EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 4, Article Number 6656
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Ranked Set Sampling-Based Statistical Inference for the Exponentiated Inverted Weibull Distribution and Its Variants: Theory, Simulation and Data-Driven Analysis

Upama Deka[1], Bhanita Das[1], Partha Jyoti Hazarika[2], Jondeep Das[3], Mohamed S. Eliwa[4,5,*], Mohamed F. Abouelenein[6]

[1] *Department of Statistics, North-Eastern Hill University, Shillong 793022, Meghalaya, India*
[2] *Department of Statistics, Dibrugarh University, Assam 786004, India*
[3] *Department of Statistics, Bhattadev University, Assam 781325, India*
[4] *Department of Statistics and Operations Research, College of Science, Qassim University, Saudi Arabia*
[5] *Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt*
[6] *Department of Insurance and Risk Management, College of Business, Imam Mohammad Ibn Saud Islamic University (IMSIU), Riyadh 11432, Riyadh, Saudi Arabia*

Abstract. When measuring a key variable is difficult or expensive, ranked set sampling provides an effective alternative for collecting data. This study investigates the estimation of parameters for the exponentiated inverted Weibull distribution using ranked set sampling and its specific forms, including extreme ranked set sampling and median ranked set sampling. Since the exponentiated inverted Weibull distribution is widely applied in the analysis of lifetime and reliability data, obtaining precise parameter estimates is essential for sound statistical inference. The research compares the maximum likelihood estimates of the distribution's parameters under different sampling schemes, namely simple random sampling, ranked set sampling, extreme ranked set sampling, and median ranked set sampling. An extensive simulation study is conducted to assess the performance of these estimation methods in terms of bias, mean squared error, and relative efficiency under a range of sampling conditions. The study shows that variations in the size of the set and sampling cycles influence the accuracy of the estimate, with ranked set sampling, especially its extreme and median forms, generally outperforming simple random sampling.

2020 Mathematics Subject Classifications: 62E99, 62E15

Key Words and Phrases: Ranked set sampling techniques, parameter estimation, computer simulation, lifetime data analysis

---

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v18i4.6656

Email addresses: mseliwa@mans.edu.eg (M. S. Eliwa)

 Copyright: © 2025 The Author(s). (CC BY-NC 4.0)

1. Introduction

In lifetime data analysis and reliability theory, it is often crucial to employ flexible probability models that may represent various hazard rate patterns, including growing, decreasing, constant, and bathtub-shaped failure rates. The Weibull distribution and extreme value distributions are often employed models in this area, valued for their analytical simplicity and versatility. However, these models demonstrate a notable limitation: they cannot adequately depict upside-down bathtub (unimodal) hazard functions, which frequently arise in particular mechanical and biological systems. To address this limitation, numerous extensions and generalizations of these models have been developed; see [1], [2], [3], [4], [5], [6], [7], [8], and [9]. These upgraded models offer more flexibility in depicting complex hazard rate patterns, leading to superior fitting and more accurate dependability assessments for practical applications. The exponentiated inverse Weibull distribution (EIWD) is an extension model that was initially described and later expanded in [10]. The EIWD offers more versatility in modeling various hazard rate forms, rendering it an effective instrument for examining pertinent and applicable data. This model generalizes the inverted Weibull distribution by augmenting its cumulative distribution function with a non-negative parameter, commonly represented as θ. Since its inception, the distribution has garnered significant interest from scholars. For example, [11], [12], [13], [14], and [15]. Furthermore, [16] employed the Morgenstern approach to develop a bivariate variant of the distribution. When a random variable X adheres to the EIWD with shape parameters $\beta > 0$ and $\theta > 0$, its cumulative distribution function (CDF) is articulated as follows:

$$F(x; \beta, \theta) = e^{-\theta x^{-\beta}}, \quad x > 0. \tag{1}$$

The following is the probability density function (pdf),

$$f(x; \beta, \theta) = \theta\beta x^{-(\beta+1)} e^{-\theta x^{-\beta}}, \quad x > 0. \tag{2}$$

The inverted Weibull distribution, which is obtained when $\theta = 1$, is generalized by the EIWD, which is a flexible lifespan model that may be applied over the course of one's entire life. The density and hazard functions are able to assume a greater number of different forms thanks to the presence of the additional exponentiation parameter, θ. On account of this, the EIWD is especially well-suited for the analysis of complicated dependability data. Figure 1 illustrates the estimated individual working duration EIWD using a number of different shapes for probability density functions (PDFs) and hazard rates. The graphs demonstrate that the PDF that is available from EIWD is skewed to the right, and the hazard rate function curve has a unimodal form that is similar to an upside-down bathtub. This shape grows to a single peak before it decreases again. EIWD is especially advantageous for systems or components in which the failure rates initially rise (early wear-out) and then fall (age survivors). For this reason, when it comes to survival or reliability studies, it is considered to be more flexible than either the Weibull distribution or the inverted Weibull distribution. Simple random sampling (SRS) is the method that is most commonly used to estimate the parameters of a distribution.

Despite the conceptual simplicity of SRS, it frequently results in estimators that have a comparatively high variance, particularly when precise data or measurements that are difficult to collect are required. As a more efficient alternative, ranked set sampling (RSS) was created to alleviate this constraint. RSS, which was first proposed by [17], In order to improve the efficiency of estimate without expanding the sample size, RSS employs either judgmental ranking or supplementary information. For the purpose of ranking sampling units, the introduction of auxiliary variables was first proposed by [18]. Refer to the following sources for additional information: [19] and [20].

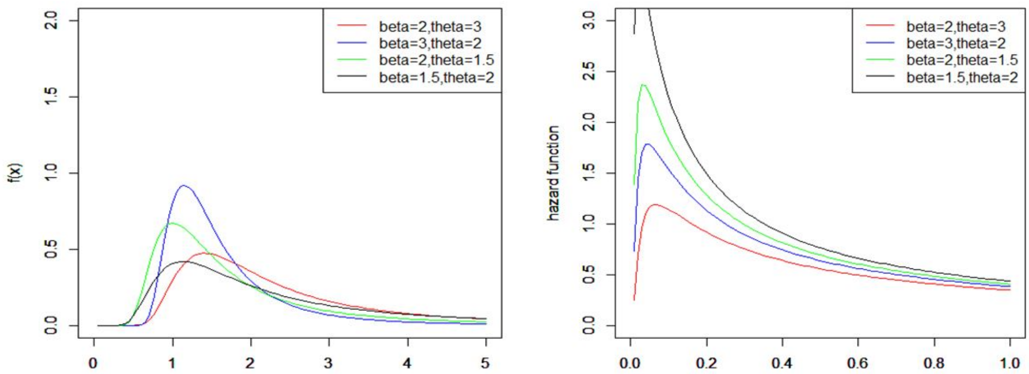

Figure 1: The PDF and hazard rate plots for the EIWD.

Since its inception, other adaptations of RSS have been suggested, including extreme ranked set sampling (ERSS) by [21], which concentrates on extreme order statistics, and median ranked set sampling (MRSS) by [22], which highlights central observations. Depending on how the data is distributed, each of these approaches has its own benefits. The RSS is a non-parametric method that has been widely researched for estimating parameters of several well-established statistical distributions. Recent research underscores the efficacy of ranked set sampling in enhancing statistical inference. For instance, [23] investigated both maximum likelihood and Bayesian estimation techniques for the Rayleigh distribution. Likewise, [24] utilized maximum likelihood methods to ascertain the shape and scale parameters of the generalized Rayleigh distribution. [25] examined the characteristics and estimation of two-parameter lifetime models under RSS, whereas [26] utilized traditional estimation techniques with RSS to analyze the inverse power Cauchy distribution. This work examines parameter estimation for the EIWD under SRS, RSS, ERSS, and MRSS, motivated by recent advancements, with the objective of developing sample strategies that enhance efficiency and accuracy in lifetime data modeling. The main aim of this research is to examine the estimation of shape parameters for the EIWD utilizing the maximum likelihood method in the context of ranked set sampling and its modified variants. The study specifically examines two prominent variations of the ranked set sampling technique: ERSS and MRSS. Standard RSS enhances efficiency by rating all units, but ERSS and MRSS are specific adaptations aimed at leveraging either the tails (ERSS) or the center (MRSS) of the distribution. Utilizing them in conjunction with RSS enables a comparison of which design yields the most efficient estimates for EIWD parameters across various data structures.

The paper is structured as follows: For the parameters β and θ of EIWD, the maximum likelihood estimation equations are derived in Section 2 using the sampling procedures. Section 3 presents the results of a thorough simulation analysis that compares the efficiency of several sampling methods. Over a variety of parameter settings and sample sizes, the assessment is based on a number of statistical metrics, such as relative efficiency, bias, and mean squared error. To further illustrate how the relative efficiency reacts to parameter value changes in the ranked set sampling schemes, this section also contains visual representations. The suggested methods are shown to be practical in Section 4 by applying them to real-world data. Section 5 provides a brief summary of the paper's findings and discusses their ramifications before calling it quits.

2. Maximum Likelihood Estimation for EIWD using Various Sampling Schemes

2.1. Estimation of parameters through SRS

If a random sample of size n is drawn from the EIWD with parameters β and θ, denoted by $x_1,\ x_2, \ldots x_n$, then the likelihood function $L(\beta, \theta)$ and the log-likelihood function $l(\beta, \theta)$ are expressed as follows:

$$L(\beta, \theta; x) = \prod_{i=1}^{n} f(x_i; \beta, \theta) = \theta^n \beta^n \prod_{i=1}^{n} x_i^{-(\beta+1)} \left(e^{-x_i^{-\beta}}\right)^{\theta}. \tag{3}$$

Then,

$$l(\beta, \theta) = n \log \theta + n \log \beta - (\beta + 1) \sum_{i=1}^{n} \log x_i - \theta \sum_{i=1}^{n} x_i^{-\beta}. \tag{4}$$

By taking the derivatives with respect to β and θ, from (4):

$$\frac{\partial l}{\partial \theta} = \frac{n}{\theta} - \sum_{i=1}^{n} x_i^{-\beta}, \tag{5}$$

$$\frac{\partial l}{\partial \beta} = \frac{n}{\beta} - \sum_{i=1}^{n} \log x_i + \theta \sum_{i=1}^{n} x_i^{-\beta} \cdot \log x_i. \tag{6}$$

From (5), the MLE of θ as a function of β is obtained as follows:

$$\hat{\theta}(\beta) = \frac{n}{\sum_{i=1}^{n} x_i^{-\beta}}. \tag{7}$$

Using (7) in (4), we have:

$$l\left(\hat{\theta}(\beta), \beta\right) = n \log \left(\frac{n}{\sum_{i=1}^{n} x_i^{-\beta}}\right) + n \log \beta - (\beta + 1) \sum_{i=1}^{n} \log x_i - n. \tag{8}$$

By maximizing (8) with respect to β, one can estimate the ML estimator of β. A numerical method is required to obtain $\hat{\theta}$ and $\hat{\beta}$. Once $\hat{\beta}$ is obtained, $\hat{\theta}$ can be obtained from (7) as $\hat{\theta}(\beta)$.

2.2. Estimation of parameters through RSS

In practice, precise measurements of all sample units may be costly or time-consuming, but ranking them based on judgment or auxiliary information is often feasible. In RSS, m sets each containing m units are drawn from the population. Units in each set are ranked without actual measurement. The i^{th} ranked unit from the i^{th} set is then measured, producing a ranked set sample of size m. After r cycles, a total of $n = mr$ observations are obtained. Let Y_{ij}; $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, r$ represent a RSS obtained from EIWD, then, Y_{ij} are independent with a density matching that of the i^{th} order statistic from a sample of size m as described

$$g_i(y_{ij}, \beta, \theta) = \frac{m!}{(i-1)!(m-i)!} f(y_{ij}, \beta, \theta) \cdot [F(y_{ij}, \beta, \theta)]^{i-1} \cdot [1 - F(y_{ij}, \beta, \theta)]^{m-i}, \tag{9}$$

where $f(y_{ij}, \beta, \theta)$ and $F(y_{ij}, \beta, \theta)$ are the pdf and cdf, respectively. The likelihood function for RSS is constructed by replacing the sample distribution with the distribution of the corresponding order statistics from equation (9) as follows:

$$\begin{aligned} L(\beta, \theta, y_{ij}) &= \prod_{j=1}^{r} \prod_{i=1}^{m} g_i(y_{ij}, \beta, \theta) \\ &= \prod_{j=1}^{r} \prod_{i=1}^{m} \frac{m! \beta \theta y_{ij}^{-(\beta+1)}}{(i-1)!(m-i)!} \left(e^{-y_{ij}^{-\beta}}\right)^{\theta} \left[\left(e^{-y_{ij}^{-\beta}}\right)^{\theta}\right]^{i-1} \left[1 - \left(e^{-y_{ij}^{-\beta}}\right)^{\theta}\right]^{m-i} \\ &= k^{mr} \beta^{mr} \theta^{mr} \prod_{j=1}^{r} \prod_{i=1}^{m} \left[y_{ij}^{-(\beta+1)} \left(e^{-y_{ij}^{-\beta}}\right)^{\theta} \left[\left(e^{-y_{ij}^{-\beta}}\right)^{\theta}\right]^{i-1} \left[1 - \left(e^{-y_{ij}^{-\beta}}\right)^{\theta}\right]^{m-i} \right], \end{aligned} \tag{10}$$

where,

$$k = \frac{m!}{(i-1)!(m-i)!}.$$

The natural logarithm of (10) can be listed as

$$\begin{aligned} l(\beta, \theta, y) = {} & mr \log k + mr \log \beta + mr \log \theta + \sum_{j=1}^{r} \sum_{i=1}^{m} \log \left(y_{ij}^{-(\beta+1)} \right) \\ & - \sum_{j=1}^{r} \sum_{i=1}^{m} \theta y_{ij}^{-\beta} - \sum_{j=1}^{r} \sum_{i=1}^{m} \theta (i-1) y_{ij}^{-\beta} + \sum_{j=1}^{r} \sum_{i=1}^{m} (m-i) \log \left[1 - \left(e^{-y_{ij}^{-\beta}}\right)^{\theta} \right]. \end{aligned} \tag{11}$$

Differentiating (11) with respect to the parameters θ and β and setting it to zero yields the respective equations,

$$\frac{\partial l}{\partial \theta} = \frac{mr}{\theta} - \sum_{j=1}^{m}\sum_{i=1}^{r} y_{ij}^{-\beta} - \sum_{j=1}^{m}\sum_{i=1}^{r}(i-1)y_{ij}^{-\beta} + \sum_{j=1}^{m}\sum_{i=1}^{r}(m-i)\frac{\left(e^{-y_{ij}^{-\beta}}\right)^{\theta} y_{ij}^{-\beta}}{1-\left(e^{-y_{ij}^{-\beta}}\right)^{\theta}}, \tag{12}$$

$$\frac{\partial l}{\partial \beta} = \frac{mr}{\beta} - \sum_{j=1}^{r}\sum_{i=1}^{m}\log y_{ij} + \sum_{j=1}^{r}\sum_{i=1}^{m}\theta y_{ij}^{-\beta}\log y_{ij} - \sum_{j=1}^{r}\sum_{i=1}^{m}\theta(i-1)y_{ij}^{-\beta}\log y_{ij}$$
$$- \sum_{j=1}^{r}\sum_{i=1}^{m}(m-i)\frac{1}{1-\left(e^{-y_{ij}^{-\beta}}\right)^{\theta}}\theta\left(e^{-y_{ij}^{-\beta}}\right)^{\theta-1} e^{-y_{ij}^{-\beta}} y_{ij}^{-\beta}\log y_{ij}. \tag{13}$$

Because these equations are nonlinear, the maximum likelihood estimators of θ and β must be obtained through numerical methods.

2.3. Estimation of parameters through ERSS

This subsection focuses on deriving the maximum likelihood equations for the parameters of the EIWD using the ERSS method. The ERSS procedure can be summarized as follows: First, m random sets, each containing m units, are drawn from the population. Ranking is performed using visual inspection or any other cost-free method. This cycle may be repeated r times to obtain a total of mr observations, which constitute the ERSS sample. Two scenarios arise depending on whether the set size m is odd or even. In the case of an even set size, from each of the first $\frac{m}{2}$ sets, the lowest-ranked unit is selected, while from each of the last $\frac{m}{2}$ sets, the highest-ranked unit is chosen. In a random sample of size m, let $X_{i(1:m)j}$ be the first order statistic and $X_{i(m:m)j}$ be the maximum order statistic. For convenience, we denote $X_{i(h:m)j}$ by P_{ij}. Let us assume,

$$P = \{X_{i(1:m)j},\ i = 1, 2, \ldots, \tfrac{m}{2};\ j = 1, 2, \ldots, r\}$$
$$\cup\ \{X_{i(m:m)j},\ i = \tfrac{m}{2}+1, \tfrac{m}{2}+2, \ldots, m;\ j = 1, 2, \ldots, r\}.$$

Then, the densities of P_{ij} are given by

$$g_1(p_{ij};\theta,\beta) = \frac{m!}{(m-1)!} f(p_{ij};\theta,\beta)\ [1 - F(p_{ij};\theta,\beta)]^{m-1}. \tag{14}$$

$$g_m(p_{ij};\theta,\beta) = \frac{m!}{(m-1)!} f(p_{ij};\theta,\beta)\left[F(p_{ij};\theta,\beta)\right]^{m-1}. \tag{15}$$

Combining the two densities, we have the likelihood function as

$$L_{\mathrm{ERSS}}(\theta,\beta,p) = \prod_{i=1}^{\frac{m}{2}}\prod_{j=1}^{r} m\, f(p_{ij};\theta,\beta)\left[1 - F(p_{ij};\theta,\beta)\right]^{m-1}$$

$$\times \prod_{i=1+\frac{m}{2}}^{m} \prod_{j=1}^{r} m\, f(p_{ij};\theta,\beta)\left[F(p_{ij};\theta,\beta)\right]^{m-1}.$$

Then, substituting the EIWD density,

$$\begin{aligned} L_{\text{ERSS}}(\theta,\beta,p) &= \prod_{j=1}^{r}\prod_{i=1}^{\frac{m}{2}} m\,\beta\theta\, p_{ij}^{-(\beta+1)}\left(e^{-p_{ij}^{-\beta}}\right)^{\theta}\left[1-\left(e^{-p_{ij}^{-\beta}}\right)^{\theta}\right]^{m-1} \\ &\quad\times \prod_{j=1}^{r}\prod_{i=\frac{m}{2}+1}^{m} m\,\beta\theta\, p_{ij}^{-(\beta+1)}\left(e^{-p_{ij}^{-\beta}}\right)^{\theta}\left[\left(e^{-p_{ij}^{-\beta}}\right)^{\theta}\right]^{m-1} \\ &= m^{mr}\,\beta^{mr}\,\theta^{mr}\prod_{j=1}^{r}\prod_{i=1}^{m} p_{ij}^{-(\beta+1)}\left(e^{-p_{ij}^{-\beta}}\right)^{\theta} \\ &\quad\times \prod_{j=1}^{r}\prod_{i=1}^{\frac{m}{2}}\left[1-\left(e^{-p_{ij}^{-\beta}}\right)^{\theta}\right]^{m-1}\prod_{j=1}^{r}\prod_{i=\frac{m}{2}+1}^{m}\left[\left(e^{-p_{ij}^{-\beta}}\right)^{\theta}\right]^{m-1}. \end{aligned} \tag{16}$$

If the set size m is odd, we select the smallest unit from the $(m-1)/2$ sets and the largest unit from the other $(m-1)/2$ sets, and from the remaining set, either the mean of the largest and smallest unit is chosen or the median of the whole set is chosen for actual measurement. We denote $X_{i(h:m)j}$ by Q_{ij}; then we can assume

$$\begin{aligned} Q = &\{X_{i(1:m)j}\ ;\ i=1,2,\ldots,(m-1)/2,\ j=1,2,\ldots,r\} \\ &\cup\{X_{i(m:m)j}\ ;\ i=(m+1)/2,(m+3)/2,\ldots,(m-1),\ j=1,2,\ldots,r\} \\ &\cup\{X_{i((m+1)/2:m)j}\ ;\ i=m,\ j=1,2,\ldots,r\}. \end{aligned}$$

Then the densities are given by

$$g_1(q_{ij};\theta,\beta) = \frac{m!}{(1-1)!(m-1)!}\, f(q_{ij};\theta,\beta)\left[F(q_{ij};\theta,\beta)\right]^{1-1}\left[1-F(q_{ij};\theta,\beta)\right]^{m-1}, \tag{17}$$

$$g_m(q_{ij};\theta,\beta) = \frac{m!}{(m-1)!(m-m)!}\, f(q_{ij};\theta,\beta)\left[F(q_{ij};\theta,\beta)\right]^{m-1}\left[1-F(q_{ij};\theta,\beta)\right]^{m-m}, \tag{18}$$

$$\begin{aligned} g_{(m+1)/2}(q_{ij};\theta,\beta) &= \frac{m!}{\left(\frac{m-1}{2}\right)!\left(\frac{m-1}{2}\right)!}\, f(q_{ij};\theta,\beta)\left[F(q_{ij};\theta,\beta)\right]^{(m+1)/2-1} \\ &\quad\times\left[1-F(q_{ij};\theta,\beta)\right]^{m-(m+1)/2}. \end{aligned} \tag{19}$$

The likelihood function can be formulated as

$$L_{\text{ERSS}}(\theta,\beta,q) = \prod_{j=1}^{r}\prod_{i=1}^{(m-1)/2} g_1(q_{ij};\theta,\beta)\prod_{j=1}^{r}\prod_{i=(m+1)/2}^{m-1} g_m(q_{ij};\theta,\beta)\prod_{j=1}^{r}\prod_{i=m}^{m} g_{(m+1)/2}(q_{ij};\theta,\beta)$$

$$
\begin{aligned}
&= \left[\frac{m!}{((m-1)/2)!^2}\right]^r m^{r(m-1)}\,\beta^{mr}\,\theta^{mr}\prod_{j=1}^{r}\prod_{i=1}^{m} q_{ij}^{-(\beta+1)}\left(e^{-q_{ij}^{-\beta}}\right)^\theta \\
&\times \prod_{j=1}^{r}\prod_{i=1}^{(m-1)/2}\left[1-\left(e^{-q_{ij}^{-\beta}}\right)^\theta\right]^{m-1} \times \prod_{j=1}^{r}\prod_{i=(m+1)/2}^{m-1}\left[\left(e^{-q_{ij}^{-\beta}}\right)^\theta\right]^{m-1} \\
&\times \prod_{j=1}^{r}\prod_{i=m}^{m}\left\{\left(e^{-q_{ij}^{-\beta}}\right)^\theta\cdot\left[1-\left(e^{-q_{ij}^{-\beta}}\right)^\theta\right]\right\}^{(m-1)/2}. \qquad (20)
\end{aligned}
$$

Since closed-form solutions for equations (16) and (20) are not available, numerical methods are used to estimate $\hat{\theta}$ and $\hat{\beta}$.

2.4. Estimation of parameters through MRSS

This section focuses on generating the maximum likelihood equations for estimating the parameters of the EIWD using the MRSS scheme. We take m random sets of size m units from the population and rank them inside each set. This selection technique can be repeated r times to obtain an MRSS with size $n = mr$. When the set size m is odd, the unit with the median rank is chosen from each group. Suppose

$$U = \{X_{i((m+1)/2:m)j}\ ;\ i = 1,2,\ldots,m;\ j = 1,2,\ldots,r\},$$

and

$$g_{(m+1)/2}(u_{ij};\theta,\beta) = \frac{m!}{\left(\frac{m+1}{2}-1\right)!\left(\frac{m+1}{2}-1\right)!} f(u_{ij};\theta,\beta)\,F(u_{ij};\theta,\beta)\left[1-F(u_{ij};\theta,\beta)\right]^{\frac{m+1}{2}}. \qquad (21)$$

Then,

$$
\begin{aligned}
L_{\text{MRSS}}(\theta,\beta,u) &= \prod_{j=1}^{r}\prod_{i=1}^{m} g_{\frac{m+1}{2}}(u_{ij};\theta,\beta) \\
&= \beta^{mr}\,\theta^{mr}\left[\frac{m!}{\{(\frac{m+1}{2})!\}^2}\right]^{mr}\prod_{j=1}^{r}\prod_{i=1}^{m} u_{ij}^{-(\beta+1)}\left(e^{-u_{ij}^{-\beta}}\right)^\theta \\
&\times\left\{\left(e^{-u_{ij}^{-\beta}}\right)^\theta\left[1-\left(e^{-u_{ij}^{-\beta}}\right)^\theta\right]\right\}^{\frac{m-1}{2}}. \qquad (22)
\end{aligned}
$$

When set size m is even, we select the $(\frac{m}{2})^{\text{th}}$ term from each of the first $\frac{m}{2}$ sets and the $(\frac{m+1}{2})^{\text{th}}$ from each of the last $\frac{m}{2}$ sets. Let

$$
\begin{aligned}
V &= \{X_{i(\frac{m}{2}:m)j}\ ;\ i = 1,2,\ldots,\frac{m}{2},\ j = 1,2,\ldots,r\} \\
&\cup \{X_{i(\frac{m}{2}+1:m)j}\ ;\ i = \frac{m}{2}+1,\ldots,m,\ j = 1,2,\ldots,r\},
\end{aligned}
$$

and

$$g_{\frac{m}{2}}(v_{ij};\theta,\beta)=\frac{m!}{(\frac{m}{2}-1)!\,(\frac{m}{2}+1-1)!}\,f(v_{ij};\theta,\beta)\left[F(v_{ij};\theta,\beta)\right]^{\frac{m}{2}-1}\left[1-F(v_{ij};\theta,\beta)\right]^{\frac{m}{2}}, \tag{23}$$

$$g_{\frac{m}{2}+1}(v_{ij};\theta,\beta)=\frac{m!}{(\frac{m}{2}+1-1)!\,(\frac{m}{2}-1)!}\,f(v_{ij};\theta,\beta)\left[F(v_{ij};\theta,\beta)\right]^{\frac{m}{2}}\left[1-F(v_{ij};\theta,\beta)\right]^{\frac{m}{2}-1}. \tag{24}$$

Then,

$$\begin{aligned} L_{\text{MRSS}}(\theta,\beta,v) &= \prod_{j=1}^{r}\prod_{i=1}^{\frac{m}{2}} g_{\frac{m}{2}}(v_{ij};\theta,\beta)\prod_{j=1}^{r}\prod_{i=1+\frac{m}{2}}^{m} g_{\frac{m}{2}+1}(v_{ij};\theta,\beta) \\ &= \left[\frac{m!}{(\frac{m}{2})!\,(\frac{m}{2}-1)!}\right]^{2mr}\beta^{mr}\,\theta^{mr}\prod_{j=1}^{r}\prod_{i=1}^{m} v_{ij}^{-(\beta+1)}\left(e^{-v_{ij}^{-\beta}}\right)^{\theta} \\ &\times\prod_{j=1}^{r}\prod_{i=1}^{\frac{m}{2}}\left[\left(e^{-v_{ij}^{-\beta}}\right)^{\theta}\right]^{\frac{m}{2}-1}\left[1-\left(e^{-v_{ij}^{-\beta}}\right)^{\theta}\right]^{\frac{m}{2}} \\ &\times\prod_{j=1}^{r}\prod_{i=\frac{m}{2}+1}^{m}\left[\left(e^{-v_{ij}^{-\beta}}\right)^{\theta}\right]^{\frac{m}{2}}\left[1-\left(e^{-v_{ij}^{-\beta}}\right)^{\theta}\right]^{\frac{m}{2}-1}. \end{aligned} \tag{25}$$

The likelihood functions (22) and (25) are used to numerically generate the ML estimates $\hat{\theta}$ and $\hat{\beta}$.

3. Simulation Results

This study carries out a simulation analysis to evaluate the performance of the maximum likelihood estimators for the parameters β and θ of the EIWD using data collected through simple random sampling, ranked set sampling, extreme ranked set sampling, and median ranked set sampling methods. The simulation study is conducted by following these steps:

i. Assume, the marginal distribution is given by $F_X(x)=u_0$, Then simulate from $U(0,1)$, and by the inversion formula we obtain a realization of X as

$$x_0=\left[\frac{-1}{\theta_1}\log(1-u_0)\right]^{-1/\beta_1}.$$

ii. We use the R software to draw SRS samples. Then, use the RSS sampling package in R to draw RSS, ERSS, and MRSS samples of the same size.

iii. The analysis is carried out using a Monte Carlo approach with 10,000 iterations, conducted for $n = 12, 15, 20, 40, 60, 100$ which represent small, moderate or large samples typically encountered in lifetime or reliability data considering various set sizes, cycle numbers and parameter values $\theta = 0.3, 0.64$ and $\beta = 1.4, 2.57$.these parameter values are chosen to ensure broad applicability of the findings.

iv. The resulting estimators are compared based on bias, mean squared error (MSE), and relative efficiency (RE), as defined below:

$$\text{Bias}(\hat{\theta}_i, \theta) = \frac{1}{N}\sum_{i=1}^{N}\left(\hat{\theta}_i - \theta\right),$$

$$\text{MSE}(\hat{\theta}_i, \theta) = \frac{1}{N}\sum_{i=1}^{N}\left(\hat{\theta}_i - \theta\right)^2,$$

$$\text{RE}(\hat{\theta}^{(1)}) = \frac{\text{MSE}(\hat{\theta})}{\text{MSE}(\hat{\theta}^{(1)})}.$$

where, $\hat{\theta}_i$ denotes the estimated parameter for the i^{th} repetition, θ represents the true population parameter, $N = 10000$; $\hat{\theta} = \hat{\theta}_{\text{SRS}}$ and $\hat{\theta}^{(1)}$ is the estimate obtained with a varied sampling method. The accompanying Tables 1 and 2 present the findings. The findings show that ERSS-based estimations of θ and β are more efficient and show fewer biases than those derived from different sampling methods. The biases and MSEs in these sampling techniques diminish as set sizes increase, while REs based on these methods improve with larger sizes of set. Figures 2 - 5 show the MSEs and efficiency of several sampling techniques graphically for $\theta = 0.3$ and $\beta = 1.4$. Figures 2 and 3 indicate that MSE values decrease as sample size n increases. The SRS consistently shows the highest MSE for both parameters β and θ , meaning it is the least efficient. RSS reduces MSE significantly compared to SRS. ERSS almost has the lowest MSE indicating that it gives the most precise estimates of β and θ. MRSS performs better than SRS but not usually as well as ERSS, sometimes between RSS and ERSS. Figures 4 and 5 indicate the relative efficiency compared each method to SRS. It is seen that ERSS has the highest relative efficiency especially at moderate sample sizes. RSS has moderate efficiency gains while MRSS shows improvements but not as consistently strong as ERSS and sometimes close to RSS.

Table 1: Table of estimator characteristics (Bias, MSE, Efficiency) for $\theta = 0.3$, $\beta = 1.4$.

n	$m;r$	Sampling	$\theta = 0.3$			$\beta = 1.4$		
			Bias($\hat{\theta}$)	MSE($\hat{\theta}$)	Eff	Bias($\hat{\beta}$)	MSE($\hat{\beta}$)	Eff
12	4;3	SRS	0.16850	0.02839	–	0.22126	0.04896	–
		RSS	0.11542	0.01700	1.66990	0.18687	0.03492	1.40209
		ERSS	0.01213	0.01263	2.24836	0.11542	0.01332	3.67497
		MRSS	0.08474	0.00718	3.95376	0.17745	0.03149	1.55477
15	5;3	SRS	0.05798	0.00336	–	0.19297	0.03724	–
		RSS	0.04002	0.00160	2.09994	0.14933	0.02230	1.66990
		ERSS	0.00984	0.00096	3.49480	0.02239	0.00841	4.42937
		MRSS	0.04626	0.00214	1.57068	0.14788	0.02187	1.70263
20	5;4	SRS	0.02473	0.00061	–	0.09940	0.00988	–
		RSS	0.01569	0.00025	2.48374	0.06591	0.00434	2.27405
		ERSS	0.00322	0.00011	5.55455	0.04707	0.00222	4.45689
		MRSS	0.01738	0.00030	2.01654	0.07505	0.00563	1.75388
40	10;4	SRS	0.01173	0.00138	–	0.01471	0.00687	–
		RSS	0.02496	0.00062	2.20690	0.04872	0.00237	2.89341
		ERSS	0.02468	0.00061	4.24832	0.03329	0.00111	6.19820
		MRSS	0.02685	0.00072	1.90674	0.06046	0.00366	1.87923
60	10;6	SRS	0.02450	0.00060	–	0.11368	0.00492	–
		RSS	0.02114	0.00024	2.54529	0.03987	0.00159	3.09533
		ERSS	0.00996	0.00023	5.06061	0.03454	0.00119	7.12612
		MRSS	0.01955	0.00030	2.00107	0.03626	0.00132	3.74363
100	10;10	SRS	0.00802	0.00006	–	0.08289	0.00687	–
		RSS	0.00546	0.00002	4.01652	0.00927	0.00146	4.70652
		ERSS	0.00239	0.00007	9.31428	0.02742	0.00075	9.12791
		MRSS	0.00292	0.00001	4.93186	0.03978	0.00158	4.33702

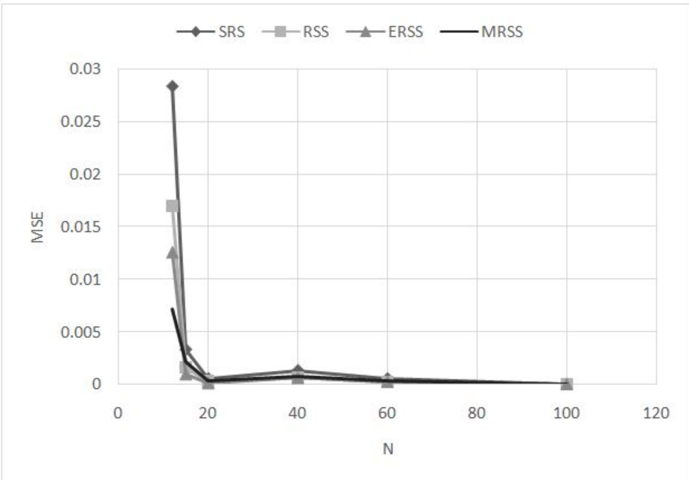

Figure 2: The MSE of θ under different sampling methods.

Table 2: Table of estimator characteristics (Bias, MSE, Efficiency) for $\theta = 0.64$ and $\beta = 2.57$.

n	$m;r$	Sampling	$\theta = 0.64$			$\beta = 2.57$		
			Bias($\hat{\theta}$)	MSE($\hat{\theta}$)	Eff	Bias($\hat{\beta}$)	MSE($\hat{\beta}$)	Eff
12	4;3	SRS	0.09905	0.00981	–	0.16899	0.02856	–
		RSS	0.09894	0.00979	1.00215	0.14235	0.02026	1.40937
		ERSS	0.04162	0.00181	5.66339	0.09339	0.00872	3.27122
		MRSS	0.05404	0.00292	3.35924	0.12026	0.01446	1.97483
15	5;3	SRS	0.02099	0.00044	–	0.15772	0.02487	–
		RSS	0.01977	0.00039	1.12532	0.10438	0.01187	1.58346
		ERSS	0.00845	0.00007	6.19718	0.08148	0.00664	3.74695
		MRSS	0.01065	0.00011	3.89381	0.13345	0.03332	2.14348
20	5;4	SRS	0.02671	0.00071	–	0.12548	0.01575	–
		RSS	0.02487	0.00062	1.15372	0.09588	0.03123	1.98364
		ERSS	0.01067	0.00011	6.25438	0.00666	0.00653	4.14572
		MRSS	0.01513	0.00286	4.01560	0.01672	0.04124	2.61899
40	10;4	SRS	0.01916	0.00010	–	0.08253	0.00106	–
		RSS	0.00888	0.00008	1.32051	0.01867	0.00216	2.04023
		ERSS	0.00191	0.00001	7.02564	0.00042	0.00002	5.38432
		MRSS	0.00254	0.00056	5.45874	0.00398	0.00018	2.78061
60	10;6	SRS	0.01658	0.00027	–	0.03523	0.00049	–
		RSS	0.01304	0.00010	2.61176	0.01782	0.00126	2.55836
		ERSS	0.00626	0.00004	7.92308	0.02223	0.00003	8.12963
		MRSS	0.01237	0.00159	5.80263	0.01343	0.00018	4.74444
100	10;10	SRS	0.00947	0.00009	–	0.01375	0.00019	–
		RSS	0.00414	0.00002	5.23529	0.00512	0.00090	4.78302
		ERSS	0.00275	0.00007	12.71428	0.00407	0.00018	9.81250
		MRSS	0.00628	0.00074	8.28205	0.00412	0.00003	7.26923

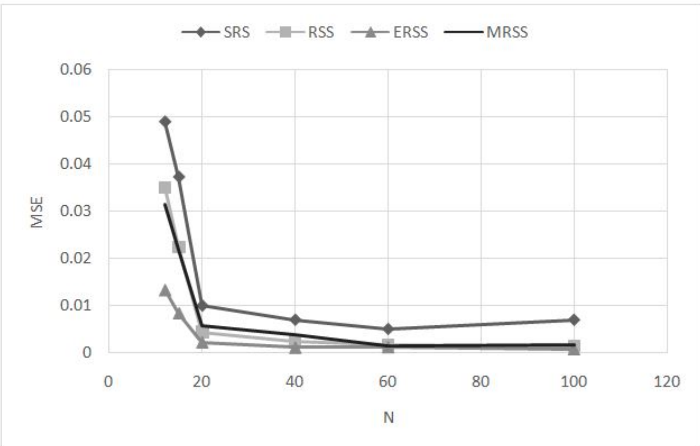

Figure 3: The MSE of β under different sampling methods.

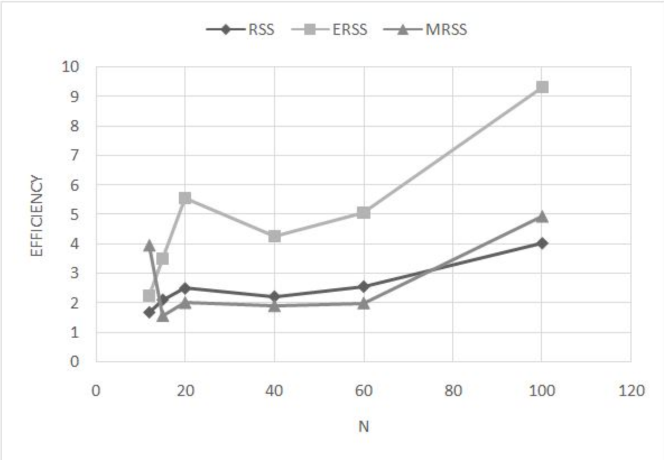

Figure 4: The efficiency of θ under different sampling methods.

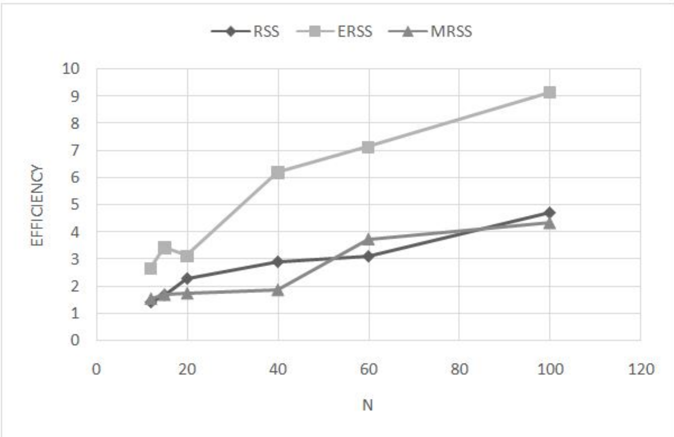

Figure 5: The efficiency of β under different sampling methods.

4. Empirical Data Evaluation: Carbon Fiber Dataset

A real-world data set is employed to demonstrate the effectiveness of the ranked set sampling, median ranked set sampling, and extreme ranked set sampling schemes in reducing the mean squared errors of the estimators compared to the traditional simple random sampling method. The data set, originally reported by [10], consists of tensile strength measurements from 100 carbon fiber samples. Dataset is given as: 3.7, 3.11, 4.42, 3.28, 3.75, 2.96, 3.39, 3.31, 3.15, 2.81, 1.41, 2.76, 3.19, 1.59, 2.17, 3.51, 1.84, 1.61, 1.57, 1.89, 2.74, 3.27, 2.41, 3.09, 2.43, 2.53, 2.81, 3.31, 2.35, 2.77, 2.68, 4.91, 1.57, 2.00, 1.17, 2.17, 0.39, 2.79, 1.08, 2.88, 2.73, 2.87, 3.19, 1.87, 2.95, 2.67, 4.20, 2.85, 2.55, 2.17, 2.97, 3.68, 0.81, 1.22, 5.08, 1.69, 3.68, 4.70, 2.03, 2.82, 2.50, 1.47, 3.22, 3.15, 2.97, 2.93, 3.33, 2.56, 2.59, 2.83, 1.36, 1.84, 5.56, 1.12, 2.48, 1.25, 2.48, 2.03, 1.61, 2.05, 3.60, 3.11, 1.69, 4.90, 3.39, 3.22, 2.55, 3.56, 2.38, 1.92, 0.98, 1.59, 1.73, 1.71, 1.18, 4.38, 0.85, 1.80, 2.12, 3.65. To explore the underlying structure and distributional characteristics of the dataset, a set of nonparametric info graphics was generated (see Figure 6).

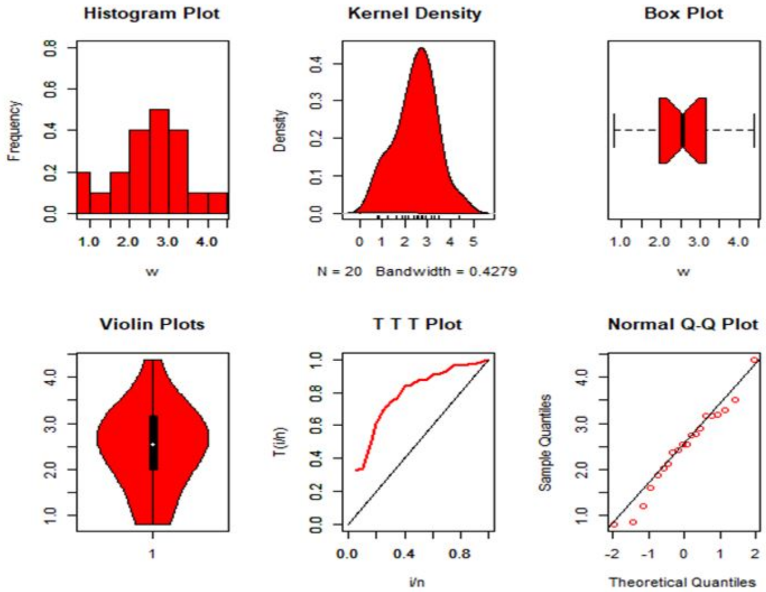

Figure 6: Non-parametric plots for Carbon Fiber dataset.

These plots visually depict the behavior of the data without making assumptions about its distribution. The mean of the dataset is 2.6114, while the median is slightly higher at 2.675, and the mode is 2.17, indicating a nearly symmetric distribution. The standard deviation is 1.0082, denoting a moderate level of dispersion from the mean. Similarly, the variance of 1.0165 corroborates the observed variability in the data. The dataset spans from a minimum value of 0.39 to a maximum of 5.56, yielding a range of 5.17 units. An analysis of the quartiles reveals that the first quartile (Q_1) is 1.84, and the third quartile (Q_3) is 3.2125, resulting in an interquartile range (IQR) of 0.68625. This measure indicates that the central 50% of the data values are fairly concentrated. The skewness of the dataset is 0.3985, suggesting a positively skewed distribution with a modest elongation in the lower tail. The kurtosis value of 0.2492 implies that the distribution has a shape close to normal, characterized by a somewhat flat peak and lighter tails compared to the standard normal distribution. Overall, the data exhibits approximate symmetry, moderate variability, with no strong evidence of extreme outliers or heavy skewness. These features provide a stable basis for further statistical modeling and inference. Additionally, an inspection of the Total Time on Test (TTT) plot indicates an increasing failure rate over time. This trend suggests that the system or process under observation becomes less reliable as time progresses, which is a critical insight for reliability analysis and lifetime modelling.

To assess the suitability of the EIWD for modelling the given dataset, the Kolmogorov–Smirnov (KS) goodness-of-fit test is employed. Based on the maximum likelihood estimation, the estimated values of the parameters θ and β are found to be 3.0855 and 1.7737, respectively, with corresponding standard errors of 0.3274 and 0.1120. The calculated KS test statistic is 0.97917, with an associated p-value of 0.2215. Since the p-value exceeds the conventional significance level (e.g., 0.05), there is no statistical evidence to reject the null hypothesis, indicating that the EIWD provides a satisfactory fit to the observed data. For the purpose of comparing different sampling strategies, a simple random sample of 20 observations is drawn from the original dataset. In the context of ranked set sampling and its modified versions, a set size of $m = 10$ and a cycle count of $r = 2$ are selected, resulting in a total sample size of $n = mr = 20$. These samples are generated using R software, as described below:

Table 3: Samples have been drawn using different sampling techniques.

SRS (size 20)										
2.74	2.88	2.76	2.41	2.55	4.38	2.55	2.38	3.28	2.12	1.61
	1.87	0.85	1.22	2.03	0.81	3.19	3.15	3.15	3.15	3.51
RSS ($m = 10$; $r = 2$)										
Cycle 1	0.81	1.57	2.55	2.67	2.53	3.33	2.05	2.97	3.19	4.42
Cycle 2	0.81	1.57	1.08	2.67	2.50	2.38	2.82	2.79	3.75	5.08
ERSS ($m = 10$; $r = 2$)										
Cycle 1	1.08	0.85	1.18	0.39	0.81	4.91	4.20	4.70	4.90	5.56
Cycle 2	0.39	1.61	1.18	0.81	1.17	4.91	3.68	3.51	5.56	4.90
MRSS ($m = 10$; $r = 2$)										
Cycle 1	3.15	2.03	2.17	2.59	2.55	2.74	1.92	2.81	2.55	2.53
Cycle 2	2.17	2.59	2.03	2.76	2.17	2.67	1.80	3.15	3.31	2.95

The SRS, RSS, MRSS, and ERSS samples are all well suited to the EIWD. Table 4 and Figure 7 show the MLEs of the parameters for each sampling technique, as well as the accompanying standard error, AIC, and BIC values. Because all four sampling strategies produce the same parametric distribution EIWD, comparing their AIC and BIC values is useful. Based on the AIC and BIC criterion, the ERSS sample is the best fit for the EIWD.

Table 4: Summary of parameter estimates under different sampling techniques.

Sampling	$(\hat{\theta}, \hat{\beta})$	Standard Errors	AIC	BIC
ERSS	$(1.30299, 1.13840)$	$(0.29484, 0.19149)$	157.7079	160.2161
RSS	$(2.90086, 1.83696)$	$(0.69678, 0.29168)$	162.7282	165.2470
MRSS	$(3.16454, 6.07463)$	$(0.76085, 0.29881)$	165.5638	166.4515
SRS	$(3.14725, 1.96397)$	$(0.76592, 0.30611)$	168.7288	171.6841

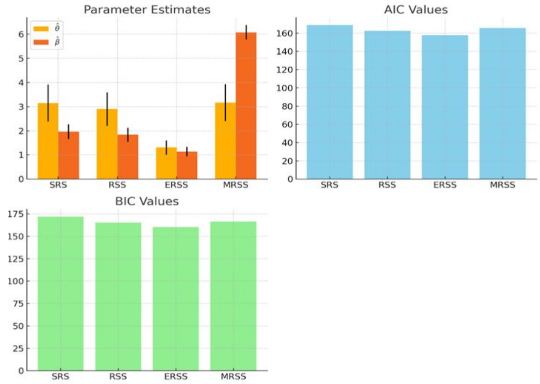

Figure 7: The goodness-of-fit analysis for Carbon Fiber dataset.

5. Conclusions

Finding the unknown EIWD parameters inside the RSS framework, especially using its two modified variants (ERSS and MRSS), was the goal of this study, which aimed to apply maximum likelihood estimation. The non-linearity of the likelihood equations hindered closed-form solutions, so extensive Monte Carlo simulations were used to test the ML estimators of the EIWD's shape and scale parameters. The investigation focused on important performance indicators such as relative efficiency, bias, and mean squared error. Estimators derived from RSS, ERSS, and MRSS routinely produced less bias and MSE than those derived from SRS, according to the results. Out of all the methods, ERSS provided the best accurate estimates with the least amount of bias and MSE, making it the most efficient. These findings demonstrated that ERSS and similar structured sampling strategies are capable of effectively managing EIWD distributions. A genuine dataset of carbon fiber materials' tensile strength was used to further validate the proposed estimation methods. Compared to previous sampling strategies, the ERSS method produced lower AIC and BIC values, which revealed that it increased performance and indicated that the model fit was superior. Potential new lines of inquiry have been uncovered by this investigation. First, other estimating techniques, such the method of moments or Bayesian methods, could be investigated using the same sampling strategies. Secondly, different distributions of lifetime or dependability that have comparable structural properties might be incorporated into the framework. Lastly, future research might investigate how well these estimators function under more complicated circumstances in high-dimensional or multivariate environments, especially when handling censored data.

- **Data Availability Statement:** The data sets are available on paper.
- **Conflicts of Interest:** The authors declare no conflict of interest.

References

[1] Y. Y. Abdelall, A. Hassan, and E. Almetwally. A new extension of the odd inverse weibull-g family of distributions: Bayesian and non-bayesian estimation with engineering applications. *Computational Journal of Mathematical and Statistical Sciences*, 3(2):359–388, 2024.

[2] S. Alkarni, A. Z. Afify, I. Elbatal, and M. Elgarhy. The extended inverse weibull distribution: Properties and applications. *Complexity*, 2020:Article ID 3297693, 2020.

[3] N. Alsadat, A. S. Hassan, M. Elgarhy, C. Chesneau, and R. E. Mohamed. An efficient stress–strength reliability estimate of the unit gompertz distribution using ranked set sampling. *Symmetry*, 15(5):1121, 2023.

[4] M. El-Morshedy, M. S. Eliwa, A. El-Gohary, and A. A. Khalil. Bivariate exponentiated discrete weibull distribution: Statistical properties, estimation, simulation, and applications. *Mathematical Sciences*, 14(1):29–42, 2020.

[5] M. El-Morshedy, M. S. Eliwa, and A. Tyagi. A discrete analogue of odd weibull-g family of distributions: Properties, classical and bayesian estimation with applications to count data. *Journal of Applied Statistics*, 49(11):2928–2952, 2021.

[6] M. S. Eliwa and M. El-Morshedy. Bivariate odd weibull-g family of distributions: Properties, bayesian and non-bayesian estimation with bootstrap confidence intervals and application. *Journal of Taibah University Science*, 14(3):331–345, 2020.

[7] M. Ç. Korkmaz, E. Altun, M. Alizadeh, and M. El-Morshedy. The log exponential-power distribution: Properties, estimations and quantile regression model. *Mathematics*, 9(21):2634, 2021.

[8] F. Willayat, N. Saud, M. Ijaz, A. Silvianita, and M. El-Morshedy. Marshall–olkin extended gumbel type-ii distribution: Properties and applications. *Complexity*, 2022:Article ID 2219570, 2022.

[9] S. M. Zaidi, M. M. Al Sobhi, M. El-Morshedy, and A. Z. Afify. A new generalized family of distributions: Properties and applications. *AIMS Mathematics*, 6(1):456–476, 2021.

[10] A. Flaih, H. Elsalloukh, E. Mendi, and M. Milanova. The exponentiated inverted weibull distribution. *Applied Mathematics and Information Sciences*, 6(2):167–171, 2012.

[11] A. Aljuaid. Estimating the parameters of an exponentiated inverted weibull distribution under type-ii censoring. *Applied Mathematical Sciences*, 7(35):1721–1736, 2013.

[12] I. Elbatal and H. Z. Muhammad. Exponentiated generalized inverse weibull distribution. *Applied Mathematical Sciences*, 8(81):3997–4012, 2014.

[13] A. S. Hassan, A. A. Marwa, H. Zaher, and E. A. Elsherpieny. Comparison of estimators for exponentiated inverted weibull distribution based on grouped data. *International Journal of Engineering Research and Applications*, 4(1):77–90, 2014.

[14] A. Saghir, S. Tazeem, and I. Ahmad. The weighted exponentiated inverted weibull distribution: Properties and application. *Journal of Informatics and Mathematical Sciences*, 9(1):137–151, 2017.

[15] J. Saran, N. Pushkarna, and K. Verma. On the record values and its predictions

from exponentiated inverted weibull distribution and associated inference. *Journal of Statistics and Management Systems*, 25(2):347–368, 2021.

[16] U. Deka, B. Das, and D. Deka. Concomitants of order statistics for bivariate exponentiated inverted weibull distribution. *Journal of Mathematical and Computational Science*, 11(5):6444–6467, 2021.

[17] G. McIntyre. A method for unbiased selective sampling, using ranked sets. *Australian Journal of Agricultural Research*, 3(4):385–390, 1952.

[18] S. L. Stokes. Ranked set sampling with concomitant variables. *Communications in Statistics – Theory and Methods*, 6(12):1207–1211, 1977.

[19] H. M. Aljohani, E. M. Almetwally, A. S. Alghamdi, and E. H. Hafez. Ranked set sampling with application of modified kies exponential distribution. *Alexandria Engineering Journal*, 60(4):4041–4046, 2021.

[20] Hassan, D. S. Metwally, H. E. Semary, S. A. Benchiha, A. M. Gemeay, and M. Elgarhy. Improved estimation based on ranked set sampling for the chris-jerry distribution with application to engineering data. *Computational Journal of Mathematical and Statistical Sciences*, 2025. In press.

[21] H. M. Samawi, M. S. Ahmed, and W. A. Abu-Dayyeh. Estimating the population mean using extreme ranked set sampling. *Biometrical Journal*, 38(5):577–586, 1996.

[22] H. A. Muttlak. Median ranked set sampling. *Journal of Applied Statistical Science*, 6:245–255, 1997.

[23] S. Dey, M. Salehi, and J. Ahmadi. Rayleigh distribution revisited via ranked set sampling. *Metron*, 75(1):69–85, 2017.

[24] M. Esemen and S. Gurler. Parameter estimation of generalized rayleigh distribution based on ranked set sample. *Journal of Statistical Computation and Simulation*, 88(4):615–628, 2017.

[25] R. A. Bantan, M. Elsehetry, A. S. Hassan, M. Elgarhy, D. Sharma, F. Jamal, and Ch. Chesneau. A two-parameter model: Properties and estimation under ranked sampling. *Mathematics*, 9:1–16, 2021.

[26] A. S. Hassan, N. Alsadat, M. Elgarhy, C. Chesneau, and R. E. Mohamed. Different classical estimation methods using ranked set sampling and data analysis for the inverse power cauchy distribution. *Journal of Radiation Research and Applied Sciences*, 16(4):100685, 2023.