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Bivariate Kind of Generalized Laguerre-Based Appell Polynomials with Applications to Special Polynomials

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Abstract. In this paper, we introduce a new generalization of Laguerre and Laguerre-based Appell polynomials and investigate their fundamental properties. We derive a recurrence relation, multiplicative and derivative operators, and differential equation by verifying quasi-monomiality. Also, the series representation and determinant representation for this novel polynomial family are established. Furthermore, we define subpolynomials within this framework, namely generalized Laguerre-Hermite Appell polynomials and establish their corresponding results. Additionally, Laguerre-Hermite-Bernoulli, Euler and Genocchi polynomials are obtained, and explore their structural and operational characteristics. The results obtained contribute to the broader study of special polynomials and their applications in mathematical physics and differential equations.

2020 Mathematics Subject Classifications: 33E20, 33C45, 33B10, 33E30, 11T23, 11B83, 11B68

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1. Introduction and preliminary results

It is well established that special polynomials in two variables provide new analytical tools for solving a broad range of partial differential equations frequently encountered in physical problems. The introduction of the two-variable Laguerre polynomials, denoted as $\mathcal{L}_n(r_1, r_2)$ [1–10], is of significant interest due to their intrinsic mathematical properties and extensive applications in physics.

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The two-variable Laguerre polynomials (2VLP) $\mathcal{L}_n(r_1, r_2)$ are characterized by the following generating function [11]:

$$e^{r_2 t} J_0(2\sqrt{r_1 t}) = \sum_{n=0}^{\infty} \mathcal{L}_n(r_1, r_2) \frac{t^n}{n!}. \tag{1}$$

where $J_0(r_1 t)$ represents the ordinary Bessel function of the first kind of order zero [12], defined as:

$$r_1^{\frac{n}{2}} J_n(2\sqrt{r_1}) = \sum_{k=0}^{\infty} \frac{(-1)^k \ (\sqrt{r_1})^k}{k!(n+k)!}. \tag{2}$$

Additionally, we note that:

$$\exp(-\alpha D_{r_1}^{-1}) = J_0(2\sqrt{\alpha r_1}), \qquad D_{r_1}^{-n}\{1\} := r_1^n/n! \tag{3}$$

is the inverse differential operator.

The class of Appell polynomial sequences [13] appears in numerous applied mathematics problems, theoretical physics, approximation theory, and other mathematical disciplines. These sequences are defined through the generating function:

$$\mathcal{R}(r_1, t) := \mathcal{R}(t) e^{r_1 t} = \sum_{n=0}^{\infty} \mathcal{R}_n(r_1) \frac{t^n}{n!}, \quad \mathcal{R}_n := R_n(0), \tag{4}$$

where $\mathcal{R}(t)$ is an analytic function at $t = 0$, expressed as:

$$\mathcal{R}(t) = \sum_{n=0}^{\infty} \mathcal{R}_n \frac{t^n}{n!}, \quad \mathcal{R}_0 \neq 0, \quad \mathcal{R}_i \ (i = 0, 1, 2, \cdots) \ \text{ being real coefficients.} \tag{5}$$

The Appell polynomials $\mathcal{R}_n(r_1)$ are explicitly given by the series expansion:

$$\mathcal{R}_n(r_1) = \sum_{k=0}^{n} \binom{n}{k} \mathcal{R}_{n-k} \ r_1^k, \quad \mathcal{R}_n^{'}(r_1) = n \ \mathcal{R}_{n-1}(r_1). \tag{6}$$

By appropriately selecting $\mathcal{R}(t)$, various members of the Appell polynomial family can be derived. These are listed in Table 1 below:

Table 1. Certain members belonging to the Appell family

S.No.	Name of polynomials	$\mathcal{R}(t)$	Generating function	Series definition
I.	Bernoulli	$\frac{t}{e^t-1}$	$\left(\frac{t}{e^t-1}\right)e^{r_1t}=\sum_{n=0}^{\infty}\mathbb{B}_n(r_1)\frac{t^n}{n!}$	$\mathbb{B}_n(r_1)=\sum_{k=0}^{n}\binom{n}{k}\mathbb{B}_k r_1^{n-k}$
	polynomials		$\left(\frac{t}{e^t-1}\right)=\sum_{n=0}^{\infty}B_n\frac{t^n}{n!}$	
	and numbers [14]		$\mathbb{B}_n(:=\mathbb{B}_n(0)=\mathbb{B}_n(1))$	
II.	Euler	$\frac{2}{e^t+1}$	$\left(\frac{2}{e^t+1}\right)e^{r_1t}=\sum_{n=0}^{\infty}\mathbb{E}_n(r_1)\frac{t^n}{n!}$	$\mathbb{E}_n(r_1)=\sum_{k=0}^{n}\binom{n}{k}\frac{\mathbb{E}_k}{2^k}\left(r_1-\frac{1}{2}\right)^{n-k}$
	polynomials		$\frac{2e^t}{e^{2t}+1}=\sum_{n=0}^{\infty}\mathbb{E}_n\frac{t^n}{n!}$	
	and numbers [14]		$\mathbb{E}_n:=2^n\mathbb{E}_n\left(\frac{1}{2}\right)$	
III.	Genocchi	$\frac{2t}{e^t+1}$	$\left(\frac{2t}{e^t+1}\right)e^{r_1t}=\sum_{n=0}^{\infty}\mathbb{G}_n(r_1)\frac{t^n}{n!}$	$\mathbb{G}_n(r_1)=\sum_{k=0}^{n}\binom{n}{k}\mathbb{G}_k r_1^{n-k}$
	polynomials		$\frac{2t}{e^t+1}=\sum_{n=1}^{\infty}\mathbb{G}_n\frac{t^n}{n!}$	
	and numbers [**?**]		$\mathbb{G}_n:=\mathbb{G}_n(0)$	

To facilitate further computations, we present the initial values of Bernoulli numbers $\mathbb{B}_n$, Euler numbers $\mathbb{E}_n$, and Genocchi numbers $\mathbb{G}_n$ in Table 2 below:

Table 2. Values of five four $\mathbb{B}_n$, $\mathbb{E}_n$ and $\mathbb{G}_n$

n	0	1	2	3	4
B_n	1	$\pm\frac{1}{2}$	$\frac{1}{6}$	0	$-\frac{1}{30}$
E_n	1	0	-1	0	5
G_n	0	1	-1	0	1

In 2012, Khan and Raza [15] introduced and analyzed a hybrid class of Laguerre-Sheffer polynomials, denoted as ${}_{\mathcal{L}}S_n(r_1,r_2)$, which are defined via the generating function:

$$\mathcal{R}(t)\ e^{r_2H(t)}\ J_0(2\sqrt{r_1H(t)})=\sum_{n=0}^{\infty}{}_{\mathcal{L}}S_n(r_1,r_2)\frac{t^n}{n!}. \tag{7}$$

Since Sheffer polynomials $S_n(r_1)$ [16] reduce to Appell polynomials $\mathcal{R}_n(r_1)$ when $H(t)=t$, choosing $H(t)=t$ in equation (7) leads to the hybrid Legendre-Appell polynomials (LeAP), defined by:

$$\mathcal{R}(t)\ e^{r_2t}\ J_0(2\sqrt{r_1t})=\sum_{n=0}^{\infty}{}_{L}\mathcal{R}_n(r_1,r_2)\frac{t^n}{n!}, \tag{8}$$

or equivalently,

$$\mathcal{R}(t)\ e^{r_2t}\ C_0(r_1t)=\sum_{n=0}^{\infty}{}_{\mathcal{L}}\mathcal{R}_n(r_1,r_2)\frac{t^n}{n!}, \tag{9}$$

where $C_0(r_1)$ denotes the 0^{th} order Bessel Tricomi function [14]. The n^{th}-order Tricomi functions $C_n(r_1)$ are defined as

$$C_n(r_1)=\sum_{k=0}^{\infty}\frac{(-1)^k r_1^k}{k!(n+k)!}. \tag{10}$$

We also note that

$$\exp(-\alpha\widehat{D}_{r_1}^{-1}) = C_0(\alpha x), \qquad \widehat{D}_{r_1}^{-n}\{1\} := \frac{r_1^n}{n!}. \tag{11}$$

Further, it can be expressed as

$$\mathcal{R}(t)\ e^{yt}\ e^{-D_{r_1}^{-1}t} = \sum_{n=0}^{\infty} {}_{\mathcal{L}}\mathcal{R}_n(r_1, r_2)\frac{t^n}{n!}. \tag{12}$$

The hybrid LAP ${}_{\mathcal{L}}\mathcal{R}_n(r_1, r_2)$ satisfies the series expansion:

$${}_{\mathcal{L}}\mathcal{R}_n(r_1, r_2) = n!\sum_{k=0}^{n} \frac{\mathcal{L}_{n-k}(r_2)r_1^k}{(n-k)!(k!)^2}. \tag{13}$$

By appropriately choosing $\mathcal{R}(t)$, various members of the hybrid LAP family can be derived. These are summarized in Table 3:

Table 3. Certain members belonging to the LAP family

S. No.	Name of hybrid polynomials	$\mathcal{R}(t)$	Generating function	Series definition
I.	Hybrid Laguerre-Bernoulli polynomials	$\frac{t}{e^t-1}$	$\left(\frac{t}{e^t-1}\right) e^{r_2 t}\ J_0(2\sqrt{r_1 t} = \sum_{n=0}^{\infty} {}_{\mathcal{L}}\mathbb{B}_n(r_1, r_2)\frac{t^n}{n!}$	${}_{\mathcal{L}}\mathbb{B}_n(r_1, r_2) = n!\sum_{k=0}^{n} \frac{(-1)^k\mathbb{B}_{n-k}(r_2)r_1^k}{(n-k)!(k!)^2}$
II.	Hybrid Laguerre-Euler polynomials	$\frac{2}{e^t+1}$	$\left(\frac{2}{e^t+1}\right) e^{r_2 t}\ J_0(2\sqrt{r_1 t} = \sum_{n=0}^{\infty} {}_{\mathcal{L}}\mathbb{E}_n(r_1, r_2)\frac{t^n}{n!}$	${}_{\mathcal{L}}\mathbb{E}_n(r_1, r_2) = n!\sum_{k=0}^{n} \frac{(-1)^k\mathbb{E}_{n-k}(r_2)r_1^k}{(n-k)!(k!)^2}$
III.	Hybrid Laguerre-Genocchi polynomials	$\frac{2t}{e^t+1}$	$\left(\frac{2t}{e^t+1}\right) e^{r_2 t}\ J_0(2\sqrt{r_1 t} = \sum_{n=0}^{\infty} {}_{\mathcal{L}}\mathbb{G}_n(r_1, r_2)\frac{t^n}{n!}$	${}_{\mathcal{L}}\mathbb{G}_n(r_1, r_2) = n!\sum_{k=0}^{n} \frac{(-1)^k\mathbb{G}_{n-k}(r_2)r_1^k}{(n-k)!(k!)^2}$

The 2-variable general polynomials (2VgP) denoted by $\mathcal{P}_n(r_1, r_2)$ are specified by generating relation [17]:

$$\exp(r_1 t)\Psi(r_2, t) = \sum_{n=0}^{\infty} \mathcal{P}_n(r_1, r_2)\frac{t^n}{n!}, \quad (\mathcal{P}_0(r_1, r_2) = 1), \tag{14}$$

where $\Psi(r_2, t)$ has (at least the formal) series expansion

$$\Psi(r_2, t) = \sum_{k=0}^{\infty} \Psi_k(r_2)\frac{t^k}{k!}, \quad (\Psi_0(r_2) \neq 0). \tag{15}$$

The foundational idea of the monomiality principle dates back to 1941 when Steffenson [18] first introduced the concept through the notion of poweroid. This approach was later refined and extended by Dattoli [19], paving the way for further advancements in the field.

The monomiality principle asserts that the operators $\widehat{\mathcal{M}}$ and $\widehat{\mathcal{P}}$ act, respectively, as multiplicative and differential operators for a given polynomial sequence $\{q_n(r_1)\}_{n\in\mathbb{N}}$. More specifically, these operators satisfy the fundamental recurrence relations:

$$q_{n+1}(r_1) = \widehat{\mathcal{M}}\{q_n(r_1)\}, \tag{16}$$

and

$$n\ q_{n-1}(r_1) = \widehat{\mathcal{P}}\{q_n(r_1)\}. \tag{17}$$

A polynomial sequence $\{q_n(r_1)\}_{n\in\mathbb{N}}$ that adheres to these operator relations is referred to as a quasi-monomial set. Such a set must also satisfy the fundamental commutation relation:

$$[\widehat{\mathcal{P}}, \widehat{\mathcal{M}}] = \widehat{\mathcal{P}}\widehat{\mathcal{M}} - \widehat{\mathcal{M}}\widehat{\mathcal{P}} = \widehat{1}, \tag{18}$$

which aligns naturally with the algebraic framework of the Weyl algebra.

If a polynomial sequence $\{q_n(r_1)\}_{n\in\mathbb{N}}$ is quasi-monomial, its defining properties can be derived directly from the characteristics of the operators $\widehat{\mathcal{M}}$ and $\widehat{\mathcal{P}}$. Specifically, the following key properties hold:

(i) The polynomials $q_n(r_1)$ satisfy a differential equation of the form:

$$\widehat{\mathcal{M}}\widehat{\mathcal{P}}\{q_n(r_1)\} = n\ q_n(r_1), \tag{19}$$

provided that $\widehat{\mathcal{M}}$ and $\widehat{\mathcal{P}}$ admit suitable differential representations.

(ii) An explicit formula for $q_n(r_1)$ can be expressed as:

$$q_n(r_1) = \widehat{\mathcal{M}}^n\ \{1\}, \tag{20}$$

with the initial condition $q_0(r_1) = 1$.

(iii) The exponential generating function of $q_n(r_1)$ is given by:

$$e^{t\widehat{\mathcal{M}}}\{1\} = \sum_{n=0}^{\infty} q_n(r_1)\,\frac{t^n}{n!} \qquad (|t| < \infty)\,, \tag{21}$$

which follows directly from equation (20). For more details, see [17, 20–27].

The operational framework outlined above has found extensive applications across various fields, including classical optics, quantum mechanics, and different branches of mathematical physics. These techniques offer robust analytical tools for studying diverse polynomial families.

Motivated by these developments, we introduce a new generalize Laguerre-based Appell polynomials ${}_{\mathcal{P}\mathcal{L}}\mathcal{R}_n(r_1, r_2, r_3)$. The structure of this paper is as follows: In Section 2, we define the new generalization of Laguerre and Laguerre-based Appell polynomials and explore their key properties, including recurrence relations, associated operators, and differential equations. Section 3 focuses on the series expansions and determinant representations of these generalized polynomials. In Section 4, we examine specific subfamilies and establish their distinctive properties. Finally, the paper concludes with some remarks summarizing our findings and future research directions.

2. The new generalization of Laguerre and Laguerre-based Appell polynomials

This section of our research paper introduces the new generalization of three-variable Laguerre-based Appell polynomials, denoted as ${}_{\mathcal{P}\mathcal{L}}\mathcal{R}_n(r_1, r_2, r_3)$. We present their series expansion, quasi-monomial property, operational formulas, and corresponding differential equations. Our study initiates with the construction of a novel generalization of three-variable Laguerre polynomials, denoted as 3VLP ${}_{\mathcal{P}}\mathcal{L}_n(r_1, r_2, r_3)$.

Utilizing the relations (1) and (14), we introduce the new generalization of three variable Laguerre polynomials ${}_{\mathcal{P}}\mathcal{L}_v(r_1, r_2, r_3)$ in the following form:

$$e^{r_1 t}\Psi(r_2, t)C_0(r_3 t) = \sum_{n=0}^{\infty} {}_{\mathcal{P}}\mathcal{L}_n(r_1, r_2, r_3)\frac{t^n}{n!}, \quad (\mathcal{P}_0(r_1, r_2) = 1). \tag{22}$$

By simplifying the left-hand side of equation (22) using equations (2) and (15), we derive the subsequent series representations for the generalized three-variable Laguerre polynomials $3VLP$ ${}_{\mathcal{P}}\mathcal{L}_n(r_1, r_2, r_3)$ holds:

$${}_{\mathcal{P}}\mathcal{L}_n(r_1, r_2, r_3) = \sum_{m=0}^{n} \binom{n}{m} \Psi_m(r_2) L_{n-m}(r_1, r_3). \tag{23}$$

We present the derived quasi-monomial identities for the three-variable Laguerre polynomials $3VLeP$, denoted as ${}_{\mathcal{P}}\mathcal{L}_n(r_1, r_2, r_3)$.

Theorem 1. *The new generalization of three variable Laguerre polynomials 3VLP* ${}_{\mathcal{P}}\mathcal{L}_n(r_1, r_2, r_3)$ *demonstrate quasi monomials properties under the following multiplicative and derivative operators:*

$$\widehat{M}_{3VgLeP} = r_1 + \frac{\Psi'(r_2, \widehat{D}_{r_1})}{\Psi(r_2, \widehat{D}_{r_1})} - n\widehat{D}_{r_3}^{-1}, \tag{24}$$

and

$$\widehat{P}_{3VgLeP} = \widehat{D}_{r_1}, \tag{25}$$

respectively.

Proof. By differentiating Equation (22) w.r.t. t on both sides, it follows that

$$\sum_{n=0}^{\infty} {}_{\mathcal{P}}\mathcal{L}_{n+1}(r_1, r_2, r_3)\frac{t^n}{n!} = xe^{r_1 t}\Psi(r_2, t)C_0(r_3 t) + \frac{\Psi'(r_2, t)}{\Psi(r_2, t)}e^{r_1 t}\Psi(r_2, t)C_0(r_3 t) + \left(\sum_{n=0}^{\infty} \frac{(-1)^n r_3^n n t^{n-1}}{([n]!)^2}\right) e^{r_1 t}\Psi(r_2, t)$$

$$\sum_{n=0}^{\infty} {}_{\mathcal{P}}\mathcal{S}_{n+1}(r_1, r_2, r_3)\frac{t^n}{n!} = \left(r_1 + \frac{\Psi'(r_2, t)}{\Psi(r_2, t)}\right) e^{r_1 t}\Psi(r_2, t)C_0(-r_3 t^2) + \left(\sum_{n=0}^{\infty} \frac{(-1)^{n+1} r_3^{n+1} t^n}{([n+1]!)^2}\right) e^{r_1 t}\Psi(r_2, t).$$

Utilizing Equation (22), we have

$$\sum_{n=0}^{\infty} {}_{\mathcal{P}}\mathcal{L}_{n+1}(r_1,r_2,r_3)\frac{t^n}{n!} = \left(r_1 + \frac{\Psi'(r_2,t)}{\Psi(r_2,t)}\right)\sum_{n=0}^{\infty} {}_{\mathcal{P}}\mathcal{L}_{n}(r_1,r_2,r_3)\frac{t^n}{n!} + \left(\sum_{n=0}^{\infty}\frac{(-1)^{n+1}r_3^{n+1}(n+1)t^n}{([n+1]!)^2}\right)e^{r_1t}\Psi(r_2,t). \quad (26)$$

Differentiating the above Equation w.r.t. r_3, we have

$$\sum_{n=0}^{\infty} D_{r_3}\ {}_{\mathcal{P}}\mathcal{L}_{n+1}(r_1,r_2,r_3)\frac{t^n}{n!} = \left(r_1 + \frac{\Psi'(r_2,t)}{\Psi(r_2,t)}\right)\sum_{n=0}^{\infty} D_{r_3}\ {}_{\mathcal{P}}\mathcal{L}_{n}(r_1,r_2,r_3)\frac{t^n}{n!} - n\sum_{n=0}^{\infty} {}_{\mathcal{P}}\mathcal{L}_{n}(r_1,r_2,r_3)\frac{t^n}{n!}. \quad (27)$$

Consequently,

$$D_{r_1}\left\{e^{r_1t}\Psi(r_2,t)C_0(r_3t)\right\} = te^{r_1t}\Psi(r_2,t)C_0(r_3t), \quad (28)$$

and $\frac{\Psi'(r_2,t)}{\Psi(r_2,t)}$ possesses power series expansion in t with $\Psi(r_2,t)$ being the invertible series of t.

Operating $D_{r_3}^{-1}$ to Equation (27) on both sides, we have

$$\sum_{n=0}^{\infty} {}_{\mathcal{P}}\mathcal{L}_{n+1}(r_1,r_2,r_3)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left(r_1 + \frac{\Psi'(r_2,t)}{\Psi(r_2,t)} - nD_{r_3}^{-1}\right){}_{\mathcal{P}}\mathcal{L}_{n}(r_1,r_2,r_3)\frac{t^n}{n!}. \quad (29)$$

In light of (16) and (29), we obtain the assertion (24).

Similarly, by applying identity (28) to (22), we get

$$D_{r_1}\left\{\sum_{n=0}^{\infty} {}_{\mathcal{P}}\mathcal{L}_{n}(r_1,r_2,r_3)\frac{t^n}{n!}\right\} = \sum_{n=1}^{\infty} {}_{\mathcal{P}}\mathcal{L}_{n-1}(r_1,r_2,r_3)\frac{t^n}{(n-1)!} \quad (30)$$

By matching the coefficients of same exponents of t on both sides of (30), it follows that

$$D_{r_1}\left\{{}_{\mathcal{P}}\mathcal{L}_{n}(r_1,r_2,r_3)\right\} = n\,{}_{\mathcal{P}}\mathcal{L}_{n-1}(r_1,r_2,r_3), \quad n \succeq 1. \quad (31)$$

Thus in view of (17) and (31), we get the assertion (25).

Theorem 2. *The following differential equations for* 3*-variable generalized Laguerre polynomials* $3VGLP$ ${}_{\mathcal{P}}\mathcal{L}_n(r_1,r_2,r_3)$ *as:*

$$\left(r_1\widehat{D}_{r_1} + \frac{\Psi'(r_2,\widehat{D}_{r_1})}{\Psi(r_2,\widehat{D}_{r_1})}\widehat{D}_{r_1} - n\widehat{D}_{r_3}^{-1}\widehat{D}_{r_1} - n\right){}_{\mathcal{P}}\mathcal{L}_{n}(r_1,r_2,r_3) = 0, \quad (32)$$

Proof. In view of equations (24) and (25) in (19), we get

$$\left(r_1 D_{r_1} + \frac{\Psi'(r_2, \widehat{D}_{r_1})}{\Psi(r_2, \widehat{D}_{r_1})} D_{r_1} - n\widehat{D}_{r_3}^{-1} D_{r_1}\right) {}_{\mathcal{P}}\mathcal{L}_n(r_1, r_2, r_3) = n \; {}_{\mathcal{P}}\mathcal{L}_n(r_1, r_2, r_3).$$

Upon solving the above equation, we get the assertion (32) of Theorem 2.2.

Remark 1. *Since* $\mathcal{P}_0(r_1, r_2) = 1$*, therefore in view of monomiality principle equation* (15)*, we have*

$${}_{\mathcal{P}}\mathcal{L}_n(r_1, r_2, r_3) = \left(r_1 + \frac{\Psi'(r_2, \widehat{D}_{r_1})}{\Psi(r_2, \widehat{D}_{r_1})} - n\widehat{D}_{r_3}^{-1}\right)^n \{1\}, \quad (\mathcal{P}_0(r_1, r_2) = 1).$$

Also, in view of equations (15), (22) and (24), we have

$$\exp\left(\widehat{M}_{3VgP}\right)\{1\} = e^{r_1 t}\Psi(r_2, t)C_0(r_3 t) = \sum_{v=0}^{\infty} {}_{\mathcal{P}}\mathcal{L}_n(r_1, r_2, r_3)\frac{t^n}{n!}. \tag{33}$$

Now, we proceed to introduce the new generalization of 3-variable Laguerre-based Appell polynomials (3VLbAP). To obtain the generating functions for the newly generalized three-variable Laguerre-based Appell polynomials, we utilize the exponential generating function associated with Appell polynomials. Thus, replacing r_1 in the left hand side of (5) by the multiplicative operator ${}_{\mathcal{P}}\mathcal{L}_n(r_1, r_2, r_3)$ given by (24) denoting the new generalization of 3-variable Laguerre-based Appell polynomials ${}_{\mathcal{PL}}\mathcal{R}_n(r_1, r_2, r_3)$, we get

$$\mathcal{R}(t)\exp\left(\widehat{M}_{3VgP}\right)\{1\} = \sum_{n=0}^{\infty} {}_{\mathcal{PL}}\mathcal{R}_n(r_1, r_2, r_3)\frac{t^n}{n!}, \tag{34}$$

which on using equation (24), we get the following two equivalent forms of ${}_{\mathcal{PL}}\mathcal{R}_n(r_1, r_2, r_3)$:

$$\mathcal{R}(t)\exp\left(r_1 + \frac{\Psi'(r_2, \widehat{D}_{r_1}}{\Psi(r_2, \widehat{D}_{r_1}} - n\widehat{D}_{r_3}^{-1}\right)\{1\} = \sum_{n=0}^{\infty} {}_{\mathcal{PL}}\mathcal{R}_n(r_1, r_2, r_3)\frac{t^n}{n!}. \tag{35}$$

Using the relation (33) in the left hand side of equation (34), the generating function for the new generalization of 3-variable Laguerre-based Appell polynomials ${}_{\mathcal{PL}}\mathcal{R}_n(r_1, r_2, r_3)$ in the following form:

$$\mathcal{R}(t)e^{r_1 t}\Psi(r_2, t)C_0(r_3 t) = \sum_{n=0}^{\infty} {}_{\mathcal{PL}}\mathcal{R}_n(r_1, r_2, r_3)\frac{t^n}{n!}. \tag{36}$$

where

$$\mathcal{R}(t) = \sum_{k=0}^{\infty} \alpha_k \frac{t^k}{k!}, \quad \alpha_0 \neq 0 \quad \Psi(r_2, t) = \sum_{k=0}^{\infty} \psi_k(r_2)\frac{t^k}{k!}, \quad \psi_0 \neq 0. \tag{37}$$

Theorem 3. *The generalized Laguerre-based Appell polynomials* ${}_{\mathcal{P}}\mathcal{L}\mathcal{R}_n(r_1,r_2,r_3)$ *satisfy the following recurrence relation:*

$$ {}_{\mathcal{P}}\mathcal{L}\mathcal{R}_{n+1}(r_1,r_2,r_3)=\sum_{k=0}^{n}\binom{n}{k}{}_{\mathcal{P}}\mathcal{L}\mathcal{R}_{n-k}(r_1,r_2,r_3)\gamma_k+r_1{}_{\mathcal{P}}\mathcal{L}\mathcal{R}_n(r_1,r_2,r_3)+\sum_{k=0}^{n}\binom{n}{k}{}_{\mathcal{P}}\mathcal{L}\mathcal{R}_{n-k}(r_1,r_2,r_3)\rho_k(r_2) $$

$$ -nD_{r_3}^{-1}{}_{\mathcal{P}}\mathcal{L}\mathcal{R}_{n-1}(r_1,r_2,r_3). \tag{38} $$

where

$$ \frac{\mathcal{R}'(t)}{\mathcal{R}(t)}=\sum_{k=0}^{\infty}\gamma_k\frac{t^k}{k!},\qquad \frac{\Psi_t(r_2,t)}{\Psi(r_2,t)}=\sum_{k=0}^{\infty}\rho_k(r_2)\frac{t^k}{k!},\qquad \Psi_t(r_2,t)=\frac{\partial}{\partial t}\Psi(r_2,t) \tag{39} $$

and $D_{r_3}^{-1}$ *is the inverse of* D_{r_3}.

Proof. By differentiating both sides of equation (36) with respect to t, we obtain the following:

$$ \sum_{n=0}^{\infty}{}_{\mathcal{P}}\mathcal{L}\mathcal{R}_{n+1}(r_1,r_2,r_3)\frac{t^n}{n!}=\frac{\mathcal{R}'(t)}{\mathcal{R}(t)}\mathcal{R}(t)e^{r_1t}\Psi(r_2,t)C_0(r_3t)+r_1\mathcal{R}(t)e^{r_1t}\Psi(r_2,t)C_0(r_3t) $$

$$ +\frac{\Psi_t(r_2,t)}{\Psi(r_2,t)}\mathcal{R}(t)e^{r_1t}\Psi(r_2,t)C_0(r_3t)+\left(\sum_{n=0}^{\infty}\frac{r_3^n nt^{n-1}}{([n]!)^2}\right)\mathcal{R}(t)e^{r_1t}\Psi(r_2,t) \tag{40} $$

Using (39), we have

$$ \sum_{n=0}^{\infty}{}_{\mathcal{P}}\mathcal{L}\mathcal{R}_{n+1}(r_1,r_2,r_3)\frac{t^n}{n!}=\left(\sum_{k=0}^{\infty}\gamma_k\frac{t^k}{k!}\right)\left(\sum_{n=0}^{\infty}{}_{\mathcal{P}}\mathcal{L}\mathcal{R}_n(r_1,r_2,r_3)\frac{t^n}{n!}\right)+\left(\sum_{k=0}^{\infty}\rho_k(r_2)\frac{t^k}{k!}\right)\left(\sum_{n=0}^{\infty}{}_{\mathcal{P}}\mathcal{L}\mathcal{R}_n(r_1,r_2,r_3)\frac{t^n}{n!}\right) $$

$$ +r_1\sum_{n=0}^{\infty}{}_{\mathcal{P}}\mathcal{L}\mathcal{R}_n(r_1,r_2,r_3)\frac{t^n}{n!}+\left(\sum_{n=0}^{\infty}\frac{r_3^n nt^{n-1}}{([n]!)^2}\right)\mathcal{R}(t)e^{r_1t}\Psi(r_2,t). \tag{41} $$

Hence, when use the Cauchy product, we obtain

$$ \sum_{n=0}^{\infty}{}_{\mathcal{P}}\mathcal{L}\mathcal{R}_{n+1}(r_1,r_2,r_3)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}{}_{\mathcal{P}}\mathcal{L}\mathcal{R}_{n-k}(r_1,r_2,r_3)\gamma_k\frac{t^n}{n!}+\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\rho_k(r_2){}_{\mathcal{P}}\mathcal{L}\mathcal{R}_{n-k}(r_1,r_2,r_3)\frac{t^n}{n!} $$

$$ +r_1\sum_{n=0}^{\infty}{}_{\mathcal{P}}\mathcal{S}\mathcal{R}_n(r_1,r_2,r_3)\frac{t^n}{n!}+\left(\sum_{n=0}^{\infty}\frac{r_3^{n+1}(n+1)t^n}{([n+1]!)^2}\right)\mathcal{R}(t)e^{r_1t}\Psi(r_2,t). \tag{42} $$

Taking the derivative of both sides of the last equation with respect to r_3, we get

$$ \sum_{n=0}^{\infty}D_{r_3}\left\{{}_{\mathcal{P}}\mathcal{L}\mathcal{R}_{n+1}(r_1,r_2,r_3)\right]\frac{t^n}{n!}=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}D_{r_3}\left\{{}_{\mathcal{P}}\mathcal{L}\mathcal{R}_{n-k}(r_1,r_2,r_3)\right\}\gamma_k\frac{t^n}{n!} $$

$$+\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\rho_k(r_2)D_{r_3}\left\{{}_{\mathcal{P}}\mathcal{L}\mathcal{R}_{n-k}(r_1,r_2,r_3)\right\}\frac{t^n}{n!}+r_1\sum_{n=0}^{\infty}D_{r_3}\left\{{}_{\mathcal{P}}\mathcal{L}\mathcal{R}_n(r_1,r_2,r_3)\right\}\frac{t^n}{n!}$$
$$-n\sum_{n=0}^{\infty}{}_{\mathcal{P}}\mathcal{L}\mathcal{R}_{n-1}(r_1,r_2,r_3)\frac{t^n}{n!}. \tag{43}$$

Applying $D_{r_3}^{-1}$ to both sides of the above equation, we get

$$\sum_{n=0}^{\infty}{}_{\mathcal{P}}\mathcal{L}\mathcal{R}_{n+1}(r_1,r_2,r_3)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}{}_{\mathcal{P}}\mathcal{L}\mathcal{R}_{n-k}(r_1,r_2,r_3)\gamma_k\frac{t^n}{n!}+\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\rho_k(r_2){}_{\mathcal{P}}\mathcal{L}\mathcal{R}_{n-k}(r_1,r_2,r_3)\frac{t^n}{n!}$$
$$+r_1\sum_{n=0}^{\infty}{}_{\mathcal{P}}\mathcal{L}\mathcal{R}_n(r_1,r_2,r_3)\frac{t^n}{n!}-n\sum_{n=0}^{\infty}D_{r_3}^{-1}{}_{\mathcal{P}}\mathcal{L}\mathcal{R}_{n-1}(r_1,r_2,r_3)\frac{t^n}{n!}. \tag{44}$$

Thus, equating the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation (44), we get assertion (38).

Theorem 4. *The generalized Laguerre-based Appell polynomials ${}_{\mathcal{P}}\mathcal{L}\mathcal{R}_n(r_1,r_2,r_3)$ satisfy the multiplicative and derivative operators as follows:*

$$\hat{M}=r_1+\frac{\mathcal{R}'(\widehat{D}_{r_1})}{\mathcal{R}(\widehat{D}_{r_1})}+\frac{\Psi'(r_2,\widehat{D}_{r_1})}{\Psi(r_2,\widehat{D}_{r_1})}-nD_{r_3}^{-1}, \tag{45}$$

and

$$\hat{P}=D_{r_1}, \tag{46}$$

respectively.

Proof. Taking the derivative with respect to t on both sides of (36), we have

$$\sum_{n=0}^{\infty}{}_{\mathcal{P}}\mathcal{L}\mathcal{R}_{n+1}(r_1,r_2,r_3)\frac{t^n}{n!}=\frac{\mathcal{R}'(t)}{\mathcal{R}(t)}\mathcal{R}(t)e^{r_1t}\Psi(r_2,t)C_0(r_3t)+r_1\mathcal{R}(t)e^{r_1t}\Psi(r_2,t)C_0(r_3t)$$
$$+\frac{\Psi'(r_2,t)}{\Psi(r_2,t)}\mathcal{R}(t)e^{r_1t}\Psi(r_2,t)C_0(r_3t)+\left(\sum_{n=0}^{\infty}\frac{(-1)^n r_3^n n t^{n-1}}{([n]!)^2}\right)\mathcal{R}(t)e^{r_1t}\Psi(r_2,t) \tag{47}$$

$$\sum_{n=0}^{\infty}{}_{\mathcal{P}}\mathcal{S}\mathcal{R}_{n+1}(r_1,r_2,r_3)\frac{t^n}{n!}=\left(r_1+\frac{\mathcal{R}'(t)}{\mathcal{R}(t)}+\frac{\Psi'(r_2,t)}{\Psi(r_2,t)}\right)\mathcal{R}(t)e^{r_1t}\Psi(r_2,t)C_0(r_3t)$$
$$+\left(\sum_{n=0}^{\infty}\frac{(-1)^{n+1}r_3^{n+1}(n+1)t^n}{([n+1]!)^2}\right)\mathcal{R}(t)e^{r_1t}\Psi(r_2,t). \tag{48}$$

By using equation (36), we get

$$\sum_{n=0}^{\infty} {}_{\mathcal{P}}{}_{\mathcal{L}}\mathcal{R}_{n+1}(r_1,r_2,r_3)\frac{t^n}{n!} = \left(r_1 + \frac{\mathcal{R}'(t)}{\mathcal{R}(t)} + \frac{\Psi'(r_2,t)}{\Psi(r_2,t)}\right)\sum_{n=0}^{\infty} {}_{\mathcal{P}}{}_{\mathcal{L}}\mathcal{R}_n(r_1,r_2,r_3)\frac{t^n}{n!}$$
$$+\left(\sum_{n=0}^{\infty} \frac{(-1)^{n+1}r_3^{n+1}(n+1)t^n}{([n+1]!)^2}\right)\mathcal{R}(t)e^{r_1 t}\Psi(r_2,t). \tag{49}$$

Taking the derivative of both sides of the last equation with respect to r_3, we get

$$\sum_{n=0}^{\infty} D_{r_3}{}_{\mathcal{P}}{}_{\mathcal{L}}\mathcal{R}_{n+1}(r_1,r_2,r_3)\frac{t^n}{n!} = \left(r_1 + \frac{\mathcal{R}'(t)}{\mathcal{R}(t)} + \frac{\Psi'(r_2,t)}{\Psi(r_2,t)}\right)\sum_{n=0}^{\infty} D_{r_3}{}_{\mathcal{P}}{}_{\mathcal{L}}\mathcal{R}_n(r_1,r_2,r_3)\frac{t^n}{n!}$$
$$-n\sum_{n=0}^{\infty} {}_{\mathcal{P}}{}_{\mathcal{L}}\mathcal{R}_n(r_1,r_2,r_3)\frac{t^n}{n!}. \tag{50}$$

Applying $D_{r_3}^{-1}$ to both sides of the above equation, we get

$$\sum_{n=0}^{\infty} {}_{\mathcal{P}}{}_{\mathcal{L}}\mathcal{R}_{n+1}(r_1,r_2,r_3)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left(r_1 + \frac{\mathcal{R}'(t)}{\mathcal{R}(t)} + \frac{\Psi'(r_2,t)}{\Psi(r_2,t)} - nD_{r_3}^{-1}\right){}_{\mathcal{P}}{}_{\mathcal{L}}\mathcal{R}_n(r_1,r_2,r_3)\frac{t^n}{n!}. \tag{51}$$

In view of (16) and (51), we get the assertion (45).
Again in view of (17) and (36), we get the assertion (46).

Theorem 5. *The generalized Laguerre-based Appell polynomials ${}_{\mathcal{P}}{}_{\mathcal{L}}\mathcal{R}_n(r_1,r_2,r_3)$ satisfies the differential equation as follows:*

$$\left(r_1 D_{r_1} + \frac{\mathcal{R}'(\widehat{D}_{r_1})}{\mathcal{R}(\widehat{D}_{r_1})}D_{r_1} + \frac{\Psi'(r_2,\widehat{D}_{r_1})}{\Psi(r_2,\widehat{D}_{r_1})}D_{r_1} - nD_{r_1}D_{r_3}^{-1} - n\right){}_{\mathcal{P}}{}_{\mathcal{L}}\mathcal{R}_n(r_1,r_2,r_3) = 0. \tag{52}$$

Proof. In view of equations (45), (46) in (19), we get the assertion (52). So, we omit the proof.

3. Series representation and determinant form

Hybrid special polynomials play a crucial role in mathematical analysis due to their rich structural properties. Their series representation provides explicit forms and recurrence relations, aiding in solving differential and functional equations. The determinant form of these polynomials offers a compact and elegant way to analyze their algebraic and combinatorial properties. It facilitates the study of orthogonality, symmetry, and transformation identities. Hybrid polynomials also bridge classical and modern polynomial families, extending their applicability in mathematical physics and engineering. Their determinant representation helps in computing higher-order coefficients efficiently. These polynomials are widely used in approximation theory and numerical analysis. Overall, they contribute significantly to both theoretical and applied mathematical research.

Theorem 6. *The 3-variable generalized Laguerre-based Appell polynomials* ${}_{\mathcal{P}\mathcal{L}}\mathcal{R}_n(r_1, r_2, r_3)$ *are defined by the series:*

$$ {}_{\mathcal{P}\mathcal{L}}\mathcal{R}_n(r_1, r_2, r_3) = \sum_{k=0}^{n} \binom{n}{k} \mathcal{R}_k \ {}_{\mathcal{P}}\mathcal{L}_{n-k}(r_1, r_2, r_3), \tag{53} $$

with $\mathcal{R}_k$ *is given by equation* (5).

Proof. In view of equation (36), we can write

$$ \sum_{v=0}^{\infty} {}_{\mathcal{P}\mathcal{L}}\mathcal{R}_n(r_1, r_2, r_3) \frac{t^n}{n!} = \mathcal{R}(t) \sum_{n=0}^{\infty} {}_{\mathcal{P}}\mathcal{L}_n(r_1, r_2, r_3) \frac{t^n}{n!}. \tag{54} $$

Using the expansion (5) of $\mathcal{R}(t)$ from the left-hand side of equation (54), we simplify and then equate the coefficients of like powers of δ on both sides of the resulting equation, leading us to assertion (53).

Theorem 7. *The generalized Laguerre-based Appell polynomials* ${}_{\mathcal{P}\mathcal{L}}\mathcal{R}_n(r_1, r_2, r_3)$ *has the following determinant representation*

$$ {}_{\mathcal{P}\mathcal{L}}\mathcal{R}_{n,q}(r_1, r_2, r_3) = \frac{(-1)^n}{(\beta_0)^{n+1}} \begin{vmatrix} 1 & {}_{\mathcal{P}}\mathcal{L}_1(r_1, r_2, r_3) & {}_{\mathcal{P}}\mathcal{L}_2(r_1, r_2, r_3) & \cdots & {}_{\mathcal{P}}\mathcal{L}_{n-1}(r_1, r_2, r_3) & {}_{\mathcal{P}}\mathcal{L}_n^{(m)}(r_1, r_2, r_3) \\ \beta_0 & \beta_1 & \beta_2 & \cdots & \beta_{n-1} & \beta_n \\ 0 & \beta_0 & \binom{2}{1}\beta_1 & \cdots & \binom{n-1}{1}\beta_{n-2} & \binom{n}{1}\beta_{n-1} \\ 0 & 0 & \beta_0 & \cdots & \binom{n-1}{1}\beta_{n-3} & \binom{n}{2}\beta_{n-2} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \\ 0 & 0 & 0 & \cdots & \beta_0 & \binom{n}{n-1}\beta_1 \end{vmatrix}, \tag{55} $$

where $\sum_{n=0}^{\infty} {}_{\mathcal{P}}\mathcal{L}_n(r_1, r_2, r_3) \frac{t^n}{n!} = e^{r_1 t}\Psi(r_2, t) C_0(r_3 t)$, $\frac{1}{\mathcal{R}(t)} = \sum_{k=0}^{\infty} \beta_k \frac{t^k}{k!}$.

Proof. Using the series representation of $\frac{1}{\mathcal{R}(t)}$ as follows:

$$ [\mathcal{R}(t)]^{-1} = \sum_{k=0}^{\infty} \beta_k \frac{t^k}{k!}, $$

using the generation function (22), we get

$$ e^{r_1 t}\Psi(r_2, t) C_0(r_3 t) = \left(\sum_{k=0}^{\infty} \beta_k \frac{t^k}{k!} \right) \left(\sum_{n=0}^{\infty} {}_{\mathcal{P}\mathcal{L}}\mathcal{R}_n(r_1, r_2, r_3) \frac{t^n}{n!} \right). $$

Hence

$$ \sum_{n=0}^{\infty} {}_{\mathcal{P}\mathcal{L}}\mathcal{R}_n(r_1, r_2, r_3) \frac{t^n}{n!} = \left(\sum_{k=0}^{\infty} \beta_k \frac{t^k}{k!} \right) \left(\sum_{n=0}^{\infty} {}_{\mathcal{P}\mathcal{L}}\mathcal{R}_n(r_1, r_2, r_3) \frac{t^n}{n!} \right). $$

Applying the Cauchy product, we have

$$\sum_{n=0}^{\infty} {}_{\mathcal{P}\mathcal{L}}\mathcal{R}_n(r_1,r_2,r_3)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\beta_k\ {}_{\mathcal{P}\mathcal{L}}\mathcal{R}_{n-k}(r_1,r_2,r_3)\frac{t^n}{n!}.$$

By comparing the coefficients of $\frac{t^n}{n!}$ from the polynomial equation, we get

$${}_{\mathcal{P}\mathcal{L}}\mathcal{R}_n(r_1,r_2,r_3) = \sum_{k=0}^{n}\binom{n}{k}\beta_k\ {}_{\mathcal{P}\mathcal{L}}\mathcal{R}_{n-k}(r_1,r_2,r_3), \quad n\in\mathbb{N}_0.$$

So, we obtain the system of equations as follows:

$${}_{\mathcal{P}}\mathcal{L}_0(r_1,r_2,r_3) = \beta_0\ {}_{\mathcal{P}\mathcal{L}}\mathcal{R}_0(r_1,r_2,r_3),$$

$${}_{\mathcal{P}}\mathcal{L}_1(r_1,r_2,r_3) = \beta_0\ {}_{\mathcal{P}\mathcal{L}}\mathcal{R}_1(r_1,r_2,r_3) + \beta_1\ {}_{\mathcal{P}\mathcal{L}}\mathcal{R}_0(r_1,r_2,r_3),$$

$${}_{\mathcal{P}}\mathcal{L}_2(r_1,r_2,r_3) = \beta_0\ {}_{\mathcal{P}\mathcal{L}}\mathcal{R}_2(r_1,r_2,r_3) + \binom{2}{1}\beta_1\ {}_{\mathcal{P}\mathcal{L}}\mathcal{R}_1(r_1,r_2,r_3) + \beta_2\ {}_{\mathcal{P}\mathcal{L}}\mathcal{R}_0(r_1,r_2,r_3),$$

$$\vdots$$

$${}_{\mathcal{P}}\mathcal{L}_{n-1}(r_1,r_2,r_3) = \beta_0\ {}_{\mathcal{P}\mathcal{L}}\mathcal{R}_{n-1}(r_1,r_2,r_3)+\binom{n-1}{1}\beta_1\ {}_{\mathcal{P}\mathcal{L}}\mathcal{R}_{n-2}(r_1,r_2,r_3)+\cdots+\beta_{n-1}\ {}_{\mathcal{P}\mathcal{L}}\mathcal{R}_0(r_1,r_2,r_3),$$

$${}_{\mathcal{P}\mathcal{L}}\mathcal{R}_n(r_1,r_2,r_3) = \beta_0\ {}_{\mathcal{P}\mathcal{L}}\mathcal{R}_n(r_1,r_2,r_3)+\binom{n}{1}\beta_1\ {}_{\mathcal{P}\mathcal{L}}\mathcal{R}_{n-1}(r_1,r_2,r_3)+\cdots+\beta_n\ {}_{\mathcal{P}\mathcal{L}}\mathcal{R}_0(r_1,r_2,r_3).$$

Applying Cramers' rule, we get

$${}_{\mathcal{P}\mathcal{L}}\mathcal{R}_n(r_1,r_2,r_3) = \frac{\begin{vmatrix} \beta_0 & 0 & ... & 0 & {}_{\mathcal{P}}\mathcal{L}_0(r_1,r_2,r_3) \\ \beta_1 & \beta_0 & ... & 0 & {}_{\mathcal{P}}\mathcal{L}_1(r_1,r_2,r_3) \\ \beta_2 & \binom{2}{1}\beta_1 & ... & 0 & {}_{\mathcal{P}}\mathcal{L}_2(r_1,r_2,r_3) \\ \beta_3 & \binom{3}{2}\beta_2 & ... & 0 & {}_{\mathcal{P}}\mathcal{L}_3(r_1,r_2,r_3) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \beta_{n-1} & \binom{n-1}{1}\beta_{n-2} & ... & \beta_0 & {}_{\mathcal{P}}\mathcal{L}_{n-1}(r_1,r_2,r_3) \\ \beta_n & \binom{n}{1}\beta_{n-1} & ... & \binom{n}{n-1}\beta_1 & {}_{\mathcal{P}}\mathcal{L}_m(r_1,r_2,r_3) \end{vmatrix}}{\begin{vmatrix} \beta_0 & 0 & ... & 0 & 0 \\ \beta_1 & \beta_0 & ... & 0 & 0 \\ \beta_2 & \binom{2}{1}\beta_1 & ... & 0 & 0 \\ \beta_3 & \binom{3}{2}\beta_2 & ... & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \beta_{n-1} & \binom{n-1}{1}\beta_{n-2} & ... & \beta_0 & 0 \\ \beta_n & \binom{n}{1}\beta_{n-1} & ... & \binom{n}{n-1}\beta_1 & \beta_0 \end{vmatrix}}.$$

By taking the transpose in the last equation, we have

$$
{}_{\mathcal{P}\mathcal{L}}\mathcal{R}_n(r_1,r_2,r_3)=\frac{1}{(\beta_0)^{n+1}}\begin{vmatrix}
\beta_0 & \beta_1 & \cdots & \beta_{n-1} & \beta_n \\
0 & \beta_0 & \cdots & \binom{n-1}{1}\beta_{n-2} & \binom{n}{1}\beta_{n-1} \\
0 & 0 & \cdots & \binom{n-1}{1}\beta_{n-3} & \binom{n}{2}\beta_{n-2} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & \beta_0 & \binom{n}{n-1}\beta_1 \\
{}_{\mathcal{P}}\mathcal{L}_0(r_1,r_2,r_3) & {}_{\mathcal{P}}\mathcal{L}_1(r_1,r_2,r_3) & \cdots & {}_{\mathcal{P}}\mathcal{L}_{n-1}(r_1,r_2,r_3) & {}_{\mathcal{P}}\mathcal{L}_n^{(m)}(r_1,r_2,r_3)
\end{vmatrix}.
$$

Thus, simple row operations are used to finish the proof.

4. Applications

This study extends the exploration of recently introduced polynomials, focusing on the examination of the generalization of three-variable Laguerre-based Appell polynomials. Specifically, when considering the case where $\Psi(r_2,t)=e^{r_2t^2}$ in the generating function (36), which results in the reduction of $3VLAP$ ${}_{\mathcal{P}\mathcal{L}}\mathcal{R}_n(r_1,r_2,r_3)$ to the Laguerre-Hermite-Appell polynomials $(LHAP)$ ${}_{\mathcal{P}\mathcal{H}}\mathcal{R}_n(r_1,r_2,r_3)$ can be characterized by a specific generating function.

$$\mathcal{R}(t)e^{r_1t+r_2t^2}C_0(r_3t)=\sum_{n=0}^{\infty}{}_{\mathcal{P}\mathcal{H}}\mathcal{R}_n(r_1,r_2,r_3)\frac{t^n}{n!}. \tag{56}$$

In other words, we note that

$${}_{\mathcal{P}\mathcal{H}}\mathcal{R}_n(r_1,r_2,r_3)=\exp\left(-\widehat{D}_{r_3}^{-1}\frac{\partial}{\partial r_1}\right)\{{}_H\mathcal{R}_n(r_1,r_2)\}=\exp\left(r_2\frac{\partial^2}{\partial r_1^2}\right)\{L\mathcal{R}_n(r_1,r_3)\}. \tag{57}$$

Theorem 8. *The three variable Laguerre-Hermite-based Appell polynomials are defined by the series:*

$${}_{\mathcal{P}\mathcal{H}}\mathcal{R}_n(r_1,r_2,r_3)=\sum_{k=0}^{n}\binom{n}{k}\mathcal{R}_k\ {}_{\mathcal{P}\mathcal{H}}{}_{n-k}(r_1,r_2,r_3). \tag{58}$$

Proof. In view of (56), we have

$$\sum_{n=0}^{\infty}{}_{\mathcal{P}\mathcal{H}}\mathcal{R}_n(r_1,r_2,r_3)\frac{t^n}{n!}=\mathcal{R}(t)\sum_{n=0}^{\infty}{}_{\mathcal{P}}\mathcal{H}_n(r_1,r_2,r_3)\frac{t^n}{n!}. \tag{59}$$

Now, by using the expansion of $\mathcal{R}(t)$ from the left-hand side of equation (59), we can simplify and equate the coefficients of like powers of t on both sides of the resulting equation to obtain assertion (58).

Next, we will demonstrate the determinant form for ${}_{\mathcal{P}\mathcal{H}}\mathcal{R}_n(r_1,r_2,r_3)$ using an approach similar to that presented in [28, 29], taking into account equation (56).

Theorem 9. *The determinant representation of 3-variable Laguerre-Hermite-Appell polynomials* ${}_{\mathcal{P}\mathcal{H}}\mathcal{R}_n(r_1, r_2, r_3)$ *of degree* n *is*

$$ {}_{\mathcal{P}\mathcal{H}}\mathcal{R}_0(r_1, r_2, r_3) = \frac{1}{\beta_0}, $$

$$ {}_{\mathcal{P}\mathcal{H}}\mathcal{R}_n(r_1, r_2, r_3) = \frac{(-1)^n}{(\beta_0)^{n+1}} \begin{vmatrix} 1 & {}_{\mathcal{P}}\mathcal{H}_1(r_1, r_2, r_3) & {}_{\mathcal{P}}\mathcal{H}_2(r_1, r_2, r_3) & \cdots & {}_{\mathcal{P}}\mathcal{H}_{n-1}(r_1, r_2, r_3) & {}_{\mathcal{P}}\mathcal{H}_n(r_1, r_2, r_3) \\ \beta_0 & \beta_1 & \beta_2 & \cdots & \beta_{n-1} & \beta_n \\ 0 & \beta_0 & \binom{2}{1}\beta_1 & \cdots & \binom{n-1}{1}\beta_{n-2} & \binom{n}{1}\beta_{n-1} \\ 0 & 0 & \beta_0 & \cdots & \binom{n-1}{1}\beta_{n-3} & \binom{n}{2}\beta_{n-2} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \\ 0 & 0 & 0 & \cdots & \beta_{0,q} & \binom{n}{n-1}\beta_1 \end{vmatrix}, \tag{60} $$

$$ \beta_n = -\frac{1}{\mathcal{R}_0}\left(\sum_{k=1}^{n}\binom{n}{k}\mathcal{R}_k\beta_{n-k}\right), \quad n = 0, 1, 2, \ldots, $$

where $\beta_0 \neq 0$, $\beta_0 = \frac{1}{\mathcal{R}_{0,q}}$ *and* ${}_{\mathcal{P}}\mathcal{H}_n(r_1, r_2, r_3)$, $n = 0, 1, 2, \ldots,$ *are the three variable q-Legendre-Hermite polynomials.*

Proof. By inserting the series forms of the new generalization of the three-variable Laguerre-Hermite polynomials into the generating function of the three-variable Laguerre-Hermite-Appell polynomials, we obtain:

$$ \mathcal{R}(t)\sum_{n=0}^{\infty} {}_{\mathcal{P}}\mathcal{H}_n(r_1, r_2, r_3)\frac{t^n}{n!} = \sum_{n=0}^{\infty} {}_{\mathcal{P}\mathcal{H}}\mathcal{R}_n(r_1, r_2, r_3)\frac{t^n}{n!}. \tag{61} $$

By multiplying

$$ \frac{1}{\mathcal{R}(t)} = \sum_{k=0}^{\infty}\beta_k\frac{t^k}{k!}, \tag{62} $$

on both sides, it follows that

$$ \sum_{n=0}^{\infty} {}_{\mathcal{P}}\mathcal{H}_n(r_1, r_2, r_3)\frac{t^n}{n!} = \sum_{k=0}^{\infty}\beta_k\frac{t^k}{k!}\sum_{n=0}^{\infty} {}_{\mathcal{P}\mathcal{H}}\mathcal{R}_n(r_1, r_2, r_3)\frac{t^n}{n!}. \tag{63} $$

Applying Cauchy product in (63) gives

$$ {}_{\mathcal{P}}\mathcal{H}_n(r_1, r_2, r_3) = \sum_{k=0}^{n}\binom{n}{k}\beta_k \; {}_{\mathcal{P}\mathcal{H}}\mathcal{R}_{n-k}(r_1, r_2, r_3). \tag{64} $$

This equality leads to a system of n equations with the unknowns $\mathcal{R}_n(r_1, r_2, r_3)$, where $n = 0, 1, 2, \ldots$

To solve this system using Cramer's rule, we note that the denominator is the determinant of a lower triangular matrix, which has a determinant of $(\beta_0)^{n+1}$. By taking the transpose of the numerator and replacing the i^{th} row with the $(i+1)^{th}$ position for $i = 1, 2, \ldots, n-1$, we obtain the desired result.

We will now demonstrate the multiplicative and derivative operators of ${}_{\mathcal{P}}\mathcal{H}\mathcal{R}_n(r_1, r_2, r_3)$. The following theorem is presented:

Theorem 10. *The generalization of Laguerre-Hermite-based Appell polynomials satisfies the multiplicative and derivative operators as follows:*

$$\hat{M} = r_1 + \frac{\mathcal{R}'(\widehat{D}_{r_1})}{\mathcal{R}(\widehat{D}_{r_1})} - n\left(r_2 + D_{r_3}^{-1}\right), \tag{65}$$

and

$$\hat{P} = D_{r_1}, \tag{66}$$

respectively.

Proof. Utilizing the derivative with respect to t on both sides of equation (56), we find

$$\sum_{n=0}^{\infty} {}_{\mathcal{P}}\mathcal{H}\mathcal{R}_{n+1}(r_1, r_2, r_3)\frac{t^n}{n!} = \frac{\mathcal{R}'(t)}{\mathcal{R}(t)}\mathcal{R}(t)e^{r_1t+r_2t^2}C_0(r_3t) + r_1\mathcal{R}(t)e^{r_1t+r_2t^2}C_0(r_3t) + 2r_2t\mathcal{R}(t)e^{r_1t+r_2t^2}C_0(r_3t)$$
$$+\left(\sum_{n=0}^{\infty}\frac{(-1)^n r_3^n n t^{n-1}}{([n]!)^2}\right)\mathcal{R}(t)e^{r_1t+r_2t^2} \tag{67}$$

$$\sum_{n=0}^{\infty} {}_{\mathcal{P}}\mathcal{H}\mathcal{R}_{n+1}(r_1, r_2, r_3)\frac{t^n}{n!} = \left(r_1 + \frac{\mathcal{R}'(t)}{\mathcal{R}(t)} - nr_2\right)\mathcal{R}(t)e^{r_1t+r_2t2}C_0(r_3t)$$
$$+\left(\sum_{n=0}^{\infty}\frac{(-1)^{n+1} r_3^{n+1}(n+1)t^n}{([n+1]!)^2}\right)\mathcal{R}(t)e^{r_1t+r_2t^2}. \tag{68}$$

By using equation (56), we get

$$\sum_{n=0}^{\infty} {}_{\mathcal{P}}\mathcal{H}\mathcal{R}_{n+1}(r_1, r_2, r_3)\frac{t^n}{n!} = \left(r_1 + \frac{\mathcal{R}'(t)}{\mathcal{R}(t)} - nr_2\right)\sum_{n=0}^{\infty} {}_{\mathcal{P}}\mathcal{H}\mathcal{R}_n(r_1, r_2, r_3)\frac{t^n}{n!}$$
$$+\left(\sum_{n=0}^{\infty}\frac{(-1)^{n+1} r_3^{n+1}(n+1)t^n}{([n+1]!)^2}\right)\mathcal{R}(t)e^{r_1t+r_2t^2}. \tag{69}$$

On differentiating both sides of the last equation with respect to r_3, we obtain:

$$\sum_{n=0}^{\infty} D_{r_3}{}_{\mathcal{P}}\mathcal{H}\mathcal{R}_{n+1}(r_1, r_2, r_3)\frac{t^n}{n!} = \left(r_1 + \frac{\mathcal{R}'(t)}{\mathcal{R}(t)} - nr_2\right)\sum_{n=0}^{\infty} D_{r_3}{}_{\mathcal{P}}\mathcal{H}\mathcal{R}_n(r_1, r_2, r_3)\frac{t^n}{n!}$$
$$-n\sum_{n=0}^{\infty} {}_{\mathcal{P}}\mathcal{H}\mathcal{R}_n(r_1, r_2, r_3)\frac{t^n}{n!}. \tag{70}$$

Applying $D_{r_3}^{-1}$ to both sides of the above equation, we get

$$\sum_{n=0}^{\infty} {}_{\mathcal{P}}\mathcal{H}\mathcal{R}_{n+1}(r_1, r_2, r_3)\frac{t^n}{n!} = \sum_{n=0}^{\infty} \left(r_1 + \frac{\mathcal{R}'(t)}{\mathcal{R}(t)} - n\left(r_2 + D_{r_3}^{-1}\right)\right) {}_{\mathcal{P}}\mathcal{H}\mathcal{R}_n(r_1, r_2, r_3)\frac{t^n}{n!}. \quad (71)$$

In view of (16) and (71), we get the assertion (65).
Again in view of (17) and (56), we get the assertion (66).

Theorem 11. *The following differential equation for* ${}_{\mathcal{P}}\mathcal{H}\mathcal{R}_n(r_1, r_2, r_3)$ *holds true:*

$$\left(r_1 D_{r_1} + \frac{\mathcal{R}'(D_{r_1})}{\mathcal{R}(D_{r_1})} - n\left(r_2 + D_{r_3}^{-1}\right) D_{r_1} - n \right) {}_{\mathcal{P}}\mathcal{H}\mathcal{R}_n(r_1, r_2, r_3) = 0. \quad (72)$$

Proof. Using (65) and (66) in (19), we get

$$\left(r_1 D_{r_1} + \frac{\mathcal{R}'(\widehat{D}_{r_1})}{\mathcal{R}(\widehat{D}_{r_1})} D_{r_1} - n\left(r_2 + D_{r_3}^{-1}\right) D_{r_1} \right) {}_{\mathcal{P}}\mathcal{H}\mathcal{R}_n(r_1, r_2, r_3) = n\ {}_{\mathcal{P}}\mathcal{H}\mathcal{R}_n(r_1, r_2, r_3). \quad (73)$$

Upon the simplification, we get the assertion (72).

5. Examples

The Appell polynomial family, defined by the parameter function $\mathcal{R}(t)$, provides solutions to specific differential equations. Different choices of $\mathcal{R}(t)$ generate various polynomials, allowing adaptability in mathematical modeling. This flexibility makes them valuable in physics, engineering, and other scientific fields. Table 1 systematically presents their generating functions, series definitions, and numerical values. Generating functions offer concise power series representations, aiding analytical manipulation. Series definitions provide formal expressions crucial for problem-solving and computation. Numerical values enhance practical understanding and facilitate real-world applications. These polynomials are widely used in probability theory, quantum mechanics, and signal processing. Their versatility allows specialized solutions for complex mathematical problems. Overall, the Appell polynomial family serves as a powerful tool in scientific research.

The Bernoulli, Euler, and Genocchi numbers are foundational in mathematics, with applications in number theory, combinatorics, algebraic geometry, and more. Bernoulli numbers appear in polynomials and the Euler-Maclaurin formula, while Euler numbers contribute to modular forms and elliptic curve theory. Genocchi numbers play a key role in combinatorial problems, graph theory, and automata theory. These numbers connect to hyperbolic secant functions and have implications in quantum field theory and signal processing. By treating them as members of the Appell family, new polynomials like the two-iterated degenerate Hermite-Appell polynomials are derived, offering rich research opportunities in their generating expressions and characteristics. As a result, different members of ${}_{\mathcal{P}}\mathcal{H}\mathcal{R}_n(r_1, r_2, r_3)$ appear as Laguerre-Hermite-based Bernoulli polynomials ${}_{\mathcal{P}\mathcal{H}}\mathbb{B}_n(r_1, r_2, r_3)$, Laguerre-Hermite-Euler polynomials ${}_{\mathcal{P}\mathcal{H}}\mathbb{E}_n(r_1, r_2, r_3)$, and Laguerre-Hermite-Genocchi polynomials ${}_{\mathcal{P}\mathcal{H}}\mathbb{G}_n(r_1, r_2, r_3)$. The following expressions can be used to cast these polynomials:

$$\frac{t}{e^t-1}e^{r_1t+r_2t^2}C_0(r_3t) = \sum_{n=0}^{\infty} {}_{\mathcal{P}\mathcal{H}}\mathbb{B}_n(r_1, r_2, r_3)\frac{t^n}{n!}, \tag{74}$$

$$\frac{2}{e^t+1}e^{r_1t+r_2t^2}C_0(r_3t) = \sum_{n=0}^{\infty} {}_{\mathcal{P}\mathcal{H}}\mathbb{E}_n(r_1, r_2, r_3)\frac{t^n}{n!}, \tag{75}$$

and

$$\frac{2t}{e^t+1}e^{r_1t+r_2t^2}C_0(r_3t) = \sum_{n=0}^{\infty} {}_{\mathcal{P}\mathcal{H}}\mathbb{G}_n(r_1, r_2, r_3)\frac{t^n}{n!}. \tag{76}$$

For instance the Laguerre-Hermite-based Bernoulli polynomials ${}_{\mathcal{P}\mathcal{H}}\mathbb{B}_n(r_1, r_2, r_3)$, Laguerre-Hermite-Euler polynomials ${}_{\mathcal{P}\mathcal{H}}\mathbb{E}_n(r_1, r_2, r_3)$, and Laguerre-Hermite-Genocchi polynomials ${}_{\mathcal{P}\mathcal{H}}\mathbb{G}_n(r_1, r_2, r_3)$ are defined by the following operational identities:

$${}_{\mathcal{P}\mathcal{H}}\mathbb{B}_n(r_1, r_2, r_3) = \exp\left(-\widehat{D}_{r_3}^{-1}\frac{\partial}{\partial r_1}\right)\{{}_H\mathbb{B}_n(r_1, r_2)\} = \exp\left(r_2\frac{\partial^2}{\partial r_1^2}\right)\{{}_L\mathbb{B}_n(r_1, r_3)\}, \tag{77}$$

$${}_{\mathcal{P}\mathcal{H}}\mathbb{E}_n(r_1, r_2, r_3) = \exp\left(-\widehat{D}_{r_3}^{-1}\frac{\partial}{\partial r_1}\right)\{{}_H\mathbb{E}_n(r_1, r_2)\} = \exp\left(r_2\frac{\partial^2}{\partial r_1^2}\right)\{{}_L\mathbb{E}_n(r_1, r_3)\}, \tag{78}$$

and

$${}_{\mathcal{P}\mathcal{H}}\mathbb{G}_n(r_1, r_2, r_3) = \exp\left(-\widehat{D}_{r_3}^{-1}\frac{\partial}{\partial r_1}\right)\{{}_H\mathbb{G}_n(r_1, r_2)\} = \exp\left(r_2\frac{\partial^2}{\partial r_1^2}\right)\{{}_L\mathbb{G}_n(r_1, r_3)\}. \tag{79}$$

Similarly, a similar method can be used to draw corresponding results for these polynomials. The monomiality principle, which looks at how polynomials behave in respect to their monomial coefficients, is one such result. Extensive study of the monomiality principle provides important information about the structure and characteristics of these polynomials.

Moreover, the polynomials' explicit expressions can be found, offering a distinct depiction of their coefficients and terms. This makes it easier to comprehend and analyze the polynomials. Examining the differential equations that these polynomials satisfy is another line of inquiry. There are relationships between these polynomials and other mathematical ideas that can be found by investigating the related differential equations. It also enables in-depth comprehension by enabling researchers to examine their behavior in a variety of scenarios.

Furthermore, determinant forms investigation for these polynomials offers an important and interesting research avenue. You can use determinants, which are mathematical entities that describe particular aspects of matrices, to make links between these polynomials and linear algebra. This endeavor creates new opportunities for further understanding and applications in a larger mathematical environment.

Furthermore, in view of expressions (60), the polynomials ${}_{\mathcal{P}\mathcal{H}}\mathbb{B}_n(r_1, r_2, r_3)$, ${}_{\mathcal{P}\mathcal{H}}\mathbb{E}_n(r_1, r_2, r_3)$ and ${}_{\mathcal{P}\mathcal{H}}\mathbb{G}_n(r_1, r_2, r_3)$ satisfy the following determinant representations:

$$
{}_{\mathcal{P}\mathcal{H}}\mathbb{B}_n(r_1,r_2,r_3) = (-1)^n \begin{vmatrix} 1 & {}_{\mathcal{P}}\mathcal{H}_1(r_1,r_2,r_3) & {}_{\mathcal{P}}\mathcal{H}_2(r_1,r_2,r_3) & \cdots & {}_{\mathcal{P}}\mathcal{H}_{n-1}(r_1,r_2,r_3) & {}_{\mathcal{P}}\mathcal{H}_n(r_1,r_2,r_3) \\ 1 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n} & \frac{1}{n+1} \\ 0 & 1 & \binom{2}{1}\frac{1}{2} & \cdots & \binom{n-1}{1}\frac{1}{n-1} & \binom{n}{1}\frac{1}{n} \\ 0 & 0 & 1 & \cdots & \binom{n-1}{2}\frac{1}{n-2} & \binom{n}{2}\frac{1}{n-1} \\ . & . & . & \cdots & . & . \\ . & . & . & \cdots & . & . \\ 0 & 0 & 0 & \cdots & 1 & \binom{n}{n-1}\frac{1}{2} \end{vmatrix}, \tag{80}
$$

$$
{}_{\mathcal{P}\mathcal{H}}\mathbb{E}_n(r_1,r_2,r_3) = (-1)^n \begin{vmatrix} 1 & {}_{\mathcal{P}}\mathcal{H}_1(r_1,r_2,r_3) & {}_{\mathcal{P}}\mathcal{H}_2(r_1,r_2,r_3) & \cdots & {}_{\mathcal{P}}\mathcal{H}_{n-1}(r_1,r_2,r_3) & {}_{\mathcal{P}}\mathcal{H}_n(r_1,r_2,r_3) \\ 1 & \frac{1}{2} & \frac{1}{2}(1)_2 & \cdots & \frac{1}{2}(1)_{n-1} & \frac{1}{2}(1)_n \\ 0 & 1 & \binom{2}{1}\frac{1}{2} & \cdots & \binom{n-1}{1}\frac{1}{2}(1)_{n-2} & \binom{n}{1}\frac{1}{2}(1)_{n-1} \\ 0 & 0 & 1 & \cdots & \binom{n-1}{2}\frac{1}{2}(1)_{n-3} & \binom{n}{2}\frac{1}{2}(1)_{n-2} \\ . & . & . & \cdots & . & . \\ . & . & . & \cdots & . & . \\ 0 & 0 & 0 & \cdots & 1 & \binom{n}{n-1}\frac{1}{2} \end{vmatrix}, \tag{81}
$$

and

$$
{}_{\mathcal{P}\mathcal{H}}\mathbb{G}_n(r_1,r_2,r_3) = (-1)^n \begin{vmatrix} 1 & {}_{\mathcal{P}}\mathcal{H}_1(r_1,r_2,r_3) & {}_{\mathcal{P}}\mathcal{H}_2(r_1,r_2,r_3) & \cdots & {}_{\mathcal{P}}\mathcal{H}_{n-1}(r_1,r_2,r_3) & {}_{\mathcal{P}}\mathcal{H}_n(r_1,r_2,r_3) \\ 0 & \frac{1}{2} & \binom{2}{1}\frac{(1)_2}{4} & \cdots & \binom{n-1}{1}\frac{(1)_{n-1}}{2(n-1)} & \binom{n}{1}\frac{(1)_n}{2n} \\ 0 & 0 & \frac{1}{2} & \cdots & \binom{n-1}{2}\frac{(1)_{n-2}}{2(n-2)} & \binom{n}{2}\frac{(1)_{n-1}}{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{2} & \binom{n}{n-1}\frac{(1)_2}{4} \end{vmatrix}. \tag{82}
$$

6. Concluding remarks

In this study, we introduced and systematically analyzed a new generalization of Laguerre and Laguerre-based Appell polynomials. Through a detailed exploration of their fundamental properties, we established recurrence relations, multiplicative and derivative operators, and a governing differential equation via quasi-monomiality. The derivation of both series and determinant representations further highlights the structural depth of this novel polynomial family. Additionally, the introduction of generalized Laguerre-Hermite Appell polynomials, along with their specific cases involving Bernoulli, Euler, and Genocchi polynomials, enriches the framework of special functions. These findings contribute to the broader understanding of polynomial sequences and their applications in mathematical physics and differential equations.

For future research, the investigation of these polynomials in the context of orthogonality and integral transforms could provide further insights into their analytical properties. Exploring their connections with fractional calculus and special function theory may yield new results with applications in approximation theory and signal processing. Furthermore, the study of their extensions in the framework of q-calculus and number theory could reveal deeper algebraic and combinatorial properties. Finally, the numerical aspects and computational implementations of these polynomials can be explored for potential applications in scientific computing and engineering.

Availability of data and materials

Not applicable.

Competing interests

The authors declare no competing interests.

Authors' contributions

All authors contributed equally to the article.

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References

[1] G Dattoli and A Torre. Operational methods and two variable laguerre polynomials. *Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur*, 132:3–9, 1998.

[2] Noor Alam, Waseem Ahmad Khan, Can Kızılateş, and Cheon Seoung Ryoo. Two-variable q-general-appell polynomials within the context of the monomiality principle. *Mathematics*, 13(5):765, 2025.

[3] Naeem Ahmad and Waseem Ahmad Khan. Insights into new generalization of q-legendre-based appell polynomials: Properties and quasi monomiality. *Mathematics*, 13(6):955, 2025.

[4] Naeem Ahmad and Waseem Ahmad Khan. A new generalization of q-laguerre-based appell polynomials and quasi-monomiality. *Symmetry*, 17(3):439, 2025.

[5] Waseem Ahmad Khan, Divesh Srivastava, and Kottakkaran Soppy Nisar. A new class of generalized polynomials associated with milne-thomsons-based poly-bernoulli polynomials. *Miskolc Mathematical Journal*, 25(2):793–803, 2024.

[6] Waseem A Khan, Jihad Younis, and Mohd Nadeem. Construction of partially degenerate laguerre–bernoulli polynomials of the first kind. *Applied Mathematics in Science and Engineering*, 30(1):362–375, 2022.

[7] T. Kanan, M. Elbes, K. Abu Maria, and M. Alia. Exploring the potential of iot-based learning environments in education. *International Journal of Advances in Soft Computing and its Applications*, 15, 2023.

[8] Duha Abu Judeh and M Abu Hammad. Applications of conformable fractional pareto probability distribution. *Int. J. Advance Soft Compu. Appl*, 14(2):115–124, 2022.

[9] Muhammad Nazam, Hassen Aydi, Mohd Salmi Noorani, and Haitham Qawaqneh. Existence of fixed points of four maps for a new generalized f-contraction and an application. *Journal of Function Spaces*, 2019(1):5980312, 2019.

[10] Haitham Qawaqneh, Mohd Salmi Noorani, Hassen Aydi, and Wasfi Shatanawi. On common fixed point results for new contractions with applications to graph and integral equations. *Mathematics*, 7(11):1082, 2019.

[11] Mohra Zayed, Waseem Ahmad Khan, Cheon Seoung Ryoo, and Ugur Duran. An exploratory study on bivariate extended q-laguerre-based appell polynomials with some applications. *AIMS Mathematics*, 10(6):12841–12867, 2025.

[12] Larry C Andrews. *Special functions of mathematics for engineers*, volume 49. Spie Press, 1998.

[13] Paul Appell. Sur une classe de polynômes. In *Annales scientifiques de l'École normale supérieure*, volume 9, pages 119–144, 1880.

[14] RP Boas Jr. Higher transcendental functions. vol. iii. based in part on notes left by harry bateman. bateman project staff, a. erdélyi, ed. mcgraw-hill, new york-london, 1955. xviii+ 292 pp. 6.50. $Science, 122(3163) : 290 - -290, 1955.$

[15] Subuhi Khan and Nusrat Raza. Monomiality principle, operational methods and family of laguerre–sheffer polynomials. *Journal of Mathematical Analysis and Applications*, 387(1):90–102, 2012.

[16] I. Sheffer. Some properties of polynomial sets of type zero. 1939.

[17] Waseem Ahmad Khan, Khidir Shaib Mohamed, Francesco Aldo Costabile, Can Kızılates, and Cheon Seoung Ryoo. Finding the q-appell convolution of certain polynomials within the context of quantum calculus. *relation*, 6:8, 2025.

[18] J. Steffensen. The poweroid, an extension of the mathematical notion of power. 1941.

[19] G Dattoli. Hermite-bessel and laguerre-bessel functions: A by-product ot the monomiality principle, advanced special functions and applications. *Proceedings of the Melfi School on Advanced Topics in Mathematics and Physics*, pages 147–164.

[20] Subuhi Khan, Mumtaz Riyasat, and Shahid Ahmad Wani. On some classes of differential equations and associated integral equations for the laguerre–appell polynomials. *Advances in Pure and Applied Mathematics*, 9(3):185–194, 2018.

[21] Waseem Ahmad Khan, Khidir Shaib Mohamed, Francesco Aldo Costabile, Shahid Ahmad Wani, and Alawia Adam. A new generalization of m th-order laguerre-based appell polynomials associated with two-variable general polynomials. *Mathematics*, 13(13):2179, 2025.

[22] Waseem Ahmad Khan, Mofareh Alhazmi, and Tabinda Nahid. A novel family of q-mittag-leffler-based bessel and tricomi functions via umbral approach. *Symmetry*, 16(12):1580, 2024.

[23] Waseem Ahmad Khan, Ugur Duran, Jihad Younis, and Cheon Seoung Ryoo. On some extensions for degenerate frobenius-euler-genocchi polynomials with applications in computer modeling. *Applied Mathematics in Science and Engineering*, 32(1):2297072, 2024.

[24] Waseem Ahmad Khan and Maryam Salem Alatawi. A note on modified degenerate changhee–genocchi polynomials of the second kind. *Symmetry*, 15(1):136, 2023.

[25] Haitham Qawaqneh, Mohd Salmi Md Noorani, Hassen Aydi, Amjed Zraiqat, and Arslan Hojat Ansari. On fixed point results in partial b-metric spaces. *Journal of Function Spaces*, 2021(1):8769190, 2021.

[26] Haitham Qawaqneh, Mohd Salmi Md Noorani, and Hassen Aydi. Some new characterizations and results for fuzzy contractions in fuzzy b-metric spaces and applications. *AIMS Mathematics*, 8(3):6682–6696, 2023.

[27] H Qawaqneh, HA Hammad, and H Aydi. Exploring new geometric contraction mappings and their applications in fractional metric spaces, aims math. 9 (2024), 521–541.

[28] Francesco Aldo Costabile, Maria Italia Gualtieri, and Anna Napoli. General bivariate appell polynomials via matrix calculus and related interpolation hints. *Mathematics*, 9(9):964, 2021.

[29] Francesco A Costabile, Maria Italia Gualtieri, and Anna Napoli. Bivariate general appell interpolation problem. *Numerical Algorithms*, 91(2):531–556, 2022.