im}(f^*)$. Consider the left-exact short sequence of morphisms in $\mathscr{A}$:

$$0 \to Y \xrightarrow{f} Z \xrightarrow{g} T,$$

which implies:

- $\mathrm{N}(f) = (0, e_{0,Y})$,
- $g \circ f = e_{Y,T}$,

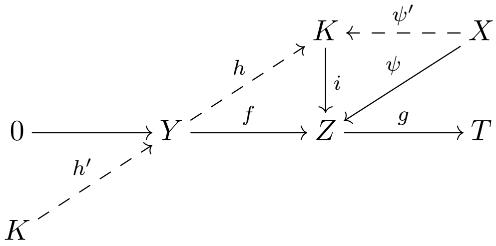
– $\text{coN}(h) = (0, e_{K,0})$, where $h : Y \to K$ such that $i \circ h = f$, with $i : K \to Z$ being the kernel of g. That is, the following diagram commutes:



We have:

$$\begin{aligned}\psi \in \text{Ker}(g^*) &\Rightarrow g^*(\psi) = e_{X,T}\\ &\Rightarrow g \circ \psi = e_{X,T}.\end{aligned}$$

Since $\text{coN}(h) = (0, e_{K,0})$, by Lemma 2, h is an epimorphism. Since $\mathscr{A}$ is a balanced category, every epimorphism is split. That is, there exists $h' : K \to Y$ such that $h \circ h' = 1_K$. Thus, the following diagram commutes:



We have:

$$\begin{aligned} i \circ h = f &\Rightarrow i \circ h \circ h' = f \circ h'\\ &\Rightarrow i \circ 1_K = f \circ h'\\ &\Rightarrow i = f \circ h'\\ &\Rightarrow i \circ \psi' = f \circ h' \circ \psi'\\ &\Rightarrow \psi = f \circ (h' \circ \psi')\\ &\Rightarrow \psi = f^*(h' \circ \psi').\end{aligned}$$

Thus, $\psi \in \text{Im}(f^*)$. Hence, $\text{Ker}(g^*) \subset \text{Im}(f^*)$.

Therefore, $\text{Hom}_{\mathscr{A}}(X, -) : \mathscr{A} \longrightarrow \text{Ab}$ is a covariant, additive, and left-exact functor.

- Show that $\text{Hom}_{\mathscr{A}}(X, -) : \mathscr{A} \longrightarrow \text{Ab}$ is exact if and only if X is a projective object in $\mathscr{A}$.

• Suppose X is a projective object in $\mathscr{A}$ and show that $\text{Hom}_{\mathscr{A}}(X, -) : \mathscr{A} \longrightarrow \text{Ab}$ is an exact functor.
Consider the following short exact sequence of morphisms in $\mathscr{A}$:

$$0 \longrightarrow Y \xrightarrow{f} Z \xrightarrow{g} T \longrightarrow 0$$

We will show that

$$\{e_{X,0}\} \longrightarrow \mathrm{Hom}_{\mathscr{A}}(X,Y) \xrightarrow{\mathrm{Hom}_{\mathscr{A}}(X,f)=f^*} \mathrm{Hom}_{\mathscr{A}}(X,Z) \xrightarrow{\mathrm{Hom}_{\mathscr{A}}(X,g)=g^*} \mathrm{Hom}_{\mathscr{A}}(X,T) \longrightarrow \{e_{X,0}\}$$

is exact. By part (i), we know that

$$\{e_{X,0}\} \longrightarrow \mathrm{Hom}_{\mathscr{A}}(X,Y) \xrightarrow{\mathrm{Hom}_{\mathscr{A}}(X,f)=f^*} \mathrm{Hom}_{\mathscr{A}}(X,Z) \xrightarrow{\mathrm{Hom}_{\mathscr{A}}(X,g)=g^*} \mathrm{Hom}_{\mathscr{A}}(X,T)$$

is left-exact. It remains to show that $\mathrm{Hom}_{\mathscr{A}}(X,g)$ is an epimorphism.
Since X is projective, for every epimorphism $g : Z \twoheadrightarrow T$ in $\mathscr{A}$ and every morphism $f : X \to T$ in $\mathscr{A}$, there exists a morphism $\phi : X \to Z$ in $\mathscr{A}$ such that $g \circ \phi = f$. This means the following diagram commutes:

$$\begin{array}{ccccc} & & X & & \\ & \overset{\phi}{\swarrow} & \downarrow f & & \\ Z & \underset{g}{\twoheadrightarrow} & T & \longrightarrow & 0 \end{array}$$

That is, for every $f \in \mathrm{Hom}_{\mathscr{A}}(X,T)$, there exists $\phi \in \mathrm{Hom}_{\mathscr{A}}(X,Z)$ such that $g \circ \phi = f = g^*(\phi)$. Hence, $\mathrm{Hom}_{\mathscr{A}}(X,g) = g^*$ is an epimorphism.
• Conversely, suppose $\mathrm{Hom}_{\mathscr{A}}(X,-) : \mathscr{A} \longrightarrow \mathrm{Ab}$ is an exact functor and show that X is a projective object in $\mathscr{A}$. We have:
The exactness of $\mathrm{Hom}_{\mathscr{A}}(X,-)$ implies that for every short exact sequence of morphisms in $\mathscr{A}$:

$$0 \longrightarrow Y \xrightarrow{f} Z \xrightarrow{g} T \longrightarrow 0\ ,$$

the sequence

$$\{e_{X,0}\} \longrightarrow \mathrm{Hom}_{\mathscr{A}}(X,Y) \xrightarrow{\mathrm{Hom}_{\mathscr{A}}(X,f)=f^*} \mathrm{Hom}_{\mathscr{A}}(X,Z) \xrightarrow{\mathrm{Hom}_{\mathscr{A}}(X,g)=g^*} \mathrm{Hom}_{\mathscr{A}}(X,T) \longrightarrow \{e_{X,0}\}$$

is exact. This means g^* is an epimorphism. Therefore, for every epimorphism $g : Z \twoheadrightarrow T$ in $\mathscr{A}$ and every morphism $f : X \to T$ in $\mathscr{A}$, there exists a morphism $\phi : X \to Z$ in $\mathscr{A}$ such that $g \circ \phi = g^*(\phi) = f$. This means the following diagram commutes:

$$\begin{array}{ccccc} & & X & & \\ & \overset{\phi}{\swarrow} & \downarrow f & & \\ Z & \underset{g}{\twoheadrightarrow} & T & \longrightarrow & 0 \end{array}$$

Hence, X is a projective object in $\mathscr{A}$.

Let $\mathscr{A}$ be a balanced abelian category and X an object in $\mathscr{A}$. Then the functor denoted by $Hom_{\mathscr{A}}(X,-) : \mathscr{A} \longrightarrow \mathrm{Ab}$ defined by:

(i) $\forall Y \in \text{Ob}(\mathscr{A})$, $Hom_{\mathscr{A}}(-,X)(Y) = Hom_{\mathscr{A}}(Y,X) \in \text{Ob}(\text{Ab})$

(ii) $\forall f \in \text{Hom}_{\mathscr{A}}(Y,Z)$,

$$Hom_{\mathscr{A}}(-,X)(f) = Hom_{\mathscr{A}}(f,X) = f^* : \begin{array}{ccc} Hom_{\mathscr{A}}(X,Y) & \longrightarrow & Hom_{\mathscr{A}}(X,Z) \\ \phi & \longmapsto & \phi \circ f \end{array}$$

is a contravariant, additive, left exact functor, and it is exact if and only if X is a injective object in $\mathscr{A}$.

Proof.

- It is evident that $\text{Hom}_{\mathscr{A}}(-,X) : \mathscr{A} \longrightarrow \text{Ab}$ is a contravariant additive functor.
- Let us show that $\text{Hom}_{\mathscr{A}}(-,X) : \mathscr{A} \longrightarrow \text{Ab}$ is a left-exact functor.
  Consider the right short exact sequence of morphisms in $\mathscr{A}$:

$$Y \xrightarrow{f} Z \xrightarrow{g} T \longrightarrow 0$$

We must show that the sequence:

$$\{e_{0,X}\} \longrightarrow \text{Hom}_{\mathscr{A}}(T,X) \xrightarrow{\text{Hom}_{\mathscr{A}}(g,X)=g^*} \text{Hom}_{\mathscr{A}}(Z,X) \xrightarrow{\text{Hom}_{\mathscr{A}}(f,X)=f^*} \text{Hom}_{\mathscr{A}}(Y,X)$$

is left-exact. It suffices to show that the kernel of g^*, $\text{Ker}\, g^*$, is zero and that $\text{Im}\, g^* = \text{Ker}\, f^*$.

• Let us show that the kernel of g^*, $\text{Ker}(\text{Hom}_{\mathscr{A}}(g,X)) = \text{Ker}\, g^*$, is zero. By Lemma 2, it suffices to show that $\text{Hom}_{\mathscr{A}}(g,X) = g^*$ is a monomorphism. By definition of $\text{Hom}_{\mathscr{A}}(g,X) = g^*$, we have:

$$\text{Hom}_{\mathscr{A}}(g,X) = g^* : \begin{array}{ccc} \text{Hom}_{\mathscr{A}}(T,X) & \longrightarrow & \text{Hom}_{\mathscr{A}}(Z,X) \\ \phi & \longmapsto & \phi \circ g \end{array}$$

Let $\phi_1, \phi_2 \in \text{Hom}_{\mathscr{A}}(T,X)$ such that $g^*(\phi_1) = g^*(\phi_2)$, i.e., $\phi_1 \circ g = \phi_2 \circ g$. We have:

$$\phi_1 \circ g = \phi_2 \circ g \implies \phi_1 = \phi_2,$$

since by hypothesis, the cokernel of g, $N(g)$, is zero, and by Lemma 2, g is an epimorphism. Thus, $\text{Hom}_{\mathscr{A}}(g,X) = g^*$ is a monomorphism.

• Let us show that $\text{Im}\, g^* = \text{Ker}\, f^*$.

- First, we show that $\text{Im}\, g^* \subset \text{Ker}\, f^*$. It suffices to show that $f^* \circ g^* = e_{\text{Hom}_{\mathscr{A}}(T,X),\text{Hom}_{\mathscr{A}}(Y,X)}$, where $e_{\text{Hom}_{\mathscr{A}}(T,X),\text{Hom}_{\mathscr{A}}(Y,X)}$ is the neutral element (zero morphism) of the abelian group
$\text{Hom}_{Ab}(\text{Hom}_{\mathscr{A}}(T,X), \text{Hom}_{\mathscr{A}}(Y,X))$. We have:

$$f^* \circ g^* : \text{Hom}_{\mathscr{A}}(T,X) \to \text{Hom}_{\mathscr{A}}(Y,X).$$

Let $\phi \in \mathrm{Hom}_{\mathscr{A}}(T, X)$. Then:

$$\begin{aligned} f^* \circ g^*(\phi) = f^*(g^*(\phi)) = f^*(\phi \circ g) = (\phi \circ g) \circ f \\ = \phi \circ (g \circ f) \quad \text{(since by hypothesis, } g \circ f = e_{Y,T}) \\ = \phi \circ e_{Y,T} = e_{Y,X}, \quad \text{because} \end{aligned}$$

for all $u \in \mathrm{Hom}_{\mathscr{A}}(Y, T)$, where $e_{Y,T}$ is the neutral element of the abelian group $\mathrm{Hom}_{\mathscr{A}}(Y, T)$, we have:

$$\begin{aligned} \phi \circ (u + e_{Y,T}) = \phi \circ (e_{Y,T} + u) = \phi \circ u &\implies \phi \circ u + \phi \circ e_{Y,T} = \phi \circ e_{Y,T} + \phi \circ u = \phi \circ u \\ &\implies \begin{cases} \phi \circ u + \phi \circ e_{Y,T} = \phi \circ u + e_{Y,X} = \phi \circ u, \\ \phi \circ e_{Y,T} + \phi \circ u = e_{Y,X} + \phi \circ u = \phi \circ u, \end{cases} \\ &\implies \phi \circ e_{Y,T} = e_{Y,X}. \end{aligned}$$

Thus, $f^* \circ g^* = e_{\mathrm{Hom}_{\mathscr{A}}(T,X),\mathrm{Hom}_{\mathscr{A}}(Y,X)}$, and therefore $\mathrm{Im}\, g^* \subset \mathrm{Ker}\, f^*$.

- Now, we show that $\mathrm{Ker}(f^*) \subset \mathrm{Im}\, g^*$. We have:

$$\mathrm{Ker}(f^*) = \{\phi \in \mathrm{Hom}_{\mathscr{A}}(Z, X) : f^*(\phi) = \phi \circ f = e_{Y,X}\}.$$

Let $\phi \in \mathrm{Ker}(f^*) = \mathrm{Ker}\, \mathrm{Hom}_{\mathscr{A}}(f, X)$. We show that $\phi \in \mathrm{Im}\, g^*$. Since $\phi \in \mathrm{Ker}(f^*)$, we have:

$$\begin{aligned} \phi \in \mathrm{Ker}(f^*) &\implies f^*(\phi) = e_{Y,X} \\ &\implies \phi \circ f = e_{Y,X}. \end{aligned}$$

Since the sequence:

$$Y \xrightarrow{f} Z \xrightarrow{g} T \longrightarrow 0$$

is right short exact, we have $g \circ f = e_{Y,T}$. Let $coN(f) = (j, P)$, and by definition of the cokernel of f, we have $j \circ f = e_{Y,P}$ and there exists a unique morphism $h_1 : P \to T$ such that $h_1 \circ j = g$. Thus, h_1 is an epimorphism. Indeed, let $u, v : R \to P$ such that $u \circ h_1 = v \circ h_1$. Then:

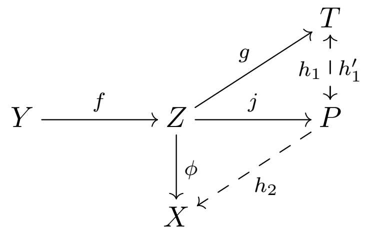
$$\begin{aligned} u \circ h_1 = v \circ h_1 &\implies (u \circ h_1) \circ j = (v \circ h_1) \circ j \\ &\implies u \circ (h_1 \circ j) = v \circ (h_1 \circ j) \\ &\implies u \circ g = v \circ g \\ &\implies u = v \quad \text{(since } g \text{ is an epimorphism).} \end{aligned}$$

Thus, h_1 is an epimorphism. Since $\mathscr{A}$ is a balanced category, h_1 is a split epimorphism, i.e., there exists a unique $h_1' : T \to P$ such that:

$$h_1 \circ h_1' = 1_T.$$

Moreover, $h_1' \circ g = j$.

Since $\phi \circ f = e_{Y,X}$, by the definition of $coN(f)$, we have $j \circ f = e_{Y,P}$, and there exists a unique $h_2 : P \to X$ such that $h_2 \circ j = \phi$. That is, the following diagram commutes:



Thus, we have:

$$\begin{aligned}\phi &= h_2 \circ j \\ &= h_2 \circ (h_1' \circ g) \quad (\text{since } h_1' \circ g = j) \\ &= (h_2 \circ h_1') \circ g \\ \phi &= g^*(h_2 \circ h_1').\end{aligned}$$

Hence, $\phi \in \mathrm{Im}(g^*)$. Therefore, $\mathrm{Ker}(f^*) \subset \mathrm{Im}\, g^*$. Thus, $\mathrm{Hom}_{\mathscr{A}}(-, X) : \mathscr{A} \longrightarrow \mathrm{Ab}$ is a contravariant, additive, and left-exact functor.

- We now show that $\mathrm{Hom}_{\mathscr{A}}(-, X) : \mathscr{A} \longrightarrow \mathrm{Ab}$ is an exact functor if and only if X is an injective object in $\mathscr{A}$.
  • Suppose X is an injective object in $\mathscr{A}$ and show that $\mathrm{Hom}_{\mathscr{A}}(-, X) : \mathscr{A} \longrightarrow \mathrm{Ab}$ is exact. Consider the short exact sequence of morphisms in $\mathscr{A}$:

$$0 \longrightarrow Y \xrightarrow{f} Z \xrightarrow{g} T \longrightarrow 0$$

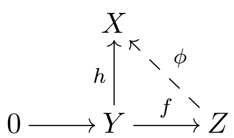
We must show that the sequence:

$$\{e_{0,X}\} \longrightarrow \mathrm{Hom}_{\mathscr{A}}(T, X) \xrightarrow{\mathrm{Hom}_{\mathscr{A}}(g,X)=g^*} \mathrm{Hom}_{\mathscr{A}}(Z, X) \xrightarrow{\mathrm{Hom}_{\mathscr{A}}(f,X)=f^*} \mathrm{Hom}_{\mathscr{A}}(Y, X) \longrightarrow \{e_{0,X}\}$$

is exact. By part 1, the sequence:

$$\{e_{0,X}\} \longrightarrow \mathrm{Hom}_{\mathscr{A}}(T, X) \xrightarrow{\mathrm{Hom}_{\mathscr{A}}(g,X)=g^*} \mathrm{Hom}_{\mathscr{A}}(Z, X) \xrightarrow{\mathrm{Hom}_{\mathscr{A}}(f,X)=f^*} \mathrm{Hom}_{\mathscr{A}}(Y, X)$$

is left-exact. It remains to show that $\mathrm{Hom}_{\mathscr{A}}(f, X) = f^*$ is an epimorphism. Since X is injective, for every monomorphism $f : Y \hookrightarrow Z$ in $\mathscr{A}$ and every morphism $h : Y \to X$ in $\mathscr{A}$, there exists a morphism $\phi : Z \to X$ in $\mathscr{A}$ such that $\phi \circ f = h$. This means the following diagram commutes:



In other words, for every $h \in \mathrm{Hom}_{\mathscr{A}}(Y, X)$, there exists $\phi \in \mathrm{Hom}_{\mathscr{A}}(Z, X)$ such that $\phi \circ f = f^*(\phi) = h$. Hence, $\mathrm{Hom}_{\mathscr{A}}(f, X)$ is an epimorphism.

• Conversely, suppose $\mathrm{Hom}_{\mathscr{A}}(-, X) : \mathscr{A} \longrightarrow \mathrm{Ab}$ is exact and show that X is an injective object in $\mathscr{A}$. Since $\mathrm{Hom}_{\mathscr{A}}(X, -)$ is exact, for every short exact sequence in $\mathscr{A}$:

$$0 \longrightarrow Y \xrightarrow{f} Z \xrightarrow{g} T \longrightarrow 0$$

the sequence:

$$\{e_{0,X}\} \longrightarrow \mathrm{Hom}_{\mathscr{A}}(T, X) \xrightarrow{\mathrm{Hom}_{\mathscr{A}}(g,X)=g^*} \mathrm{Hom}_{\mathscr{A}}(Z, X) \xrightarrow{\mathrm{Hom}_{\mathscr{A}}(f,X)=f^*} \mathrm{Hom}_{\mathscr{A}}(Y, X) \longrightarrow \{e_{0,X}\}$$

is exact. This means f^* is an epimorphism. Thus, for every monomorphism $f : Y \hookrightarrow Z$ in $\mathscr{A}$ and every morphism $h : Y \to X$ in $\mathscr{A}$, there exists a morphism $\phi : Z \to X$ in $\mathscr{A}$ such that $\phi \circ f = f^*(\phi) = h$. This means the following diagram commutes:

$$\begin{array}{ccccc} & & X & & \\ & & \uparrow h & \nwarrow \phi & \\ 0 & \longrightarrow & Y & \xrightarrow{f} & Z \end{array}$$

Hence, X is an injective object in $\mathscr{A}$. Thus, $\mathrm{Hom}_{\mathscr{A}}(-, X) : \mathscr{A} \longrightarrow \mathrm{Ab}$ is an exact functor if and only if X is an injective object in $\mathscr{A}$.

3. Exactness of Functors $\mathbf{Hom}_{\mathbf{Comp}(\mathscr{A})}(X, -)$ and $\mathbf{Hom}_{\mathbf{Comp}(\mathscr{A})}(-, X)$

Let $\mathscr{A}$ be a balanced abelian category and X an object in $\mathscr{A}$. Then:

[label=.]The functor $\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X, -) : \mathrm{Comp}(\mathscr{A}) \to \mathrm{Comp}(Ab)$ is a covariant, additive, and left-exact functor. The functor $\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X, -) : \mathrm{Comp}(\mathscr{A}) \to \mathrm{Comp}(Ab)$ is exact if and only if X is a projective object in $\mathscr{A}$.

Proof.

[label=.]Let us show that the functor $\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X, -) : \mathrm{Comp}(\mathscr{A}) \to \mathrm{Comp}(Ab)$ is covariant, additive, and left-exact. It is evident that $\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X, -)$ is covariant and additive. Let us show that the functor $\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X, -) : \mathrm{Comp}(\mathscr{A}) \to \mathrm{Comp}(Ab)$ is left-exact.

Consider the left short exact sequence of morphisms in $\mathrm{Comp}(\mathscr{A})$:

$$(0) \longrightarrow (Y, \alpha) \xrightarrow{f} (Z, \beta) \xrightarrow{g} (T, \theta)$$

where $\mathscr{A}$ is a balanced abelian category. We must show that

$$\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X, -)((0)) \longrightarrow \mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X, -)((Y, \alpha))^{f^*} \longrightarrow$$

$$\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X, -)((Z, \beta)) \xrightarrow{g^*} \mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X, -)((T, \theta))$$

is a left short exact sequence of morphisms in $\mathrm{Comp}(Ab)$. We have the following diagram:

$$\begin{array}{ccccccc}
\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X,-)((0)): \dots \longrightarrow & 0_{\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X,0)} & \longrightarrow & 0_{\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X,0)} & \longrightarrow & 0_{\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X,0)} & \longrightarrow \dots \\
\downarrow & \downarrow & & \downarrow & & \downarrow & \\
\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X,-)((Y,\alpha)): \dots \rightarrow & \mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X,Y_n) & \xrightarrow{\alpha_n^*} & \mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X,Y_{n+1}) & \xrightarrow{\alpha_{n+1}^*} & \mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X,Y_{n+2}) & \rightarrow \dots \\
f^*\downarrow & f_n^*\downarrow & & f_{n+1}^*\downarrow & & f_{n+2}^*\downarrow & \\
\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X,-)((Z,\beta)): \dots \rightarrow & \mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X,Z_n) & \xrightarrow{\beta_n^*} & \mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X,Z_{n+1}) & \xrightarrow{\beta_{n+1}^*} & \mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X,Z_{n+2}) & \rightarrow \dots \\
g^*\downarrow & g_n^*\downarrow & & g_{n+1}^*\downarrow & & g_{n+2}^*\downarrow & \\
\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X,-)((T,\theta)): \dots \rightarrow & \mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X,T_n) & \xrightarrow{\theta_n^*} & \mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X,T_{n+1}) & \xrightarrow{\theta_{n+1}^*} & \mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X,T_{n+2}) & \rightarrow \dots
\end{array}$$

By Theorem 2, for every integer $n \in \mathbb{Z}$, the sequence

$$0_{\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X,0)} \longrightarrow \mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X,Y_n) \xrightarrow{\mathrm{Hom}_{\mathscr{A}}(X,f)=f_n^*} \mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X,Z_n) \xrightarrow{\mathrm{Hom}_{\mathscr{A}}(X,g)=g_n^*} \mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X,T_n)$$

is a left short exact sequence of morphisms in $\mathrm{Comp}(Ab)$.
Hence,

$$\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X,-)((0)) \longrightarrow \mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X,-)((Y,\alpha)) \xrightarrow{f}$$

$$\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X,-)((Z,\beta)) \xrightarrow{g} \mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X,-)((T,\theta))$$

is a left short exact sequence of morphisms in $\mathrm{Comp}(Ab)$.
Items i.a and i.b imply that the functor $\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X,-): \mathrm{Comp}(\mathscr{A}) \to \mathrm{Comp}(Ab)$ is covariant, additive, and left-exact. Let us show that the functor $\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X,-): \mathrm{Comp}(\mathscr{A}) \to \mathrm{Comp}(Ab)$ is exact if and only if X is a projective object in $\mathscr{A}$. Suppose that the functor $\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X,-): \mathrm{Comp}(\mathscr{A}) \to \mathrm{Comp}(Ab)$ is exact and show that X is a projective object in $\mathscr{A}$. We have:
$\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X,-): \mathrm{Comp}(\mathscr{A}) \to \mathrm{Comp}(Ab)$ being exact implies that for every short exact sequence of morphisms in $\mathrm{Comp}(\mathscr{A})$, $(0) \longrightarrow (Y,\alpha) \xrightarrow{f} (Z,\beta) \xrightarrow{g} (T,\theta) \longrightarrow (0)$ the sequence

$$\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X,-)((0)) \longrightarrow \mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X,-)((Y,\alpha)) \xrightarrow{f^*}$$

$$\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X,-)((Z,\beta)) \xrightarrow{g^*} \mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X,-)((T,\theta)) \longrightarrow \mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X,-)((0))$$

is a short exact sequence of morphisms in $\mathrm{Comp}(Ab)$. This means the following diagram commutes:

$$\begin{array}{ccccccccc}
\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(-,X)((0)) : ... & \longrightarrow & 0_{\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X,0)} & \longrightarrow & 0_{\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X,0)} & \longrightarrow & 0_{\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X,0)} & \longrightarrow & ... \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X,-)((Y,\alpha)) : ... & \longrightarrow & \mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X,Y_n) & \xrightarrow{\alpha_n^*} & \mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X,Y_{n+1}) & \xrightarrow{\alpha_{n+1}^*} & \mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X,Y_{n+2}) & \longrightarrow & ... \\
f^*\downarrow & & f_n^*\downarrow & & f_{n+1}^*\downarrow & & f_{n+2}^*\downarrow & & \\
\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X,-)((Z,\beta)) : ... & \longrightarrow & \mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X,Z_n) & \xrightarrow{\beta_n^*} & \mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X,Z_{n+1}) & \xrightarrow{\beta_{n+1}^*} & \mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X,Z_{n+2}) & \longrightarrow & ... \\
g^*\downarrow & & g_n^*\downarrow & & g_{n+1}^*\downarrow & & g_{n+2}^*\downarrow & & \\
\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X,-)((T,\theta)) : ... & \longrightarrow & \mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X,T_n) & \xrightarrow{\theta_n^*} & \mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X,T_{n+1}) & \xrightarrow{\theta_{n+1}^*} & \mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X,T_{n+2}) & \longrightarrow & ... \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(-,X)((0)) : ... & \longrightarrow & 0_{\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X,0)} & \longrightarrow & 0_{\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X,0)} & \longrightarrow & 0_{\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X,0)} & \longrightarrow & ...
\end{array}$$

We have for every integer $n \in \mathbb{Z}$, g_n^* is an epimorphism. Therefore, for every epimorphism $g_n : Z_n \twoheadrightarrow T_n$ in $\mathscr{A}$ and every morphism $f_n : X \to T_n$ in $\mathscr{A}$, there exists a morphism $\phi_n : X \to Z_n$ in $\mathscr{A}$ such that $g_n \circ \phi_n = g^*(\phi_n) = f_n$. This means the following diagram commutes:

$$\begin{array}{ccccc}
 & \swarrow^{\phi_n} & X & & \\
 & & \downarrow f_n & & \\
Z_n & \xrightarrow[g_n]{} & T_n & \longrightarrow & 0
\end{array}$$

Hence, X is a projective object in $\mathscr{A}$. Suppose that X is a projective object of $\mathscr{A}$ and let us show that the functor $\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X,-) : \mathrm{Comp}(\mathscr{A}) \to \mathrm{Comp}(Ab)$ is an exact functor. Let the following short exact sequence of morphisms in $\mathscr{A}$:

$$(0) \longrightarrow (Y,\alpha) \xrightarrow{f} (Z,\beta) \xrightarrow{g} (T,\theta) \longrightarrow (0)$$

We show that

$$\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X,-)((0)) \longrightarrow \mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X,-)((Y,\alpha)) \xrightarrow{f^*}$$

$$\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X,-)((Z,\beta)) \xrightarrow{g^*} \mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X,-)((T,\theta)) \longrightarrow \mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X,-)((0))$$

is a short exact sequence of morphisms in $\mathrm{Comp}(Ab)$. By i., we have:

$$\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X,-)((0)) \longrightarrow \mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X,-)((Y,\alpha)) \xrightarrow{f^*}$$

$$\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X,-)((Z,\beta)) \xrightarrow{g^*} \mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X,-)((T,\theta))$$

is a left short exact sequence of morphisms in $\mathrm{Comp}(Ab)$. Thus, it remains to show that for every n, $\mathrm{Hom}_{\mathscr{A}}(X, g_n)$ is an epimorphism. Since X is projective, by Theorem 2, for every epimorphism $g_n : Z_n \twoheadrightarrow T_n$ in $\mathscr{A}$ and every morphism $f_n : X \to T_n$ in

$\mathscr{A}$, there exists a morphism $\phi_n : X \to Z_n$ in $\mathscr{A}$ such that $g_n \circ \phi_n = f_n$. This means that the following diagram commutes:

$$\begin{array}{ccccc} & & X & & \\ & \swarrow^{\phi_n} & \downarrow{\scriptstyle f_n} & & \\ Z_n & \xrightarrow[g_n]{} & T_n & \longrightarrow & 0 \end{array}$$

That is, for every $f_n \in \mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X, T_n)$, there exists $\phi_n \in \mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X, Z_n)$ such that $g_n \circ \phi_n = f_n = g_n^*(\phi_n)$. Hence, $\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X, g_n) = g_n^*$ is an epimorphism for every integer n in $\mathbb{Z}$. Therefore, the functor $\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X, -) : \mathrm{Comp}(\mathscr{A}) \to \mathrm{Comp}(Ab)$ is exact. Thus, (ii.1) and (ii.2) imply that the functor $\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X, -) : \mathrm{Comp}(\mathscr{A}) \to \mathrm{Comp}(Ab)$ is exact if and only if X is a projective object in $\mathscr{A}$.

Let $\mathscr{A}$ be a balanced abelian category and X an object in $\mathscr{A}$. Then:

[label=.]the functor $\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(-, X) : \mathrm{Comp}(\mathscr{A}) \to \mathrm{Comp}(Ab)$ is a contravariant, additive, and left-exact functor; the functor $\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(-, X) : \mathrm{Comp}(\mathscr{A}) \to \mathrm{Comp}(Ab)$ is exact if and only if X is a injective object in $\mathscr{A}$.

Proof.

i. Let us show that the functor $\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(-, X) : \mathrm{Comp}(\mathscr{A}) \to \mathrm{Comp}(Ab)$ is contravariant, additive, and left exact.

i.1 It is evident that $\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(-, X)$ is contravariant and additive.

i.2 Let us show that the functor $\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(-, X) : \mathrm{Comp}(\mathscr{A}) \to \mathrm{Comp}(Ab)$ is left exact.
Let

$$(Y, \alpha) \xrightarrow{f} (Z, \beta) \xrightarrow{g} (T, \theta) \longrightarrow (0)$$

be a right short exact sequence of morphisms in $\mathrm{Comp}(\mathscr{A})$, where $\mathscr{A}$ is a balanced abelian category. We show that

$$\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(-, X)((0)) \longrightarrow \mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(-, X)((T, \theta)) \xrightarrow{g^*}$$

$$\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(-, X)((Z, \beta)) \xrightarrow{f^*} \mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(-, X)((Y, \alpha))$$

is a left short exact sequence of morphisms in $\mathrm{Comp}(Ab)$. Consider the following diagram:

$$\begin{array}{ccccccccc}
\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(-,X)((0)): ... & \longrightarrow & 0_{\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(0,X)} & \longrightarrow & 0_{\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(0,X)} & \longrightarrow & 0_{\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(0,X)} & \longrightarrow & ... \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(-,X)((T,\theta)): ... & \longrightarrow & \mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(T_n,X) & \xrightarrow{\theta_n^*} & \mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(T_{n+1},X) & \xrightarrow{\theta_{n+1}^*} & \mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(T_{n+2},X) & \longrightarrow & ... \\
g^*\downarrow & & g_n^*\downarrow & & g_{n+1}^*\downarrow & & g_{n+2}^*\downarrow & & \\
\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(-,X)((Z,\beta)): ... & \longrightarrow & \mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(Z_n,X) & \xrightarrow{\beta_n^*} & \mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(Z_{n+1},X) & \xrightarrow{\beta_{n+1}^*} & \mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(Z_{n+2},X) & \longrightarrow & ... \\
f^*\downarrow & & f_n^*\downarrow & & f_{n+1}^*\downarrow & & f_{n+2}^*\downarrow & & \\
\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(-,X)((Y,\alpha)): ... & \longrightarrow & \mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(Y_n,X) & \xrightarrow{\alpha_n^*} & \mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(Y_{n+1},X) & \xrightarrow{\alpha_{n+1}^*} & \mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(Y_{n+2},X) & \longrightarrow & ...
\end{array}$$

Now, by Theorem 2, for every integer n in $\mathbb{Z}$, the sequence

$$\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(0,X) \longrightarrow \mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(T_n,X) \xrightarrow{g_n^*} \mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(Z_n,X) \xrightarrow{f_n^*} \mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(Y_n,X)$$

is a left short exact sequence of morphisms in $\mathrm{Comp}(Ab)$. Hence

$$\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(-,X)((0)) \longrightarrow \mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(-,X)((T,\theta)) \xrightarrow{g^*}$$

$$\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(-,X)((Z,\beta)) \xrightarrow{f^*} \mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(-,X)((Y,\alpha))$$

is a left short exact sequence of morphisms in $\mathrm{Comp}(Ab)$. Therefor i.1 and i.2 imply that the functor $\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(-,X) : \mathrm{Comp}(\mathscr{A}) \to \mathrm{Comp}(Ab)$ is contravariant, additive, and left exact.

ii Let us show that the functor $\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(-,X) : \mathrm{Comp}(\mathscr{A}) \to \mathrm{Comp}(Ab)$ is exact if and only if X is an injective object in $\mathscr{A}$.

ii.1 Suppose that the functor $\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(-,X) : \mathrm{Comp}(\mathscr{A}) \to \mathrm{Comp}(Ab)$ is exact and show that X is an injective object in $\mathscr{A}$. We have:
$\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X,-) : \mathrm{Comp}(\mathscr{A}) \to \mathrm{Comp}(Ab)$ being exact implies that for every short exact sequence of morphisms in $\mathrm{Comp}(\mathscr{A})$, $(0) \longrightarrow (Y,\alpha) \xrightarrow{f} (Z,\beta) \xrightarrow{g} (T,\theta) \longrightarrow (0)$ then

$$\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(-,X)((0)) \longrightarrow \mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(-,X)((T,\theta)) \xrightarrow{g^*}$$

$$\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(-,X)((Z,\beta)) \xrightarrow{f^*} \mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(-,X)((Y,\alpha)) \longrightarrow \mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(-,X)((0))$$

is a short exact sequence of morphisms in $\mathrm{Comp}(Ab)$. This means that the following diagram is commutative:

$$\begin{array}{ccccccccc}
\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(-,X)((0)) : ... & \longrightarrow & 0_{\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(0,X)} & \longrightarrow & 0_{\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(0,X)} & \longrightarrow & 0_{\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(0,X)} & \longrightarrow \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(-,X)((T,\theta)) : ... & \longrightarrow & \mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(T_n,X) & \xrightarrow{\theta_n^*} & \mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(T_{n+1},X) & \xrightarrow{\theta_{n+1}^*} & \mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(T_{n+2},X) & \longrightarrow \\
g^*\downarrow & & g_n^*\downarrow & & g_{n+1}^*\downarrow & & g_{n+2}^*\downarrow & \\
\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(-,X)((Z,\beta)) : ... & \longrightarrow & \mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(Z_n,X) & \xrightarrow{\beta_n^*} & \mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(Z_{n+1},X) & \xrightarrow{\beta_{n+1}^*} & \mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(Z_{n+2},X) & \longrightarrow \\
f^*\downarrow & & f_n^*\downarrow & & f_{n+1}^*\downarrow & & f_{n+2}^*\downarrow & \\
\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(-,X)((Y,\alpha)) : ... & \longrightarrow & \mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(Y_n,X) & \xrightarrow{\alpha_n^*} & \mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(Y_{n+1},X) & \xrightarrow{\alpha_{n+1}^*} & \mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(Y_{n+2},X) & \longrightarrow \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(-,X)((0)) : ... & \longrightarrow & 0_{\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(0,X)} & \longrightarrow & 0_{\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(0,X)} & \longrightarrow & 0_{\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(0,X)} & \longrightarrow
\end{array}$$

We have for every integer n in $\mathbb{Z}$, f_n^* is an epimorphism. Therefore, for every monomorphism $f_n : Y_n \hookrightarrow Z_n$ in $\mathscr{A}$ and every morphism $h_n : Y_n \to X$ in $\mathscr{A}$, there exists a morphism $\phi_n : Z_n \to X$ in $\mathscr{A}$ such that $\phi_n \circ f_n = f_n^*(\phi_n) = h_n$. This means that the following diagram commutes:

$$\begin{array}{ccccc}
 & & X & & \\
 & & h_n \uparrow & \nwarrow \phi_n & \\
O & \longrightarrow & Y_n & \xrightarrow{f_n} & Z_n
\end{array}$$

Hence, X is an injective object of $\mathscr{A}$.

ii.2 Suppose that X is an injective object in $\mathscr{A}$ and show that the functor $\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(-,X) : \mathrm{Comp}(\mathscr{A}) \to \mathrm{Comp}(Ab)$ is exact.

Let the short exact sequence of morphisms in $\mathscr{A}$ $(0) \to (Y,\alpha) \xrightarrow{f} (Z,\beta) \xrightarrow{g} (T,\theta) \to (0)$ and show that

$$\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(-,X)((0)) \longrightarrow \mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(-,X)((T,\theta)) \xrightarrow{g^*}$$

$$\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(-,X)((Z,\beta)) \xrightarrow{f^*} \mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(-,X)((Y,\alpha)) \longrightarrow \mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(-,X)((0))$$

is a short exact sequence of morphisms in $\mathrm{Comp}(Ab)$. By (i), we have:

$$\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(-,X)((0)) \longrightarrow \mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(-,X)((T,\theta)) \xrightarrow{g^*}$$

$$\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(-,X)((Z,\beta)) \xrightarrow{f^*} \mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(-,X)((Y,\alpha))$$

is a left short exact sequence of morphisms in $\mathrm{Comp}(Ab)$. Thus, it remains to show that for every n, $\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(f,X) = f^*$ is an epimorphism. By Theorem 2, for every integer n in $\mathbb{Z}$, since X is injective, for every monomorphism $f_n : Y_n \hookrightarrow Z_n$ in $\mathscr{A}$ and every morphism $h_n : Y_n \to X$ in $\mathscr{A}$, there exists a morphism $\phi_n : Z_n \to X$

in $\mathscr{A}$ such that $\phi_n \circ f_n = h_n$. This means that the following diagram commutes:

$$\begin{array}{ccccc} & & X & & \\ & & \uparrow h_n & \nwarrow \phi_n & \\ O & \longrightarrow & Y_n & \xrightarrow{f_n} & Z_n \end{array}$$

That is, for every $h_n \in \mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(Y_n, X)$, there exists $\phi_n \in \mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(Z_n, X)$ such that $\phi_n \circ f_n = f_n^*(\phi_n) = h_n$. Hence, $\mathrm{Hom}_{\mathscr{A}}(f, X)$ is an epimorphism. Therefore, the functor $\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(-, X) : \mathrm{Comp}(\mathscr{A}) \to \mathrm{Comp}(Ab)$ is exact. Thus, (ii.1) and (ii.2) imply that the functor $\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(-, X) : \mathrm{Comp}(\mathscr{A}) \to \mathrm{Comp}(Ab)$ is exact if and only if X is an injective object in $\mathscr{A}$.

4. Exactness of Homological Functors of Degree n: $\tilde{H}_n(X, -)$ and $\tilde{H}_n(-, X)$

Consider the homological functor $H_n : \mathrm{Comp}(Ab) = \mathrm{Comp}(\mathbb{Z} - \mathrm{Mod}) \longrightarrow \mathrm{Ab}$ which is a special case of the homological functor $H_n : \mathrm{Comp}(\text{A-Mod}) \longrightarrow \mathrm{Ab}$ for all $n \in \mathbb{Z}$. That is, H_n is a covariant additive functor. [$\tilde{H}_n(X, -)$]
Let $\mathscr{A}$ be a balanced abelian category and X a projective object in $\mathscr{A}$. Then homological functor of degree n ($n \in \mathbb{Z}$), denoted $\tilde{H}_n(X, -) = H_n \circ \mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X, -)$ where H_n : $\mathrm{Comp}(Ab) \longrightarrow \mathrm{Ab}$, $\tilde{H}_n(X, -) : \mathrm{Comp}(\mathscr{A}) \longrightarrow \mathrm{Ab}$ is defined by:

(i) for any complex sequence in $\mathrm{Comp}(\mathscr{A})$ $(\beta_n : Y_n \to Y_{n+1})_{n\in\mathbb{Z}}$ denoted (Y, β), we associate
$\tilde{H}_n(X, -)((Y, \beta)) = (H_n \circ \mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X, -))((Y, \beta)) = \mathrm{Ker}\beta^*_{n+1}/\mathrm{Im}\beta^*_n \quad \forall n \in \mathbb{Z}$;

(ii) for any complex sequence $(Y, \beta) = (\beta_n : Y_n \to Y_{n+1})_{n\in\mathbb{Z}}$ in $\mathrm{Comp}(\mathscr{A})$, any complex sequence $(Z, \alpha) = (\alpha_n : Z_n \to Z_{n+1})_{n\in\mathbb{Z}}$ in $\mathrm{Comp}(\mathscr{A})$, and any complex chain $f : (Y, \beta) \to (Z, \alpha) = (f_n : Y_n \longrightarrow Z_n)_{n\in\mathbb{Z}}$ in $\mathrm{Comp}(\mathscr{A})$ denoted $f : (Y, \beta) \longrightarrow (Z, \alpha)$, we associate:

$$\begin{array}{rccc} \tilde{H}_n(X, -)(f) : & (H_n \circ \mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X, -))((Y, \beta)) & \longrightarrow & (H_n \circ \mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X, -))((Z, \alpha)) \\ & \overline{g_n} & \longmapsto & \overline{f_n(g_n)} \end{array}$$

is a covariant additive functor.

Proof.
We know that the homology functor $H_n : \mathrm{Comp}(\text{A-Mod}) \longrightarrow \mathrm{Ab}$ is defined by:

(i) for any complex sequence in Comp(A-Mod) $(\beta_n : Y_n \to Y_{n+1})_{n\in\mathbb{Z}}$ denoted (Y, β), we associate

$$H_n(Y, \beta) = \mathrm{Ker}\beta_{n+1}/\mathrm{Im}\beta_n \quad \forall n \in \mathbb{Z};$$

(ii) for any complex sequence $(Y,\beta) = (\beta_n : Y_n \to Y_{n+1})_{n\in\mathbb{Z}}$ in Comp(A-Mod), any complex sequence $(Z,\alpha) = (\alpha_n : Z_n \to Z_{n+1})_{n\in\mathbb{Z}}$ in Comp(A-Mod), and any complex chain $f : (Y,\beta) \to (Z,\alpha) = (f_n : Y_n \longrightarrow Z_n)_{n\in\mathbb{Z}}$ in Comp(A-Mod) denoted $f : (Y,\beta) \longrightarrow (Z,\alpha)$, we associate:

$$\begin{array}{rccc} H_n(f): & H_n(Y,\beta) & \longrightarrow & H_n(Z,\alpha) \\ & \overline{g_n} & \longmapsto & \overline{f_n(g_n)} \end{array}$$

and H_n is a covariant additive functor. Since Comp(Ab) = Comp$(\mathbb{Z}-$ Mod) is a special case of Comp(A-Mod), the functor $\tilde{H}_n$ is covariant. By Theorem 3, HomComp$(\mathscr{A})(X,-)$ is covariant. Now, the composition of two covariant functors is covariant, so $\tilde{H}_n(X,-) : \text{Comp}(\mathscr{A}) \longrightarrow \text{Ab}$ is well-defined and is a covariant functor. By Proposition 1 and Theorem 2, $\tilde{H}_n(X,-)$ is an additive functor. Hence, $\tilde{H}_n(X,-)$ is a covariant additive functor. And H_n is a covariant additive functor. Since Comp(Ab) = Comp$(\mathbb{Z}-$ Mod) is a special case of Comp(A-Mod), the functor $\tilde{H}_n(X,-)$ is well-defined and is a covariant additive functor.

Let $(0) \longrightarrow (Y,\alpha) \xrightarrow{f} (Z,\beta) \xrightarrow{g} (T,\gamma) \longrightarrow (0)$ be a short exact sequence of morphisms in Comp$(\mathscr{A})$, where X is a projective object in $\mathscr{A}$ and $\mathscr{A}$ is a balanced abelian category. Then:

(i) the morphism of connection associated to the covariant functor $\tilde{H}_n(X,-)$ is defined by:

$$\begin{array}{rccc} \lambda_n : & \tilde{H}_n(X,-)((T,\gamma)) & \longrightarrow & \tilde{H}_{n+1}(X,-)((Y,\alpha)) \\ & \overline{k_{n+1}} & \longmapsto & \overline{f^{*-1}_{n+2}(\beta^*_{n+1}(g^{*-1}_{n+1}(k_{n+1})))} \end{array} \quad \forall n \in \mathbb{Z};$$

(ii) the sequence

$$\cdots \longrightarrow \tilde{H}_n(X,-)((Y,\alpha)) \xrightarrow{\tilde{H}_n(X,-)(f)} \tilde{H}_n(X,-)((Z,\beta)) \xrightarrow{\tilde{H}_n(X,-)(g)} \tilde{H}_n((T,\gamma)) \xrightarrow{\lambda_n}$$

$$\tilde{H}_{n+1}(X,-)((Y,\alpha)) \xrightarrow{\tilde{H}_{n+1}(X,-)(f)} \tilde{H}_{n+1}(X,-)((Z,\beta)) \xrightarrow{\tilde{H}_{n+1}(X,-)(g)} \tilde{H}_{n+1}(X,-)((T,\gamma)) \xrightarrow{\lambda_{n+1}} \cdots$$

is a long exact sequence of abelian group morphisms. That is, for all $n \in \mathbb{Z}$:

$$\begin{cases} \text{Im}(\tilde{H}_n(X,-)(f)) = \text{Ker}(\tilde{H}_n(X,-)(g)) \\ \text{Im}(\tilde{H}_n(X,-)(g)) = \text{Ker}(\lambda_n) \\ \text{Im}(\lambda_n) = \text{Ker}(\tilde{H}_{n+1}(X,-)(f)). \end{cases}$$

Proof.

(i) We know that if the homological functor $H_n : \text{Comp(A-Mod)} \longrightarrow \text{Ab}$ is covariant, then for every short exact sequence of morphisms in Comp(A-Mod)

$(0) \longrightarrow (Y,\alpha) \xrightarrow{f} (Z,\beta) \xrightarrow{g} (T,\gamma) \longrightarrow (0)$ the connecting morphism is defined by

$$\begin{array}{rccc} \lambda_n: & H_n((T,\gamma)) & \longrightarrow & \dfrac{H_{n+1}((Y,\alpha))}{} \\ & \overline{k_{n+1}} & \longmapsto & \overline{f_{n+2}^{-1}(\beta_{n+1}(g_{n+1}^{-1}(k_{n+1})))} \end{array} \quad \forall n \in \mathbb{Z}.$$

Since Comp(Ab) = Comp($\mathbb{Z}$ − Mod) is a special case of Comp(A-Mod), by Theorem 3 if X is a projective object in $\mathscr{A}$ where $\mathscr{A}$ is a balanced abelian category, then

$$\begin{array}{rccc} \lambda_n: & \tilde{H}_n(X,-)((T,\gamma)) & \longrightarrow & \tilde{H_{n+1}}(X,-)((Y,\alpha)) \\ & \overline{k_{n+1}} & \longmapsto & \overline{f_{n+2}^{*-1}(\beta_{n+1}^{*}(g_{n+1}^{*-1}(k_{n+1})))} \end{array} \quad \forall n \in \mathbb{Z}$$

is well-defined with $H_n : \mathrm{Comp(Ab)} \longrightarrow Ab$.

(ii) According to Theorem 3, if X is a projective object in $\mathscr{A}$, where $\mathscr{A}$ is a balanced abelian category, then $\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X,-)$ transforms any complex sequence in Comp($\mathscr{A}$) into a complex sequence in Comp(Ab). However, H_n transforms every short exact sequence in Comp(A-Mod) into a long exact sequence of morphisms in Ab. In particular, H_n transforms any complex sequence of morphisms in Comp(Ab) = Comp($\mathbb{Z}$ − Mod) into a long exact sequence of morphisms in Ab. Now, $\tilde{H}_n(X,-) = H_n \circ \mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X,-)$, where $H_n : \mathrm{Comp}(Ab) \longrightarrow \mathrm{Ab}$, and $\tilde{H}_n(X,-) : \mathrm{Comp}(\mathscr{A}) \longrightarrow \mathrm{Ab}$. Thus, the sequence

$$\ldots \longrightarrow \tilde{H}_n(X,-)((Y,\alpha)) \xrightarrow{\tilde{H}_n(X,-)(f)} \tilde{H}_n(X,-)((Z,\beta)) \xrightarrow{\tilde{H}_n(X,-)(g)} \tilde{H}_n((T,\gamma)) \xrightarrow{\lambda_n}$$

$$\tilde{H}_{n+1}(X,-)((Y,\alpha)) \xrightarrow{\tilde{H}_{n+1}(X,-)(f)} \tilde{H}_{n+1}(X,-)((Z,\beta)) \xrightarrow{\tilde{H}_{n+1}(X,-)(g)} \tilde{H}_{n+1}(X,-)((T,\gamma)) \xrightarrow{\lambda^{n+1}(-,X)} \ldots$$

is a long exact sequence of abelian group morphisms.

[$\tilde{H}_n(-,X)$]

Let $\mathscr{A}$ be a balanced abelian category and X an injective object in $\mathscr{A}$. Then the homological functor of degree n ($n \in \mathbb{Z}$), denoted $\tilde{H}_n(-,X) = H_n \circ \mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(-,X)$, where $H_n : \mathrm{Comp}(Ab) \longrightarrow \mathrm{Ab}$, $\tilde{H}_n(-,X) : \mathrm{Comp}(\mathscr{A}) \longrightarrow \mathrm{Ab}$ is defined by:

(i) For any complex $(Y,\beta) = (\beta_n : Y_n \to Y_{n+1})_{n\in\mathbb{Z}}$ in Comp($\mathscr{A}$), we associate:

$$\tilde{H}_n(-,X)((Y,\beta)) = H_n(\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(-,X))((Y,\beta)) = \mathrm{Ker}\beta_{n+1}^{*}/\mathrm{Im}\beta_n^{*} \quad \forall n \in \mathbb{Z}$$

(ii) For any complex morphism $f : (Y,\beta) \to (Z,\alpha) = (f_n : Y_n \to Z_n)_{n\in\mathbb{Z}}$, we associate:

$$\begin{array}{rccc} \tilde{H}_n(f,X): & \tilde{H}_n(-,X)((Z,\alpha)) & \longrightarrow & \tilde{H}_n(-,X)((Y,\beta)) \\ & \overline{g_n} & \longmapsto & \overline{g_n(f_n)} \end{array}$$

is a contravariant additive functor.

Proof.

We know that the homology functor $H_n : \mathrm{Comp(A\text{-}Mod)} \longrightarrow \mathrm{Ab}$ is defined by:

(i) For any complex $(Y,\beta) = (\beta_n : Y_n \to Y_{n+1})_{n\in\mathbb{Z}}$ in Comp(A-Mod), we associate:

$$H_n((Y,\beta)) = \mathrm{Ker}\beta_{n+1}/\mathrm{Im}\beta_n \quad \forall n \in \mathbb{Z}$$

(ii) For any complex morphism $f : (Y,\beta) \to (Z,\alpha)$, we associate:

$$H_n(f): \begin{array}{ccc} H_n((Z,\alpha)) & \longrightarrow & H_n((Y,\beta)) \\ \overline{g_n} & \longmapsto & \overline{g_n(f_n)} \end{array}$$

Since H_n is a covariant additive functor, and $\mathrm{Comp}(Ab) = \mathrm{Comp}(\mathbb{Z} - \mathrm{Mod})$ is a special case of Comp(A-Mod), the functor $\tilde{H}_n$ is covariant. According to Theorem 3, $\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(-,X)$ is contravariant. However, the composition of a covariant functor and a contravariant functor is contravariant, so $\tilde{H}_n(-,X) : \mathrm{Comp}(\mathscr{A}) \longrightarrow \mathrm{Ab}$ is well-defined and is a contravariant functor. By Proposition 1 and Theorem 2, $\tilde{H}_n(-,X)$ is an additive functor. Thus, $\tilde{H}_n(-,X)$ is a contravariant additive functor.

Let $(0) \longrightarrow (Y,\alpha) \xrightarrow{f} (Z,\beta) \xrightarrow{g} (T,\gamma) \longrightarrow (0)$ be a short exact sequence in $\mathrm{Comp}(\mathscr{A})$, where $\mathscr{A}$ is a balanced abelian category and X an injective object in $\mathscr{A}$. Then:

(i) the morphism of connection associated to the contravariant functor $\tilde{H}_n(-,X)$ is defined by:

$$\delta_n : \begin{array}{ccc} \tilde{H}_n(-,X)((Y,\alpha)) & \longrightarrow & \tilde{H}_{n+1}(-,X)((T,\gamma)) \\ \overline{k_{n+1}} & \longmapsto & \overline{g^{*-1}_{n+2}(\beta^*_{n+1}(f^{*-1}_{n+1}(k_{n+1})))} \end{array}, \quad \forall n \in \mathbb{Z}$$

(ii) The sequence

$$\cdots \longrightarrow \tilde{H}_n(-,X)((T,\gamma)) \xrightarrow{\tilde{H}_n(-,X)(g)} \tilde{H}_n(-,X)((Z,\beta)) \xrightarrow{\tilde{H}_n(-,X)(f)} \tilde{H}_n(-,X)((Y,\alpha))$$

$$\xrightarrow{\delta_n} \tilde{H}_{n+1}(-,X)((T,\gamma)) \xrightarrow{\tilde{H}_{n+1}(-,X)(g)} \tilde{H}_{n+1}(-,X)((Z,\beta)) \xrightarrow{\tilde{H}_{n+1}(-,X)(f)} \tilde{H}_{n+1}(-,X)((Y,\alpha)) \xrightarrow{\delta_{n+1}} \cdots$$

is a long exact sequence in Ab. That is ($\forall n \in \mathbb{Z}$):

$$\begin{cases} \mathrm{Im}(\tilde{H}_n(-,X)(g)) = \mathrm{Ker}(\tilde{H}_n(-,X)(f)) \\ \mathrm{Im}(\tilde{H}_n(-,X)(f)) = \mathrm{Ker}(\delta_n) \\ \mathrm{Im}(\delta_n) = \mathrm{Ker}(\tilde{H}_{n+1}(-,X)(g)) \end{cases}$$

Proof.

(i) We know that if the homological functor H_n : Comp(A-Mod) $\longrightarrow$ Ab is covariant, then for every short exact sequence of morphisms in Comp(A-Mod)

$(0) \longrightarrow (Y,\alpha) \xrightarrow{f} (Z,\beta) \xrightarrow{g} (T,\gamma) \longrightarrow (0)$ the connecting morphism is defined by

$$\begin{array}{llll} \lambda_n: & H_n((Y,\alpha)) & \longrightarrow & \dfrac{H_{n+1}((T,\gamma))}{g_{n+2}^{-1}(\beta_{n+1}(f_{n+1}^{-1}(k_{n+1})))} \quad \forall n \in \mathbb{Z} \\ & \overline{k_{n+1}} & \longmapsto & \end{array}$$

Since Comp(Ab) = Comp($\mathbb{Z}$ − Mod) is a special case of Comp(A-Mod), by Theorem 3 if X is an injective object in $\mathscr{A}$ where $\mathscr{A}$ is a balanced abelian category, then by definition of $\tilde{H}_n(-,X)$:

$$\begin{array}{llll} \delta_n: & \tilde{H}_n(-,X)((Y,\alpha)) & \longrightarrow & \dfrac{\tilde{H}_{n+1}(-,X)((T,\gamma))}{g_{n+2}^{*-1}(\beta_{n+1}^{*}(f_{n+1}^{*-1}(k_{n+1})))} \quad \forall n \in \mathbb{Z} \\ & \overline{k_{n+1}} & \longmapsto & \end{array}$$

is well-defined with H_n : Comp(Ab) $\longrightarrow$ Ab.

(ii) According to Theorem 3, if X is an injective object in $\mathscr{A}$, where $\mathscr{A}$ is a balanced abelian category, then $\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(-,X)$ transforms any complex sequence in Comp($\mathscr{A}$) into a complex sequence in Comp(Ab). However, H_n transforms every short exact sequence in Comp(A-Mod) into a long exact sequence of morphisms in Ab. In particular, H_n transforms any complex sequence of morphisms in Comp(Ab) = Comp($\mathbb{Z}$ − Mod) into a long exact sequence of morphisms in Ab. Now, $\tilde{H}_n(-,X) = H_n \circ \mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(-,X)$, where H_n : Comp(Ab) $\longrightarrow$ Ab, and $\tilde{H}_n(-,X)$: Comp($\mathscr{A}$) $\longrightarrow$ Ab. Thus, the sequence

$$\cdots \longrightarrow \tilde{H}_n(-,X)((T,\gamma)) \xrightarrow{\tilde{H}_n(-,X)(g)} \tilde{H}_n(-,X)((Z,\beta)) \xrightarrow{\tilde{H}_n(-,X)(f)} \tilde{H}_n(-,X)((Y,\alpha))$$

$$\xrightarrow{\lambda_n} \tilde{H}_{n+1}(-,X)((T,\gamma)) \xrightarrow{\tilde{H}_{n+1}(-,X)(g)} \tilde{H}_{n+1}(-,X)((Z,\beta)) \xrightarrow{\tilde{H}_{n+1}(-,X)(f)} \tilde{H}_{n+1}(-,X)((Y,\alpha)) \longrightarrow \cdots$$

is a long exact sequence of abelian group morphisms.

5. Conclusion

In this article, by using the exacteness of we have shown the exactness of the functors $\mathrm{Hom}_{\mathscr{A}}(X,-)$,
$\mathrm{Hom}_{\mathscr{A}}(-,X)$: Comp($\mathscr{A}$) $\to$ Comp(Ab) and the exactness of the functors $\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X,-)$, $\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(-,X)$: Comp($\mathscr{A}$) $\to$ Comp(Ab) where $\mathscr{A}$ is a balanced abelian category. We then constructed the additive covariant homological functor: $\tilde{H}_n(X,-) = H_n \circ \mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X,-)$ and the additive contravariant homological functor: $\tilde{H}_n(-,X) = H_n \circ \mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(-,X)$ where H_n : Comp(Ab) $\longrightarrow$ Ab and $\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(X,-)$, $\mathrm{Hom}_{\mathrm{Comp}(\mathscr{A})}(-,X)$: Comp($\mathscr{A}$) $\longrightarrow$

Comp(Ab) $\forall n \in \mathbb{Z}$. Moreover, we constructed the connecting morphism $\lambda_n : \tilde{H}_n(X,-)((T,\gamma)) \to \tilde{H}_{n+1}(X,-)((Y,\alpha))$ associated to the covariant functor $\tilde{H}_n(X,-)$, by showing how the functor $\tilde{H}_n(X,-)$ transforms a short exact sequence of morphisms in Comp$(\mathscr{A})$ into a long exact sequence of morphisms in Ab, where X is an projective object of $\mathscr{A}$ and $\mathscr{A}$ is a balanced abelian category. Then we again constructed the connecting morphism $\delta_n : \tilde{H}_n(-,X)((Y,\alpha)) \to \tilde{H}_{n+1}(-,X)((T,\gamma))$ associated to the contravariant functor $\tilde{H}_n(-,X)$ also showing how the functor $\tilde{H}_n(-,X)$ transforms a short exact sequence of morphisms in Comp$(\mathscr{A})$ into a long exact sequence of morphisms in Ab, where X is an injective object of $\mathscr{A}$ and $\mathscr{A}$ is a balanced abelian category.

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