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Certain Properties and Characterizations of Δ_h-Truncated Exponential Based Hermite Polynomials

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Abstract. This article introduces a novel class of Δ_h-truncated exponential-based Hermite polynomials and examine their fundamental properties and structural identities. We derive generating functions, recurrence relations, and explicit formulas, along with summation identities. The study further uncovers connections with the monomiality principle, offering insights into their underlying algebraic framework. In addition, an operational formalism is developed, and symmetric identities are established to enhance the theoretical foundation of these polynomials.

2020 Mathematics Subject Classifications: 33C45, 33E20, 33B10, 33E30, 11T23.

Key Words and Phrases: Monomiality principle, Δ_h-truncated exponential Hermite Appell polynomials, explicit forms, Symmetry identities.

1. Introduction and preliminaries

Polynomial families form a foundational pillar in applied mathematics, given their multifaceted characterizations—ranging from orthogonality and generating functions to differential expressions, operational techniques, integral representations, and recurrence relations. Due to their versatile nature and diverse applications, their generalizations

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and extensions continue to garner significant attention across mathematical and physical sciences. These formulations are vital not only in providing series expansions for transcendental functions in mathematical physics but also in shaping computational and analytical techniques. Notable studies in this domain include extensive developments in p-adic analysis, q-analysis, and umbral calculus (see, for example, [1–5]).

Among these developments, two-variable special polynomials have emerged as highly potent tools, facilitating the derivation of efficient and elegant identities and offering pathways to novel classes of special polynomials. The inception of two-variable Appell polynomials by Bretti *et al.* [1] via iterated isomorphism marked a pivotal moment, followed by the construction and study of two-variable truncated exponential, Hermite, Legendre, and Laguerre polynomials in works such as [6–13].

Despite their relevance in areas such as quantum mechanics and optics, the TEP (TEP) remains relatively underexplored. Initially defined by Andrews [14], these polynomials are given by:

$$e_n(\xi_1) = \sum_{k=0}^{n} \frac{\xi_1^k}{k!}, \tag{1}$$

with the limiting behavior

$$\lim_{n\to\infty} e_n(\xi_1) = e^{\xi_1}.$$

A detailed investigation of their properties was later initiated by Dattoli *et al.* [8].

A key identity for the TEP follows from their integral representation:

$$e_n(\xi_1) = \frac{1}{n!} \int_0^\infty e^{-\xi} (\xi_1 + \xi)^n \, \mathrm{d}\xi, \tag{2}$$

derived using the classical gamma integral:

$$n! = \int_0^\infty e^{-\xi} \, \xi^n \, \mathrm{d}\xi. \tag{3}$$

The ordinary generating function of $e_n(\mu_1)$ is expressed as [8]:

$$\sum_{n=0}^{\infty} e_n(\xi_1) t^n = \frac{e^{\xi_1 t}}{1-t} \quad (t \in \mathbb{C}, \ |t| < 1) \, . \tag{4}$$

A significant extension of the TEP to two variables was established by Dattoli *et al.* [8], where they proved particularly useful in problems involving integrals of special functions and physical models.

The generating function for the two-variable version is:

$$\sum_{n=0}^{\infty} {}_{[2]}e_n(\xi_1, \mu_2) t^n = \frac{e^{\xi_1 t}}{1 - \xi_2 t^2}, \tag{5}$$

and the corresponding explicit representation reads:

$${}_{[2]}e_n(\xi_1, \xi_2) = \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{\xi_2^k \, \xi_1^{n-2k}}{(n-2k)!}. \tag{6}$$

The generalized form involving higher-order polynomials is given by:

$$\sum_{n=0}^{\infty} {}_{[s]}e_n(\xi_1, \xi_2)t^n = \frac{e^{\xi_1 t}}{1 - \xi_2 t^s}, \tag{7}$$

with explicit form:

$${}_{[s]}e_n(\xi_1, \xi_2) = \sum_{k=0}^{\left[\frac{n}{s}\right]} \frac{\xi_2^k \ \xi_1^{n-sk}}{(n - sk)!}. \tag{8}$$

It is easy to verify from expressions (4), (5), and (7) that:

$${}_{[2]}e_n(\xi_1, \xi_2) = e_n^{(2)}(\xi_1, \xi_2), \quad e_n(\xi_1) = e_n^{(1)}(\xi_1, 1).$$

A direct consequence of this formalism links the Chebyshev polynomials of the second kind $U_n(\mu_2)$ to the TEP:

$$U_n(\xi_2) = {}_{[2]}e_n(0, \xi_2), \tag{9}$$

whose generating function is well-known [14]:

$$\sum_{n=0}^{\infty} U_n(\xi_1)t^n = \frac{1}{1 - 2\xi_1 t + t^2}, \quad (|t| < 1, \ \xi_1 \leq 1)\,. \tag{10}$$

In the operational calculus framework, the multiplicative and derivative operators for the TEP are identified as:

$$\widehat{\mathcal{M}}_{e^{(s)}} = \xi_1 + s\xi_2 D_{\xi_2} \xi_2 D_{\xi_1}^{s-1}, \tag{11}$$

$$\widehat{\mathcal{P}}_{e^{(s)}} = D_{\xi_2}, \tag{12}$$

signifying that ${}_{[s]}e_n(\xi_1, \xi_2)$ form a quasi-monomial sequence [2].

This formalism has been further expanded by composing TEP with Appell-type structures. Khan [15] introduced the truncated exponential-based Appell polynomials through:

$$\sum_{n=0}^{\infty} {}_{[s]}e_n(\xi_1, \xi_2)t^n = \mathcal{A}(t)\frac{e^{\xi_1 t}}{1 - \xi_2 t^s}, \tag{13}$$

where $\mathcal{A}(t)$ denotes the Appell-type generating function.

The origin of the monomiality principle dates back to Steffenson's poweroid method in 1941 [16], later refined by Dattoli [7]. A polynomial sequence $\{q_n(\xi_1)\}$ is quasi-monomial if:

$$q_{n+1}(\xi_1) = \widehat{\mathcal{M}}\{q_n(\xi_1)\}, \tag{14}$$

$$n \ q_{n-1}(\xi_1) = \widehat{\mathcal{P}}\{q_n(\xi_1)\}, \tag{15}$$

and the operators satisfy the Weyl algebra:

$$[\widehat{\mathcal{P}}, \widehat{\mathcal{M}}] = \widehat{1}. \tag{16}$$

The quasi-monomial property yields key operational identities:

$$\widehat{\mathcal{M}}\widehat{\mathcal{P}}\{q_n(\xi_1)\} = n\ q_n(\xi_1), \tag{17}$$

$$q_n(\xi_1) = \widehat{\mathcal{M}}^n\{1\}, \tag{18}$$

$$e^{t\widehat{\mathcal{M}}}\{1\} = \sum_{n=0}^{\infty} q_n(\xi_1)\frac{t^n}{n!}, \tag{19}$$

as outlined in [2, 7, 8, 17–19].

In recent years, the introduction of Δ_h-type generalizations has expanded the horizon of special polynomial theory. These constructions utilize the forward difference operator:

$$\Delta_h[g](\xi_1) = g(\xi_1 + h) - g(\xi_1), \tag{20}$$

and its higher-order form:

$$\Delta_h^i[g](\xi_1) = \sum_{l=0}^{i}(-1)^{i-l}\binom{i}{l}g(\xi_1 + lh), \tag{21}$$

where $\Delta_h^0 = I$ (identity), $\Delta_h^1 = \Delta_h$.

The Δ_h-Appell polynomials $\mathbb{A}_n^{[h]}(\xi_1)$ are introduced via the generating function [20]:

$$\mathcal{A}(h;t)(1 + ht)^{\frac{\xi_1}{h}} = \sum_{n=0}^{\infty} \mathcal{A}_n^{[h]}(\xi_1)\frac{t^n}{n!}, \tag{22}$$

with the condition:

$$\mathcal{A}(h;t) = \sum_{n=0}^{\infty} \mathcal{A}_{n,h}\frac{t^n}{n!}, \quad \mathcal{A}_{0,h} \neq 0. \tag{23}$$

The Stirling numbers of the first kind, $\mathcal{S}_1(n, m)$, are crucial in expressing rising factorials, as given by the relation:

$$(\xi_1)_n = \sum_{m=0}^{n} \mathcal{S}_1(n, m)\mu_1^m, \tag{24}$$

where $(\xi_1)_0 = 1$ and $(\xi_1)_n = \xi(\xi_1 - 1)\cdots(\xi_1 - n + 1)$.

The Stirling numbers $\mathcal{S}_1(n, m)$ can be represented by the following generating function [21–23]:

$$\frac{1}{m!}(\log(1 + t))^m = \sum_{n=m}^{\infty} \mathcal{S}_1(n, m)\frac{t^n}{n!}, \quad (m \geq 0). \tag{25}$$

For $n \geq 0$, the Δ_h Stirling numbers of the first kind are defined by:

$$\frac{1}{k!}(\log_h(1 + t))^k = \sum_{n=k}^{\infty} \mathcal{S}_{1,h}(n, k)\frac{t^n}{n!}, \quad (k \geq 0). \tag{26}$$

It is important to note that $\lim_{h\to 0}\mathcal{S}_{1,h}(n,k)=\mathcal{S}_1(n,k)$.

The degenerate Hermite polynomials are defined by [24]

$$(1+ht)^{\frac{\xi_1}{h}}(1+ht^2)^{\frac{\xi_2}{h}}=\sum_{n=0}^{\infty}\mathcal{H}_n^{[h]}(\xi_1,\xi_2)\frac{t^n}{n!}. \tag{27}$$

Note that

$$\lim_{h\to 0}\mathcal{H}_n^{[h]}(\xi_1,\xi_2)=\mathcal{H}_n(\xi_1,\xi_2),$$

where $\mathcal{H}_n(\xi_1,\xi_2)$ is called the 2-variable Hermite polynomials (see [7]).

A generalized falling factorial sum $\sigma_k(n;h)$ can be defined by the generating function [24–26]:

$$\frac{(1+ht)^{\frac{(n+1)}{h}}-1}{(1+ht)^{\frac{1}{h}}-1}=\sum_{k=0}^{\infty}\sigma_k(n;h)\frac{t^k}{k!}. \tag{28}$$

Note that $\lim_{h\to 0}\sigma_k(n;h)=S_k(n)$.

This article is organized as follows. In Section 2, we introduce the novel class of Δ_h-truncated exponential-based Hermite polynomials and derive their generating functions, recurrence relations, and explicit formulas. Section 3 is devoted to the derivation of summation identities related to these polynomials. In Section 4, we explore their connection with the monomiality principle and develop an operational formalism that highlights their algebraic structure. Section 5 presents symmetric identities to further enrich the theoretical framework. Finally, concluding remarks are provided.

2. Δ_h-Truncated exponential-based Hermite polynomials

This section introduces a new family of three-variable Δ_h-truncated exponential-based Hermite polynomials and examines their foundational properties. It significantly enhances the current understanding of polynomial theory and suggests promising directions for further study. The derivation of the generating function for ${}_{e^{(r)}}\mathcal{H}_n^{[h]}(\xi_1,\xi_2,\xi_3)$ plays a pivotal role in revealing the structure and analytical behavior of these polynomials. This construction not only aids in uncovering key identities and recurrence relations but also enriches their connection to broader mathematical frameworks. To initiate this, we derive the generating function for ${}_{e^{(r)}}\mathcal{H}_n^{[h]}(\xi_1,\xi_2,\xi_3)$ by establishing the following result:

Theorem 1. *The generating function associated with the three-variable Δ_h-truncated exponential-based Hermite polynomials ${}_{e^{(r)}}\mathcal{H}_n^{[h]}(\xi_1,\xi_2,\xi_3)$ is expressed as follows:*

$$\frac{1}{1-\xi_3 t^r}(1+ht)^{\frac{\xi_1}{h}}(1+ht^2)^{\frac{\xi_2}{h}}=\sum_{n=0}^{\infty}{}_{e^{(r)}}\mathcal{H}_n^{[h]}(\xi_1,\xi_2,\xi_3)\frac{t^n}{n!}. \tag{29}$$

Proof. Consider the expansion of $\frac{1}{1-\xi_3 t^r}(1+ht)^{\frac{\xi_1}{h}}(1+ht^2)^{\frac{\xi_2}{h}}$ around $\xi_1 = \xi_2 = \xi_3 = 0$ using a Newton series for finite differences. By analyzing the product expansion of the functions $(1+ht)^{\frac{\xi_1}{h}}$ and $(1+ht^2)^{\frac{\xi_2}{h}}$ with respect to powers of t, we identify the coefficients of $\frac{t^n}{n!}$ as the polynomials ${}_{e^{(r)}}\mathcal{H}_n^{[h]}(\xi_1, \xi_2, \xi_3)$, which are defined in equation (29). This confirms the generating function for the three-variable Δ_h-truncated exponential-based Hermite polynomials ${}_{e^{(r)}}\mathcal{H}_n^{[h]}(\xi_1, \xi_2, \xi_3)$.

Theorem 2. *For the three-variable Δ_h-Truncated exponential-based Hermite polynomials ${}_{e^{(r)}}\mathcal{H}_n^{[h]}(\xi_1, \xi_2, \xi_3)$, the following relations hold:*

$$\begin{aligned} \frac{{}_{\xi_1}\Delta_h}{h}\, {}_{e^{(r)}}\mathcal{H}_n^{[h]}(\xi_1, \xi_2, \xi_3) &= n\ {}_{e^{(r)}}\mathcal{H}_{n-1}^{[h]}(\xi_1, \xi_2, \xi_3) \\ \frac{{}_{\xi_2}\Delta_h}{h}\, {}_{e^{(r)}}\mathcal{H}_n^{[h]}(\xi_1, \xi_2, \xi_3) &= n(n-1)\ {}_{e^{(r)}}\mathcal{H}_{n-2}^{[h]}(\xi_1, \xi_2, \xi_3). \end{aligned} \tag{30}$$

Proof. Differentiating equation (29) with respect to ξ_1 and applying expression (20), we get

$$\begin{aligned} {}_{\xi_1}\Delta_h \frac{1}{1-\xi_3 t^r}(1+ht)^{\frac{\xi_1}{h}}(1+ht^2)^{\frac{\xi_2}{h}} &= \frac{1}{1-\xi_3 t^r}(1+ht)^{\frac{\xi_1+1}{h}}(1+ht^2)^{\frac{\xi_2}{h}} - \frac{1}{1-\xi_3 t^r}(1+ht)^{\frac{\xi_1}{h}}(1+ht^2)^{\frac{\xi_2}{h}} \\ &= (1+ht-1)\frac{1}{1-\xi_3 t^r}(1+ht)^{\frac{\xi_1}{h}}(1+ht^2)^{\frac{\xi_2}{h}} \\ &= ht\ \frac{1}{1-\xi_3 t^r}(1+ht)^{\frac{\xi_1}{h}}(1+ht^2)^{\frac{\xi_2}{h}}. \end{aligned} \tag{31}$$

Substituting the right-hand side of equation (29) into (31), we obtain

$${}_{\xi_1}\Delta_h \sum_{n=0}^{\infty} {}_{e^{(r)}}\mathcal{H}_n^{[h]}(\xi_1, \xi_2, \xi_3)\frac{t^n}{n!} = h\sum_{n=0}^{\infty} {}_{e^{(r)}}\mathcal{H}_n^{[h]}(\xi_1, \xi_2, \xi_3)\frac{t^{n+1}}{n!}. \tag{32}$$

Now, replacing $n \to n-1$ on the right-hand side of the above expression and comparing the coefficients of like powers of t, we deduce the identities in (30).

We now derive an explicit expression satisfied by the three-variable Δ_h-Truncated exponential-based Hermite polynomials ${}_{e^{(r)}}\mathcal{H}_n^{[h]}(\xi_1, \xi_2, \xi_3)$ as follows:

Theorem 3. *For the three-variable Δ_h-Truncated exponential-based Hermite polynomials ${}_{e^{(r)}}\mathcal{H}_n^{[h]}(\xi_1, \xi_2, \xi_3)$, the following expression holds:*

$$ {}_{e^{(r)}}\mathcal{H}_n^{[h]}(\xi_1,\xi_2,\xi_3) = \sum_{d=0}^{[\frac{\xi_1}{h}]} \binom{n}{d} \binom{\frac{\xi_1}{h}}{d} \ h^d \ {}_{e^{(r)}}\mathcal{H}_{n-d}^{[h]}(0,\xi_2,\xi_3). \tag{33} $$

Proof. Consider the expansion of the generating function (29):

$$ \frac{1}{1-\xi_3 t^r}(1+ht)^{\frac{\xi_1}{h}}(1+ht^2)^{\frac{\xi_2}{h}} \{1\} = \sum_{d=0}^{[\frac{\xi_1}{h}]} \binom{\frac{\xi_1}{h}}{d} \frac{(ht)^d}{d!} \sum_{\phi=0}^{\infty} {}_{e^{(r)}}\mathcal{H}_n^{[h]}(0,\xi_2,\xi_3)\frac{t^n}{n!} \tag{34} $$

This leads to the equivalent form:

$$ \sum_{\phi=0}^{\infty} {}_{e^{(r)}}\mathcal{H}_n^{[h]}(\xi_1,\xi_2,\xi_3)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\sum_{d=0}^{[\frac{\xi_1}{h}]} \binom{\frac{\xi_1}{h}}{d} h^d \ {}_{e^{(r)}}\mathcal{H}_n^{[h]}(0,\xi_2,\xi_3)\frac{t^{n+d}}{n! \ d!}. \tag{35} $$

Now, replacing $n \to n-d$ on the right-hand side of the above expression, we get:

$$ \sum_{n=0}^{\infty} {}_{e^{(r)}}\mathcal{H}_n^{[h]}(\xi_1,\xi_2,\xi_3)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\sum_{d=0}^{[\frac{\xi_1}{h}]} \binom{\frac{\xi_1}{h}}{d} h^d \ {}_{e^{(r)}}\mathcal{H}_{n-d}^{[h]}(0,\xi_2,\xi_3)\frac{t^{n}}{(n-d)! \ d!}. \tag{36} $$

Multiplying and dividing the right-hand side of equation (36) by $n!$ and then comparing coefficients of like powers of t on both sides yields the required identity in (33).

Theorem 4. *For the three-variable Δ_h-Truncated exponential-based Hermite polynomials ${}_{e^{(r)}}\mathcal{H}_n^{[h]}(\xi_1,\xi_2,\xi_3)$, the following series representations hold:*

$$ {}_{e^{(r)}}\mathcal{H}_n^{[h]}(\xi_1,\xi_2,\xi_3) = n! \sum_{k=0}^{[\frac{n}{2}]} \frac{(-\frac{\xi_2}{h})_k(-h)^k e_{n-2k}^{(r)}(\xi_1,\xi_3;h)}{k!(n-2k)!}, \tag{37} $$

and

$$ {}_{e^{(r)}}\mathcal{H}_n^{[h]}(\xi_1,\xi_2,\xi_3) = n! \sum_{k=0}^{[\frac{n}{r}]} \frac{\xi_3^k \mathcal{H}_{n-rk}^{[h]}(\xi_1,\xi_2)}{(n-rk)!}. \tag{38} $$

Proof. By utilizing equations (7), (27), and (29), the results in (37) and (38) follow directly.

3. Summation formulae

In this section, we present concise summation formulae for special two-variable polynomials. These expressions offer efficient methods to compute sums and explore structural properties, aiding in analysis across combinatorics, probability, and mathematical physics. We now establish the following results.

Theorem 5. *Let* ${}_{e^{(r)}}\mathcal{H}_n^{[h]}(\xi_1,\xi_2,\xi_3)$ *denote the* Δ_h*-Truncated exponential-based Hermite polynomials of order* r*. Then, the following identity holds:*

$$ {}_{e^{(r)}}\mathcal{H}_n^{[h]}(\xi_1+1,\xi_2,\xi_3)=\sum_{k=0}^{n}\binom{n}{k}{}_{e^{(r)}}\mathcal{H}_{n-k}^{[h]}(\xi_1,\xi_2,\xi_3)(-\frac{1}{h})_k(-h)^k. \tag{39}$$

Proof. Utilizing the generating function (29), we proceed as follows:

$$\sum_{n=0}^{\infty}{}_{e^{(r)}}\mathcal{H}_n^{[h]}(\xi_1+1,\xi_2,\xi_3)\frac{t^n}{n!}-\sum_{n=0}^{\infty}{}_{e^{(r)}}\mathcal{H}_n^{[h]}(\xi_1,\xi_2,\xi_3)\frac{t^n}{n!}=\frac{1}{1-\xi_3t^r}(1+ht)^{\frac{\xi_1}{h}}(1+ht^2)^{\frac{\xi_2}{h}}\left((1+ht)^{\frac{1}{h}}-1\right)$$

$$=\sum_{n=0}^{\infty}{}_{e^{(r)}}\mathcal{H}_n^{[h]}(\xi_1,\xi_2,\xi_3)\frac{t^n}{n!}\left(\sum_{k=0}^{\infty}(-\frac{1}{h})_k(-h)^k\frac{t^k}{k!}-1\right)$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}{}_{e^{(r)}}\mathcal{H}_{n-k}^{[h]}(\xi_1,\xi_2,\xi_3)(-\frac{1}{h})_k(-h)^k\frac{t^n}{n!}-\sum_{n=0}^{\infty}{}_{e^{(r)}}\mathcal{H}_n^{[h]}(\xi_1,\xi_2,\xi_3)\frac{t^n}{n!} \tag{40}$$

By equating the coefficients of powers of t on both sides, the identity (39) is established.

Next, we derive the explicit expressions satisfied by the bivariate Δ_h-Truncated exponential-based Hermite polynomials ${}_{e^{(r)}}\mathcal{H}_n^{[h]}(\xi_1,\xi_2,\xi_3)$ of order r, through the following result:

Theorem 6. *For the three-variable* Δ_h*-Truncated exponential-based Hermite polynomials* ${}_{e^{(r)}}\mathcal{H}_n^{[h]}(\xi_1,\xi_2,\xi_3)$ *of order* r*, the following structural identities are valid:*

$$ {}_{e^{(r)}}\mathcal{H}_n^{[h]}(\xi_1,\xi_2,\xi_3)=\sum_{k=0}^{n}\binom{n}{k}{}_{e^{(r)}}\mathcal{H}_{n-k}^{[h]}(0,\xi_2,\xi_3)\left(-\frac{\xi_1}{h}\right)_k(-h)^k, \tag{41}$$

$$ {}_{e^{(r)}}\mathcal{H}_n^{[h]}(\xi_1,\xi_2,\xi_3)=\sum_{k=0}^{n}\binom{n}{k}{}_{e^{(r)}}\mathcal{H}_{n-k}^{[h]}(\xi_1-p,\xi_2,\xi_3)\left(-\frac{p}{h}\right)_k(-h)^k, \tag{42}$$

$$ {}_{e^{(r)}}\mathcal{H}_n^{[h]}(\xi_1+s,\xi_2,\xi_3)=\sum_{k=0}^{n}\binom{n}{k}{}_{e^{(r)}}\mathcal{H}_{n-k}^{[h]}(\xi_1,\xi_2,\xi_3)\left(-\frac{s}{h}\right)_k(-h)^k, \tag{43}$$

$$ {}_{e^{(r)}}\mathcal{H}_n^{[h]}(\xi_1,\xi_2,\xi_3)=\sum_{l=0}^{n}\sum_{k=0}^{l}\binom{n}{l}{}_{e^{(r)}}\mathcal{H}_{n-l}^{[h]}(0,\xi_2,\xi_3)\left(\frac{\xi_1}{h}\right)^k h^l S_1(l,k), \tag{44}$$

$$ {}_{e^{(r)}}\mathcal{H}_n^{[h]}(\xi_1,\xi_2,\xi_3)=\sum_{l=0}^{n}\sum_{k=0}^{l}\binom{n}{l}{}_{e^{(r)}}\mathcal{H}_{n-l}^{[h]}(0,\xi_2,\xi_3)\left(\frac{\xi_1}{h}\right)_{k,h} h^l S_{1,h}(l,k). \tag{45}$$

Proof. We expand the generating function (29) in the following way:

$$\left(\sum_{n=0}^{\infty} {}_{e^{(r)}}\mathcal{H}_n^{[h]}(0,\xi_2,\xi_3)\frac{t^n}{n!}\right)\left(\sum_{k=0}^{\infty}\left(-\frac{\xi_1}{h}\right)_k (-h)^k\frac{t^k}{k!}\right)=\sum_{n=0}^{\infty} {}_{e^{(r)}}\mathcal{H}_n^{[h]}(\xi_1,\xi_2,\xi_3)\frac{t^n}{n!},$$

Applying the Cauchy product on the left-hand side and comparing like powers of t, equation (39) is validated. In a similar fashion, identities (40) and (41) follow directly.

Furthermore, from equation (29), we have:

$$\sum_{n=0}^{\infty} {}_{e^{(r)}}\mathcal{H}_n^{[h]}(\xi_1,\xi_2,\xi_3)\frac{t^n}{n!}=\frac{1}{1-\xi_3 t^r}e^{\log((1+ht)^{\frac{\xi_1}{h}})}(1+ht^2)^{\frac{\xi_2}{h}}. \tag{46}$$

Employing equation (1), the expression becomes:

$$\sum_{n=0}^{\infty} {}_{e^{(r)}}\mathcal{H}_n^{[h]}(\xi_1,\xi_2,\xi_3)\frac{t^n}{n!}=\left(\sum_{n=0}^{\infty} {}_{e^{(r)}}\mathcal{H}_n^{[h]}(0,\xi_2,\xi_3)\frac{t^n}{n!}\right)\left(\sum_{k=0}^{\infty}\left(\frac{\xi_1}{h}\right)^k\frac{\log(1+ht)^k}{k!}\right), \tag{47}$$

Substituting equation (25) leads to:

$$\sum_{n=0}^{\infty} {}_{e^{(r)}}\mathcal{H}_n^{[h]}(\xi_1,\xi_2,\xi_3)\frac{t^n}{n!}=\left(\sum_{n=0}^{\infty} {}_{e^{(r)}}\mathcal{H}_n^{[h]}(0,\xi_2,\xi_3)\frac{t^n}{n!}\right)\left(\sum_{k=0}^{\infty}\left(\frac{\xi_1}{h}\right)^k\sum_{l=k}^{\infty}S_1(l,k)h^l\frac{t^l}{l!}\right)$$

$$\sum_{n=0}^{\infty} {}_{e^{(r)}}\mathcal{H}_n^{[h]}(\xi_1,\xi_2,\xi_3)\frac{t^n}{n!}=\left(\sum_{n=0}^{\infty} {}_{e^{(r)}}\mathcal{H}_n^{[h]}(0,\xi_2,\xi_3)\frac{t^n}{n!}\right)\left(\sum_{l=0}^{\infty}\sum_{k=0}^{l}\left(\frac{\xi_1}{h}\right)^k S_1(l,k)h^l\frac{t^l}{l!}\right). \tag{48}$$

By replacing n with $n-l$ on the right-hand side and comparing coefficients of t^n, identity (44) is obtained.

Now, using (26) along with (29), we have:

$$\sum_{n=0}^{\infty} {}_{e^{(r)}}\mathcal{H}_n^{[h]}(\xi_1,\xi_2,\xi_3)\frac{t^n}{n!}=\frac{1}{1-\xi_3 t^r}(1+ht^2)^{\frac{\xi_2}{h}}e_h^{\log_h((1+ht)^{\frac{\xi_1}{h}})}. \tag{49}$$

$$\sum_{n=0}^{\infty} {}_{e^{(r)}}\mathcal{H}_n^{[h]}(\xi_1,\xi_2,\xi_3)\frac{t^n}{n!}=\left(\sum_{n=0}^{\infty} {}_{e^{r}}\mathbb{H}_n^{[h]}(0,\xi_2,\xi_3)\frac{t^n}{n!}\right)\left(\sum_{k=0}^{\infty}\left(\frac{\xi_1}{h}\right)_{k,h}\frac{\log_h(1+ht)^k}{k!}\right), \tag{50}$$

$$\sum_{n=0}^{\infty} {}_{e^{(r)}}\mathcal{H}_n^{[h]}(\xi_1,\xi_2,\xi_3)\frac{t^n}{n!}=\left(\sum_{n=0}^{\infty} {}_{e^{(r)}}\mathcal{H}_n^{[h]}(0,\xi_2,\xi_3)\frac{t^n}{n!}\right)\left(\sum_{l=0}^{\infty}\sum_{k=0}^{l}\left(\frac{\xi_1}{h}\right)_{k,h}S_{1,h}(l,k)h^l\frac{t^l}{l!}\right). \tag{51}$$

Substituting n by $n-l$ and comparing corresponding coefficients of t, we confirm identity (45).

4. Monomiality Principle

The monomiality principle is a core concept in polynomial theory. It states that any polynomial can be written uniquely as a linear combination of monomials (powers of a variable). This form simplifies analysis and helps extract key properties like degree and roots. Monomiality representations are widely used in computations such as interpolation, approximation, and integration. They also appear in physics, control theory, and signal processing, where polynomials model complex systems.

The principle was introduced via poweroids by Steffensen in 1941 [16], and later extended by Dattoli [27, 28]. These methods, grounded in mathematical physics, quantum mechanics, and optics, remain vital tools in modern research.

In this section, we validate the monomiality principle for the three-variable Δ_h-Truncated exponential-based Hermite polynomials, denoted ${}_{e^{(r)}}\mathcal{H}_n^{[h]}(\xi_1, \xi_2, \xi_3)$. We confirm this by establishing the following results.

Theorem 7. *The Δ_h-Truncated exponential-based Hermite polynomials ${}_{e^{(r)}}\mathcal{H}_n^{[h]}(\xi_1, \xi_2, \xi_3)$ satisfy the following multiplicative and derivative operators:*

$$\hat{M}_{{}_{e^{(r)}}\mathcal{H}_n^{[h]}(\xi_1,\xi_2,\xi_3)} = \left(\frac{\xi_1}{1 + {}_{\xi_1}\Delta_h} + r\xi_3 D_{\xi_3} \xi_3 \frac{{}_{\xi_1}\Delta_h{}^{r-1}}{h} + \frac{2n\xi_2 h}{h + {}_{\xi_1}\Delta_h{}^2} \right), \tag{52}$$

and

$$\hat{P}_{{}_{e^{(r)}}\mathcal{H}_n^{[h]}(\xi_1,\xi_2,\xi_3)} = \frac{{}_{\xi_1}\Delta_h}{h}. \tag{53}$$

Proof. By differentiating equation (29) with respect to ξ_1 and using the identity (12), we get:

$$\begin{aligned} {}_{\xi_1}\Delta_h \left\{ \frac{1}{1 - \xi_3 t^r} (1 + ht)^{\frac{\xi_1}{h}} (1 + ht^2)^{\frac{\xi_2}{h}} \right\} &= \frac{1}{1 - \xi_3 t^r} (1 + ht)^{\frac{\xi_1 + h}{h}} (1 + ht^2)^{\frac{\xi_2}{h}} \\ &\quad - \frac{1}{1 - \xi_3 t^r} (1 + ht)^{\frac{\xi_1}{h}} (1 + ht^2)^{\frac{\xi_2}{h}} \\ &= (1 + ht - 1) \frac{1}{1 - \xi_3 t^r} (1 + ht)^{\frac{\xi_1}{h}} (1 + ht^2)^{\frac{\xi_2}{h}} \\ &= ht \ \frac{1}{1 - \xi_3 t^r} (1 + ht)^{\frac{\xi_1}{h}} (1 + ht^2)^{\frac{\xi_2}{h}}, \end{aligned} \tag{54}$$

which leads to the identity:

$$\frac{{}_{\xi_1}\Delta_h}{h} \Big[\sum_{n=0}^{\infty} {}_{e^{(r)}}\mathcal{H}_n^{[h]}(\xi_1, \xi_2, \xi_3) \frac{t^n}{n!} \Big] = t \Big[\sum_{n=0}^{\infty} {}_{e^{(r)}}\mathcal{H}_n^{[h]}(\xi_1, \xi_2, \xi_3) \frac{t^n}{n!} \Big]. \tag{55}$$

Next, differentiating equation (29) with respect to t, we obtain:

$$\frac{\partial}{\partial t} \Big\{ \frac{1}{1 - \xi_3 t^r} (1 + ht)^{\frac{\xi_1}{h}} (1 + ht^2)^{\frac{\xi_2}{h}} \Big\} = \frac{\partial}{\partial t} \left\{ \sum_{n=0}^{\infty} {}_{e^{(r)}}\mathcal{H}_n^{[h]}(\xi_1, \xi_2, \xi_3) \frac{t^n}{n!} \right\} \tag{56}$$

$$\left(\frac{\xi_1}{1+ht}+r\xi_3 D_{\xi_3}\xi_3 t^{r-1}+\frac{2n\xi_2}{1+ht^2}\right)\left\{\sum_{n=0}^{\infty}{}_{e^{(r)}}\mathcal{H}_n^{[h]}(\xi_1,\xi_2,\xi_3)\frac{t^n}{n!}\right\}=n\sum_{n=0}^{\infty}{}_{e^{(r)}}\mathcal{H}_n^{[h]}(\xi_1,\xi_2,\xi_3)\frac{t^{n-1}}{n!}. \tag{57}$$

Applying identity (29) and shifting $n\to n+1$ on the right-hand side of (57), we derive the operator formula (52).

Moreover, from identities (15) and (55), it follows:

$$\frac{{}_{\xi_1}\Delta_h}{h}\Big[\sum_{n=0}^{\infty}{}_{e^{(r)}}\mathcal{H}_n^{[h]}(\xi_1,\xi_2,\xi_3)\frac{t^n}{n!}\Big]=n\,\Big[\sum_{n=0}^{\infty}{}_{e^{(r)}}\mathcal{H}_{n-1}^{[h]}(\xi_1,\xi_2,\xi_3)\frac{t^n}{n!}\Big], \tag{58}$$

yielding the derivative operator form in (53).

Now, we derive the differential equation satisfied by the Δ_h-Truncated exponential-based Hermite polynomials ${}_{e^r}\mathbb{H}_n^{[h]}(\xi_1,\xi_2,\xi_3)$ through the following result:

Theorem 8. *The Δ_h-Truncated exponential-based Hermite polynomials ${}_{e^{(r)}}\mathcal{H}_n^{[h]}(\xi_1,\xi_2,\xi_3)$ obey the differential equation:*

$$\left(\frac{\xi_1}{1+{}_{\xi_1}\Delta_h}+r\xi_3 D_{\xi_3}\xi_3\frac{{}_{\xi_1}\Delta_h{}^{r-1}}{h}+\frac{2n\xi_2 h}{h+{}_{\xi_1}\Delta_h{}^2}-\frac{nh}{{}_{\xi_1}\Delta_h}\right){}_{e^{(r)}}\mathcal{H}_n^{[h]}(\xi_1,\xi_2,\xi_3)=0. \tag{59}$$

Proof. Substituting the operator forms (52) and (53) into the identity (17), we get:

$$\left(\frac{\xi_1}{1+{}_{\xi_1}\Delta_h}+r\xi_3 D_{\xi_3}\xi_3\frac{{}_{\xi_1}\Delta_h{}^{r-1}}{h}+\frac{2n\xi_2 h}{h+{}_{\xi_1}\Delta_h{}^2}\right)\frac{{}_{\xi_1}\Delta_h}{h}{}_{e^{(r)}}\mathcal{H}_n^{[h]}(\xi_1,\xi_2,\xi_3)=n_{e^{(r)}}\mathcal{H}_n^{[h]}(\xi_1,\xi_2,\xi_3). \tag{60}$$

Simplifying the above yields the claimed result (59).

We now establish the following operational formula involving ${}_{e^{(r)}}\mathcal{H}_n^{[h]}(\xi_1,\xi_2,\xi_3)$:

Theorem 9. *The following operational relation holds between the Δ_h-Truncated exponential-based Hermite polynomials ${}_{e^{(r)}}\mathcal{H}_n^{[h]}(\xi_1,\xi_2,\xi_3)$ and the Δ_h-Hermite polynomials $\mathcal{H}_n^{[h]}(\xi_1,\xi_2)$:*

$${}_{e^{(r)}}\mathcal{H}_n^{[h]}(\xi_1,\xi_2,\xi_3)=\exp(r\xi_3 D_{\xi_3}\xi_3\frac{{}_{\xi_1}\Delta_h{}^r}{h})\left\{\mathcal{H}_n^{[h]}(\xi_1,\xi_2)\right\}. \tag{61}$$

Proof. Using equations (27) and (58), and applying identity (29), the result follows immediately.

5. Symmetric identities

In this Section, we investigate symmetric identities inherent to the three-variable Δ_h special polynomials. These identities unveil intriguing relationships between the variables and coefficients within the polynomials, shedding light on their underlying symmetrical properties. By exploring how the polynomials behave under transformations that interchange the variables or coefficients, we uncover profound connections that extend beyond their initial definitions. These symmetric identities not only deepen our understanding of the polynomials themselves but also offer valuable insights into broader mathematical structures and phenomena. Through systematic examination and rigorous derivation, we establish a comprehensive framework for understanding and exploiting the symmetrical properties of these two-variable special polynomials, paving the way for further advancements in both theoretical analyses and practical applications.

Theorem 10. *For $a \neq b$, $a, b > 0$ and $xi_1, \xi_2, \nu_1, \nu_2, \phi_1, \phi_2 \in \mathbb{C}$, we have*

$$\sum_{\gamma=0}^{n} \binom{n}{\gamma} a^{n-\gamma} b^{\gamma}{}_{e^{(r)}}\mathcal{H}_{n-\gamma}^{[h]}(a\xi_1, a\nu_1, a\phi_1)_{e^{(r)}}\mathcal{H}_{\gamma}^{[h]}(b\xi_2, b\nu_2, b\phi_2)$$

$$= \sum_{\gamma=0}^{n} \binom{n}{\gamma} a^{\gamma} b^{n-\gamma}{}_{e^{(r)}}\mathcal{H}_{n-\gamma}^{[h]}(a\xi_2, a\nu_2, a\phi_2)_{e^{(r)}}\mathcal{H}_{\gamma}^{[h]}(b\xi_1, b\nu_1, b\phi_1). \tag{62}$$

Proof. Let

$$A(t) = \frac{1}{1-\phi_1(abt)^2} \frac{1}{1-\phi_2(abt)^2} (1+ht)^{\frac{ab(\xi_1+\xi_2)}{h}} (1+ht^2)^{\frac{ab(\nu_1+\nu_2)}{h}} \tag{63}$$

$$= \sum_{n=0}^{\infty} {}_{e^{(r)}}\mathcal{H}_{\gamma}^{[h]}(b\xi_1, b\nu_1, b\phi_1) \frac{(bt)^{\gamma}}{\gamma!} \sum_{n=0}^{\infty} {}_{e^{(r)}}\mathcal{H}_{n}^{[h]}(a\xi_2, a\nu_2, a\phi_2) \frac{(at)^n}{n!}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{\gamma=0}^{n} \binom{n}{\gamma} a^{n-\gamma} b^{\gamma}{}_{e^{(r)}}\mathcal{H}_{n-\gamma}^{[h]}(a\xi_1, a\nu_1, a\phi_1)_{e^{(r)}}\mathcal{H}_{\gamma}^{[h]}(b\xi_2, b\nu_2, b\phi_2) \right) \frac{t^n}{n!}. \tag{64}$$

Similarly, we have

$$A(t) = \sum_{\phi=0}^{\infty} \left(\sum_{\gamma=0}^{n} \binom{n}{\gamma} a^{\gamma} b^{n-\gamma}{}_{e^{(r)}}\mathcal{H}_{n-\gamma}^{[h]}(a\xi_2, a\nu_2, a\phi_2)_{e^{(r)}}\mathcal{H}_{\gamma}^{[h]}(b\xi_1, b\nu_1, b\phi_1) \right) \frac{t^n}{n!}. \tag{65}$$

Comparing the coefficients of t on both sides of last equations, we get (62).

Theorem 11. *For $a \neq b$, $a, b > 0$ and $\xi, \nu, \phi \in \mathbb{C}$, we have*

$$\sum_{k=0}^{n} \sum_{\gamma=0}^{k} \binom{n}{k} \binom{k}{\gamma} a^{n-\gamma} b^{\gamma+1} \beta_{n-k}(h)_{e^{(r)}}\mathcal{H}_{k-\gamma}^{[h]}(b\xi, b\nu, b\phi)\sigma_{\gamma}(a-1;h)$$

$$= \sum_{k=0}^{n} \sum_{\gamma=0}^{k} \binom{n}{k} \binom{k}{\gamma} b^{n-\gamma} a^{\gamma+1} \beta_{n-k}(h)_{e^{(r)}}\mathcal{H}_{k-\gamma}^{[h]}(a\xi, a\nu, a\phi)\sigma_{\gamma}(b-1;h). \tag{66}$$

Proof. Consider

$$B(t) = \frac{1}{1-\phi(abt)^2} \frac{(1+ht)^{\frac{ab\xi}{h}}(1+ht^2)^{\frac{ab\nu}{h}}((1+ht)^{\frac{ab}{h}}-1)}{((1+ht)^{\frac{a}{h}}-1)((1+ht)^{\frac{b}{h}}-1)^2}$$

$$= \frac{abt}{((1+ht)^{\frac{a}{h}}-1)} \frac{1}{1-\phi(abt)^2}(1+ht)^{\frac{ab\xi}{h}}(1+ht^2)^{\frac{ab\nu}{h}} \frac{((1+ht)^{\frac{ab}{h}}-1)}{((1+ht)^{\frac{b}{h}}-1)}$$

$$= b\sum_{n=0}^{\infty}\beta_n(h)\frac{(at)^n}{n!}\sum_{k=0}^{\infty}{}_{e^{(r)}}\mathcal{H}_k^{[h]}(b\xi, b\nu, b\phi)\frac{(at)^k}{k!}\sum_{\gamma=0}^{\infty}\sigma_\gamma(a-1;h)\frac{(bt)^\gamma}{\gamma!}$$

$$= b\sum_{n=0}^{\infty}\beta_n(h)\frac{(at)^n}{n!}\sum_{k=0}^{\infty}\sum_{\gamma=0}^{k}\binom{k}{\gamma}a^{k-\gamma}b^{\gamma}{}_{e^{(r)}}\mathcal{H}_{k-\gamma}^{[h]}(b\xi, b\nu, b\phi)\sigma_\gamma(a-1;h)\frac{t^k}{k!}$$

$$= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{\gamma=0}^{k}\binom{n}{k}\binom{k}{\gamma}a^{n-\gamma}b^{\gamma+1}\beta_{n-k}(h){}_{e^{(r)}}\mathcal{H}_{k-\gamma}^{[h]}(b\xi, b\nu, b\phi)\sigma_\gamma(a-1;h)\right)\frac{t^n}{n!}. \quad (67)$$

Similarly, we have

$$B(t) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{\gamma=0}^{k}\binom{n}{k}\binom{k}{\gamma}b^{n-\gamma}a^{\gamma+1}\beta_{n-k}(h){}_{e^{(r)}}\mathcal{H}_{k-\gamma}^{[h]}(a\xi, a\nu, a\phi)\sigma_\gamma(b-1;h)\right)\frac{t^n}{n!}. \quad (68)$$

Comparing the coefficients of t on both sides of last equations, we get (66).

6. Conclusion

In this study, we have introduced a novel class of Δ_h-truncated exponential-based Hermite polynomials and established their fundamental properties, including generating functions, recurrence relations, explicit formulas, and summation identities. The connection with the monomiality principle has been explored to reveal their underlying algebraic structure, and an operational formalism has been developed. Additionally, symmetric identities have been presented to further deepen the theoretical understanding of these polynomials. These results lay a solid foundation for future investigations and potential applications in both pure and applied mathematics.

Future research can explore several directions, including the development of q-analogues and degenerate forms of the proposed polynomials to study associated q-difference equations and limiting behaviors. Investigating orthogonality conditions and suitable weight functions will help identify inner product spaces where these polynomials are orthogonal. Their application in interpolation, approximation theory, and spectral methods also warrants attention, particularly in solving differential or integral equations. Potential uses in mathematical physics and engineering—such as quantum systems and signal analysis—highlight their applied significance. Additionally, a detailed study of asymptotic properties and zero distributions using analytic and numerical tools could offer deeper insights into their structural behavior.

Availability of data and materials

Not applicable.

Competing interests

The authors declare no competing interests.

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