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On the Boundedness of the Intrinsic Square Function on Continual Herz Spaces with Variable Exponents

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Abstract. Our aim in this paper is to prove the boundedness of the intrinsic square function on Herz spaces with variable exponents. Firstly we define the Lebesgue spaces with variable exponent, Herz spaces with variable exponents and some basic notations. Then we give some basic lemmas and definition of continual Herz spaces. Finally we obtain the boundedness of the intrinsic square function on continual Herz spaces under some proper assumptions.

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1. Introduction

Consider an open set H of $\mathbb{R}^n$ and a measurable function $q(\cdot) : H \to [1, \infty)$. Assume that the following condition holds,

$$1 \leq q_-(H) \leq q_+(H) < \infty, \tag{1.1}$$

where

i) $q_- := \operatorname*{essinf}_{h \in H} \; q(H)$

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ii) $q_+ := \operatorname*{esssup}_{h\in H} q(H)$.

Let $q'(h) = \frac{q(h)}{q(h)-1}$ denotes the conjugate exponent of $q(h)$. We denote by $\mathcal{P}(\mathbb{R}^n)$ the set of all measurable function satisfying (1.1). For measurable function f, Lebesgue space $L^{q(\cdot)}(H)$ is given by

$$L_{q(\cdot)}(f) = \int_G |f(h|^{q(h)}dh < \infty,$$

with the norm defined by,

$$\|f\|_{L^{p(\cdot)}(H)} = \inf\left\{\eta > 0 : L_{q(\cdot)}\left(\frac{f}{\eta}\right) \leq 1\right\}.$$

Note that $L^{p(\cdot)}(H)$ is the Banach function space.

In the last two decades it was evident that classical function spaces are no longer appropriate for studying a number of modern problems arising in many mathematical models of applied sciences. It thus became necessary to introduce and study new function spaces. Such spaces are: variable exponent Lebesgue and Sobolev spaces, grand function spaces, Morrey-type spaces, amalgam spaces, Herz spaces, their hybrid variants, etc (see e.g., the monographs [1], [2], [3], [28] and references therein dedicated to new function spaces). Morrey spaces describe local regularity more precisely than Lebesgue spaces. For more results in Herz spaces see [45–48]. As a result, one can use Morrey spaces widely not only in Harmonic Analysis but also in the theory of PDEs. We refer to the recent monographs [4] for Morrey-type spaces and applications. For more results on variable exponent function spaces see [5–9, 39, 40].

Herz spaces are indeed important in the field of harmonic analysis and partial differential equations, particularly as substitutes for Hardy spaces in certain contexts. This substitution is particularly useful when dealing with non-translation invariant singular integral operators. Herz spaces continue to play a significant role in various mathematical contexts, including the characterization of multipliers on Hardy spaces and in the regularity theory for elliptic and parabolic equations in divergence form, see [10, 11]. Herz space has undergone significant advancements, proving to be highly valuable in the field of Harmonic analysis. For more results in variable exponent function spaces see [35–44]. The classical versions of the homogeneous and non-homogeneous Herz spaces are defined as:

Izuki introduced the idea of Herz spaces $\dot{K}^{\kappa}_{p(\cdot),q}(\mathbb{R}^n)$ and $K^{\kappa}_{p(\cdot),q}(\mathbb{R}^n)$, where the exponent p was a variable and norms are defined as,

for $p \in [1,\infty)$, $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $\kappa \in \mathbb{R}$. The Herz space $\dot{K}^{\kappa}_{p,q(\cdot)}(\mathbb{R}^n)$ of homogeneous type is given by

$$\dot{K}^{\kappa}_{p,q(\cdot)}(\mathbb{R}^n) = \left\{g \in L^{q(\cdot)}_{\text{loc}}(\mathbb{R}^n \setminus \{0\}) : \|g\|_{\dot{K}^{\kappa}_{p,q(\cdot)}(\mathbb{R}^n)} < \infty\right\}, \tag{1.2}$$

where

$$\|g\|_{\dot{K}^{\kappa}_{p,q(\cdot)}(\mathbb{R}^n)} = \left(\sum_{i=-\infty}^{i=\infty} \|2^{i\kappa} g \mathbf{1}_i\|^p_{L^{q(\cdot)}} \right)^{\frac{1}{p}}.$$

The non-homogeneous version of Herz spaces $K^{\kappa}_{p(\cdot),q}(\mathbb{R}^n)$ are defined below.

Let $\kappa \in \mathbb{R}$, $p \in [1, \infty)$ and $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. The Herz space $K^{\kappa}_{p,q(\cdot)}(\mathbb{R}^n)$ of non-homogeneous type is given by

$$K^{\kappa}_{p,q(\cdot)}(\mathbb{R}^n) = \left\{ g \in L^{q(\cdot)}_{\text{loc}}(\mathbb{R}^n \setminus \{0\}) : \|g\|_{K^{\kappa}_{p,q(\cdot)}(\mathbb{R}^n)} < \infty \right\}, \tag{1.3}$$

where

$$\|g\|_{K^{\kappa}_{p,q(\cdot)}(\mathbb{R}^n)} = \|g\|_{L^{q(\cdot)}(\mathcal{O}(0,1))} + \left(\sum_{\tau=-\infty}^{\infty} \|2^{i\kappa} g \mathbf{1}_i\|^p_{L^{q(\cdot)}} \right)^{\frac{1}{p}}.$$

An important focus within Herz spaces involves investigating the boundedness of sublinear operators. This exploration has led to studies motivated by this topic, culminating in the establishment of sublinear operator boundedness in Herz spaces with variable exponents, as demonstrated in [12].

In [13], an alternative method was employed to define variable exponent Herz spaces. A notable aspect of this method involves substituting the discrete L^p-norm (Lebesgue-norm) with the continuous L^p-norm relative to Haar measure, streamlining and clarifying the proofs. These spaces are called variable continual Herz spaces. For the boundedness results on some operators on continual Herz spaces see [15–17]. In [18], authors defined the variable Herz-Morrey spaces $M\dot{K}^{\kappa,\lambda}_{p,q(\cdot)}(\mathbb{R}^n)$, and obtained the estimates of sublinear operators within these spaces. It's worth mentioning that Herz-Morrey spaces with variable exponents serve as extensions of Herz spaces with variable exponents. Additionally, in [19], the boundedness of higher-order commutators of fractional integrals on Herz-Morrey spaces can be verified. Our findings not only consolidate and build on prior discoveries, but also offer novel applications to the regularity solutions of some elliptic PDEs with smooth boundaries. The investigation of the fractional Hardy operator, belongs to one of the hot topics in the area of PDEs because of its wide-ranging interest to various fields in Mathematics and Physics. For instance, it is motivated by physical models related to relativistic Schrödinger operator with Coulomb potential (see [22, 23]) and by the study of Hardy inequalities and Hardy-Lieb-Thirring inequalities (see, e.g., [20–26]).

The intrinsic square function S_ζ holds significant importance in function spaces. Izuki obtained the boundedness of S_ζ on weighted variable Herz spaces under certain appropriate conditions, as detailed in [27]. This study investigates the boundedness of the intrinsic square function on continuous Herz spaces with variable exponents. The article is structured into four sections: the first section serves as an introduction, the second section presents fundamental definitions and lemmas, the concept of continuous Herz spaces is defined in part three, and the final section delves into examining the boundedness of the intrinsic square function on variable continuous Herz spaces with variable exponents. For more results see [29–34].

We will use following notations in the paper:

Notations

(i) $\mathcal{O}(y,s)$ denotes a ball with center at y and radius s ;

(ii) $\mathcal{T}(\tau,t)$ *denotes the spherical layer such that*

(iii) $\mathcal{T}\tau,t) := \mathcal{O}(0,t)\setminus\mathcal{O}(0,\tau) = \{y\in\mathbb{R}^n : \tau < |y| < t\}$;

(iv) $\mathcal{T}_m := \mathcal{T}(2^{m-1},2^m)$;

(v) $\mathbf{1}_E(y)$ is the characteristic function of a set E;

(vi) $\mathbb{R}_{\mu+} := (\mu,\infty)$, where $\mu \geq 0$;

(vii) $\mathbf{1}_{\tau,t}(y) = \mathbf{1}_{\mathcal{T}_{\tau,t}}(y)$;

(viii) $d\tau/\tau$ represents the Haar measure on $\mathbb{R}_+$;

(ix) $\mathbb{N}$ denotes the set of natural numbers;

(x) $\mathbb{N}_0 = \mathbb{N}\cup\{0\}$;

(xi) $\mathbb{Z}$ denotes the set of all integers;

(xii) *For two non-nagative functions* f *and* g, $f \leq g$ *we mean* $f \leq Cg$,

(xiii) C denote the positive constant.

2. Preliminaries

Next for compact subsets $K \subset H$, the space $L^{q(\cdot)}_{\text{loc}}(H)$ is given as

$$L^{p(\cdot)}_{\text{loc}}(H) := \left\{K : K \in L^{q(\cdot)}(K)\right\}.$$

Let $x,y\in H$ with $|x-y| \leq \frac{1}{2}$ and $C(q)$ is not depending on x,y. We have the log-condition,

$$|q(x)-q(y)| \leq \frac{C(q)}{-\ln|x-y|}. \tag{2.1}$$

We say that $q(\cdot)$ satisfies log decay condition at infinity if there exists $q_\infty \in (1,\infty)$, such that

$$|q(x)-q_\infty| \leq \frac{C}{\ln(e+|x|)}. \tag{2.2}$$

Similarly, we say that $q(\cdot)$ satisfies log decay condition at origin if there exists $q_0 \in (1,\infty)$, such that

$$|q(x)-q_0| \leq \frac{C}{\ln|x|}, |x| \leq \frac{1}{2}. \tag{2.3}$$

With respect to classes of variable exponents used in this paper, we adopt the following notation:

(i) The set $\mathfrak{P}^{\log} = \mathfrak{P}^{\log}(H)$ comprises all functions $q \in L^{\infty}(H)$ that fulfill both (1.1) and (2.1).

(ii) When H is unbounded, $\mathfrak{P}\infty(H)$ and $\mathfrak{P}0,\infty(H)$ are subsets of $L^{\infty}(H)$. The functions in these sets take values in the interval $[1,\infty)$ and satisfy condition (2.2), and (2.2) and (2.3), respectively.

(iii) $\mathfrak{P}^{\log}_{\infty}(H)$ is the set consists of exponents that satisfy the condition (2.1);

(iv) The notation $\mathbb{R}\mu+$ represents the set of all non-negative real numbers. $M_{\infty}(\mathbb{R}_{\mu+})$ is the class consists of functions defined on the domain $\mathbb{R}\mu+$ that have certain properties. Specifically, these functions are of the form $g(t) = \text{constant} + g_0(t)$, where $g_0(t)$ is a function belonging to the class $\mathfrak{P}_{\infty}(\mathbb{R}_{\mu+})$, where $H = \mathbb{R}_{\mu+}, \mu \geq 0$.

(v) When $H = \mathbb{R}+$ (or when $\mu = 0$), $\mathcal{M}0,\infty(\mathbb{R}+)$ represents the class of functions defined on the positive real line $\mathbb{R}+$ that belong to the class $\mathcal{M}\infty(\mathbb{R}_{+})$ and meet a decay criterion at the origin ($y = 0$). The decay condition implies that these functions are bounded by a logarithmic term as y approaches zero. Specifically, for $|y| \leq \frac{1}{2}$, the function satisfies the inequality $|f(y) - f_0| \leq \frac{C}{\ln |y|}$ for some real numbers f_0 and C. We also write $f_0 = f(0)$, $f_{\infty} = f(\infty)$ in this case;

(vi) $\mathfrak{P}_{0,\infty}(\mathbb{R}+)$ is a subclass of functions within the class $\mathcal{M}_{0,\infty}(\mathbb{R}+)$ that have values in the interval $[1,\infty)$. In other words, these are the functions from $\mathcal{M}_{0,\infty}(\mathbb{R}+)$ that satisfy the given conditions and have their values constrained within the specified range, with values in $[1,\infty)$.

Hölder's inequality in variable Lebesgue spaces are stated as

$$\|fg\|_{r(\cdot)} \leq \|f\|_{p(\cdot)}\|g\|_{q(\cdot)},$$

where we define r as $\frac{1}{r(i)} = \frac{1}{p(i)} + \frac{1}{q(i)}$ for every $i \in H$, and $p, q, r \in \mathcal{P}(\mathbb{R}^n)$

The set $\mathcal{B}(\mathbb{R}^n)$ is comprised of $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ that fulfill the requirement that M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.

Lemma 1. *[13] Suppose* $0 < s \leq 1$, $t_0 \geq 1$, $t_{\infty} \geq 1$, $p \in \mathfrak{P}_{0,\infty}(\mathbb{R}^n)$ *and* $D > 1$ *such that* D *but not on* s*. Then, for we have:*

$$\frac{1}{t_0} s^{\frac{n}{p(0)}} \leq \|\mathbf{1}_{\mathcal{T}_{s,D_s}}\|_{p(\cdot)} \leq t_0 s^{\frac{n}{p(0)}}. \tag{2.4}$$

Similarly, for $s \geq 1$*, we get:*

$$\frac{1}{t_{\infty}} s^{\frac{n}{p_{\infty}}} \leq \|\mathbf{1}_{\mathcal{T}_{s,D_s}}\|_{p(\cdot)} \leq t_{\infty} s^{\frac{n}{p_{\infty}}}. \tag{2.5}$$

Definition 1. *If $\mathbb{R}_{\mu+}, \mu \geq 0$ and $\frac{dt}{t}$ denotes the Haar measure, norm of Lebesgue spaces with Haar measure is defined as,*

$$\|g\|_{L^{q(\cdot)}(\mathbb{R}_{\mu+};\frac{dt}{t})} = \inf\left\{\gamma > 0 : \int\limits_{\mu}^{\infty} \left|\frac{g(t)}{\gamma}\right|^{q(t)} \frac{dt}{t} \leq 1\right\}.$$

Let

$$\mathcal{H}\omega(j) = \int\limits_{0}^{\infty} \mathcal{K}(\frac{j}{\theta})\omega(\theta)\frac{d\theta}{\theta}. \tag{2.6}$$

This is an integral operator referred to as the Mellin convolution operator, employing Haar measure $\frac{d\theta}{\theta}$ and featuring a kernel homogeneous of order 0.

Lemma 2. *(see [13]) Let $p \in \mathfrak{P}_{0,\infty}(\mathbb{R}_+)$ and $p(0) = p(\infty)$. The operator $\mathcal{H}$ is bounded on $L^{p(\cdot)}(\mathbb{R}_+; \frac{dt}{t})$ if*

$$\int\limits_{0}^{\infty} |\mathcal{H}(t)|^s \frac{dt}{t} < \infty \ \textit{ when } s = 1 \ \ \& \ s = s_0, \tag{2.7}$$

for $\frac{1}{s_0} = 1 - \frac{1}{p_-} + \frac{1}{p_+}$.

Lemma 3. *(see [13]) For every measurable function Ω, the following relations*

$$\int_{2b<|x|<j} |\Omega(x)|dx = \frac{1}{\ln 2}\int_b^t \frac{d\theta}{\theta}\int_{max(2b,\theta)<|x|<min(j,2\theta)} |\Omega(x)|dx, \ j > 2\kappa > 0, \tag{2.8}$$

and

$$\int_{|x|\geq 2j} |\Omega(x)|dx = \frac{1}{\ln 2}\int_t^\infty \frac{d\theta}{\theta} \int\limits_{max(\theta,2j)<|x|<2\theta} |\Omega(x)|dx, \ j > 0. \tag{2.9}$$

These conditions are valid provided that the integrals on the left-hand side of the above expressions exist.

3. Continual Herz spaces with variable exponent

Definition 2. *We can define continual Herz spaces with variable exponents $H^{p(\cdot),q(\cdot),\kappa(\cdot)}_{\mu,\delta}(\mathbb{R}^n)$ by its norm,*

$$\|g\|_{H^{p(\cdot),q(\cdot),\kappa(\cdot)}_{\mu,\delta}(\mathbb{R}^n)} := \|g\|_{L^{p(\cdot)}(\mathcal{O}(0,\gamma\mu+\theta))} + \|t^{\kappa(t)}\|g\mathbf{1}_{R_{\gamma t,\delta t}}\|_{L^{p(\cdot)}}\|_{L^{q(\cdot)}((\gamma\mu,\infty);\frac{dt}{t}} < \infty. \tag{3.1}$$

These lemmas are already proved in [13].

Lemma 4. *Let $4 \leq \mathbb{R} < \infty$ and $0 < \rho < 2$*

$$\|g\|_{L^{p(\cdot)}(\mathcal{O}(0,R)\backslash\mathcal{O}(0,2+\theta))} \leq C(\rho, R)\|j^{\kappa(j)}\|g\mathbf{1}_{j,2j}\|_{L^{p(\cdot)}}\|_{L^q((2,\infty);\frac{dj}{j})}.$$

Lemma 5. *Let $1 \leq p_- \leq p(x) \leq p_+ < \infty$ holds,*

$$\|g\|_{H^{p(\cdot),q,\kappa(\cdot)}_{\mu,\delta}} \approx \|g\|_{L^{p(\cdot)}(\mathcal{O}(0,\gamma\mu+\theta))} + \|t^{\kappa_\infty}\|g\mathbf{1}_{\mathcal{T}_{\gamma t,\delta t}}\|_{L^{p(\cdot)}}\|_{L^q(\mathbb{R}_{+.}\mu;\frac{dt}{t})}, \mu > 0, \tag{3.2}$$

and

$$\|g\|_{H^{p(\cdot),q,\kappa(\cdot)}_{0,\delta}} \approx \|t^{\kappa(0)}(1+t)^{\kappa_\infty-\kappa(0)}\|g\mathbf{1}_{R_{\gamma t,\delta t}}\|_{L^{p(\cdot)}}\|_{L^{q(\cdot)}(\mathbb{R}_+;\frac{dt}{t})}, \tag{3.3}$$

then equivalences of norms given above are valids, if $\kappa \in M^{\log}_{\infty}(\mathbb{R}_{+,\mu})$ for (3.2) and $\kappa \in M^{\log}_{0,\infty}(\mathbb{R}_+)$ in the case of (3.3).

4. Boundedness result on continual Herz spaces with variable exponent

In this section, we demonstrate the boundedness of an intrinsic square function on continuous Herz spaces with variable exponent. Initially, we will introduce the intrinsic square function $S_\zeta f(x)$.

Definition 3. *Let $x \in \mathbb{R}^n$ and $\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times (0,\infty)$ we define a set,*

$$\Gamma(x) :\equiv \{(y,t) \in \mathbb{R}^{n+1}_+ : |x-y| < t\}.$$

Let $0 < \zeta \leq 1$, then by $\mathcal{C}_\zeta$ we denote the functions ϕ defined on $\mathbb{R}^n$ satisfying the following conditions:

(i) supp $\phi \subset \{|x| \leq 1\}$,

(ii) $\int\limits_{\mathbb{R}^n} \phi(x)dx = 0$,

(iii) $|\phi(x) - \phi(x')| \leq |x-x'|^\zeta$ for $x, x' \in \mathbb{R}^n$.

For each $(y,t) \in \mathbb{R}^{n+1}+$, we denote $\phi t(y) = t^{-n}\phi(y/t)$. Let $g \in L^1_{\text{loc}}(\mathbb{R}^n)$, then $A_\zeta g(y,t) := \sup\limits_{\phi\in\mathcal{C}_\zeta} |g * \phi_t(y)|$, where $(y,t) \in \mathbb{R}^{n+1}_+$. Then the intrinsic square function with order ζ is given as

$$S_\zeta g(x) := \left(\int \int\limits_{\Gamma(x)} A_\zeta g(y,t)^2 \frac{dydt}{t^{n+1}}\right)^{1/2}.$$

The boundedness of S_ζ in variable exponent Lebesgue spaces $L^{p(\cdot)}$ is discussed in detail in [14]. In the subsequent theorem, we establish the boundedness of the intrinsic square function in variable exponent continuous Herz spaces $H^{p(\cdot),q(\cdot),\kappa(\cdot)}_{\mu;(\gamma,\delta)}(\mathbb{R}^n)$. Here, the parameters $p(\cdot)$, $q(\cdot)$, and $\kappa(\cdot)$ are variable, subject to the condition that S_ζ is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$. It's important to mention that when the constants q are considered, the norms within the continuous Herz space become equivalent for various values of γ and δ, particularly when $0 < \gamma < \delta$. Nonetheless, this equality doesn't persist when q varies and is not constant.

4.1. Non-Homogeneous Herz space(when $\mu = 0$)

Theorem 4.1. Suppose that $p \in \mathfrak{P}_\infty^{\log}(\mathbb{R}^n)$, $q \in \mathfrak{P}_\infty^{\log}(\mu, \infty)$ with $1 < p_- < p_+ < \infty$, $1 < q_- \leq q_+ < \infty$, then every intrinsic square function is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$, is also bounded on the continualHerz spaces with variable exponent $H^{p(\cdot),q(\cdot),\kappa(\cdot)}_{\mu;(\gamma,\delta)}(\mathbb{R}^n)$, if

$$-\frac{n}{p_\infty} < \kappa_\infty < \frac{n}{p'_\infty}. \tag{4.1}$$

Proof. Norms of $H^{p(\cdot),q(\cdot),\kappa(\cdot)}_{\mu;(\gamma,\delta)}(\mathbb{R}^n)$ are equal for different values of $\mu > 0$, and $\delta > 0$ when q constant, so for simplicity we choose $\delta = 2$ and $\mu = 2$,

$$\|g\|_{H^{p(\cdot),q,\kappa(\cdot)}_{2,2}} = N(g)^{\kappa}_{p,q} + \|g\|_{L^{p(\cdot)}(\mathcal{O}(0,2+\theta))} \tag{4.2}$$

where we can define,

$$N(g)^{\kappa}_{p,q}+ = \|t^{\kappa_\infty} \|g\mathbf{1}_{\mathcal{T}_{t,2t}}\|_{L^{p(\cdot)}}\|_{L^q((2,\infty),\frac{dt}{t})}. \tag{4.3}$$

We first estimate $\|S_\beta g\|_{L^{p(\cdot)}(\mathcal{O}(0,2+\theta))}$, where we choose $\theta \in (0,1)$, and estimate for $S_\zeta g$ follows from the boundedness on $L^{p(\cdot)}$ of $S_\zeta g$, so we have

$$\|S_\zeta g\|_{L^{p(\cdot)}(\mathcal{O}(0,2+\theta))} \leq C\|g\|_{L^{p(\cdot)}(\mathcal{O}(0,2+\theta))} \leq C\|g\|_{H^{p(\cdot),q,\kappa(\cdot)}_{2,2}}.$$

Now to find the estimate of $N(S_\zeta g)_{p^*,q,\kappa}$ term, we will split the functions $g(\ell)$ as

$$g(\ell) = f_0(\ell) + f_t(\ell) + g_t(\ell) + h_t(\ell)$$

where

$$f_0(\ell) = g(\ell)\mathbf{1}_{\mathcal{O}(0,1)}(\ell), \qquad f_t(\ell) = g(\ell)\mathbf{1}_{\mathcal{O}(0,\frac{t}{2})\setminus\mathcal{O}(0,1)}(\ell)$$
$$g_t(\ell) = g(\ell)\mathbf{1}_{\mathcal{O}(0,8t)\setminus\mathcal{O}(0,\frac{t}{2})}(\ell), \qquad h_t(\ell) = g(\ell)\mathbf{1}_{R^n\setminus\mathcal{O}(0,8t)}(\ell),$$

now we have pointwise inequality,

$$|S_\zeta g(\ell)| \leq |S_\zeta f_0(\ell)| + |S_\zeta f_t(\ell)| + |S_\zeta g_t(\ell)| + |S_\zeta h_t(\ell)|.$$

Assume that $\phi \in \mathcal{C}_\zeta$, $k \in \mathbb{Z}$, $\ell \in \mathcal{T}_k$ and $(y,t) \in \Gamma(\ell)$, then we have

$$\begin{aligned} |g(\mathbf{1}_l) * \phi_t(y)| = & \left| \int_{\mathcal{T}_l} \phi_t(y) g(\ell) d\ell \right| \\ \leq & Ct^{-n} \int_{\{\ell \in \mathcal{T}_l : |y-\ell| < t\}} |g(\ell)| d\ell. \end{aligned}$$

Let $\ell \in \mathcal{T}_l$ with $|y-\ell|<t$ we obtain

$$\begin{aligned} t =&\frac{1}{2}(t+t) > \frac{1}{2}(|\ell-y|+|y-\ell|) \geq \frac{1}{2}|\ell-y| \geq \frac{1}{2}(|\ell|-|y|) \\ \geq&\frac{1}{2}(|\ell|-2^t) \geq \frac{1}{2}(|\ell|-2^{k-2}) \geq \frac{1}{2}(|\ell|-2^{-1}|\ell|) = \frac{|\ell|}{4}. \end{aligned}$$

As a result we get

$$\begin{aligned} &|S_\zeta(g\mathbf{1}_l)(\ell)| \\ &= \left(\int\int_{\Gamma(\ell)} \left(\sup_{\phi\in\mathcal{C}_\zeta} |g\mathbf{1}_l * \phi_t(y)|^2 \frac{dydt}{t^{n+1}} \right)^2 \right)^{1/2} \\ &\leq C \left(\int_{\frac{|\ell|}{4}}^{\infty} \int_{\{y:|\ell-y|<t\}} \left(\frac{1}{t^n} \int_{\{\ell\in\mathcal{T}_l:|y-\ell|<t\}} |g(\ell)|d\ell \right)^2 \frac{dydt}{t^{n+1}} \right)^{1/2} \\ &\leq C \left(\int_{\mathcal{T}_l} g(\ell)d\ell \right) \left(\int_{\frac{|\ell|}{4}}^{\infty} \left(\int_{\{y:|\ell-y|<t\}} dy \right) \frac{dt}{t^{3n+1}} \right)^{1/2} \\ &= C \left(\int_{\mathcal{T}_l} g(\ell)d\ell \right) \left(\int_{\frac{|\ell|}{4}}^{\infty} \frac{dt}{t^{2n+1}} \right)^{1/2} \\ &= C \left(\int_{\mathcal{T}_l} g(\ell)d\ell \right) |\ell|^{-n}. \end{aligned}$$

Estimation of $S_\zeta f_0(\ell)$. For $\ell \in \mathcal{O}(0,1)$, $\ell \in \mathcal{T}_{t,2t}$

$$|S_\zeta(f_0)(\ell)| \leq Ct^{-n} \int_{\mathcal{O}(0,1)} g(\ell)d\ell \leq Ct^{-n}\|f_0\|_{p(\cdot)}\|\mathbf{1}_{\mathcal{O}(0,1)}\|_{p'(\cdot)} \leq Ct^{-n}\|f_0\|_{p(\cdot)}.$$

$$\begin{aligned} N(S_\zeta f_0)_{p,q,\kappa} &\leq C\|t^{\kappa_\infty - n}\|\mathbf{1}_{\mathcal{T}_{t,2t}}\|_{L^{p(\cdot)}}\|_{L^q((2,\infty);\frac{dt}{t})}\|f_0\|_{L^{p(\cdot)}} \\ &\leq C\|t^{\kappa_\infty - \frac{n}{p'_\infty}}\|_{L^q((2,\infty);\frac{dt}{t})}\|f_0\|_{L^{p(\cdot)}} \\ &\leq C\|f_0\|_{L^{p(\cdot)}}. \end{aligned}$$

Estimation of $S_\zeta f_t(\ell)$. Let $\ell \in \mathcal{T}_{t,2t}$

$$|S_\zeta f_t(\ell)| \leq C \int_{\mathcal{O}(0,\frac{t}{2})\setminus\mathcal{O}(0,1)} t^{-n} g(\ell)d\ell.$$

In this context, Utilizing Hölder's inequality alongside Lemma (2), we derive,

$$\begin{aligned}
|S_\beta f_t(\ell)| &\leq \frac{C}{t^n} \int\limits_{1<|y|<\frac{t}{2}} g(\ell) d\ell \\
&\leq \frac{C}{t^n} \int\limits_1^t \frac{d\rho}{\rho} \int\limits_{\frac{\rho}{2}<|y|<\rho} g(\ell) d\ell \\
&\leq \frac{C}{t^n} \int\limits_1^t \frac{d\rho}{\rho} \|g \mathbf{1}_{\mathcal{T}_{\frac{\rho}{2},\rho}}\|_{p(\cdot)} \|\mathbf{1}_{\mathcal{T}_{\frac{\rho}{2},\rho}}\|_{p'(\cdot)} \\
&\leq \frac{C}{t^n} \int\limits_1^t \|g \mathbf{1}_{\mathcal{T}_{\frac{\rho}{2},\rho}}\|_{p(\cdot)} \rho^{\frac{n}{p'_\infty}-1} d\rho.
\end{aligned}$$

$$\begin{aligned}
t^{\kappa_\infty} \| |S_\zeta f_t(\ell).\mathbf{1}_{\mathcal{T}_{t,2t}}\|_{p(\cdot)} &\leq C t^{\kappa_\infty - n + \frac{n}{p_\infty}} \int\limits_1^t \|g \mathbf{1}_{\mathcal{T}_{\frac{\rho}{2},\rho}}\|_{p(\cdot)} \rho^{\frac{n}{p'_\infty}-1} d\rho \\
&\leq C t^{\kappa_\infty + \frac{n}{p'_\infty}} \int\limits_1^t \|g \mathbf{1}_{\mathcal{T}_{\frac{\rho}{2},\rho}}\|_{p(\cdot)} \rho^{\frac{n}{p'_\infty}-1} d\rho \\
&\leq \int\limits_1^t \left(\frac{t}{\rho}\right)^{\kappa_\infty - \frac{n}{p'_\infty}} \omega(\rho) \frac{d\rho}{\rho},
\end{aligned}$$

where $\omega(\rho) = \rho^{\kappa_\infty} \|g \mathbf{1}_{R_{\frac{\rho}{2},\rho}}\|_{p(\cdot)}$. Define $\mathcal{K}(t)$ as

$$\mathcal{K}(t) = \begin{cases} t^{\kappa_\infty - \frac{n}{p'_\infty}}, & t > 1, \\ 0, & 0 < t < 1, \end{cases} \tag{4.4}$$

by defining, the operator $\mathcal{H}\omega(t) = \int\limits_0^\infty \mathcal{K}(\frac{t}{\rho}) \omega(\rho) \frac{d\rho}{\rho}$, as a result we get

$$\begin{aligned}
N\left(S_\zeta f_t\right)_{p,q,\kappa} &\leq \|\mathcal{H}\omega\|_{L^q((2,\infty));\frac{dt}{t})} \\
&\leq \|\omega\|_{L^q((2,\infty));\frac{dt}{t})} \\
&\leq \|t^{\kappa_\infty} \|g \mathbf{1}_{\mathcal{T}_{t,2t}}\|_{p(\cdot)}\|_{L^q((1,2));\frac{dt}{t})} + \|t^{\kappa_\infty} \|g \mathbf{1}_{R_{t,2t}}\|_{p(\cdot)}\|_{L^q((2,\infty));\frac{dt}{t})} \\
&\leq \|g\|_{L^{p(\cdot)}(\mathcal{O}(0,4)\setminus\mathcal{O}(0,1))} + \|t^{\kappa_\infty} \|g \mathbf{1}_{\mathcal{T}_{t,2t}}\|_{p(\cdot)}\|_{L^q((2,\infty));\frac{dt}{t})} \\
&\leq \|g\|_{H_2^{p(\cdot),q,\kappa(\cdot)}}.
\end{aligned}$$

By employing ([13], Corollary 4.5), the inequality $\kappa_\infty - \frac{n}{p'_\infty} \le 0$, incorporating embedding in the last inequality $\mathcal{O}(0,4)\setminus\mathcal{O}(0,1) \subset \mathcal{O}(0,2+\theta)\cup\mathcal{O}(0,4)\setminus\mathcal{O}(0,2+\theta)$, and with the assistance of Lemma (4).

Estimation of $S_\zeta g_t(\ell)$. By the $L^{p(\cdot)} \longrightarrow L^{p(\cdot)}$ boundedness of S_ζ, we obtain

$$\|S_\zeta g_t)\mathbf{1}_{R_{t,2t}}\|_{p(\cdot)} \le \|g_t\|_{L^{p(\cdot)}} \le \sum_{j=-1}^{2} \|g\mathbf{1}_{R_{2^j t,2^{j+1}t}}\|_{p(\cdot)}.$$

$$\begin{aligned} N(S_\zeta g_t)_{p,q,\kappa} &\le C\|t^{\kappa_\infty}\|g\mathbf{1}_{R_{\frac{t}{2},t}}\|_{p(\cdot)}\|_{L^q((2,\infty);\frac{dt}{t})} + \|t^{\kappa_\infty}\|g\mathbf{1}_{R_{t,2t}}\|_{p(\cdot)}\|_{L^q((2,\infty);\frac{dt}{t})} \\ &\le C\|t^{\kappa_\infty}\|g\mathbf{1}_{R_{t,2t}}\|_{p(\cdot)}\|_{L^{q(\cdot)}((1,2);\frac{dt}{t})} + \|t^{\kappa_\infty}\|g\mathbf{1}_{R_{t,2t}}\|_{p(\cdot)}\|_{L^q((2,\infty);\frac{dt}{t})} \\ &\le \|g\|_{H_2^{p(\cdot),q,\kappa(\cdot)}} \end{aligned}$$

here we used the facts,

$$\|t^{\kappa_\infty}\|g\mathbf{1}_{\mathcal{T}_{2^j t,2^{j+1}t}}\|_{p(\cdot)}\|_{L^q((2,\infty);\frac{dt}{t})} \le \|t^{\kappa_\infty}\|g\mathbf{1}_{R_{t,2t}}\|_{p(\cdot)}\|_{L^q((2,\infty);\frac{dt}{t})}, j=1,2,$$

and $\|g\mathbf{1}_{R_{t,2t}}\|_{L^{p(\cdot)}} \le \|g\mathbf{1}_{R_{0,4}}\|_{p(\cdot)}$.

Estimation of $S_\zeta h_t(\ell)$. Since $\ell \in \mathcal{T}_{t,2t}$, and Hölder's inequality, we get

$$\begin{aligned} |S_\beta h_t(\ell)| &\le \frac{C}{\rho^n}\int_{|y>8t|} g(\ell)d\ell \\ &\le \frac{C}{\rho^n}\int_1^t \frac{d\rho}{\rho}\int_{\frac{\rho}{2}<|y|<\rho} g(\ell)d\ell \\ &\le \frac{C}{t^n}\int_{4t}^{\infty}\frac{d\rho}{\rho}\|g\mathbf{1}_{\mathcal{T}_{\rho,2\rho}}\|_{p(\cdot)}\|\mathbf{1}_{\mathcal{T}_{\rho,2\rho}}\|_{p'(\cdot)} \\ &\le \frac{C}{t^n}\int_1^t \|g\mathbf{1}_{\rho,2\rho}\|_{p(\cdot)}\rho^{\frac{n}{p'_\infty}-1}d\rho. \end{aligned}$$

$$\begin{aligned} t^{\kappa_\infty}\||S_\zeta h_t(\ell).\mathbf{1}_{\mathcal{T}_{t,2t}}\|_{L^{p(\cdot)}} &\le Ct^{\kappa_\infty+\frac{n}{p_\infty}}\int_{4t}^{\infty}\|g\mathbf{1}_{\mathcal{T}_{\rho,2\rho}}\|_{p(\cdot)}\rho^{\frac{n}{p'_\infty}-1}d\rho \\ &\le \int_t^\infty \left(\frac{t}{\rho}\right)^{\kappa_\infty+\frac{n}{p_\infty}}\omega(\rho)\frac{d\rho}{\rho}, \end{aligned}$$

where $\omega(\rho) = \rho^{\kappa_\infty}\|g\mathbf{1}_{\rho,2\rho}\|_{p(\cdot)}\mathbf{1}_{(2,\infty)}$, above inequality is the Hardy type inequality, by using the fact $\kappa_\infty + n/p_\infty > 0$ and Lemma (2), get

$$\|t^{\kappa_\infty}\|\mathbf{1}_{R_{t,2t}}S_\zeta h_t(y)\|_{p(\cdot)}\|_{L^q((2,\infty);\frac{dt}{t}} \le C\|\omega\|_{L^q((2,\infty);\frac{dt}{t}} \le \|g\|_{H_2^{p(\cdot),q,\kappa(\cdot)}}.$$

Now, let's explore the scenario where q varies. Once again, we set $\mu = 2$, $\gamma = 1$, and $\delta = 2$. The proof follows a similar pattern, thus we'll omit the details. Our objective is to establish that

$$\|S_\zeta g\|_{L^{p^*(\cdot)}(\mathcal{O}(0,2+\theta))} + \mathcal{N}^{p^*,q,\kappa}_{1,2}(S_\zeta g) \leq \|g\|_{L^{p(\cdot)}(\mathcal{O}(0,2+\theta))} + \mathcal{N}^{p,q,\kappa}_{\lambda',\delta'}(g)$$

with $\lambda' < 1$, $\delta' > 2$,

$$\mathcal{N}^{p,q,\kappa}_{\lambda,\delta}(g) := \|t^{\kappa_\infty}\|g\mathbf{1}_{\mathcal{T}_{\lambda t,\delta t}}\|_{p(\cdot)}\|_{L^{q(\cdot)}((2,\infty),\frac{dt}{t})}.$$

The estimation for $\|S_\zeta g\|_{L^{p(\cdot)}(\mathcal{O}(0,2+\theta))}$ can be calculated similarly as we done in the case of constant q. To estimate $\mathcal{N}\,(S_\zeta g)^{p^*,q,\kappa}_{1,2}$, by splitting the function $g(\ell)$ as

$$g(\ell) = f_0(\ell) + f_t(\ell) + g_t(\ell) + h_t(\ell)$$

where

$$f_0(\ell) = g(\ell)\mathbf{1}_{\mathcal{O}(0,\frac{1}{2})}(\ell), \qquad f_t(\ell) = g(\ell)\mathbf{1}_{\mathcal{O}(0,\gamma' t)\setminus\mathcal{O}(0,\frac{1}{2}))}(\ell)$$

$$g_t(\ell) = g(\ell)\mathbf{1}_{\mathcal{O}(\delta' t)\setminus\mathcal{O}(0,\gamma' t)}, \qquad h_t(\ell) = g(\ell)\mathbf{1}_{\mathbb{R}^n\setminus\mathcal{O}(0,\delta' t)},$$

then

$$|S_\zeta g(\ell)| \leq |S_\zeta(f_0)(\ell)| + |S_\zeta(f_t)(\ell)| + |S_\zeta(g)t)(\ell)| + |S_\zeta(h_t)(\ell)|.$$

Estimation of $S_\zeta f_0$**.** It can be treated similarly as in the case of constant q.
Estimation of $S_\zeta f_t$**.** By the same estimation as in the case of constant q, and considering $|\ell - y| > (1-\gamma')t$ we get

$$\begin{aligned}&|S_\zeta f_t(\ell)|\\ &\leq Ct^{-n}\int_{\frac{1}{2}}^{t}\|g\mathbf{1}_{R_{\frac{\rho}{2},\rho}}\|_{p(\cdot)}\rho^{n/p'_\infty-1}d\rho,\end{aligned}$$

which yields

$$\begin{aligned}&t^{\kappa_\infty}\|S_\zeta f_t(\ell).\mathbf{1}_{\mathcal{T}_{t,2t}}\|_{L^{p^*(\cdot)}}\\ &\leq Ct^{\kappa_\infty+\frac{n}{p'_\infty}}\int_{1/2}^{t}\|g\mathbf{1}_{\mathcal{T}_{\frac{\rho}{2},\rho}}\|_{p(\cdot)}\rho^{\frac{n}{p'_\infty}-1}d\rho\\ &\leq \int_{1/2}^{t}\left(\frac{t}{\rho}\right)^{\kappa_\infty-\frac{n}{p'_\infty}}\psi(\rho)\frac{d\rho}{\rho},\end{aligned}$$

where $\psi(\rho) = \rho^{\kappa_\infty}\|\mathbf{1}_{\mathcal{T}_{\rho,2\rho}}\|_{p(\cdot)}$. Left hand side of above equation is a Hardy type operator and by using Lemma (2) we get

$$\mathcal{N}^{p^*,q,\kappa}_{1,2}(S_\zeta f_t) \leq \|\Gamma\psi\|_{L^{q(\cdot)}((2,\infty);\frac{dt}{t})} \leq \|\psi\|_{L^{q(\cdot)}((2,\infty);\frac{dt}{t})} \leq \|g\|_{H^{p(\cdot),q(\cdot),\kappa(\cdot)}_{2;(\gamma',\delta')}(\mathbb{R}^n)},$$

by following same reasoning as in the case of constant q.
Estimation of $S_\zeta g_t$***.*** By the boundedness of the operator S_ζ in the space $L^{p(\cdot)}(\mathbb{R}^n) \to L^{p^*(\cdot)}(\mathbb{R}^n)$ we obtain

$$\|(S_\zeta g_t)\mathbf{1}_{\mathcal{T}_{t,2t}}\|_{p(\cdot)} \leq C\|g_t\|_{p(\cdot)} = C\|g\mathbf{1}_{R\gamma' t,\delta' t}\|_{p(\cdot)}$$

which implies

$$\mathcal{N}_{1,2}^{p^*,q,\kappa}(S_\zeta g_t) \leq C\|g\mathbf{1}_{R\gamma' t,\delta' t}\|_{H^{p(\cdot),q(\cdot),\kappa(\cdot)}_{2;(\gamma',\delta')}(\mathbb{R}^n)}.$$

Estimation of $S_\zeta h_t$***.*** The estimate of S_ζ can be obtained similarly from the constant q case, we get

$$\|t^{\kappa_\infty}\|\mathbf{1}_{R_{t,2t}}S_\zeta h_t(\ell)\|_{p(\cdot)}\|_{L^{q(\cdot)}((2,\infty);\frac{dt}{t})} \leq C\|g\|_{H^{p(\cdot),q(\cdot),\kappa(\cdot)}_{2;(\gamma',\delta')}(\mathbb{R}^n)}.$$

By taking into consideration we obtained our desired result.

5. Conclusion

This manuscript contributes significantly to the field of mathematical analysis by introducing new results within the framework of continual Herz spaces. The study deepens our understanding of boundedness of the intrinsic square function on continual Herz spaces by building on previous findings. The application of these results to establish the existence of the regularity solutions of some elliptic PDEs with smooth boundaries in these spaces. There are a number of exciting avenues for further study in continual weighted Herz-Morrey spaces in the future. Extending these results to two weighted continual Herz-Morrey spaces with variable exponents in solving PDEs within such spaces could be particularly valuable.

6. Ethics declarations

Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Ethics approval and consent to participate

This manuscript has not and will not be submitted to more than one journal for simultaneous consideration. The submitted work is original and will not be published elsewhere.

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References

[1] L. Diening, P. Harjulehto, P. H"ast"o and M. Ružička, Lebesgue and Sobolev Spaces with Variable Exponents, *Lecture Notes in Mathematics.* Vol. 2017, Springer, Heidelberg (2011).

[2] V. Kokilashvili, A. Meskhi, H. Rafeiro and S. Samko, Integral Operators in Non-Standard Function Spaces. Vol. 1: Variable Exponent Lebesgue and Amalgam Spaces. Oper. Theory Adv. Appl. 248, *Birkhuser/Springer, Cham, 2016.*

[3] V. Kokilashvili, A. Meskhi, H. Rafeiro and S. Samko, Integral Operators in Non-Standard Function Spaces. Vol. 2: Variable Exponent H"older, Morrey-Campanato and Grand Spaces. Theory Adv. Appl. 249, Birkhuser/Springer, Cham*, 2016.*

[4] Y. Sawano, G. Di Fazio, D. I. Hakim, Morrey Spaces Introduction and Applications to Integral Operators and PDE's, volumes I, II, CRC Press, Taylor and Francis, 2020*.*

[5] M. Sultan, B. Sultan, A. Hussain, Grand Herz–Morrey Spaces with variable exponent, Math. Notes. **114** *(5) (2023), 957–977.*

[6] A. Hussain, G. Gao, Multilinear singular integrals and commutators on Herz space with variable exponent, ISRN Math. Anal. **2014** *(2014), 1-10.*

[7] A. Hussain, I. Khan. and A. Mohamed, Variable Her-Morrey estimates for rough fractional Hausdorff operator, J. Inequal. Appl. **33** *(2024).*

[8] A. Ajaib, and A. Hussain, Weighted CBMO estimates for commutators of matrix Hausdorff operator on the Heisenberg group. Open Math. **18***(1) (2020), 496-511.*

[9] J. Younas, A. Hussain, H. Alhazmi, A.F. Aljohani, I. Khan, BMO estimates for commutators of the rough fractional Hausdorff operator on grand-variable-Herz-Morrey spaces, AIMS Math. **9***(9) (2024), 23434-23448.*

[10] S. Lu, D. Yang, G. Hu, Herz Type Spaces and Their Applications, Science Press. *Beijing (2008)*

[11] M. Ragusa, Homogeneous Herz spaces and regularity results, Nonlinear Anal. ***71*** *(2009), 1909–1914.*

[12] M. Izuki, Boundedness of sublinear operators on Herz spaces with variable exponent and application to wavelet characterization, Anal. Math. ***36****(1) (2010), 33-50.*

[13] S. Samko, Variable exponent Herz spaces, Mediterr. J. Math. ***10****(4) (2013), 2007-2025.*

[14] K.P. Ho, Intrinsic square functions on Morrey and block spaces with variable exponents, Bull. Malays. Math. Sci. Soc. ***40****(3) (2017), 995-1010.*

[15] H. Rafeiro and S. Samko, Riesz potential operator in continual variable ex ponents Herz spaces, Math. Nachr. ***288****(4) (2015), 465-475.*

*[16] A. Meskhi, H. Rafeiro, and M.A. Zaighum, On the boundedness of Marcinkiewicz integrals on continual variable exponent Herz spaces,*Georgian Math. J. ***26****(1) (2019), 105-116.*

[17] *M. Sultan and B. Sultan, A note on the boundedness of Marcinkiewicz integral operator on continual Herz-Morrey spaces,* Filomat. **39** *(6) (2025), 2017-2027.*

[18] *M. Izuki, Boundedness of vector-valued sublinear operators on Herz-Morrey spaces with variable exponents,* Math. Sci. J ***13****(10) (2009), 243-253.*

[19] *J.L. Wu, W.J. Zhao, Boundedness for Higher Order Commutators of Fractional Integrals on Variable Exponent Herz-Morrey Spaces.* Mediterr. J. Math. ***14*** *(2017) 198.*

[20] *R.L. Frank, E.H. Lieb, and Seiringer, R. Hardy–Lieb–Thirring inequalities for fractional Schr"odinger operators, J. Amer. Math. Soc. 21(4)(2008), 925–950.*

[21] *R.L. Frank, A simple proof of Hardy-Lieb-Thirring inequalities, Comm. Math. Phys. 290(2009), 789–800.*

[22] *R.L. Frank, E. Lieb and R. Seiringer, Stability of relativistic matter with magnetic fields for nuclear charges up to the critical value, Comm. Math. Phys. 275(2)(2007), 479–489.*

[23] *N. Ghoussoub, F. Robert, S. Shakerian. and M. Zhao, Mass and asymptotics associated to fractional Hardy-Schr"odinger operators in critical regimes, Comm. Part. Diff. Eq. 43(6)(2018), 859–892.*

[24] *K. Tzirakis, Sharp trace Hardy-Sobolev inequalities and fractional Hardy-Sobolev inequalities, J. Funct. Anal. 270(2016), 4513–4539.*

[25] *B. Hasan, and S. Yalçın, Convolution and Coefficient Estimates for (p, q)-Convex Harmonic Functions Associated with Subordination, Journal of Function Spaces, 2022(1) (2022), 5317797.*

[26] *B. Hasan, q-Analogue of a New Subclass of Harmonic Univalent Functions Associated with Subordination, Symmetry, 14(4) (2022), 708.*

[27] *M. Izuki and T. Noi, An intrinsic square function on weighted Herz spaces with variable exponents,* J. Math. Inequal. ***11****(3) (2017), 799-816.*

[28] *B. Sultan, Atomic decomposition of anisotropic grand variable weighted Herz spaces and applications. J. Pseudo-Differ. Oper. Appl. 16, 53 (2025).*

[29] *B. Sultan, F. Azmi, M. Sultan, M. Mehmood, N. Mlaiki, Boundedness of Riesz Potential Operator on Grand Herz-Morrey Spaces,* Axioms. *11(11), 2022, 583.*

[30] *B. Sultan, M. Sultan, M. Mehmood, F. Azmi, M.A. Alghafli, N. Mlaiki. Boundedness of fractional integrals on grand weighted Herz spaces with variable exponent,* AIMS Math. *8(1), (2023), 752-764. doi: 10.3934/math.2023036.*

[31] *B. Sultan, F. Azmi, M. Sultan, T. Mahmood, N. Mlaiki, N. Souayah. Boundedness of Fractional Integrals on Grand Weighted Herz- Morrey Spaces with Variable Exponent.* Frac. Fract. *6(11), (2022), 660-670. https://doi.org/10.3390/fractalfract6110660*

[32] *B. Sultan, M. Sultan, Q. Q. Zhang, N. Mlaiki, Boundedness of Hardy operators on grand variable weighted Herz spaces, AIMS Math. 8(10) (2023), 24515–24527.*

[33] *B. Sultan, M. Sultan, Boundedness of commutators of rough Hardy operators on grand variable Herz spaces, Forum Math. (2023), https://doi .org /10 .1515 /forum -2023 -0152.*

[34] *B. Sultan, M. Sultan, I. Khan, On Sobolev theorem for higher commutators of fractional integrals in grand variable Herz spaces, Commun. Nonlinear Sci. Numer. Simul. 126 (2023).*

[35] M. Sultan, B. Sultan, A. Khan, T. Abdeljawad, Boundedness of Marcinkiewicz integral operator of variable order in grand Herz-Morrey spaces, AIMS Math., **8***(9) (2023), 22338–22353.*

[36] B. Sultan, M. Sultan, Boundedness of higher order commutators of Hardy operators on grand Herz-Morrey spaces, Bull. Sci. Math., **190** *(2024).*

[37] M. Sultan, B. Sultan, Boundedness of sublinear operators on grand central Orlicz-Morrey spaces, Bull. Sci.math. **205** *(2025), 103704*

[38] M. Sultan, B. Sultan, λ-Central Musielak–Orlicz–Morrey spaces, Arab. J. Math. **14** *(2025), 357–363.*

[39] B. Sultan, M. Sultan, A. Hussain, Boundedness of the Bochner–Riesz operators on the weighted Herz–Morrey type Hardy spaces, Complex Anal. Oper. Theory. **19** *(49) (2025).*

[40] B. Sultan, A. Hussain, M. Sultan, Chracterization of generalized Campanato spaces with variable exponents via fractional integrals, J. Pseudo-Differ. Oper. Appl. **16** *(22) (2025).*

[41] B. Sultan, M. Sultan, A. Khan, T. Abdeljawad, Boundedness of commutators of variable Marcinkiewicz fractional integral operator in grand variable Herz spaces, J. Inequal. Appl. **2024** *(2024), 93.*

[42] B. Sultan, M. Sultan, Sobolev-type Theorem for commutators of Hardy operators in grand Herz spaces, Ukr. Math. J. **76** *(2024), 1196–1213.*

[43] M. Sultan, B. Sultan, R.E. Castillo, Weighted composition operator on Gamma spaces with variable exponent, J. Pseudo-Differ. Oper. Appl. **15***(46) (2024).*

[44] M. Sultan, B. Sultan, A note on the boundedness of higher order commutators on fractional integrals in grand variable Herz-Morrey spaces, Kragujev. J. Math. **50***(7) (2026), 1063-1080.*

[45] A. Hussain, Naqash Sarfraz and F. Gurbuz, Sharp weak bounds for p-adic Hardy operators on p-adic linear spaces, Commun.Fac. Sci.Univ.Ank.Ser. A1 Math. Stat. 71(4) (2022), 919–929.

[46] A. Hussain, N. Sarfraz et al., The boundedness of commutators of rough p-adic fractional Hardy type operators on Herz-type spaces, J. Inequal. Appl. (2021) (2021), 123.

[47] A. Hussain, N. Sarfraz, Ilyas Khan, and A.M. Alqahtani, Estimates for Commutators of Bilinear Fractional p-Adic Hardy Operator on Herz-Type Spaces, J. Funct. Spaces (2021) ID 6615604.

[48] A. Hussain and A. Ajaib, Some weighted inequalities for Hausdorff operator and commutators, J. Inequal. Appl. 2018, (2018)6.