EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 3, Article Number 6659
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

On Fixed Points in Complex Valued Controlled S-Metric Spaces and an Application

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Abstract. In this paper, we present complex valued controlled S-metric spaces, a new generalisation of controlled S-metric spaces. This generalization is also a new extension of the notion of a complex valued S_b-metric space, which differs from the complex valued extended S_b-metric space. Moreover, in this newly generalized notion, we derive certain fixed point results along with an example. Many results from the existing literature are also derived as corollaries of our main results. As an application of our result, we demonstrate the existence of a solution of a Volterra integral equation.

2020 Mathematics Subject Classifications: 47H10, 54H25

Key Words and Phrases: Fixed point, Complex valued controlled S-metric spaces, Volterra integral equation

1. Introduction

The framework of fixed point theory serves as a fundamental tool in understanding the behavior of mappings and has found extensive applications in diverse branches of

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DOI: https://doi.org/10.29020/nybg.ejpam.v18i3.6659

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mathematics, including numerical analysis, optimization, and mathematical modeling, see [1–4, 4–7]. Fréchet [8] initially put forth the metric space idea in 1906. Subsequently, numerous researchers have extended the concept of metric space by altering the metric function and weakening different conditions (see, for example, [9–14]). In 1993, Czerwik [12] introduced the notion of a b-metric space as an extension of the concept of a metric space. To broaden the definition of b-metric space, Kamran et al. [15] presented the notion of an extended b-metric space in the year 2017. Mlaiki et al. [16] further introduced controlled metric space, which is a novel type of extended b-metric space.

Sedghi et al. [17] presented S-metric spaces in 2012 as a generalization of G-metric spaces [18] and D^*-metric spaces [19]. Several fixed point theorems for S-metric spaces were also found by them. In 2019, Rezaee et al. [20] introduced a new class of generalized metric spaces, called partial S-metric spaces. In 2017, Rohen et al. [21] modified the definition of S_b-metric introduced by Souayan and Mlaiki [22], and proved some coupled common fixed point theorems in S_b-metric spaces. The concept of controlled-S metric spaces was first developed by Gangwar et al. [23] in 2023. Recently, Azmi [24] extended the idea of a controlled S-metric type space to present the concept of a triple controlled S-metric type space, characterized by three control functions: β, μ, and γ. They also proved that fixed points of multivalued mappings exist within the context of controlled S-metric spaces.

The concept of complex valued metric spaces was introduced by Azam et al. [25] in 2011. They established certain fixed point theorems for a pair of mappings involving a contraction condition expressed through a rational function. Additionally, Kang et al. [26] introduced the idea of complex valued G-metric spaces and defined contraction mappings in this domain. Further, Mlaiki [27] presented the notion of a new metric space, named, a complex valued S-metric space and established the existence and the uniqueness of a common fixed point for two self mappings. Recently, Ozgur [28] presented the notion of complex valued G_b-metric spaces and presented Kannan fixed point theorem and Banach contraction principle in this setting. In 2017, Priyobarta et al. [29] defined complex valued S_b-metric spaces and proved very interesting fixed point theorems.

Motivated by the existing research in the fields of fixed point theory, complex-valued metric spaces and controlled S-metric spaces, the notion of complex valued controlled S-metric spaces (CVCS-metric spaces) is introduced as a means of incorporating complex-valued functions and providing a more flexible framework for addressing the challenges posed by complex-valued systems. This research article aims to present and analyze a series of novel fixed point results within the context of this specialized setting. This extension has proved to be instrumental in capturing the intricacies of complex-valued dynamics and has facilitated the study of various properties, such as convergence, continuity, and completeness, within this complex-valued setting. By establishing crucial fixed point theorems and exploring their implications in the broader context of CVCS-metric spaces, we aim to provide a deeper understanding of the underlying dynamics and behaviors of mappings within this complex domain.

2. Preliminaries

If $\mathbb{C}$ denotes the set of complex numbers and $\varpi_1, \varpi_2 \in \mathbb{C}$, then the partial order $\precsim$ is defined on $\mathbb{C}$ in the following manner:

$$\varpi_1 \prec \varpi_2 \text{ if and only if } Re(\varpi_1) < Re(\varpi_2), Im(\varpi_1) < Im(\varpi_2)$$

and

$$\varpi_1 \precsim \varpi_2 \text{ if and only if } Re(\varpi_1) \leq Re(\varpi_2), Im(\varpi_1) \leq Im(\varpi_2).$$

We also express $\varpi_1 \precsim \varpi_2$ if any one of the following conditions is met:

(i) $Im(\varpi_1) < Im(\varpi_2)$ and $Re(\varpi_1) = Re(\varpi_2)$,

(ii) $Im(\varpi_1) = Im(\varpi_2)$ and $Re(\varpi_1) < Re(\varpi_2)$,

(iii) $Im(\varpi_1) = Im(\varpi_2)$ and $Re(\varpi_1) = Re(\varpi_2)$.

Note that

$$0 \precsim \varpi_1 \precnsim \varpi_2 \text{ implies } |\varpi_1| < |\varpi_2|;$$

and

$$\varpi_1 \precsim \varpi_2, \varpi_2 \prec \varpi_3 \text{ implies } \varpi_1 \prec \varpi_3.$$

Now, we recall some definitions as well as established lemmas which are outlined in the references.

Definition 1. *[27] Consider a nonempty set Γ. Then a function $S : \Gamma \times \Gamma \times \Gamma \to \mathbb{C}$ is a complex valued S-metric on Γ if it satisfies the following conditions for all $\varpi, \rho, q, t \in \Gamma$:*
(i) $S(\varpi, \rho, q) \succeq 0$,
(ii) $S(\varpi, \rho, q) = 0$ *if and only if* $\varpi = \rho = q$,
(iii) $S(\varpi, \rho, q) \precsim S(\varpi, \varpi, t) + S(\rho, \rho, t) + S(q, q, t)$.
A complex valued S-metric space is denoted by (Γ, S).

Definition 2. *[27] Consider a complex valued S-metric space (Γ, S).*

(i) A sequence $\{\tau_\omega\} \in \Gamma$ converges to τ if and only if for all ϵ such that $0 \prec \epsilon \in \mathbb{C}$ there exists a natural number ω_0 such that for all $\omega \geq \omega_0$, we have $S(\tau_\omega, \tau_\omega, \tau) \precsim \epsilon$ and it is denoted by lim $\lim_{\omega \to +\infty} \tau_\omega = \tau$.

(ii) A sequence $\tau_\omega \in \Gamma$ is called a Cauchy sequence if for all ϵ such that $0 \prec \epsilon \in \mathbb{C}$ there exists a natural number ω_0 such that for all $\omega, \varrho \geq \omega_0$, we have $S(\tau_\omega, \tau_\omega, \tau_\varrho) \prec \epsilon$.

(iii) A complex valued S-metric space (Γ, S) is called complete if every Cauchy sequence in Γ is convergent.

Lemma 1. *[27] Consider a complex valued S-metric space (Γ, S) and a sequence $\{\tau_\omega\}$ in Γ. Then $\{\tau_\omega\}$ is said to converge to τ if and only if $|S(\tau_\omega, \tau_\omega, \tau)| \to 0$ as $\omega \to +\infty$.*

Lemma 2. *[27] Consider a complex valued S-metric space (Γ, S) and a sequence $\{\tau_\omega\}$ in Γ. Then $\{\tau_\omega\}$ is a Cauchy sequence if and only if $|S(\tau_\omega, \tau_\omega, \tau_{\omega+\varrho})| \to 0$ as $\omega, \varrho \to +\infty$.*

Lemma 3. *[27] Consider a complex valued S-metric space (Γ, S), then $S(\varsigma, \varsigma, \tau) = S(\tau, \tau, \varsigma)$ for all $\tau, \varsigma \in \Gamma$.*

3. Fixed Point Results

The need to generalize S-metric spaces to handle complex-valued distances under additional control conditions motivates the introduction of the concept of a CVCS-metric space as follows.

Definition 3. *Consider a set* $\Gamma \neq \emptyset$ *and let* $S : \Gamma \times \Gamma \times \Gamma \to \mathbb{C}$ *and* $\alpha : \Gamma \times \Gamma \times \Gamma \to [1, +\infty)$ *be two functions adhering to the following conditions for all* $\tau, \varsigma, \rho, \varpi \in \Gamma$*:*
$(CVCS_1)$ $0 \precsim S(\tau, \varsigma, \rho)$;
$(CVCS_2)$ $S(\tau, \varsigma, \rho) = 0$ *if and only if* $\tau = \varsigma = \rho$;
$(CVCS_3)$ $S(\tau, \varsigma, \rho) \precsim \alpha(\tau, \tau, \varpi)S(\tau, \tau, \varpi) + \alpha(\varsigma, \varsigma, \varpi)S(\varsigma, \varsigma, \varpi) + \alpha(\rho, \rho, \varpi)S(\rho, \rho, \varpi)$.
Then, the tuple (Γ, S, α) *is called a CVCS-metric space.*

Definition 2.2, Lemma 2.3, 2.4 and 2.5 can be stated and proved in a similar way for a CVCS-metric space.

Definition 4. *Consider a CVCS-metric space* (Γ, S, α).

(i) A sequence $\{\tau_\omega\} \in \Gamma$ *is CVCS-convergent to* τ *if and only if for all* ϵ *such that* $0 \prec \epsilon \in \mathbb{C}$ *there exists a natural number* ω_0 *such that for all* $\omega \geq \omega_0$*, we have* $S(\tau_\omega, \tau_\omega, \tau) \precsim \epsilon$ *and it is denoted by* $\lim \lim_{\omega \to +\infty} \tau_\omega = \tau$.

(ii) A sequence $\tau_\omega \in \Gamma$ *is called a CVCS-Cauchy sequence if for all* ϵ *such that* $0 \prec \epsilon \in \mathbb{C}$ *there exists a natural number* ω_0 *such that for all* $\omega, \varrho \geq \omega_0$*, we have* $S(\tau_\omega, \tau_\omega, \tau_\varrho) \prec \epsilon$.

(iii) A CVCS-metric space (Γ, S, α) *is called complete if every CVCS-Cauchy sequence in* Γ *is convergent.*

Lemma 4. *Consider a CVCS-metric space* (Γ, S, α) *and a sequence* $\{\tau_\omega\}$ *in* Γ*. Then* $\{\tau_\omega\}$ *is said to be CVCS-convergent to* τ *if and only if* $|S(\tau_\omega, \tau_\omega, \tau)| \to 0$ *as* $\omega \to +\infty$.

Lemma 5. *Consider a CVCS-metric space* (Γ, S, α) *and a sequence* $\{\tau_\omega\}$ *in* Γ*. Then* $\{\tau_\omega\}$ *is a CVCS-Cauchy sequence if and only if* $|S(\tau_\omega, \tau_\omega, \tau_{\omega+\varrho})| \to 0$ *as* $\omega, \varrho \to +\infty$.

Lemma 6. *Consider a CVCS-metric space* (Γ, S, α)*, then* $S(\varsigma, \varsigma, \tau) = S(\tau, \tau, \varsigma)$ *for all* $\tau, \varsigma \in \Gamma$.

Theorem 1. *Consider a CVCS-metric space* (Γ, S, α) *and a sequence* $\{\tau_\omega\}$ *in* Γ*. Then,* $\{\tau_\omega\}$ *is a CVCS-Cauchy sequence if and only if* $|S(\tau_\omega, \tau_\varrho, \tau_l)| \to 0$ *as* $\omega, \varrho, l \to +\infty$.

Proof. The proof is on the similar lines as in Theorem 3.3 of [29].

Theorem 2. *Consider a complete CVCS-metric space* (Γ, S, α)*. Define a function* $f : \Gamma \to \Gamma$ *such that*

$$S(f\tau, f\tau, f\varsigma) \precsim \theta S(\tau, \tau, \varsigma), \tag{1}$$

where $\theta \in (0,1)$. *For* $\tau_0 \in \Gamma$, *choose a sequence* $\tau_\omega = f(\tau_{\omega-1}), \omega \in \mathbb{N}$. *Suppose that*

$$\sup_{\varrho \geq 1} \lim_{i \to +\infty} \frac{\alpha(\tau_{i+1}, \tau_{i+1}, \tau_{i+2}).\alpha(\tau_\varrho, \tau_\varrho, \tau_{i+1})}{\alpha(\tau_i, \tau_i, \tau_{i+1})} < \frac{1}{2\theta}, \tag{2}$$

and $\lim_{\omega \to +\infty} \alpha(\tau_\omega, \tau_\omega, \tau_{\omega+1})$ *exists. Then there is a unique fixed point of* f.

Proof. Consider an arbitrary element τ_0 of Γ. Define a sequence $\{\tau_\omega\}$ in Γ by $\tau_\omega = f(\tau_{\omega-1}) = f^\omega(\tau_0)$. Let $\tau_\omega \neq \tau_{\omega+1}$ for all ω. From (1), we obtain

$$\begin{aligned} S(\tau_\omega, \tau_\omega, \tau_{\omega+1}) &\precsim \theta S(\tau_{\omega-1}, \tau_{\omega-1}, \tau_\omega) \\ &\precsim \theta^2 S(\tau_{\omega-2}, \tau_{\omega-2}, \tau_{\omega-1}) \\ &\vdots \\ &\precsim \theta^\omega S(\tau_0, \tau_0, \tau_1). \end{aligned}$$

Considering $\varrho > \omega$ and using triangle inequality, we obtain

$$\begin{aligned} S(\tau_\varrho, \tau_\varrho, \tau_\omega) &\precsim \alpha(\tau_\varrho, \tau_\varrho, \tau_{\omega+1}) S(\tau_\varrho, \tau_\varrho, \tau_{\omega+1}) + \alpha(\tau_\varrho, \tau_\varrho, \tau_{\omega+1}) S(\tau_\varrho, \tau_\varrho, \tau_{\omega+1}) \\ &+ \alpha(\tau_\omega, \tau_\omega, \tau_{\omega+1}) S(\tau_\omega, \tau_\omega, \tau_{\omega+1}) \\ &\precsim 2\alpha(\tau_\varrho, \tau_\varrho, \tau_{\omega+1}) S(\tau_\varrho, \tau_\varrho, \tau_{\omega+1}) + \alpha(\tau_\omega, \tau_\omega, \tau_{\omega+1}) S(\tau_\omega, \tau_\omega, \tau_{\omega+1}). \end{aligned}$$

Once more, applying the triangle inequality to $S(u_\varrho, u_\varrho, u_{\omega+1})$, we obtain

$$\begin{aligned} S(\tau_\varrho, \tau_\varrho, \tau_\omega) &\precsim 2\alpha(\tau_\varrho, \tau_\varrho, \tau_{\omega+1})(2\alpha(\tau_\varrho, \tau_\varrho, \tau_{\omega+2}) S(\tau_\varrho, \tau_\varrho, \tau_{\omega+2}) \\ &+ \alpha(\tau_{\omega+1}, \tau_{\omega+1}, \tau_{\omega+2}) S(\tau_{\omega+1}, \tau_{\omega+1}, \tau_{\omega+2})) + \alpha(\tau_\omega, \tau_\omega, \tau_{\omega+1}) S(\tau_\omega, \tau_\omega, \tau_{\omega+1}) \\ &\precsim \alpha(\tau_\omega, \tau_\omega, \tau_{\omega+1}) S(\tau_\omega, \tau_\omega, \tau_{\omega+1}) \\ &+ 2\alpha(\tau_\varrho, \tau_\varrho, \tau_{\omega+1}) \alpha(\tau_{\omega+1}, \tau_{\omega+1}, \tau_{\omega+2}) S(\tau_{\omega+1}, \tau_{\omega+1}, \tau_{\omega+2}) \\ &+ 2^2 \alpha(\tau_\varrho, \tau_\varrho, \tau_{\omega+1})\ \alpha(\tau_\varrho, \tau_\varrho, \tau_{\omega+2}) S(\tau_\varrho, \tau_\varrho, \tau_{\omega+2}). \end{aligned}$$

Proceeding with the continuous application of triangle inequality in the same manner, we obtain

$$\begin{aligned} S(\tau_\varrho, \tau_\varrho, \tau_\omega) &\precsim \alpha(\tau_\omega, \tau_\omega, \tau_{\omega+1}) S(\tau_\omega, \tau_\omega, \tau_{\omega+1}) \\ &+ \sum_{i=\omega+1}^{\varrho-2} 2^{i-\omega} \prod_{j=\omega+1}^{i} \alpha(\tau_\varrho, \tau_\varrho, \tau_j) \alpha(\tau_i, \tau_i, \tau_{i+1}) S(\tau_i, \tau_i, \tau_{i+1}) \\ &+ 2^{\varrho-\omega-1} \prod_{k=\omega+1}^{\varrho-1} \alpha(\tau_\varrho, \tau_\varrho, \tau_k) S(\tau_\varrho, \tau_\varrho, \tau_{\varrho-1}) \\ &\precsim \alpha(\tau_\omega, \tau_\omega, \tau_{\omega+1})(\theta^\omega S(\tau_0, \tau_0, \tau_1)) \\ &+ \sum_{i=\omega+1}^{\varrho-1} 2^{i-\omega} \prod_{j=\omega+1}^{i} \alpha(\tau_\varrho, \tau_\varrho, \tau_j) \alpha(\tau_i, \tau_i, \tau_{i+1})(\theta^i S(\tau_0, \tau_0, \tau_1)). \end{aligned}$$

Now, consider the series $S_\varrho = \sum_{i=\omega}^{\varrho-1} 2^{i-\omega} \prod_{j=\omega}^{i} \alpha(\tau_\varrho, \tau_\varrho, \tau_j)\alpha(\tau_i, \tau_i, \tau_{i+1})(\theta^i)$. This series converges by ratio test under the condition:

$$\sup_{\varrho \geq i} \lim_{i\to+\infty} \frac{\alpha(\tau_{i+1},\tau_{i+1},\tau_{i+2}).\alpha(\tau_\varrho,\tau_\varrho,\tau_{i+1})}{\alpha(\tau_i,\tau_i,\tau_{i+1})} < \frac{1}{2\theta}.$$

Letting $\varrho, \omega \to +\infty$, we obtain

$$\lim_{\varrho,\omega\to+\infty} S(\tau_\varrho, \tau_\varrho, \tau_\omega) = 0.$$

By $CVCS_3$, we have

$$S(\tau_\omega, \tau_\varrho, \tau_l) \precsim \alpha(\tau_\varrho, \tau_\varrho, \tau_\omega)S(\tau_\varrho, \tau_\varrho, \tau_\omega) + \alpha(\tau_l, \tau_l, \tau_\omega)S(\tau_l, \tau_l, \tau_\omega)$$

for all $\omega, \varrho, l \in \mathbb{N}$. Thus,

$$|S(\tau_\omega, \tau_\varrho, \tau_l)| \leq \alpha(\tau_\varrho, \tau_\varrho, \tau_\omega)|S(\tau_\varrho, \tau_\varrho, \tau_\omega)| + \alpha(\tau_l, \tau_l, \tau_\omega)|S(\tau_l, \tau_l, \tau_\omega)|.$$

Taking limit $\omega, \varrho, l \to +\infty$, we obtain $|S(\tau_\omega, \tau_\varrho, \tau_l)| \to 0$. So $\{\tau_\omega\}$ is a CVCS-Cauchy sequence. Completeness of (Γ, S, α) gives us that there is an element $\varsigma^* \in \Gamma$ such that $\{\tau_\omega\}$ is CVCS-convergent to ς^*. Now, we'll prove that $f(\varsigma^*) = \varsigma^*$. Consider,

$$S(\tau_{\omega+1}, \tau_{\omega+1}, f(\varsigma^*)) \precsim \theta S(\tau_\omega, \tau_\omega, \varsigma^*),$$

that is,

$$|S(\tau_{\omega+1}, \tau_{\omega+1}, f(\varsigma^*))| \leq \theta|S(\tau_\omega, \tau_\omega, \varsigma^*)|.$$

Letting $\omega \to +\infty$, we obtain $f(\varsigma^*) = \varsigma^*$. Now towards the end, the uniqueness will be proved. Let τ^* be some other fixed point of f. Consider

$$S(\varsigma^*, \varsigma^*, \tau^*) = S(f(\varsigma^*), f(\varsigma^*), f(\tau^*)) \precsim \theta S(\varsigma^*, \varsigma^*, \tau^*),$$

that is,

$$|S(f(\varsigma^*), f(\varsigma^*), f(\tau^*))| \leq \theta|S(\varsigma^*, \varsigma^*, \tau^*)|.$$

This implies that $|S(\varsigma^*, \varsigma^*, \tau^*)| \leq 0$. Thus, $\varsigma^* = \tau^*$, that is, τ^* is the unique fixed point of f.

Example 1. *Consider the complex plane $\Gamma = \mathbb{C}$ equipped with a controlled function $\alpha : \Gamma^3 \to [1, +\infty)$ defined by*

$$\alpha(\tau, \varsigma, \varpi) = 1 + \frac{|\tau| + |\varsigma| + |\varpi|}{1 + |\tau| + |\varsigma| + |\varpi|},$$

and a complex-valued S-metric $S : \Gamma^3 \to \mathbb{C}$ given by

$$S(\tau, \varsigma, \varpi) = \max\left\{|\tau - \varpi|, |\varsigma - \varpi|, \frac{|\tau - \varsigma|}{2}\right\}.$$

Then (Γ, S, α) *is a complete CVCS-metric space. Define the mapping* $f : \Gamma \to \Gamma$ *as* $f(\tau) = \frac{\tau}{5}$*. Now, for any* $\tau, \varsigma \in \Gamma$,

$$S(f\tau, f\tau, f\varsigma) = \max\left\{\left|\frac{\tau}{3} - \frac{\varsigma}{3}\right|, \frac{|\tau - \varsigma|}{2}\right\} = \frac{|\tau - \varsigma|}{3} \precsim \theta S(\tau, \tau, \varsigma),$$

for any $\theta \in (\frac{1}{5}, 1)$.
Also, let $\tau_0 = 1$ *and* $\tau_\omega = f^\omega(\tau_0) = \frac{1}{5^\omega}$*. Then,*

$$\alpha(\tau_\omega, \tau_\omega, \tau_{\omega+1}) = 1 + \frac{\frac{2}{5^\omega} + \frac{1}{5^{\omega+1}}}{1 + \frac{2}{5^\omega} + \frac{1}{5^{\omega+1}}} \to 1 \quad as \quad \omega \to +\infty,$$

$$and \ \sup_{\varrho \geq 1} \lim_{\omega \to +\infty} \frac{\alpha(\tau_{\omega+1}, \tau_{\omega+1}, \tau_{\omega+2})\alpha(\tau_\varrho, \tau_\varrho, \tau_{\omega+1})}{\alpha(\tau_\omega, \tau_\omega, \tau_{\omega+1})} = 1 < \frac{1}{2 \cdot \frac{1}{3}} = \frac{3}{2}.$$

Thus, (2) *holds and the sequence* $\{\tau_\omega\}$ *converges to* 0*, and* $f(0) = 0$ *is the unique fixed point.*

Corollary 1. *Let* (Γ, S) *be a complete complex valued* S*-metric space and* $f : \Gamma \to \Gamma$ *be a mapping, such that for* $\tau, \varsigma \in \Gamma$

$$S_H(f(\tau), f(\tau), f(\varsigma)) \precsim \theta S(\tau, \tau, \varsigma),$$

where $\theta \in (0, 1)$*. Then,* f *has a unique fixed point.*

Proof. The required result is obtained by taking $\alpha(\tau, \varsigma, \varpi) = 1$ and following the same procedures as in the preceding theorem.

Theorem 3. *Let* (Γ, S, α) *be a complete CVCS-metric space and* $f : \Gamma \to \Gamma$ *satisfy the following for every* $\tau, \varsigma \in \Gamma$*:*

$$S(f(\tau), f(\tau), f(\varsigma)) \precsim aS(\tau, \tau, f(\tau)) + bS(\varsigma, \varsigma, f(\varsigma)), \tag{3}$$

where $a, b \in (0, 1)$ *with* $a + b < 1$*. For* $\tau_0 \in \Gamma$*, choose a sequence* $\tau_\omega = f(\tau_{\omega-1}), \omega \in \mathbb{N}$*. Suppose that*

$$\sup_{\varrho \geq 1} \lim_{i \to +\infty} \frac{\alpha(\tau_{i+1}, \tau_{i+1}, \tau_{i+2}).\alpha(\tau_\varrho, \tau_\varrho, \tau_{i+1})}{\alpha(\tau_i, \tau_i, \tau_{i+1})} < \frac{1 - b}{2a}, \tag{4}$$

and $\lim_{\omega \to +\infty} \alpha(\tau_\omega, \tau_\omega, \tau_{\omega+1})$ *exists. Then,* f *admits a unique fixed point.*

Proof. Consider an arbitrary $\tau_0 \in \Gamma$. Define a sequence $\{\tau_\omega\}$ in Γ by $\tau_\omega = f(\tau_{\omega-1}) = f^\omega(\tau_0)$. Suppose that $\tau_\omega \neq \tau_{\omega+1}$ for all ω. From (3), we obtain

$$S(\tau_\omega, \tau_\omega, \tau_{\omega+1}) \precsim aS(\tau_{\omega-1}, \tau_{\omega-1}, \tau_\omega) + bS(\tau_\omega, \tau_\omega, \tau_{\omega+1})$$

that is

$$S(\tau_\omega, \tau_\omega, \tau_{\omega+1}) \precsim \frac{a}{1 - b} S(\tau_{\omega-1}, \tau_{\omega-1}, \tau_\omega)$$

$$\precsim \frac{a^2}{(1-b)^2} S(\tau_{\omega-2}, \tau_{\omega-2}, f\tau_{\omega-1})$$
$$\vdots$$
$$\precsim \frac{a^{\omega}}{(1-b)^{\omega}} S(\tau_0, \tau_0, \tau_1).$$

Let $\varrho > \omega$, where $\varrho, \omega \in \mathbb{N}$. Using the triangle inequality, we obtain

$$\begin{aligned} S(\tau_\varrho, \tau_\varrho, \tau_\omega) &\precsim \alpha(\tau_\varrho, \tau_\varrho, \tau_{\omega+1}) S(\tau_\varrho, \tau_\varrho, \tau_{\omega+1}) + \alpha(\tau_\varrho, \tau_\varrho, \tau_{\omega+1}) S(\tau_\varrho, \tau_\varrho, \tau_{\omega+1}) \\ &+ \alpha(\tau_\omega, \tau_\omega, \tau_{\omega+1}) S(\tau_\omega, \tau_\omega, \tau_{\omega+1}) \\ &= 2\alpha(\tau_\varrho, \tau_\varrho, \tau_{\omega+1}) S(\tau_\varrho, \tau_\varrho, \tau_{\omega+1}) + \alpha(\tau_\omega, \tau_\omega, \tau_{\omega+1}) S(\tau_\omega, \tau_\omega, \tau_{\omega+1}). \end{aligned}$$

Again, using the triangle inequality on $S(\tau_\varrho, \tau_\varrho, \tau_{\omega+1})$, one writes

$$\begin{aligned} S(\tau_\varrho, \tau_\varrho, \tau_\omega) &\precsim 2\alpha(\tau_\varrho, \tau_\varrho, \tau_{\omega+1})(2\alpha(\tau_\varrho, \tau_\varrho, \tau_{\omega+2}) S(\tau_\varrho, \tau_\varrho, \tau_{\omega+2}) \\ &+ \alpha(\tau_{\omega+1}, \tau_{\omega+1}, \tau_{\omega+2}) S(\tau_{\omega+1}, \tau_{\omega+1}, \tau_{\omega+2})) \\ &+ \alpha(\tau_\omega, \tau_\omega, \tau_{\omega+1}) S(\tau_\omega, \tau_\omega, \tau_{\omega+1}) \\ &= \alpha(\tau_\omega, \tau_\omega, \tau_{\omega+1}) S(\tau_\omega, \tau_\omega, \tau_{\omega+1}) \\ &+ 2\alpha(\tau_\varrho, \tau_\varrho, \tau_{\omega+1}) \alpha(\tau_{\omega+1}, \tau_{\omega+1}, \tau_{\omega+2}) S(\tau_{\omega+1}, \tau_{\omega+1}, \tau_{\omega+2}) \\ &+ 2^2 \alpha(\tau_\varrho, \tau_\varrho, \tau_{\omega+1}) \ \alpha(\tau_\varrho, \tau_\varrho, \tau_{\omega+2}) S(\tau_\varrho, \tau_\varrho, \tau_{\omega+2}). \end{aligned}$$

Using triangle inequality again in a same way, we obtain

$$\begin{aligned} S(\tau_\varrho, \tau_\varrho, \tau_\omega) &\precsim \alpha(\tau_\omega, \tau_\omega, \tau_{\omega+1}) S(\tau_\omega, \tau_\omega, \tau_{\omega+1}) \\ &+ \sum_{i=\omega+1}^{\varrho-2} 2^{i-\omega} \prod_{j=\omega+1}^{i} \alpha(\tau_\varrho, \tau_\varrho, \tau_j) \alpha(\tau_i, \tau_i, \tau_{i+1}) S(\tau_i, \tau_i, \tau_{i+1}) \\ &+ 2^{\varrho-\omega-1} \prod_{k=\omega+1}^{\varrho-1} \alpha(\tau_\varrho, \tau_\varrho, \tau_k) S(\tau_{\varrho-1}, \tau_{\varrho-1}, \tau_\varrho) \\ &\precsim \alpha(\tau_\omega, \tau_\omega, \tau_{\omega+1}) \left(\frac{a}{1-b}\right)^{\omega} S(\tau_0, \tau_0, \tau_1) \\ &+ \sum_{i=\omega+1}^{\varrho-1} 2^{i-\omega} \prod_{j=\omega+1}^{i} \alpha(\tau_\varrho, \tau_\varrho, \tau_j) \alpha(\tau_i, \tau_i, \tau_{i+1}) \left(\frac{a}{1-b}\right)^{i} S(\tau_0, \tau_0, \tau_1). \end{aligned}$$

As $\omega, \varrho \to +\infty$, one has

$$\sum_{i=\omega+1}^{\varrho-1} 2^{i-\omega} \prod_{j=\omega+1}^{i} \alpha(\tau_\varrho, \tau_\varrho, \tau_j) \alpha(\tau_i, \tau_i, \tau_{i+1}) \left(\frac{a}{1-b}\right)^{i} \to 0,$$

if $\sup_{\varrho\geq 1} \lim\limits_{i\to+\infty} \frac{\alpha(\tau_{i+1},\tau_{i+1},\tau_{i+2}).\alpha(\tau_\varrho,\tau_\varrho,\tau_{i+1})}{\alpha(\tau_i,\tau_i,\tau_{i+1})} < \frac{1-b}{2a}$. For $\varrho > n$, we obtain

$$\lim_{\varrho,\omega\to+\infty} S(\tau_\varrho,\tau_\varrho,\tau_\omega) = 0.$$

By $CVCS_3$, we have

$$S(\tau_\omega,\tau_\varrho,\tau_l) \precsim \alpha(\tau_\varrho,\tau_\varrho,\tau_\omega)S(\tau_\varrho,\tau_\varrho,\tau_\omega) + \alpha(\tau_l,\tau_l,\tau_\omega)S(\tau_l,\tau_l,\tau_\omega)$$

for all $\omega,\varrho,l \in \mathbb{N}$. Thus,

$$|S(\tau_\omega,\tau_\varrho,\tau_l)| \leq \alpha(\tau_\varrho,\tau_\varrho,\tau_\omega)|S(\tau_\varrho,\tau_\varrho,\tau_\omega)| + \alpha(\tau_l,\tau_l,\tau_\omega)|S(\tau_l,\tau_l,\tau_\omega)|.$$

Considering the limit as $\omega,\varrho,l \to +\infty$, we have $|S(\tau_\omega,\tau_\varrho,\tau_l)| \to 0$. So $\{\tau_\omega\}$ is a CVCS-Cauchy sequence. Completeness of (Γ,S,α) gives us that there is an element $\varsigma^* \in \Gamma$ such that $\{\tau_\omega\}$ is CVCS-convergent to ς^*. Now, we'll prove that $f(\varsigma^*) = \varsigma^*$. Consider,

$$S(\tau_{\omega+1},\tau_{\omega+1},f(\varsigma^*)) \precsim aS(\tau_\omega,\tau_\omega,\tau_{\omega+1}) + bS(\varsigma^*,\varsigma^*,f(\varsigma^*))$$

that is,

$$|S(\tau_{\omega+1},\tau_{\omega+1},f(\varsigma^*))| \leq a|S(\tau_\omega,\tau_\omega,\tau_{\omega+1})| + b|S(\varsigma^*,\varsigma^*,f(\varsigma^*))|.$$

Letting $\omega \to +\infty$, we obtain $|S(\varsigma^*,\varsigma^*,f(\varsigma^*))| \leq b|S(\varsigma^*,\varsigma^*,f(\varsigma^*))|$, which implies $f(\varsigma^*) = \varsigma^*$. Now towards the end, the uniqueness will be proved. Let τ^* be some other fixed point of f. Consider

$$S(\varsigma^*,\varsigma^*,\tau^*) = S(f(\varsigma^*),f(\varsigma^*),f(\tau^*)) \precsim aS(\varsigma^*,\varsigma^*,f(\varsigma^*)) + bS(\tau^*,\tau^*,f(\tau^*)),$$

that is,

$$|S(f(\varsigma^*),f(\varsigma^*),f(\tau^*))| \leq 0.$$

This implies that $|S(\varsigma^*,\varsigma^*,\tau^*)| \leq 0$. Thus, $\varsigma^* = \tau^*$, i.e., τ^* is the unique fixed point of f.

Corollary 2. *Let (Γ,S,α) be a complete complex valued S-metric space. Let $f : \Gamma \to \Gamma$ be a mapping such that, for $\tau,\varsigma \in \Gamma$,*

$$S(f(\tau),f(\tau),f(\varsigma)) \precsim aS(\tau,\tau,f(\tau)) + bS(\varsigma,\varsigma,f(\varsigma)), \tag{5}$$

where $a \in [0,\frac{1}{2})$, with $2a+b<1$. Then, there exists a fixed point for f in Γ.

Proof. If we take $\alpha(\tau,\varsigma,\varpi) = 1$ and proceed with the same steps as outlined in the proof of Theorem 3, the corollary is proved.

Theorem 4. *Let (Γ,S,α) be a complete complex valued S-metric space. Let f and g be two self mappings on Γ that meet the contraction condition given below:*

$$S(f\tau,f\tau,g\varsigma) \precsim \gamma S(\tau,\tau,\varsigma)+\beta\frac{S(\tau,\tau,f\tau)S(\varsigma,\varsigma,g\varsigma)}{2\alpha(g\varsigma,g\varsigma,\tau)S(\tau,\tau,g\varsigma)+\alpha(f\tau,f\tau,\varsigma)S(\varsigma,\varsigma,f\tau)+\alpha(\varsigma,\varsigma,\tau)S(\tau,\tau,\varsigma)}$$

for all $\tau, \varsigma \in \Gamma$ *such that* $\tau \neq \varsigma, S(\tau,\tau,g\varsigma)+S(\varsigma,\varsigma,f\tau)+S(\tau,\tau,\varsigma) \neq 0$ *for any two nonnegative real numbers* γ, β *satisfying the condition* $\gamma+\beta<1$ *or* $S(f\tau,f\tau,g\varsigma)=0$ *if* $S(\tau,\tau,g\varsigma)+S(\varsigma,\varsigma,f\tau)+S(\tau,\tau,\varsigma)=0$. *Suppose that*

$$\sup_{\varrho\geq 1}\lim_{i\to+\infty}\frac{\alpha(\tau_{i+1},\tau_{i+1},\tau_{i+2}).\alpha(\tau_\varrho,\tau_\varrho,\tau_{i+1})}{\alpha(\tau_i,\tau_i,\tau_{i+1})}<\frac{1}{2(\gamma+\beta)}, \tag{6}$$

and $\lim_{\omega\to+\infty}\alpha(\tau_\omega,\tau_\omega,\tau_{\omega+1})$ *exists. Then there exists a unique common fixed point for* f *and* g.

Proof. Let $\tau_0\in\Gamma$ and $\tau_{2\kappa+1}=f\tau_{2\kappa}, \tau_{2\kappa+2}=g\tau_{2\kappa+1}, \kappa\in\{0,1,2,\}$. Thus,

$$\begin{aligned}
&S(\tau_{2\kappa+1},\tau_{2\kappa+1},\tau_{2\kappa+2})=S(f\tau_{2\kappa},f\tau_{2\kappa},g\tau_{2\kappa+1})\\
&\precsim \gamma S(\tau_{2\kappa},\tau_{2\kappa},\tau_{2\kappa+1})\\
&+\frac{\beta S(\tau_{2\kappa},\tau_{2\kappa},f\tau_{2\kappa})S(\tau_{2\kappa+1},\tau_{2\kappa+1},g\tau_{2\kappa+1})}{(2\alpha(g\tau_{2\kappa+1},g\tau_{2\kappa+1},\tau_{2\kappa})S(\tau_{2\kappa},\tau_{2\kappa},g\tau_{2\kappa+1})+\alpha(f\tau_{2\kappa},f\tau_{2\kappa},\tau_{2\kappa+1})S(\tau_{2\kappa+1},\tau_{2\kappa+1},f\tau_{2\kappa})+\alpha(\tau_{2\kappa+1},\tau_{2\kappa+1},\tau_{2\kappa})S(\tau_{2\kappa},\tau_{2\kappa},\tau_{2\kappa+1}))}\\
&=\gamma S(\tau_{2\kappa},\tau_{2\kappa},\tau_{2\kappa+1})\\
&+\frac{\beta S(\tau_{2\kappa},\tau_{2\kappa},\tau_{2\kappa+1})S(\tau_{2\kappa+1},\tau_{2\kappa+1},\tau_{2\kappa+2})}{(2\alpha(\tau_{2\kappa+2},\tau_{2\kappa+2},\tau_{2\kappa})S(\tau_{2\kappa},\tau_{2\kappa},\tau_{2\kappa+2})+\alpha(\tau_{2\kappa+1},\tau_{2\kappa+1},\tau_{2\kappa+1})S(\tau_{2\kappa+1},\tau_{2\kappa+1},\tau_{2\kappa+1})+\alpha(\tau_{2\kappa+1},\tau_{2\kappa+1},\tau_{2\kappa})S(\tau_{2\kappa},\tau_{2\kappa},\tau_{2\kappa+1}))}.
\end{aligned}$$

Hence,

$$\begin{aligned}
|S(\tau_{2\kappa+1},\tau_{2\kappa+1},\tau_{2\kappa+2})| &\leq \gamma|S(\tau_{2\kappa},\tau_{2\kappa},\tau_{2\kappa+1})|\\
&+\frac{\beta|S(\tau_{2\kappa},\tau_{2\kappa},\tau_{2\kappa+1})||S(\tau_{2\kappa+1},\tau_{2\kappa+1},\tau_{2\kappa+2})|}{(|2\alpha(\tau_{2\kappa+2},\tau_{2\kappa+2},\tau_{2\kappa})S(\tau_{2\kappa},\tau_{2\kappa},\tau_{2\kappa+2})+\alpha(\tau_{2\kappa+1},\tau_{2\kappa+1},\tau_{2\kappa})S(\tau_{2\kappa},\tau_{2\kappa},\tau_{2\kappa+1})|)}.
\end{aligned}$$

By $CVCS_3$ and Lemma 3, we see that

$$\begin{aligned}
|S(\tau_{2\kappa+1},\tau_{2\kappa+1},\tau_{2\kappa+2})| &= |S(\tau_{2\kappa+2},\tau_{2\kappa+2},\tau_{2\kappa+1})|\\
&\leq |2\alpha(\tau_{2\kappa+2},\tau_{2\kappa+2},\tau_{2\kappa})S(\tau_{2\kappa+2},\tau_{2\kappa+2},\tau_{2\kappa})\\
&+\alpha(\tau_{2\kappa+1},\tau_{2\kappa+1},\tau_{2\kappa})S(\tau_{2\kappa+1},\tau_{2\kappa+1},\tau_{2\kappa})|\\
&= |2\alpha(\tau_{2\kappa+2},\tau_{2\kappa+2},\tau_{2\kappa})S(\tau_{2\kappa},\tau_{2\kappa},\tau_{2\kappa+2})\\
&+\alpha(\tau_{2\kappa+1},\tau_{2\kappa+1},\tau_{2\kappa})S(\tau_{2\kappa},\tau_{2\kappa},\tau_{2\kappa+1})|.
\end{aligned}$$

Thus,

$$\begin{aligned}
|S(\tau_{2\kappa+1},\tau_{2\kappa+1},\tau_{2\kappa+2})| &\leq \gamma|S(\tau_{2\kappa},\tau_{2\kappa},\tau_{2\kappa+1})|+\beta|S(\tau_{2\kappa},\tau_{2\kappa},\tau_{2\kappa+1})|\\
&=(\gamma+\beta)|S(\tau_{2\kappa},\tau_{2\kappa},\tau_{2\kappa+1})|.
\end{aligned}$$

Likewise, we obtain

$$|S(\tau_{2\kappa+2},\tau_{2\kappa+2},\tau_{2\kappa+3})|=(\gamma+\beta)|S(\tau_{2\kappa+1},\tau_{2\kappa+1},\tau_{2\kappa+2})|.$$

Therefore,

$$|S(\tau_\omega,\tau_\omega,\tau_{\omega+1})|\leq(\gamma+\beta)|S(\tau_{\omega-1},\tau_{\omega-1},\tau_\omega|\leq ...\leq(\gamma+\beta)^\omega|S(\tau_0,\tau_0,\tau_1).$$

Hence, for any $\varrho > \omega$, we have

$$\begin{aligned} S(\tau_\varrho, \tau_\varrho, \tau_\omega) &\precsim 2\alpha(\tau_\varrho, \tau_\varrho, \tau_{\omega+1})(2\alpha(\tau_\varrho, \tau_\varrho, \tau_{\omega+2})S(\tau_\varrho, \tau_\varrho, \tau_{\omega+2}) \\ &+ \alpha(\tau_{\omega+1}, \tau_{\omega+1}, \tau_{\omega+2})S(\tau_{\omega+1}, \tau_{\omega+1}, \tau_{\omega+2})) \\ &+ \alpha(\tau_\omega, \tau_\omega, \tau_{\omega+1})S(\tau_\omega, \tau_\omega, \tau_{\omega+1}) \\ &= \alpha(\tau_\omega, \tau_\omega, \tau_{\omega+1})S(\tau_\omega, \tau_\omega, \tau_{\omega+1}) \\ &+ 2\alpha(\tau_\varrho, \tau_\varrho, \tau_{\omega+1})\alpha(\tau_{\omega+1}, \tau_{\omega+1}, \tau_{\omega+2})S(\tau_{\omega+1}, \tau_{\omega+1}, \tau_{\omega+2}) \\ &+ 2^2\alpha(\tau_\varrho, \tau_\varrho, \tau_{\omega+1})\ \alpha(\tau_\varrho, \tau_\varrho, \tau_{\omega+2})S(\tau_\varrho, \tau_\varrho, \tau_{\omega+2}). \end{aligned}$$

Using triangle inequality again in a same way, we obtain

$$\begin{aligned} S(\tau_\varrho, \tau_\varrho, \tau_\omega) &\precsim \alpha(\tau_\omega, \tau_\omega, \tau_{\omega+1})S(\tau_\omega, \tau_\omega, \tau_{\omega+1}) \\ &+ \sum_{i=\omega+1}^{\varrho-2} 2^{i-\omega} \prod_{j=\omega+1}^{i} \alpha(\tau_\varrho, \tau_\varrho, \tau_j)\alpha(\tau_i, \tau_i, \tau_{i+1})S(\tau_i, \tau_i, \tau_{i+1}) \\ &+ 2^{\varrho-\omega-1} \prod_{k=\omega+1}^{\varrho-1} \alpha(\tau_\varrho, \tau_\varrho, \tau_k)S(\tau_{\varrho-1}, \tau_{\varrho-1}, \tau_\varrho) \\ &\precsim \alpha(\tau_\omega, \tau_\omega, \tau_{\omega+1})(\gamma+\beta)^\omega S(\tau_0, \tau_0, \tau_1) \\ &+ \sum_{i=\omega+1}^{\varrho-1} 2^{i-\omega} \prod_{j=\omega+1}^{i} \alpha(\tau_\varrho, \tau_\varrho, \tau_j)\alpha(\tau_i, \tau_i, \tau_{i+1})(\gamma+\beta)^i S(\tau_0, \tau_0, \tau_1). \end{aligned}$$

As $\omega, \varrho \to +\infty$, one has

$$\sum_{i=\omega+1}^{\varrho-1} 2^{i-\omega} \prod_{j=\omega+1}^{i} \alpha(\tau_\varrho, \tau_\varrho, \tau_j)\alpha(\tau_i, \tau_i, \tau_{i+1})(\gamma+\beta)^i \to 0,$$

if $\sup_{\varrho \geq 1} \lim\limits_{i \to +\infty} \frac{\alpha(\tau_{i+1}, \tau_{i+1}, \tau_{i+2}).\alpha(\tau_\varrho, \tau_\varrho, \tau_{i+1})}{\alpha(\tau_i, \tau_i, \tau_{i+1})} < \frac{1}{2(\gamma+\beta)}$. For $\varrho > \omega$, we obtain

$$\lim_{\varrho, \omega \to +\infty} S(\tau_\omega, \tau_\omega, \tau_\varrho) = 0.$$

By $CVCS_3$, we have

$$S(\tau_\omega, \tau_\varrho, \tau_l) \precsim \alpha(\tau_\varrho, \tau_\varrho, \tau_\omega)S(\tau_\varrho, \tau_\varrho, \tau_\omega) + \alpha(\tau_l, \tau_l, \tau_\omega)S(\tau_l, \tau_l, \tau_\omega)$$

for all $\omega, \varrho, l \in \mathbb{N}$. Thus,

$$|S(\tau_\omega, \tau_\varrho, \tau_l)| \leq \alpha(\tau_\varrho, \tau_\varrho, \tau_\omega)|S(\tau_\varrho, \tau_\varrho, \tau_\omega)| + \alpha(\tau_l, \tau_l, \tau_\omega)|S(\tau_l, \tau_l, \tau_\omega)|.$$

Considering the limit as $\omega, \varrho, l \to +\infty$, we have $|S(\tau_\omega, \tau_\varrho, \tau_l)| \to 0$. So $\{\tau_\omega\}$ is a CVCS-Cauchy sequence. Completeness of (Γ, S, α) gives us that there is an element $\varsigma^* \in \Gamma$ such that $\{\tau_\omega\}$ is CVCS-convergent to ς^*. Now, we'll prove that $f(\varsigma^*) = g(\varsigma^*) = \varsigma^*$. Assume that $f\varsigma^* \neq \varsigma^*$. Thus, $0 \prec \varpi = S(\varsigma^*, \varsigma^*, f\varsigma^*)$. Therefore,

$$\begin{aligned}
\varpi &\precsim 2\alpha(\varsigma^*, \varsigma^*, \tau_{2\kappa+2})S(\varsigma^*, \varsigma^*, \tau_{2\kappa+2}) + \alpha(\tau_{2\kappa+2}, \tau_{2\kappa+2}, f\varsigma^*)S(\tau_{2\kappa+2}, \tau_{2\kappa+2}, f\varsigma^*)\\
&\precsim 2\alpha(\varsigma^*, \varsigma^*, \tau_{2\kappa+2})S(\varsigma^*, \varsigma^*, \tau_{2\kappa+2}) + \alpha(\tau_{2\kappa+2}, \tau_{2\kappa+2}, f\varsigma^*)S(g\tau_{2\kappa+1}, g\tau_{2\kappa+1}, f\varsigma^*)\\
&\precsim 2\alpha(\varsigma^*, \varsigma^*, \tau_{2\kappa+2})S(\varsigma^*, \varsigma^*, \tau_{2\kappa+2}) + \gamma\alpha(\tau_{2\kappa+2}, \tau_{2\kappa+2}, f\varsigma^*)S(\tau_{2\kappa+1}, \tau_{2\kappa+1}, \varsigma^*)\\
&+ \frac{\beta\alpha(\tau_{2\kappa+2}, \tau_{2\kappa+2}, f\varsigma^*)S(\tau_{2\kappa+1}, \tau_{2\kappa+1}, \tau_{2\kappa+2})S(\varsigma^*, \varsigma^*, f\varsigma^*)}{2\alpha(f\varsigma^*, f\varsigma^*, \tau_{2\kappa+1})S(\tau_{2\kappa+1}, \tau_{2\kappa+1}, f\varsigma^*) + \alpha(\tau_{2\kappa+2}, \tau_{2\kappa+2}, \varsigma^*)S(\varsigma^*, \varsigma^*, \tau_{2\kappa+2}) + \alpha(\varsigma^*, \varsigma^*, \tau_{2\kappa+1})S(\tau_{2\kappa+1}, \tau_{2\kappa+1}, \varsigma^*)}.
\end{aligned}$$

Hence,

$$\begin{aligned}
|\varpi| &\leq 2\alpha(\varsigma^*, \varsigma^*, \tau_{2\kappa+2})|S(\varsigma^*, \varsigma^*, \tau_{2\kappa+2})| + \gamma\alpha(\tau_{2\kappa+2}, \tau_{2\kappa+2}, f\varsigma^*)|S(\tau_{2\kappa+1}, \tau_{2\kappa+1}, \varsigma^*)|\\
&+ \frac{\beta\alpha(\tau_{2\kappa+2}, \tau_{2\kappa+2}, f\varsigma^*)|S(\tau_{2\kappa+1}, \tau_{2\kappa+1}, \tau_{2\kappa+2})||z|}{2\alpha(f\varsigma^*, f\varsigma^*, \tau_{2\kappa+1})|S(\tau_{2\kappa+1}, \tau_{2\kappa+1}, f\varsigma^*)| + \alpha(\tau_{2\kappa+2}, \tau_{2\kappa+2}, \varsigma^*)|S(\varsigma^*, \varsigma^*, \tau_{2\kappa+2})| + \alpha(\varsigma^*, \varsigma^*, \tau_{2\kappa+1})|S(\tau_{2\kappa+1}, \tau_{2\kappa+1}, \varsigma^*)|}.
\end{aligned}$$

Our assumption on ϖ is contradicted by the fact that the right-hand side of the preceding inequality tends to 0 as $\kappa \to +\infty$. Consequently, it is possible to demonstrate that $f\varsigma^* = \varsigma^*$ and that $g\varsigma^* = \varsigma^*$. As a result, f and g have a common fixed point. In order to demonstrate uniqueness assume that there is another common fixed point of f and g, say ρ^*. Therefore,

$$\begin{aligned}
S(\varsigma^*, \varsigma^*, \rho^*) &= S(f\varsigma^*, f\varsigma^*, g\rho^*)\\
&\precsim \gamma S(\varsigma^*, \varsigma^*, \rho^*)\\
&+ \beta\frac{S(\varsigma^*, \varsigma^*, f\varsigma^*)S(\rho^*, \rho^*, g\rho^*)}{2\alpha(g\rho^*, g\rho^*, \varsigma^*)S(\varsigma^*, \varsigma^*, g\rho^*) + \alpha(f\varsigma^*, f\varsigma^*, \rho^*)S(\rho^*, \rho^*, f\varsigma^*) + \alpha(\rho^*, \rho^*, \varsigma^*)S(\varsigma^*, \varsigma^*, \rho^*)}\\
&= \gamma S(\varsigma^*, \varsigma^*, \rho^*).
\end{aligned}$$

This implies that $|S(\varsigma^*, \varsigma^*, \rho^*)| \leq \gamma|S(\varsigma^*, \varsigma^*, \rho^*)|$. However, since $\gamma < 1$, we can thus conclude that $S(\varsigma^*, \varsigma^*, \rho^*) = 0$, and as a result, $\varsigma^* = \rho^*$ as intended. To complete our proof, suppose that for every natural number κ, if we obtain

$$S(\tau_{2\kappa}, \tau_{2\kappa}, g\tau_{2\kappa+1}) + S(\tau_{2\kappa+1}, \tau_{2\kappa+1}, f\tau_{2\kappa}) + S(\tau_{2\kappa}, \tau_{2\kappa}, \tau_{2\kappa+1}) = 0,$$

then $S(f\tau_{2\kappa}, f\tau_{2\kappa}, g\tau_{2\kappa+1}) = 0$, which implies $\tau_{2\kappa} = f\tau_{2\kappa} = \tau_{2\kappa+1} = g\tau_{2\kappa+1} = \tau_{2\kappa+2}$. Therefore, $\tau_{2\kappa+1} = f\tau_{2\kappa} = \tau_{2\kappa}$, hence there exist ω_1, ϱ_1 such that $\omega_1 = f\varrho_1 = \varrho_1$. Similarly, there exist ω_2, ϱ_2 such that $\omega_2 = g\varrho_2 = \varrho_2$. We know that

$$S(\varrho_1, \varrho_1, g\varrho_2) + S(\varrho_2, \varrho_2, f\varrho_1) + S(\varrho_1, \varrho_1, \varrho_2) = 0.$$

We deduce that $S(f\varrho_1, f\varrho_1, g\varrho_2) = 0$, which implies that $\omega_1 = f\varrho_1 = g\varrho_2 = \omega_2$. Therefore, $\omega_1 = f\varrho_1 = f\omega_1$. Similarly, we get $\omega_2 = g\varrho_2 = g\omega_2$. Since $\omega_1 = \omega_2$, we deduce that $f\omega_1 = g\omega_1 = \omega_1$. As a result, f and g have a common fixed point, namely ω_1. To prove uniqueness, assume that f and g have τ and ς as common fixed points. We know that $S(\tau, \tau, g\varsigma) + S(\varsigma, \varsigma, fu) + S(\tau, \tau, \varsigma) = 0$. Thus, $S(\tau, \tau, \varsigma) = S(f\tau, f\tau, g\varsigma) = 0$ which implies that $\tau = \varsigma$ as required.

Corollary 3. *Let (Γ, S, α) be a complete complex valued S-metric space. Let f and g be two self mappings on Γ that meet the contraction condition given below:*

$$S(f\tau, f\tau, g\varsigma) \precsim \gamma S(\tau,\tau,\varsigma) + \frac{\beta S(\tau,\tau,f\tau)S(\varsigma,\varsigma,g\varsigma)}{b(2S(\tau,\tau,g\varsigma)+S(\varsigma,\varsigma,f\tau)+S(\tau,\tau,\varsigma))}$$

for all $\tau,\varsigma \in \Gamma$ such that $\tau \neq \varsigma, S(\tau,\tau,g\varsigma)+S(\varsigma,\varsigma,f\tau)+S(\tau,\tau,\varsigma) \neq 0$, where γ,β are two real numbers which are non- negative and satisfy the condition $\gamma+\beta < \frac{1}{b}$ or $S(f\tau,f\tau,g\varsigma) = 0$ if $S(\tau,\tau,g\varsigma)+S(\varsigma,\varsigma,f\tau)+S(\tau,\tau,\varsigma) = 0$. Then there exists a unique common fixed point for f and g.

Proof. If we take $\alpha(\tau,\varsigma,\varpi) = b$ (b is some constant) and proceed with the same steps as outlined in the proof of Theorem 4, the corollary can be proved.

Corollary 4. *Let (Γ, S, α) be a complete complex valued S-metric space. Let f and g be two self mappings on Γ that meet the contraction condition given below:*

$$S(f\tau, f\tau, g\varsigma) \precsim \gamma S(\tau,\tau,\varsigma) + \frac{\beta S(\tau,\tau,f\tau)S(\varsigma,\varsigma,g\varsigma)}{2S(\tau,\tau,g\varsigma)+S(\varsigma,\varsigma,f\tau)+S(\tau,\tau,\varsigma)}$$

for all $\tau,\varsigma \in \Gamma$ such that $\tau \neq \varsigma, S(\tau,\tau,g\varsigma)+S(\varsigma,\varsigma,f\tau)+S(\tau,\tau,\varsigma) \neq 0$, where γ,β are two real numbers which are non- negative and satisfy the condition $\gamma+\beta < 1$ or $S(f\tau,f\tau,g\varsigma) = 0$ if $S(\tau,\tau,g\varsigma)+S(\varsigma,\varsigma,f\tau)+S(\tau,\tau,\varsigma) = 0$. Then there exists a unique common fixed point for f, g.

Proof. If we take $\alpha(\tau,\varsigma,\varpi) = 1$ and proceed with the same steps as outlined in the proof of Theorem 4, the corollary can be proved.

4. An application

We examine a nonlinear Volterra-type integral equation defined for complex-valued functions

$$u(\tau) = \tau^2 + \frac{i}{4}\int_0^{\tau} \frac{u(\varsigma)^3}{1+|u(\varsigma)|}\, d\varsigma, \quad \tau \in [0,1], \tag{7}$$

where $u(\tau) \in \mathbb{C}$ is the unknown function and the kernel $K(\tau,\varsigma,u) = \frac{iu^3}{4(1+|u|)}$ is Lipschitz continuous in u. Let Γ be the space of continuous functions that map from $[0,1]$ to the complex numbers. We define the operator $\mathcal{T}: \Gamma \to \Gamma$ by

$$\mathcal{T}(u(\tau)) := \tau^2 + \frac{i}{4}\int_0^{\tau} \frac{u(\varsigma)^3}{1+|u(\varsigma)|}\, d\varsigma.$$

Consider a closed ball $B_R(0) = \{u \in \Gamma : \|u\|_\infty \leq R\}$ for some $R > 0$. Also, define the metric as

$$S(u,v,w) = \max_{\tau\in[0,1]} \left\{ |u(\tau)-w(\tau)|, |v(\tau)-w(\tau)|, \frac{|u(\tau)-v(\tau)|}{2} \right\}$$

and the control function as

$$\alpha(u,v,w) = 1 + \|u\|_\infty^2 + \|v\|_\infty + \|w\|_\infty^3.$$

Theorem 5. *The integral equation* (7) *admits a unique solution* $u^* \in \Gamma$ *under the stated assumptions.*

Proof. For $u, v \in B_R(0)$ with $R = 0.35$, we have

$$\begin{aligned}
|\mathcal{T}(u(\tau)) - \mathcal{T}(v(\tau))| &\le \frac{1}{4}\int_0^\tau \left| \frac{u^3}{1+|u|} - \frac{v^3}{1+|v|} \right| d\varsigma \\
&\le \frac{1}{4}\int_0^\tau \left(3R^2 + R^3\right)|u-v|\, d\varsigma \quad \left(\text{using } |K(\tau,\varsigma,u) - K(\tau,\varsigma,v)| \le \left(3R^2+R^3\right)|u-v|\right) \\
&\le \frac{3(0.35)^2 + (0.35)^3}{4} \max_{\tau\in[0,1]} |u(\tau) - v(\tau)| \\
&< \theta S(u,u,v),
\end{aligned}$$

where $\theta = 0.11$ and hence, the contraction condition is satisfied.
For the Picard iterates $\tau_\omega = \mathcal{T}^\omega \tau_0$, we obtain

$$\|\tau_\omega\|_\infty \le 1 + \frac{(0.35)^3}{4(1+0.35)} \approx 0.0079.$$

That is,

$$\alpha(\tau_\omega, \tau_\omega, \tau_{\omega+1}) \le 1 + (0.0079)^2 + 0.0079 + (0.0079)^3 \approx 1.008.$$

Thus,

$$\sup_{\varrho\ge 1} \lim_{\omega\to+\infty} \frac{\alpha(\tau_{\omega+1}, \tau_{\omega+1}, \tau_{\omega+2})\alpha(\tau_\varrho, \tau_\varrho, \tau_{\omega+1})}{\alpha(\tau_\omega, \tau_\omega, \tau_{\omega+1})} \le 1.008 < \frac{1}{2\times 0.11} \approx 4.54$$

By Theorem 2, $\mathcal{T}$ admits a unique fixed point in Γ, i.e., the integral equation has a unique solution.

5. Conclusion

In this paper, we introduced the concept of CVCS-metric spaces, which generalizes both complex valued S-metrics and controlled S-metric spaces. Within this new framework, we established several fixed point theorems under various contractive conditions. Our results not only extend existing theorems from the literature but also unify them under a broader and more flexible setting. Furthermore, we demonstrated the applicability of our theoretical results by proving the existence and uniqueness of a solution to a nonlinear Volterra integral equation involving complex-valued functions. These findings open new avenues for future research, particularly in analyzing nonlinear problems and integral equations within complex metric frameworks. Potential directions include extending these results to multivalued mappings and exploring their implications in applied mathematics and computational analysis.

Authors' Contributions

All authors contribute equally in this paper.

Conflict of interest

The authors declare that they have no conflict of interest.

Acknowledgements

The authors acknowledge the financial support from Al-Zaytoonah University of Jordan, Amman 11733, Jordan.

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