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On Bell-based Frobenius-type Eulerian Polynomials and Their Applications

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Abstract. This article considers a new class of generalized Bell-based Frobenius-type Eulerian polynomials of two variables. Also, diverse properties and formulae for these new polynomials are investigated and analyzed. Then, some symmetric identities and implicit summation formulae are improved. The stack of zeros and surface representations of generalized Bell-based Frobenius-type Eulerian polynomials are given for some particular values of the parameters.

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1. Introduction and preliminaries

Some different ways, such as recurrence relations, classical and exponential generating functions, p-adic integrals, explicit formulae, and so on, are used to represent the definitions of special polynomials and numbers. The most significant applications of special polynomials are involved in the theory of finite differences, analytic number theory, classical analysis, and statistics. The most famous special polynomials are Bernoulli [1], Hermite [2], Euler [3], Bell [4], Frobenius-Euler [5], Eulerian [6] and so on and see the references cited therein. The Eulerian polynomials appear in combinatorial mathematics and play an important role in the theory and applications of mathematics. Thus, many number theory and combinatorics experts have extensively studied their properties and obtained diverse interesting results [7]. In recent years, Bell-based Bernoulli polynomials of order α in [8], Bell-based Appell polynomials of order α in [8], and Bell-based Frobenius-type Eulerian polynomials in complex variables in [9] have been considered, and several properties, applications, and relations have been investigated. By motivating and inspiring the above studies, in this work, we consider Bell-based Frobenius-type Eulerian polynomials of order α, and we then derive multifarious relations and identities, including some summation formulas and derivative properties. Also, we investigate some implicit

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summation formulas and symmetric identities for Bell-based Frobenius-type Eulerian polynomials of order α. Furthermore, we provide the stack of zeros and surface representations of a generalized Bell-based Frobenius-type Eulerian polynomials for some particular values of the parameters.

The Apostol-type Bernoulli, Euler and Genocchi polynomials of order α are defined by [10–12]:

$$\left(\frac{\zeta}{\mu e^{\zeta}-1}\right)^{\alpha} e^{\sigma\zeta} = \sum_{\varepsilon=0}^{\infty} \mathbb{B}_{\varepsilon}^{(\alpha)}(\sigma;\mu)\frac{\zeta^{\varepsilon}}{\varepsilon!} \text{ for } \mid \zeta+\log\mu \mid < 2\pi, \tag{1}$$

$$\left(\frac{2}{\mu e^{\zeta}+1}\right)^{\alpha} e^{\sigma\zeta} = \sum_{\varepsilon=0}^{\infty} \mathbb{E}_{\varepsilon}^{(\alpha)}(\sigma;\mu)\frac{\zeta^{\varepsilon}}{\varepsilon!} \text{ for } \mid \zeta+\log\mu \mid < \pi, \tag{2}$$

and

$$\left(\frac{2\zeta}{\mu e^{\zeta}+1}\right)^{\alpha} e^{\sigma\zeta} = \sum_{\varepsilon=0}^{\infty} \mathbb{G}_{\varepsilon}^{(\alpha)}(\sigma;\mu)\frac{\zeta^{\varepsilon}}{\varepsilon!} \text{ for } \mid \zeta+\log\mu \mid < \pi, \tag{3}$$

respectively.

It is seen that

$$\mathbb{B}_{\varepsilon}^{(\alpha)}(0;\mu) := \mathbb{B}_{\varepsilon}^{(\alpha)}(\mu), \mathbb{E}_{\varepsilon}^{(\alpha)}(0;\mu) := \mathbb{E}_{\varepsilon}^{(\alpha)}(\mu) \text{ and } \mathbb{G}_{\varepsilon}^{(\alpha)}(0;\mu) := \mathbb{G}_{\varepsilon}^{(\alpha)}(\mu),$$

are the corresponding numbers of the Apostol-type Bernoulli, Euler, and Genocchi polynomials of order α, respectively. Also, note that

$$\mathbb{B}_{\varepsilon}^{(1)}(\sigma;\mu) := \mathbb{B}_{\varepsilon}(\sigma;\mu), \quad \mathbb{E}_{\varepsilon}^{(1)}(\sigma;\mu) := \mathbb{E}_{\varepsilon}(\sigma;\mu) \quad \text{and} \quad \mathbb{G}_{\varepsilon}^{(1)}(\sigma;\mu) := \mathbb{G}_{\varepsilon}(\sigma;\mu).$$

For $\varepsilon \geq 0$, the first kind of Stirling numbers are defined by the following summation formula [13–17]:

$$(\sigma)_{\varepsilon} = \sum_{\theta=0}^{\varepsilon} S_1(\varepsilon,\theta)\sigma^{\theta}, \tag{4}$$

where $(\sigma)_0 = 1$, and $(\sigma)_{\varepsilon} = \sigma(\sigma-1)\cdots(\sigma-\varepsilon+1)$ $(\varepsilon \geq 1)$. Also $S_1(\varepsilon,\delta)$ can be represented by the following generation function:

$$\frac{1}{\delta!}(\log(1+\zeta))^{\delta} = \sum_{\varepsilon=\delta}^{\infty} S_1(\varepsilon,\delta)\frac{\zeta^{\varepsilon}}{\varepsilon!} \quad (\delta \geq 0). \tag{5}$$

Similar to that of $S_1(\varepsilon,\delta)$, for $\varepsilon \geq 0$, the second kind of Stirling numbers are defined in the following two different ways [18, 19]:

$$\sigma^{\varepsilon} = \sum_{\delta=0}^{\varepsilon} S_2(\varepsilon,\delta)(\sigma)_{\delta}. \tag{6}$$

and

$$\frac{1}{\delta!}(e^{\zeta}-1)^{\delta} = \sum_{\varepsilon=\delta}^{\infty} S_2(\varepsilon,\delta)\frac{\zeta^{\varepsilon}}{\varepsilon!}. \tag{7}$$

The second kind r-Stirling numbers $S_r(\varepsilon,\delta)$ are given by [9, 20]:

$$\frac{1}{\delta!}e^{r\zeta}(e^{\zeta}-1)^{\delta} = \sum_{\varepsilon=\delta}^{\infty} S_r(\varepsilon+r,\delta+r)\frac{\zeta^{\varepsilon}}{\varepsilon!}. \tag{8}$$

The second kind r-Whitney numbers $W_{\tau,r}(\varepsilon,\delta)$, for any positive integer τ, are provided by [21]:

$$\frac{1}{\tau^{\delta}\delta!}e^{r\zeta}(e^{\tau\sigma}-1)^{\delta} = \sum_{\varepsilon=\delta}^{\infty} \mathbb{W}_{\tau,r}(\varepsilon,\delta)\frac{\zeta^{\varepsilon}}{\varepsilon!}. \tag{9}$$

The Bell polynomials $Bel_\varepsilon(\sigma)$ are defined by [4]:

$$e^{\sigma(e^\zeta-1)}=\sum_{\varepsilon=0}^{\infty}Bel_\varepsilon(\sigma)\frac{\zeta^\varepsilon}{\varepsilon!}. \tag{10}$$

When $\sigma=1$, $Bel_\varepsilon=Bel_\varepsilon(1),(\varepsilon\geq 0)$ are called the Bell numbers. By (7) and (10), we observe that

$$Bel_\varepsilon(\sigma)=\sum_{\delta=0}^{\varepsilon}S_2(\varepsilon,\delta)\sigma^\delta \quad (\varepsilon\geq 0). \tag{11}$$

Recently, Duran *et al.* [8] introduced the partially degenerate Bell-based Bernoulli polynomials of of the ${}_{Bel}\mathbb{B}_\varepsilon^{(\alpha)}(\sigma;\rho)$:

$$\left(\frac{\zeta}{e^\zeta-1}\right)^\alpha e^{\sigma\zeta+\eta(e^\zeta-1)}=\sum_{\varepsilon=0}^{\infty}{}_{Bel}\mathbb{B}_\varepsilon^{(\alpha)}(\sigma;\rho)\frac{\zeta^\varepsilon}{\varepsilon!}. \tag{12}$$

For $\alpha=0$ in (12), we get the bivariate Bell polynomials, [8].

$$e^{\sigma\zeta+\rho(e^\zeta-1)}=\sum_{\varepsilon=0}^{\infty}Bel_\varepsilon(\sigma;\rho)\frac{\zeta^\varepsilon}{\varepsilon!}. \tag{13}$$

Let $\mu\in\mathbb{C}$ with $\mu\neq 1$, $\alpha\in\mathbb{C}$ and $\varsigma\in\mathbb{R}$. The Frobenius-type Eulerian polynomials $\mathbb{A}_\varepsilon^{(\alpha)}(\sigma;\mu)$ of order α are given by [6]:

$$\left(\frac{1-\mu}{e^{\zeta(\mu-1)}-\mu}\right)^\alpha e^{\sigma\zeta}=\sum_{\varepsilon=0}^{\infty}\mathbb{A}_\varepsilon^{(\alpha)}(\sigma;\mu)\frac{\zeta^\varepsilon}{\varepsilon!},\quad \left|\zeta+\ln\left(\frac{\mu-1}{\mu}\right)\right|<2\pi. \tag{14}$$

At the point $\sigma=0$, $\mathbb{A}_\varepsilon^{(\alpha)}(0;\mu):=\mathbb{A}_\varepsilon^{(\alpha)}(\mu)$ are called the Frobenius-type Eulerian numbers of order α. By (14), we observe that

$$\mathbb{A}_\varepsilon^{(\alpha)}(\sigma;\mu)=\sum_{\nu=0}^{\varepsilon}\binom{\varepsilon}{\nu}\mathbb{A}_\nu^{(\alpha)}(\mu)\sigma^{\varepsilon-\nu}, \tag{15}$$

and

$$\mathbb{A}_\varepsilon^{(\alpha)}(\sigma;\mu)=(\mu-1)^\varepsilon\mathbb{H}_\varepsilon^{(\alpha)}\left(\frac{\sigma}{\mu-1}|\mu\right), \tag{16}$$

where $\mathbb{H}_\varepsilon^{(\alpha)}(\sigma|\mu)$ are the ε^{th} Frobenius-Euler polynomials of order α [10].

Recently, Khan et al.[16] considered Bell-based Frobenius-type sine-Eulerian polynomials and Bell-based Frobenius-type cosine-Eulerian polynomials, respectively, as follows:

$$\left(\frac{1-\mu}{e^{\zeta(\mu-1)}-\mu}\right)^\alpha e^{\sigma\zeta+\rho(e^\zeta-1)}sin\chi\zeta=\sum_{\varepsilon=0}^{\infty}{}_{Bel}\mathbb{A}_\varepsilon^{(\alpha,s)}(\sigma,\rho,\chi;\mu)\frac{\zeta^\varepsilon}{\varepsilon!}$$

and

$$\left(\frac{1-\mu}{e^{\zeta(\mu-1)}-\mu}\right)^\alpha e^{\sigma\zeta+\rho(e^\zeta-1)}cos\chi\zeta=\sum_{\varepsilon=0}^{\infty}{}_{Bel}\mathbb{A}_\varepsilon^{(\alpha,c)}(\sigma,\rho,\chi;\mu)\frac{\zeta^\varepsilon}{\varepsilon!}. \tag{17}$$

Then, they derived several properties and relations, and also they gave some applications of these polynomials, [4, 7].

2. Properties of Bell-based Frobenius-type Eulerian polynomials

Here, we consider Bell-based Frobenius-type Eulerian polynomials of order α and then we analyze some properties.

Definition 1. *Let $\mu \in \mathbb{C}$ with $\mu \neq 1$, and $\sigma, \rho \in \mathbb{R}$. The Bell-based Frobenius-type Eulerian polynomials ${}_{Bel}\mathbb{A}_{\varepsilon}^{(\alpha)}(\sigma, \rho; \mu)$ of order α are defined by*

$$\left(\frac{1-\mu}{e^{\zeta(\mu-1)}-\mu}\right)^{\alpha} e^{\sigma\zeta+\rho(e^{\zeta}-1)} = \sum_{\varepsilon=0}^{\infty} {}_{Bel}\mathbb{A}_{\varepsilon}^{(\alpha)}(\sigma, \rho; \mu)\frac{\zeta^{\varepsilon}}{\varepsilon!}, \quad \left|\zeta + \ln\left(\frac{\mu-1}{\mu}\right)\right| < 2\pi. \tag{18}$$

Remark 1. *On picking $\sigma = 0$ in (18), we get a new type of Bell-based Frobenius-type Eulerian polynomials ${}_{Bel}\mathbb{A}_{\varepsilon}^{(\alpha)}(\rho; \mu)$ of order α as:*

$$\left(\frac{1-\mu}{e^{\zeta(\mu-1)}-\mu}\right)^{\alpha} e^{\rho(e^{\zeta}-1)} = \sum_{\varepsilon=0}^{\infty} {}_{Bel}\mathbb{A}_{\varepsilon}^{(\alpha)}(\rho; \mu)\frac{\zeta^{\varepsilon}}{\varepsilon!}. \tag{19}$$

Remark 2. *Upon laying $\rho = 0$ in (18), the Bell-based Frobenius-type Eulerian polynomials ${}_{Bel}\mathbb{A}_{\varepsilon}^{(\alpha)}(\sigma, \rho; \mu)$ of order α reduces to familiar Frobenius-type Eulerian polynomials $\mathbb{A}_{\varepsilon}^{(\alpha)}(\sigma; \mu)$ of order α in (14).*

Theorem 1. *For $\varepsilon \geq 0$, we have*

$${}_{Bel}\mathbb{A}_{\varepsilon}^{(\alpha)}(\sigma, \rho; \mu) = \sum_{u=0}^{\varepsilon} \binom{\varepsilon}{u} \mathbb{A}_{u}^{(\alpha)}(\mu) Bel_{\varepsilon-u}(\sigma; \rho), \tag{20}$$

$${}_{Bel}\mathbb{A}_{\varepsilon}^{(\alpha)}(\sigma, \rho; \mu) = \sum_{u=0}^{\varepsilon} \binom{\varepsilon}{u} \mathbb{A}_{u}^{(\alpha)}(\sigma; \lambda) Bel_{\varepsilon-u}(\rho), \tag{21}$$

$${}_{Bel}\mathbb{A}_{\varepsilon}^{(\alpha)}(\sigma, \rho; \mu) = \sum_{u=0}^{\varepsilon} \binom{\varepsilon}{u} {}_{Bel}\mathbb{A}_{u}^{(\alpha)}(\rho; \lambda)\sigma^{\varepsilon-u}. \tag{22}$$

Proof. By (10), (12), (14), and (18) and using the Cauchy product rule, we readily get the representations (20)-(22).

Theorem 2. *For $\varepsilon \geq 0$, we have*

$${}_{Bel}\mathbb{A}_{\varepsilon}^{(\alpha+\beta)}(\sigma + w, \rho + u; \mu) = \sum_{s=0}^{\varepsilon} \binom{\varepsilon}{s} {}_{Bel}\mathbb{A}_{s}^{(\beta)}(u, w; \mu) {}_{Bel}\mathbb{A}_{\varepsilon-s}^{(\alpha)}(\sigma, \rho; \mu). \tag{23}$$

Proof. Using (14) and (18), we have

$$\sum_{\varepsilon=0}^{\infty} {}_{Bel}\mathbb{A}_{\varepsilon}^{(\alpha+\beta)}(\sigma + w, \rho + u; \mu)\frac{\zeta^{\varepsilon}}{\varepsilon!} = \left(\frac{1-\mu}{e^{\zeta(\mu-1)}-\mu}\right)^{\alpha+\beta} e^{(\sigma+u)\zeta+(\rho+w)(e^{\zeta}-1)}$$

$$= \sum_{\varepsilon=0}^{\infty} {}_{Bel}\mathbb{A}_{\varepsilon}^{(\alpha)}(\sigma, \rho; \mu)\frac{\zeta^{\varepsilon}}{\varepsilon!} \sum_{s=0}^{\infty} {}_{Bel}\mathbb{A}_{s}^{(\beta)}(u, w; \lambda)\frac{\zeta^{s}}{s!}$$

$$= \sum_{\varepsilon=0}^{\infty} \sum_{s=0}^{\varepsilon} \binom{\varepsilon}{s} {}_{Bel}\mathbb{A}_{s}^{(\beta)}(u, w; \mu) {}_{Bel}\mathbb{A}_{\varepsilon-s}^{(\alpha)}(\sigma, \rho; \mu)\frac{\zeta^{\varepsilon}}{\varepsilon!}, \tag{24}$$

which means the asserted result (23).

Remark 3. *For $u = \beta = 0$ in Theorem 2, we get*

$$ {}_{Bel}\mathbb{A}_{\varepsilon}^{(\alpha)}(\sigma + w, \rho; \mu) = \sum_{s=0}^{\varepsilon} \binom{\varepsilon}{s} \mathbb{A}_{\varepsilon-s}^{(\alpha)}(\sigma; \mu) Bel_s(\rho; w). \tag{25} $$

Theorem 3. *The following differentiation formulas hold:*

$$ \frac{\partial\, {}_{Bel}\mathbb{A}_{\varepsilon}^{(\alpha)}(\sigma, \rho; \mu)}{\partial \sigma} = \varepsilon\, {}_{Bel}\mathbb{A}_{\varepsilon-1}^{(\alpha)}(\sigma, \rho; \mu), \tag{26} $$

$$ \frac{\partial\, {}_{Bel}\mathbb{A}_{\varepsilon}^{(\alpha)}(\sigma, \rho; \mu)}{\partial \rho} = {}_{Bel}\mathbb{A}_{\varepsilon}^{(\alpha)}(\sigma + 1, \rho; \mu) - {}_{Bel}\mathbb{A}_{\varepsilon}^{(\alpha)}(\sigma, \rho; \mu). \tag{27} $$

Proof. By (18), we have

$$ \begin{aligned} \sum_{\varepsilon=1}^{\infty} \frac{\partial}{\partial \sigma} {}_{Bel}\mathbb{A}_{\sigma}^{(\alpha)}(\sigma, \rho; \mu) \frac{\zeta^{\varepsilon}}{\varepsilon!} = \frac{\partial}{\partial \sigma} \left(\frac{1-\mu}{e^{\zeta(\mu-1)} - \mu} \right)^{\alpha} e^{\sigma\zeta + \rho(e^{\zeta}-1)} &= \left(\frac{1-\mu}{e^{\zeta(\mu-1)} - \mu} \right)^{\alpha} \frac{\partial}{\partial \sigma} e^{\sigma\zeta + \rho(e^{\zeta}-1)} \\ &= \left(\frac{1-\mu}{e^{\zeta(\mu-1)} - \mu} \right)^{\alpha} \zeta e^{\sigma\zeta + \eta(e^{\zeta}-1)} \\ = \sum_{\varepsilon=1}^{\infty} {}_{Bel}\mathbb{A}_{\varepsilon-1}^{(\alpha)}(\sigma, \rho; \mu) \frac{\zeta^{\varepsilon}}{\varepsilon!}, \end{aligned} \tag{28} $$

which gives the claimed result (26). Again, using (18), we note that

$$ \begin{aligned} \sum_{\varepsilon=0}^{\infty} \frac{\partial}{\partial \rho} {}_{Bel}\mathbb{A}_{\varepsilon}^{(\alpha)}(\sigma, \rho; \mu) \frac{\zeta^{\varepsilon}}{\varepsilon!} = \frac{\partial}{\partial \rho} \left(\frac{1-\mu}{e^{\zeta(\mu-1)} - \mu} \right)^{\alpha} e^{\sigma\zeta + \rho(e^{\zeta}-1)} &= \left(\frac{1-\mu}{e^{\zeta(\mu-1)} - \mu} \right)^{\alpha} \frac{\partial}{\partial \rho} e^{\sigma\zeta + \rho(e^{\zeta}-1)} \\ &= \left(\frac{1-\mu}{e^{\zeta(\mu-1)} - \mu} \right)^{\alpha} e^{\sigma\zeta + \rho(e^{\zeta}-1)} (e^{\zeta} - 1) \\ = \sum_{\varepsilon=0}^{\infty} {}_{Bel}\mathbb{A}_{\varepsilon}^{(\alpha)}(\sigma + 1, \rho; \mu) \frac{\zeta^{\varepsilon}}{\varepsilon!} - \sum_{\varepsilon=0}^{\infty} {}_{Bel}\mathbb{A}_{\varepsilon}^{(\alpha)}(\sigma, \rho; \mu) \frac{\zeta^{\varepsilon}}{\varepsilon!}, \end{aligned} \tag{29} $$

which provides the asserted result (27).

Theorem 4. *Let $\varepsilon \geq 0$. Then*

$$ \begin{aligned} &(2\mu - 1) \sum_{\delta=0}^{\varepsilon} \binom{\varepsilon}{\delta} \mathbb{A}_{\delta}(\sigma; \mu)\, {}_{Bel}\mathbb{A}_{\varepsilon-\delta}(\sigma, \rho; 1 - \mu) \\ &= \lambda\, {}_{Bel}\mathbb{A}_{\varepsilon}(\sigma, \rho; \mu) - (1 - \mu)\, {}_{Bel}\mathbb{A}_{\varepsilon}(\sigma, \rho; 1 - \mu). \end{aligned} \tag{30} $$

Proof. We set

$$ \frac{(2\mu - 1)}{(e^{\zeta(\mu-1)} - \mu)(e^{\zeta(\mu-1)} - (1 - \mu))} = \frac{1}{e^{\zeta(\mu-1)} - \mu} - \frac{1}{e^{\zeta(\mu-1)} - (1 - \mu)}. $$

We observe that

$$ (2\mu-1) \frac{(1 - \mu)e^{\sigma\zeta}(1 - (1 - \mu))e^{\rho(e^{\zeta}-1)}}{(e^{\zeta(\mu-1)} - \mu)(e^{\zeta(\mu-1)} - (1 - \mu))} = \frac{(1 - \mu)e^{\rho(e^{\zeta}-1)}\mu e^{\sigma\zeta}}{e^{\zeta(\mu-1)} - \mu} - \frac{(1 - \mu)e^{\rho(e^{\zeta}-1)}\mu e^{\sigma\zeta}(1 - (1 - \mu))}{e^{\zeta(\mu-1)} - (1 - \mu)}, $$

and then

$$ \begin{aligned} &(2\mu - 1) \left(\sum_{\delta=0}^{\infty} \mathbb{A}_{\delta}(\sigma; \mu) \frac{\zeta^{\delta}}{\delta!} \right) \left(\sum_{\varepsilon=0}^{\infty} {}_{Bel}\mathbb{A}_{\varepsilon}(\rho; 1 - \mu) \frac{\zeta^{\varepsilon}}{\varepsilon!} \right) \\ &= \mu \sum_{\varepsilon=0}^{\infty} {}_{Bel}\mathbb{A}_{\varepsilon}(\sigma, \rho; \mu) \frac{\zeta^{\varepsilon}}{\varepsilon!} - (1 - \mu) \sum_{\varepsilon=0}^{\infty} {}_{Bel}\mathbb{A}_{\varepsilon}(\sigma, \rho; 1 - \mu) \frac{\zeta^{\varepsilon}}{\varepsilon!}, \end{aligned} $$

which implies the desired result.

Theorem 5. *For $\varepsilon \geq 0$, we have*

$$\mu_{Bel}\mathbb{A}_{\varepsilon}(\sigma,\rho;\mu)=\sum_{\delta=0}^{\varepsilon}\left(\begin{array}{c}\varepsilon\\ \delta\end{array}\right){}_{Bel}\mathbb{A}_{\varepsilon-\delta}(\sigma,\rho;\mu)(1-\mu)^{\delta}-(1-\mu)Bel_{\varepsilon}(\sigma;\rho). \tag{31}$$

Proof. Consider

$$\frac{\mu}{(e^{\zeta(\mu-1)}-\mu)e^{\zeta(\mu-1)}}=\frac{1}{(e^{\zeta(\mu-1)}-\mu)}-\frac{1}{e^{\zeta(\mu-1)}}.$$

We find

$$\frac{\mu(1-\mu)e^{\sigma\zeta+\rho(e^{\zeta}-1)}}{(e^{\zeta(\mu-1)}-\mu)e^{\zeta(\mu-1)}}=\frac{(1-\mu)e^{\sigma\zeta+\rho(e^{\zeta}-1)}}{(e^{\zeta(\mu-1)}-\mu)}-\frac{(1-\mu)e^{\sigma\zeta+\rho(e^{\zeta}-1)}}{e^{\zeta(\mu-1)}}$$

$$\mu\sum_{\varepsilon=0}^{\infty}{}_{Bel}\mathbb{A}_{\varepsilon}(\sigma,\rho;\mu)\frac{\zeta^{\varepsilon}}{\varepsilon!}=\sum_{\varepsilon=0}^{\infty}{}_{Bel}\mathbb{A}_{\varepsilon}(\sigma,\rho;\mu)\frac{\zeta^{\varepsilon}}{\varepsilon!}\sum_{\delta=0}^{\infty}(1-\mu)^{\delta}\frac{\zeta^{\delta}}{\delta!}-(1-\mu)\sum_{\varepsilon=0}^{\infty}Bel_{\varepsilon}(\sigma;\rho)\frac{\zeta^{\varepsilon}}{\varepsilon!}. \tag{32}$$

Therefore, by (32), we get (31).

Theorem 6. *Let $\varepsilon \geq 0$. Then*

$${}_{Bel}\mathbb{A}_{\varepsilon}^{(\alpha)}(\sigma,\rho;\mu)=\frac{1}{1-\mu}\sum_{\delta=0}^{\varepsilon}\left(\begin{array}{c}\varepsilon\\ \delta\end{array}\right)\left[\mathbb{A}_{\varepsilon-\delta}(\mu){}_{Bel}\mathbb{A}_{\delta}^{(\alpha)}((1-\mu)\sigma,\rho;\mu)-\mu\mathbb{A}_{\varepsilon-\delta}(\mu){}_{Bel}\mathbb{A}_{\delta}^{(\alpha)}(\sigma,\rho;\mu)\right]. \tag{33}$$

Proof. In (18), we have

$$\begin{aligned}\sum_{\varepsilon=0}^{\infty}{}_{Bel}\mathbb{A}_{\varepsilon}^{(\alpha)}(\sigma,\rho;\mu)\frac{\zeta^{\varepsilon}}{\varepsilon!}&=\left(\frac{1-\mu}{e^{\zeta(\mu-1)}-\mu}\right)\left(\frac{e^{\zeta(\mu-1)}-\mu}{1-\mu}\right)\left(\frac{1-\mu}{e^{\zeta(\mu-1)}-\mu}\right)^{\alpha}e^{\sigma\zeta+\rho(e^{\zeta}-1)}\\ &=\frac{1}{1-\mu}\left[\left(\frac{1-\mu}{e^{\zeta(\mu-1)}-\mu}\right)e^{(\mu-1)\zeta}\left(\frac{1-\mu}{e^{\zeta(\mu-1)}-\mu}\right)^{\alpha}e^{\sigma\zeta+\rho(e^{\zeta}-1)}\right.\\ &\qquad\left.-\mu\left(\frac{1-\mu}{e^{\zeta(\mu-1)}-\mu}\right)\left(\frac{1-\mu}{e^{\zeta(\mu-1)}-\mu}\right)^{\alpha}e^{\sigma\zeta+\rho(e^{\zeta}-1)}\right]\\ &=\frac{1}{1-\mu}\left[\sum_{\varepsilon=0}^{\infty}\mathbb{A}_{\varepsilon}(\mu)\frac{\zeta^{\varepsilon}}{\varepsilon!}\sum_{\delta=0}^{\infty}{}_{Bel}\mathbb{A}_{\delta}^{(\alpha)}((\mu-1)\sigma,\rho;\mu)\frac{\zeta^{\delta}}{\delta!}-\mu\sum_{\varepsilon=0}^{\infty}\mathbb{A}_{\varepsilon}(\mu)\frac{\zeta^{\varepsilon}}{\varepsilon!}\sum_{\delta=0}^{\infty}{}_{Bel}\mathbb{A}_{\delta}^{(\alpha)}(\sigma,\rho;\mu)\frac{\zeta^{\delta}}{\delta!}\right].\end{aligned} \tag{34}$$

By (18) and (34), we obtain (33).

Theorem 7. *Let $\varepsilon \geq 0$. Then*

$${}_{Bel}\mathbb{A}_{\varepsilon}^{(\alpha)}(\sigma,\rho;\mu)=\sum_{s=0}^{\varepsilon}\sum_{\delta=0}^{s}\binom{\varepsilon}{s}(\sigma)_{\delta}S_2(s,\delta){}_{Bel}\mathbb{A}_{\varepsilon}^{(\alpha)}(\sigma;\mu). \tag{35}$$

Proof. By (18), we note that

$$\begin{aligned}\sum_{\varepsilon=0}^{\infty}{}_{Bel}\mathbb{A}_{\varepsilon}^{(\alpha)}(\sigma,\rho;\mu)\frac{\zeta^{\varepsilon}}{\varepsilon!}&=\left(\frac{1-\mu}{e^{\zeta(\mu-1)}-\mu}\right)^{\alpha}e^{\rho(e^{\zeta}-1)}[e^{\zeta}-1+1]^{\sigma}=\left(\frac{1-\mu}{e^{\zeta(\mu-1)}-\mu}\right)^{\alpha}e^{\rho(e^{\zeta}-1)}\sum_{\delta=0}^{\infty}(\sigma)_{\delta}\frac{(e^{\zeta}-1)^{\delta}}{\delta!}\\ &=\sum_{\varepsilon=0}^{\infty}{}_{Bel}\mathbb{A}_{\varepsilon}^{(\alpha)}(\rho;\mu)\frac{\zeta^{\varepsilon}}{\varepsilon!}\sum_{s=0}^{\infty}\sum_{\delta=0}^{s}(\sigma)_{\delta}S_2(s,\delta)\frac{\zeta^{s}}{s!}\\ &=\sum_{\varepsilon=0}^{\infty}\left(\sum_{s=0}^{\varepsilon}\sum_{\delta=0}^{s}\binom{\varepsilon}{s}(\sigma)_{\delta}S_2(s,\delta){}_{Bel}\mathbb{A}_{\varepsilon}^{(\alpha)}(\rho;\mu)\right)\frac{\zeta^{\varepsilon}}{\varepsilon!}.\end{aligned} \tag{36}$$

In view of (18) and (36), we get (35).

3. Summation formulae

Here, we investigate several implicit formulas and symmetric relations for Bell-based Frobenius-type Eulerian polynomials of order α.

Theorem 8. *The following formula is correct:*

$$ {}_{Bel}\mathbb{A}^{(\alpha)}_{h+f}(\sigma,\rho;\mu) = \sum_{\varepsilon,s=0}^{h,f} \begin{pmatrix} f \\ s \end{pmatrix} \begin{pmatrix} h \\ \varepsilon \end{pmatrix} (\sigma-\zeta)^{\varepsilon+s} {}_{Bel}\mathbb{A}^{(\alpha)}_{h+f-\varepsilon-s}(\zeta,\eta;\mu). \tag{37} $$

Proof. By replacing z by $z+w$ in (18), we see that

$$ \left(\frac{1-\mu}{e^{(\mu-1)(z+w)}-\mu}\right)^{\alpha} e^{\eta(e^{z+w}-1)} = e^{-\xi(z+w)} \sum_{h,f=0}^{\infty} {}_{Bel}\mathbb{A}^{(\alpha)}_{h+f}(\sigma,\rho;\mu)\frac{z^h}{h!}\frac{w^f}{f!}. \tag{38} $$

Also by replacing ξ by ζ in (38), we acquire

$$ e^{-\zeta(z+w)} \sum_{h,f=0}^{\infty} {}_{Bel}\mathbb{A}^{(\alpha)}_{h+f}(\zeta,\rho;\mu)\frac{z^h}{h!}\frac{w^f}{f!} = \left(\frac{1-\mu}{e^{(\mu-1)(z+w)}-\mu}\right)^{\alpha} e^{\rho(e^{z+w}-1)} \tag{39} $$

$$ e^{(\sigma-\zeta)(z+w)} \sum_{h,f=0}^{\infty} {}_{Bel}\mathbb{A}^{(\alpha)}_{h+f}(\zeta,\rho;\mu)\frac{z^h}{h!}\frac{w^f}{f!} = \sum_{h,f=0}^{\infty} {}_{Bel}A^{(\alpha)}_{h+f}(\sigma,\rho;\mu)\frac{z^h}{h!}\frac{w^f}{f!} \tag{40} $$

$$ \sum_{N=0}^{\infty} \frac{[(\sigma-\zeta)(z+w)]^N}{N!} \sum_{h,f=0}^{\infty} {}_{Bel}\mathbb{A}^{(\alpha)}_{h+f}(\zeta,\eta;\mu)\frac{z^h}{h!}\frac{w^f}{f!} = \sum_{h,f=0}^{\infty} {}_{Bel}\mathbb{A}^{(\alpha)}_{h+f}(\sigma,\rho;\mu)\frac{z^h}{h!}\frac{w^f}{f!}. \tag{41} $$

Using the formula [16]

$$ \sum_{\mathbb{N}=0}^{\infty} f(\mathbb{N})\frac{(\zeta+\eta)^{\mathbb{N}}}{\mathbb{N}!} = \sum_{\varepsilon,\delta=0}^{\infty} f(\varepsilon+\delta)\frac{\zeta^{\varepsilon}}{\varepsilon!}\frac{\eta^{\delta}}{\delta!}, \tag{42} $$

we then obtain

$$ \sum_{j,s=0}^{\infty} \frac{(\xi-\zeta)^{j+s}z^j w^s}{j!s!} \sum_{h,f=0}^{\infty} {}_{Bel}\mathbb{A}^{(\alpha)}_{h+f}(\zeta,\eta;\mu)\frac{z^h}{h!}\frac{w^f}{f!} = \sum_{h,f=0}^{\infty} {}_{Bel}\mathbb{A}^{(\alpha)}_{h+f}(\sigma,\rho;\mu)\frac{z^h}{h!}\frac{w^f}{f!}. \tag{43} $$

Thus we have

$$ \sum_{h,f=0}^{\infty} \sum_{\varepsilon,s=0}^{h,f} \frac{(\sigma-\zeta)^{\varepsilon+s}}{\varepsilon!s!} {}_{Bel}\mathbb{A}^{(\alpha)}_{h+f-\varepsilon-s}(\zeta,\rho;\mu)\frac{z^h}{(h-j)!}\frac{w^f}{(f-s)!} = \sum_{h,f=0}^{\infty} {}_{Bel}\mathbb{A}^{(\alpha)}_{h+f}(\sigma,\rho;\mu)\frac{z^h}{h!}\frac{w^f}{f!}, \tag{44} $$

which implies the claimed result.

Remark 4. *Permitting $f=0$ in (37), we get*

$$ {}_{Bel}\mathbb{A}^{(\alpha)}_{\theta}(\sigma,\rho;\mu) = \sum_{\varepsilon=0}^{h} \begin{pmatrix} h \\ \varepsilon \end{pmatrix} (\sigma-\zeta)^{\varepsilon} {}_{Bel}\mathbb{A}^{(\alpha)}_{h-\varepsilon}(\zeta,\rho;\mu) \quad (\varepsilon \geq 0). \tag{45} $$

Remark 5. *Replace $\sigma \to \sigma+\zeta$ and setting $\rho=0$ in (37), we obtain*

$$_{Bel}\mathbb{A}^{(\alpha)}_{h+f}(\sigma+\zeta;\mu)=\sum_{\varepsilon,\delta=0}^{h,f}\left(\begin{array}{c}f\\ \delta\end{array}\right)\left(\begin{array}{c}h\\ \varepsilon\end{array}\right)\sigma^{\varepsilon+s}{}_{Bel}\mathbb{A}^{(\alpha)}_{h+f-\varepsilon-\delta}(\zeta;\mu). \tag{46}$$

Again, by permitting $\sigma=0$ in (37), we obtain

$$_{Bel}\mathbb{A}^{(\alpha)}_{h+f}(\rho;\mu)=\sum_{\varepsilon,s=0}^{h,f}\left(\begin{array}{c}f\\ \delta\end{array}\right)\left(\begin{array}{c}h\\ \varepsilon\end{array}\right)(-\zeta)^{\varepsilon+\delta}{}_{Bel}\mathbb{A}^{(\alpha)}_{h+f-\varepsilon-\delta}(\zeta,\rho;\mu).$$

Theorem 9. *Let* $\varepsilon\geq 0$. *Then*

$$_{Bel}\mathbb{A}^{(\alpha+1)}_{\varepsilon}(\sigma,\rho;\mu)=\sum_{d=0}^{\varepsilon}\left(\begin{array}{c}\varepsilon\\ d\end{array}\right)\mathbb{A}_{\varepsilon-d}(\mu)\,{}_{Bel}\mathbb{A}^{(\alpha)}_{d}(\sigma,\rho;\mu). \tag{47}$$

Proof. By (18), we have

$$\frac{1-\mu}{e^{(\mu-1)\zeta}-\mu}\left(\frac{1-\mu}{e^{(\mu-1)\zeta}-\mu}\right)^{\alpha}e^{\sigma\zeta+\rho(e^{\zeta}-1)}=\frac{1-\mu}{e^{(\mu-1)\zeta}-\mu}\sum_{d=0}^{\infty}{}_{Bel}\mathbb{A}^{(\alpha)}_{d}(\sigma,\rho;\mu)\frac{\zeta^{d}}{d!}$$

$$\left(\frac{1-\mu}{e^{(\mu-1)\zeta}-\mu}\right)^{\alpha+1}e^{\sigma\zeta+\rho(e^{\zeta}-1)}=\frac{1-\mu}{e^{(\mu-1)\zeta}-\mu}\sum_{d=0}^{\infty}{}_{Bel}\mathbb{A}^{(\alpha)}_{d}(\sigma,\rho;\mu)\frac{\zeta^{d}}{d!}$$

$$=\sum_{\varepsilon=0}^{\infty}\mathbb{A}_{\varepsilon}(\mu)\frac{\zeta^{\varepsilon}}{\varepsilon!}\sum_{d=0}^{\infty}{}_{Bel}\mathbb{A}^{(\alpha)}_{d}(\sigma,\rho;\mu)\frac{\zeta^{d}}{d!}$$

$$=\sum_{\varepsilon=0}^{\infty}\left(\sum_{d=0}^{\varepsilon}\binom{\varepsilon}{d}\mathbb{A}_{\varepsilon-d}(\mu)\,{}_{Bel}\mathbb{A}^{(\alpha)}_{d}(\sigma,\rho;\mu)\right)\frac{\zeta^{\varepsilon}}{\varepsilon!},$$

which implies the result (47).

Theorem 10. *Let* $\varepsilon\geq 0$. *Then*

$$_{Bel}\mathbb{A}^{(\alpha)}_{\varepsilon}(\sigma+1,\rho;\mu)=\sum_{\delta=0}^{\varepsilon}\left(\begin{array}{c}\varepsilon\\ \delta\end{array}\right){}_{Bel}\mathbb{A}^{(\alpha)}_{\delta}(\sigma,\rho;\mu). \tag{48}$$

Proof. Using definition (18), we have

$$\sum_{\varepsilon=0}^{\infty}{}_{Bel}\mathbb{A}^{(\alpha)}_{\varepsilon}(\sigma+1,\rho;\mu)\frac{\zeta^{\varepsilon}}{\varepsilon!}-\sum_{\varepsilon=0}^{\infty}{}_{Bel}\mathbb{A}^{(\alpha)}_{\varepsilon}(\sigma,\rho;\mu)\frac{\zeta^{\varepsilon}}{\varepsilon!}$$

$$=\left(\frac{1-\mu}{e^{(\mu-1)\zeta}-\mu}\right)^{\alpha}e^{\sigma\zeta+\rho(e^{\zeta}-1)}(e^{\zeta}-1)$$

$$=\left(\sum_{\delta=0}^{\infty}{}_{Bel}\mathbb{A}^{(\alpha)}_{\delta}(\sigma,\rho;\mu)\frac{\zeta^{\delta}}{\delta!}\right)\left(\sum_{\varepsilon=0}^{\infty}\frac{\zeta^{\varepsilon}}{\varepsilon!}\right)-\sum_{\varepsilon=0}^{\infty}{}_{Bel}\mathbb{A}^{(\alpha)}_{\varepsilon}(\sigma,\rho;\mu)\frac{\zeta^{\varepsilon}}{\varepsilon!}$$

$$=\sum_{\varepsilon=0}^{\infty}\sum_{\delta=0}^{\varepsilon}\binom{\varepsilon}{\delta}{}_{Bel}\mathbb{A}^{(\alpha)}_{\delta}(\sigma,\rho;\mu)\frac{\zeta^{\varepsilon}}{\varepsilon!}-\sum_{\varepsilon=0}^{\infty}{}_{Bel}\mathbb{A}^{(\alpha)}_{\varepsilon}(\sigma,\rho;\mu)\frac{\zeta^{\varepsilon}}{\varepsilon!},$$

which gives the claimed result (48).

Theorem 11. *Let* $\varepsilon \geq 0$*. Then*

$$ {}_{Bel}\mathbb{A}_{\varepsilon}^{(\alpha)}(\sigma,\rho;\mu)=\sum_{\delta=0}^{\varepsilon}\sum_{\theta=\vartheta}^{\varepsilon}\begin{pmatrix}\alpha+\theta-1\\ \theta\end{pmatrix}\theta!\begin{pmatrix}\varepsilon\\ \vartheta\end{pmatrix}S_2(\vartheta,\theta)(\mu-1)^{\vartheta-\theta}Bel_{\varepsilon-\vartheta}(\sigma;\rho). \tag{49} $$

Proof. In (18), we have

$$ \sum_{\varepsilon=0}^{\infty}{}_{Bel}\mathbb{A}_{\varepsilon}^{(\alpha)}(\sigma,\rho;\mu)\frac{\zeta^{\varepsilon}}{\varepsilon!}=\left(\frac{1-\mu}{e^{\zeta(\mu-1)}-\mu}\right)^{\alpha}e^{\sigma\zeta+\rho(e^{\zeta}-1)}=e^{\sigma\zeta+\rho(e^{\zeta}-1)}\left(1+\frac{e^{\zeta(\mu-1)}-1}{1-\mu}\right)^{-\alpha} $$
$$ =\sum_{\theta=0}^{\infty}\begin{pmatrix}-\alpha\\ \theta\end{pmatrix}\left(\frac{e^{\zeta(\mu-1)}-1}{1-\mu}\right)^{\theta}\sum_{\varepsilon=0}^{\infty}Bel_{\varepsilon}(\sigma;\rho)\frac{\zeta^{\varepsilon}}{\varepsilon!} $$
$$ =\sum_{\varepsilon=0}^{\infty}\left(\sum_{\theta=0}^{\varepsilon}\sum_{\vartheta=\theta}^{\varepsilon}\begin{pmatrix}\alpha+\theta-1\\ \theta\end{pmatrix}\theta!\begin{pmatrix}\varepsilon\\ \vartheta\end{pmatrix}S_2(\vartheta,\theta)(\mu-1)^{\vartheta-\theta}Bel_{\varepsilon-\vartheta}(\sigma;\rho)\right)\frac{\zeta^{\varepsilon}}{\varepsilon!}, $$

which yields the result (49).

Theorem 12. *Let* $\varepsilon \geq 0$*. Then*

$$ {}_{Bel}\mathbb{A}_{\varepsilon}^{(\alpha)}(\sigma,\rho;\mu)=\sum_{\delta=0}^{\infty}\sum_{\theta=0}^{\delta}\binom{\delta}{\theta}Bel_{\varepsilon}(\sigma+(\mu-1)\theta;\rho)\frac{(\alpha)_{\delta}}{\delta!}(\mu-1)^{-\alpha}(-1)^{\delta-\theta}. \tag{50} $$

Proof. By (18), we have

$$ \sum_{\varepsilon=0}^{\infty}{}_{Bel}\mathbb{A}_{\varepsilon}^{(\alpha)}(\sigma,\rho;\mu)\frac{\zeta^{\varepsilon}}{\varepsilon!}=\left(\frac{1-\mu}{e^{\zeta(\mu-1)}-\mu}\right)^{\alpha}e^{\sigma\zeta+\rho(e^{\zeta}-1)} $$
$$ =e^{\sigma\zeta+\rho(e^{\zeta}-1)}\left(1-\frac{e^{\zeta(\mu-1)}-1}{\mu-1}\right)^{-\alpha}=e^{\sigma\zeta+\rho(e^{\zeta}-1)}\left(\frac{\mu-e^{\zeta(\mu-1)}}{\mu-1}\right)^{-\alpha}=e^{\sigma\zeta+\rho(e^{\zeta}-1)}\sum_{\delta=0}^{\infty}(\alpha)_{\delta}\frac{1}{\delta!}\left(\frac{e^{\zeta(\mu-1)}-1}{\mu-1}\right)^{\delta} $$
$$ =e^{\sigma\zeta+\rho(e^{\zeta}-1)}\sum_{\delta=0}^{\infty}(\alpha)_{\delta}\frac{1}{\delta!}(\mu-1)^{-\alpha}\sum_{\theta=0}^{\delta}\binom{\delta}{\theta}e^{\zeta(\mu-1)\theta}(-1)^{\delta-\theta}\sum_{\theta=0}^{\infty}\sum_{\delta=0}^{\theta}(\alpha)_{\delta}(\mu-1)^{\theta-\delta}S_l^{\delta}\left(\frac{\sigma}{\mu-1}\right)\frac{\zeta^{\theta}}{\theta!} $$
$$ =\sum_{\varepsilon=0}^{\infty}\sum_{\delta=0}^{\infty}\sum_{\theta=0}^{\delta}\binom{\delta}{\theta}Bel_{\varepsilon}(\sigma+(\mu-1)\theta;\rho)\frac{(\alpha)_{\delta}}{\delta!}(\mu-1)^{-\alpha}(-1)^{\delta-\theta}\frac{\zeta^{\varepsilon}}{\varepsilon!} $$
$$ =\sum_{\varepsilon=0}^{\infty}\left(\sum_{\theta=0}^{\varepsilon}\sum_{\delta=0}^{\theta}\binom{\varepsilon}{\theta}(\alpha)_{\delta}(\mu-1)^{\theta-\delta}S_l^{\delta}\left(\frac{\sigma}{\mu-1}\right)Bel_{\varepsilon-\theta}(\varsigma)\right)\frac{\zeta^{\varepsilon}}{\varepsilon!}, $$

which means the claimed formula (50).

Theorem 13. *Let* $\varepsilon \geq 0$*. Then*

$$ {}_{Bel}\mathbb{A}_{\varepsilon}^{(\alpha)}(\sigma,\rho;\mu)=\frac{1}{\varepsilon+1}\sum_{\delta=0}^{\varepsilon+1}\begin{pmatrix}\varepsilon+1\\ \delta\end{pmatrix}\left(\mu\sum_{\theta=0}^{\delta}\begin{pmatrix}\delta\\ \theta\end{pmatrix}\mathbb{B}_{\delta-\theta}(\sigma;\mu)-\mathbb{B}_{\theta}(\sigma;\mu)\right){}_{Bel}\mathbb{A}_{\varepsilon-\delta+1}^{(\alpha)}(0,\rho;\mu). \tag{51} $$

Proof. By (1) and (18), we have

$$ \sum_{\varepsilon=0}^{\infty}{}_{Bel}\mathbb{A}_{\varepsilon}^{(\alpha)}(\sigma,\rho;\mu)\frac{\zeta^{\varepsilon}}{\varepsilon!}=\left(\frac{1-\mu}{e^{\zeta(\mu-1)}-\mu}\right)^{\alpha}e^{\sigma\zeta+\rho(e^{\zeta}-1)}\left(\frac{\zeta}{\mu e^{\zeta}-1}\right)\left(\frac{\mu e^{\zeta}-1}{\zeta}\right) $$

$$= \frac{1}{\zeta}\left(\mu\sum_{\varepsilon=0}^{\infty}{}_{Bel}\mathbb{A}_{\varepsilon}^{(\alpha)}(0,\rho;\mu)\frac{\zeta^{\varepsilon}}{\varepsilon!}\sum_{\delta=0}^{\infty}\mathbb{B}_{\delta}(\sigma;\mu)\frac{\zeta^{\delta}}{\delta!}\sum_{\theta=0}^{\infty}\frac{\zeta^{\theta}}{\theta!}-\sum_{\varepsilon=0}^{\infty}{}_{Bel}\mathbb{A}_{\varepsilon}^{(\alpha)}(0,\rho;\mu)\frac{\zeta^{\varepsilon}}{\varepsilon!}\sum_{\delta=0}^{\infty}\mathbb{B}_{\delta}(\sigma;\mu)\frac{\zeta^{\delta}}{\delta!}\right), \tag{52}$$

which yields the claimed result (51).

Theorem 14. *Let $\varepsilon \geq 0$. Then*

$$_{Bel}\mathbb{A}_{\varepsilon}^{(\alpha)}(\sigma,\rho;\mu) = \frac{1}{2}\sum_{\delta=0}^{\varepsilon}\binom{\varepsilon}{\delta}\left(\mu\sum_{\theta=0}^{\delta}\binom{\delta}{\theta}\mathbb{E}_{\delta-\theta}(\sigma;\mu)+\mathbb{E}_{\theta}(\sigma;\mu)\right){}_{Bel}\mathbb{A}_{\varepsilon-\delta}^{(\alpha)}(0,\rho;\mu). \tag{53}$$

Proof. From (2) and (18), we have

$$\sum_{\varepsilon=0}^{\infty}{}_{Bel}\mathbb{A}_{\varepsilon}^{(\alpha)}(\sigma,\rho;\mu)\frac{\zeta^{\varepsilon}}{\varepsilon!} = \left(\frac{1-\mu}{e^{\zeta(\mu-1)}-\mu}\right)^{\alpha}e^{\sigma\zeta+\rho(e^{\zeta}-1)}\left(\frac{2}{\mu e^{\zeta}+1}\right)\left(\frac{\mu e^{\zeta}+1}{2}\right)$$

$$= \frac{1}{2}\left(\mu\sum_{\varepsilon=0}^{\infty}{}_{Bel}\mathbb{A}_{\varepsilon}^{(\alpha)}(0,\rho;\mu)\frac{\zeta^{\varepsilon}}{\varepsilon!}\sum_{\delta=0}^{\infty}\mathbb{E}_{\delta}(\sigma;\mu)\frac{\zeta^{\delta}}{\delta!}\sum_{\theta=0}^{\infty}\frac{\zeta^{\theta}}{\theta!}+\sum_{\varepsilon=0}^{\infty}{}_{Bel}\mathbb{A}_{\varepsilon}^{(\alpha)}(0,\rho;\mu)\frac{\zeta^{\varepsilon}}{\varepsilon!}\sum_{\delta=0}^{\infty}\mathbb{E}_{\delta}(\sigma;\mu)\frac{\zeta^{\delta}}{\delta!}\right), \tag{54}$$

which yields the result (53).

Theorem 15. *Let $\varepsilon \geq 0$. Then*

$$_{Bel}\mathbb{A}_{\varepsilon}^{(\alpha)}(\sigma,\rho;\mu) = \frac{1}{2(\varepsilon+1)}\sum_{\delta=0}^{\varepsilon+1}\binom{\varepsilon+1}{\delta}\left(\mu\sum_{\theta=0}^{\delta}\binom{\delta}{\theta}\mathbb{G}_{\delta-\theta}(\sigma;\mu)+\mathbb{G}_{\theta}(\sigma;\mu)\right){}_{Bel}\mathbb{A}_{\varepsilon-\delta+1}^{(\alpha)}(0,\rho;\mu). \tag{55}$$

Proof. By (3) and (18), we have

$$\sum_{\varepsilon=0}^{\infty}{}_{Bel}\mathbb{A}_{\varepsilon}^{(\alpha)}(\sigma,\rho;\mu)\frac{\zeta^{\varepsilon}}{\varepsilon!} = \left(\frac{1-\mu}{e^{\zeta(\mu-1)}-\mu}\right)^{\alpha}e^{\sigma\zeta+\rho(e^{\zeta}-1)}\left(\frac{2\zeta}{\mu e^{\zeta}+1}\right)\left(\frac{\mu e^{\zeta}+1}{2\zeta}\right)$$

$$= \frac{1}{2\zeta}\left(\mu\sum_{\varepsilon=0}^{\infty}{}_{Bel}\mathbb{A}_{\varepsilon}^{(\alpha)}(0,\rho;\mu)\frac{\zeta^{\varepsilon}}{\varepsilon!}\sum_{\delta=0}^{\infty}\mathbb{G}_{\delta}(\sigma;\mu)\frac{\zeta^{\delta}}{\delta!}\sum_{\theta=0}^{\infty}\frac{\zeta^{\theta}}{\theta!}+\sum_{\varepsilon=0}^{\infty}{}_{Bel}\mathbb{A}_{\varepsilon}^{(\alpha)}(0,\rho;\mu)\frac{\zeta^{\varepsilon}}{\varepsilon!}\sum_{\delta=0}^{\infty}\mathbb{G}_{\delta}(\sigma;\mu)\frac{\zeta^{\delta}}{\delta!}\right),$$

which implies the claimed result (55).

4. Symmetric Identities

Here, we give some symmetric identities of the Bell-based Frobenius-type Eulerian polynomials of order α.

Theorem 16. *Let $\varepsilon \geq 0$ and $a, b, > 0$ with $a \neq b$. Then*

$$\sum_{\delta=0}^{\varepsilon}\binom{\varepsilon}{\delta}b^{\delta}a^{\varepsilon-\delta}{}_{Bel}\mathbb{A}_{\varepsilon-\delta}^{(\alpha)}(b\sigma,\rho;\mu){}_{Bel}\mathbb{A}_{\delta}^{(\alpha)}(a\sigma,\rho;\mu) = \sum_{\delta=0}^{\varepsilon}\binom{\varepsilon}{\delta}a^{\delta}b^{\varepsilon-\delta}{}_{Bel}\mathbb{A}_{\varepsilon-\delta}^{(\alpha)}(a\sigma,\rho;\mu){}_{Bel}\mathbb{A}_{\delta}^{(\alpha)}(b\sigma,\rho;\mu). \tag{56}$$

Proof. Let

$$A(\zeta)=\left(\frac{(1-\mu)^2}{(e^{(\mu-1)a\zeta}-\mu)(e^{(\mu-1)b\zeta}-\mu)}\right)^{\alpha}e^{2ab\sigma\zeta+\rho(e^{a\zeta}-1)+\rho(e^{b\zeta}-1)}, \tag{57}$$

$$A(\zeta)=\sum_{\varepsilon=0}^{\infty}{}_{Bel}\mathbb{A}_{\varepsilon}^{(\alpha)}(b\sigma,\rho;\mu)\frac{(a\zeta)^{\varepsilon}}{\varepsilon!}\sum_{\delta=0}^{\infty}{}_{Bel}\mathbb{A}_{\delta}^{(\alpha)}(a\sigma,\rho;\mu)\frac{(b\zeta)^{\delta}}{\delta!}$$

$$=\sum_{\varepsilon=0}^{\infty}\left(\sum_{\delta=0}^{\varepsilon}\binom{\varepsilon}{\delta}b^{\delta}a^{\varepsilon-\delta}{}_{Bel}\mathbb{A}_{\varepsilon-\delta}^{(\alpha)}(b\sigma,\rho;\mu){}_{Bel}\mathbb{A}_{\delta}^{(\alpha)}(a\sigma,\rho;\mu)\right)\frac{\zeta^{\varepsilon}}{\varepsilon!}.$$

Similarly, we can show that

$$A(\zeta)=\sum_{\varepsilon=0}^{\infty}{}_{Bel}\mathbb{A}_{\varepsilon}^{(\alpha)}(a\sigma,\rho;\mu)\frac{(b\zeta)^{\varepsilon}}{\varepsilon!}\sum_{\delta=0}^{\infty}{}_{Bel}\mathbb{A}_{\delta}^{(\alpha)}(b\sigma,\rho;\mu)\frac{(a\zeta)^{\delta}}{\delta!}$$

$$=\sum_{\varepsilon=0}^{\infty}\left(\sum_{\delta=0}^{\varepsilon}\binom{\varepsilon}{\delta}a^{\delta}b^{\varepsilon-\delta}{}_{Bel}\mathbb{A}_{\varepsilon-\delta}^{(\alpha)}(a\sigma,\rho;\mu){}_{Bel}\mathbb{A}_{\delta}^{(\alpha)}(b\sigma,\rho;\mu)\right)\frac{\zeta^{\varepsilon}}{\varepsilon!},$$

which implies the claimed result (56).

Remark 6. *For $\alpha=1$ in (56), we have*

$$\sum_{\delta=0}^{\varepsilon}\binom{\varepsilon}{\delta}b^{\delta}a^{\varepsilon-\delta}{}_{Bel}\mathbb{A}_{\varepsilon-\delta}(b\sigma,\rho;\mu){}_{Bel}\mathbb{A}_{\delta}(a\sigma,\rho;\mu)=\sum_{\delta=0}^{\varepsilon}\binom{\varepsilon}{\delta}a^{\delta}b^{\varepsilon-\delta}{}_{Bel}\mathbb{A}_{\varepsilon-\delta}(a\sigma,\rho;\mu){}_{Bel}\mathbb{A}_{\delta}(b\sigma,\rho;\mu). \tag{58}$$

Theorem 17. *Let $\varepsilon\geq 0$. Then*

$$\sum_{\delta=0}^{\varepsilon}\binom{\varepsilon}{\delta}\sum_{\theta=0}^{a-1}\sum_{\vartheta=0}^{b-1}(-\mu)^{\theta+\vartheta}a^{\varepsilon-\delta}b^{\delta\delta}{}_{Bel}\mathbb{A}_{\varepsilon-\delta}^{(\alpha)}\left(b\sigma+\frac{b}{a}\theta+\vartheta,\rho;\mu\right){}_{Bel}\mathbb{A}_{k}^{(\alpha)}(a\sigma,\rho;\mu)$$
$$=\sum_{\delta=0}^{\varepsilon}\binom{\varepsilon}{\delta}\sum_{\theta=0}^{b-1}\sum_{\vartheta=0}^{a-1}(-\lambda)^{\theta+\vartheta}b^{\varepsilon-\delta}a^{\delta}{}_{Bel}\mathbb{A}_{\varepsilon-\delta}^{(\alpha)}\left(a\sigma+\frac{a}{b}\theta+\vartheta,\rho;\mu\right){}_{Bel}\mathbb{A}_{\delta}^{(\alpha)}(b\sigma,\rho;\mu). \tag{59}$$

Proof. Let

$$B(\zeta)=\left(\frac{(1-\mu)^2}{(e^{(\mu-1)a\zeta}-\mu)(e^{(\mu-1)b\zeta}-\mu)}\right)^{\alpha}\frac{\left(1-\left(-\mu e^{b\zeta}\right)^{a}\right)\left(1-\left(-\mu e^{a\zeta}\right)^{b}\right)}{(\mu e^{a\zeta}+1)(\mu e^{b\zeta}+1)}e^{2ab\sigma\zeta+\rho(e^{a\zeta}-1)+\rho(e^{b\zeta}-1)}.$$

Then, we have

$$B(\zeta)=\left(\frac{1-\mu}{e^{(\mu-1)a\zeta}-\mu}\right)^{\alpha}e^{ab\sigma\zeta+\rho(e^{a\zeta}-1)}\left(\frac{1-\left(-\mu e^{b\zeta}\right)^{a}}{\mu e^{b\zeta}+1}\right)\left(\frac{1-\mu}{e^{(\mu-1)b\zeta}-\mu}\right)^{\alpha}$$
$$\times\left(\frac{1-\left(-\mu e^{a\zeta}\right)^{b}}{\mu e^{a\zeta}+1}\right)e^{ab\sigma\zeta+\rho(e^{b\zeta}-1)}$$

$$=\left(\frac{1-\mu}{e^{(\mu-1)a\zeta}-\mu}\right)^{\alpha}e^{ab\sigma\zeta+\rho(e^{a\zeta}-1)}\sum_{\theta=0}^{a-1}(-\mu)^{\theta}e^{b\zeta\theta}\left(\frac{1-\mu}{e^{(\mu-1)b\zeta}-\mu}\right)^{\alpha}e^{ab\sigma\zeta+\rho(e^{b\zeta}-1)}\sum_{\varepsilon=0}^{b-1}(-\mu)^{\vartheta}e^{a\zeta\vartheta}$$

$$= \left(\frac{1-\mu}{e^{(\mu-1)a\zeta}-\mu}\right)^{\alpha} e^{\rho(e^{a\zeta}-1)} \sum_{\theta=0}^{a-1}\sum_{\vartheta=0}^{b-1}(-\mu)^{\theta+\vartheta} e^{(b\sigma+\frac{b}{a}\theta+\vartheta)a\zeta} \sum_{\delta=0}^{\infty} {}_{Bel}\mathbb{A}_{\delta}^{(\alpha)}(a\sigma,\rho;\mu)\frac{(b\zeta)^{\delta}}{\delta!}$$

$$= \sum_{\varepsilon=0}^{\infty}\sum_{\theta=0}^{a-1}\sum_{\vartheta=0}^{b-1}(-\mu)^{\theta+\vartheta} {}_{Bel}\mathbb{A}_{\varepsilon}^{(\alpha)}\left(b\sigma+\frac{b}{a}\theta+\vartheta,\rho;\mu\right)\frac{(a\zeta)^{\varepsilon}}{\varepsilon!}\sum_{\delta=0}^{\infty} {}_{Bel}\mathbb{A}_{\delta}^{(\alpha)}(a\sigma,\rho;\mu)\frac{(b\zeta)^{\delta}}{(\delta)!}$$

$$= \sum_{\varepsilon=0}^{\infty}\sum_{\delta=0}^{\varepsilon}\binom{\varepsilon}{\delta}\sum_{\theta=0}^{a-1}\sum_{\vartheta=0}^{b-1}(-\mu)^{\theta+\vartheta}a^{\varepsilon-\delta}b^{\delta} {}_{Bel}\mathbb{A}_{\varepsilon-\delta}^{(\alpha)}\left(b\sigma+\frac{b}{a}\theta+\vartheta,\rho;\mu\right) \times {}_{Bel}\mathbb{A}_{\delta}^{(\alpha)}(a\sigma,\rho;\mu)\frac{\zeta^{\varepsilon}}{\varepsilon!}. \quad (60)$$

On the other hand, we have

$$B(\zeta) = \sum_{\epsilon=0}^{\infty}\sum_{\delta=0}^{\varepsilon}\binom{\varepsilon}{\delta}\sum_{\theta=0}^{b-1}\sum_{\vartheta=0}^{a-1}(-\mu)^{\theta+\vartheta}b^{\varepsilon-\delta}a^{\delta} {}_{Bel}\mathbb{A}_{\varepsilon-\delta}^{(\alpha)}\left(a\sigma+\frac{a}{b}\theta+\vartheta,\rho;\mu\right) \times {}_{Bel}\mathbb{A}_{\delta}^{(\alpha)}(b\sigma,\rho;\mu)\frac{\zeta^{\varepsilon}}{\varepsilon!}. \quad (61)$$

By (60) and (61), we arrive at the desired result (59).

5. Stack of zeros and surface representations of Bell-based Frobenius-type Eulerian polynomials

Certain zero values of the Bell-based Frobenius-type Eulerian polynomials are discussed, and some graphical representations are presented in this section.

Let's remember the definition of Bell-based Frobenius-type Eulerian polynomials from (18):

$$\left(\frac{1-\mu}{e^{\zeta(\mu-1)}-\mu}\right)^{\alpha} e^{\sigma\zeta+\rho(e^{\zeta}-1)} = \sum_{\varepsilon=0}^{\infty} {}_{Bel}\mathbb{A}_{\varepsilon}^{(\alpha)}(\sigma,\rho;\mu)\frac{\zeta^{\varepsilon}}{\varepsilon!} \quad \left|\zeta+\ln\left(\frac{\mu-1}{\mu}\right)\right| < 2\pi.)$$

The first few polynomials of ${}_{Bel}\mathbb{A}_{\varepsilon}^{(\alpha)}(\sigma,\rho;\mu)$ are as follows:

$$
\begin{aligned}
{}_{Bel}\mathbb{A}_0^{(\alpha)}(\sigma,\rho;\mu) &= 1,\\
{}_{Bel}\mathbb{A}_1^{(\alpha)}(\sigma,\rho;\mu) &= \alpha+\rho+\sigma,\\
{}_{Bel}\mathbb{A}_2^{(\alpha)}(\sigma,\rho;\mu) &= \alpha^2+\alpha\mu+\rho+2\alpha\rho+\rho^2+2\alpha\sigma+2\rho\sigma+\sigma^2,\\
{}_{Bel}\mathbb{A}_3^{(\alpha)}(\sigma,\rho;\mu) &= \alpha^3+\alpha\mu+3\alpha^2\mu+\alpha\mu^2+\rho+3\alpha\rho+3\alpha^2\rho+3\alpha\mu\rho\\
&\quad+3\rho^2+3\alpha\rho^2+\rho^3+3\alpha^2\sigma+3\alpha\mu\sigma+3\rho\sigma\\
&\quad+6\alpha\rho\sigma+3\rho^2\sigma+3\alpha\sigma^2+3\rho\sigma^2+\sigma^3,\\
{}_{Bel}\mathbb{A}_4^{(\alpha)}(\sigma,\rho;\mu) &= -\alpha+\alpha^4+\frac{24\alpha}{(1-\mu)^4}-\frac{36\alpha}{(1-\mu)^3}+\frac{14\alpha}{(1-\mu)^2}-\frac{\alpha}{1-\mu}-10\alpha\mu+4\alpha^2\mu\\
&\quad+6\alpha^3\mu-\frac{96\alpha\mu}{(1-\mu)^4}+\frac{144\alpha\mu}{(1-\mu)^3}-\frac{56\alpha\mu}{(1-\mu)^2}+\frac{4\alpha\mu}{1-\mu}-7\alpha\mu^2+7\alpha^2\mu^2\\
&\quad+\frac{144\alpha\mu^2}{(1-\mu)^4}-\frac{216\alpha\mu^2}{(1-\mu)^3}+\frac{84\alpha\mu^2}{(1-\mu)^2}-\frac{6\alpha\mu^2}{1-\mu}-\frac{96\alpha\mu^3}{(1-\mu)^4}+\frac{144\alpha\mu^3}{(1-\mu)^3}\\
&\quad-\frac{56\alpha\mu^3}{(1-\mu)^2}+\frac{4\alpha\mu^3}{1-\mu}+\frac{24\alpha\mu^4}{(1-\mu)^4}-\frac{36\alpha\mu^4}{(1-\mu)^3}+\frac{14\alpha\mu^4}{(1-\mu)^2}-\frac{\alpha\mu^4}{1-\mu}+\rho\\
&\quad+4\alpha\rho+6\alpha^2\rho+4\alpha^3\rho+10\alpha\mu\rho+12\alpha^2\mu\rho+4\alpha\mu^2\rho\\
&\quad+7\rho^2+12\alpha\rho^2+6\alpha^2\rho^2+6\alpha\mu\rho^2+6\rho^3+4\alpha\rho^3+\rho^4+4\alpha^3\sigma\\
&\quad+4\alpha\mu\sigma+12\alpha^2\mu\sigma+4\alpha\mu^2\sigma+4\rho\sigma+12\alpha\rho\sigma+12\alpha^2\rho\sigma+12\alpha\mu\rho\sigma\\
&\quad+12\rho^2\sigma+12\alpha\rho^2\sigma+4\rho^3\sigma+6\alpha^2\sigma^2+6\alpha\mu\sigma^2+6\rho\sigma^2+12\alpha\rho\sigma^2\\
&\quad+6\rho^2\sigma^2+4\alpha\sigma^3+4\rho\sigma^3+\sigma^4.
\end{aligned}
$$

We obtain the zeros of the Bell-based Frobenius-type Eulerian polynomials of order α, namely the solutions of ${}_{Bel}\mathbb{A}_{\varepsilon}^{(\alpha)}(\sigma,\rho;\mu)$, by making use of technology, given in Figure 1 for $1 \leq \varepsilon \leq 20$:

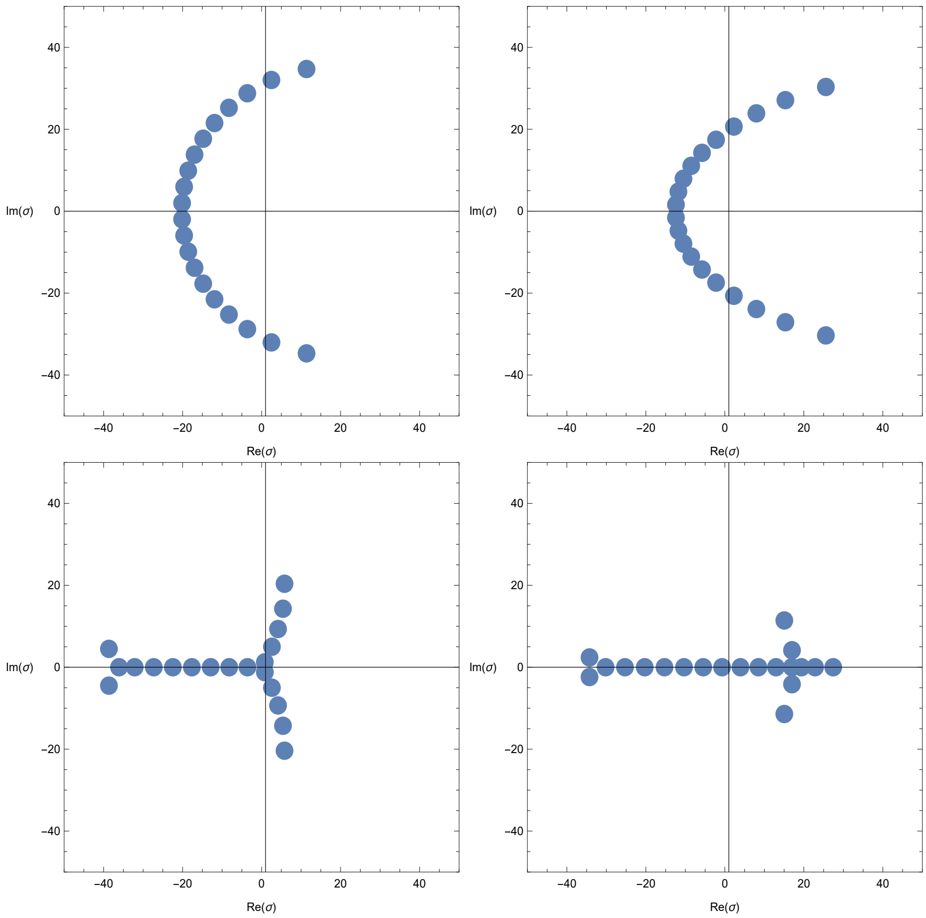

Figure 1: Zeros of ${}_{Bel}\mathbb{A}_{\varepsilon}^{(\alpha)}(\sigma,\rho;\mu)$

Figure 1 (top-left) shows the zeros of ${}_{Bel}\mathbb{A}_{\varepsilon}^{(5)}(\sigma,5;3)$ for $1 \leq \varepsilon \leq 20$. Figure 1 (top-right) shows the zeros of ${}_{Bel}\mathbb{A}_{\varepsilon}^{(5)}(\sigma,-5;3)$ for $1 \leq \varepsilon \leq 20$. Figure 1 (bottom-left) shows the zeros of ${}_{Bel}\mathbb{A}_{\varepsilon}^{(5)}(\sigma,5;-3)$ for $1 \leq \varepsilon \leq 20$. Figure 1 (bottom-right) shows the zeros of ${}_{Bel}\mathbb{A}_{\varepsilon}^{(5)}(\sigma,-5;-3)$ for $1 \leq \varepsilon \leq 20$.

Stacks of zeros of the generalized Bell-based Frobenius-type Eulerian polynomials ${}_{Bel}\mathbb{A}_{\varepsilon}^{(\alpha)}(\sigma,\rho;\mu)$ of order α for $1 \leq \varepsilon \leq 20$, creating a 3D structure, are given in Figure 2:

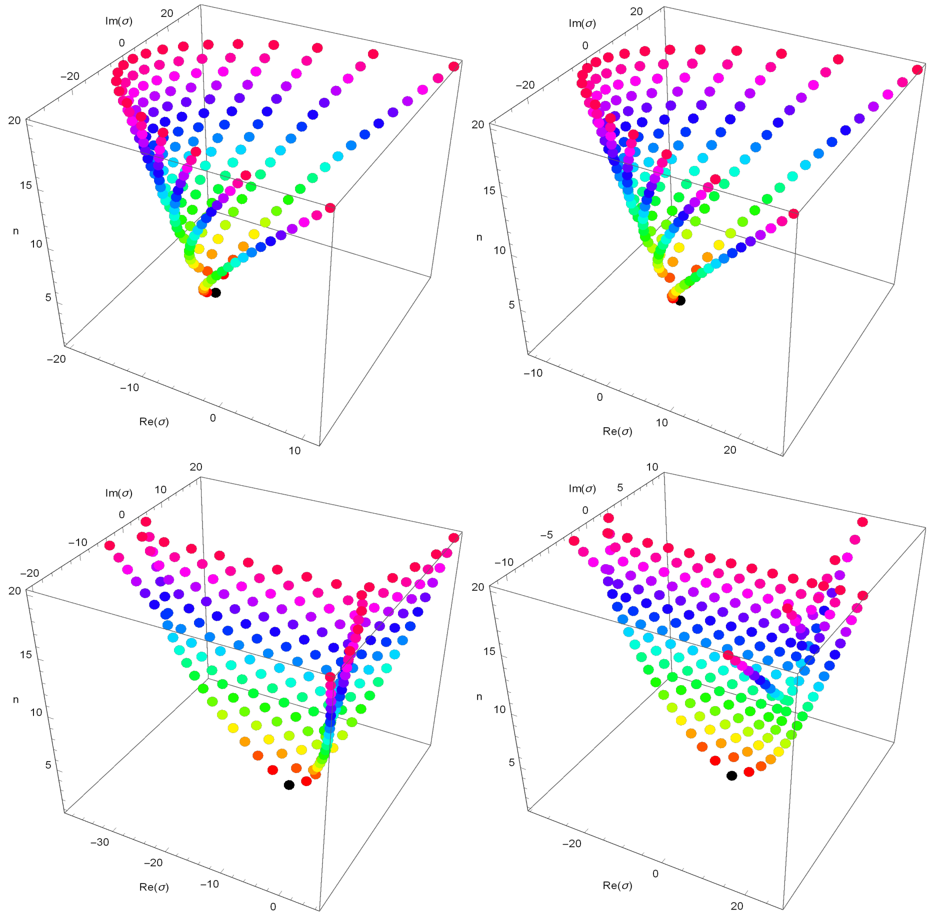

Figure 2: Solutions of ${}_{Bel}\mathbb{A}_{\varepsilon}^{(\alpha)}(\sigma,\rho;\mu)=0$

Figure 2 (top-left) shows the zeros of ${}_{Bel}\mathbb{A}_{\varepsilon}^{(5)}(\sigma,5;3)$ for $1 \leq \varepsilon \leq 20$. Figure 2 (top-right) shows the zeros of ${}_{Bel}\mathbb{A}_{\varepsilon}^{(5)}(\sigma,-5;3)$ for $1 \leq \varepsilon \leq 20$. Figure 2 (bottom-left) shows the zeros of ${}_{Bel}\mathbb{A}_{\varepsilon}^{(5)}(\sigma,5;-3)$ for $1 \leq \varepsilon \leq 20$. Figure 2 (bottom-right) shows the zeros of ${}_{Bel}\mathbb{A}_{\varepsilon}^{(5)}(\sigma,-5;-3)$ for $1 \leq \varepsilon \leq 20$.

Plots of real zeros of the generalized Bell-based Frobenius-type Eulerian polynomials ${}_{Bel}\mathbb{A}_{\varepsilon}^{(\alpha)}(\sigma,\rho;\mu)$ of order α for $1 \leq \varepsilon \leq 20$ are provided by Figure 3:

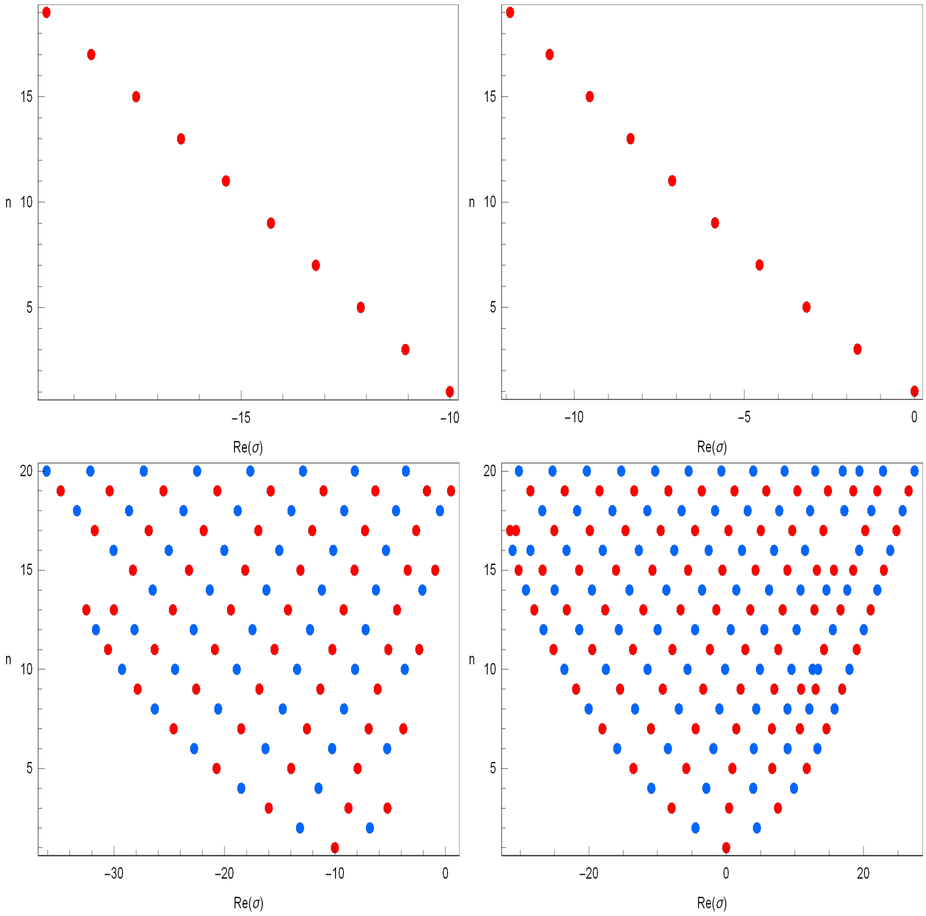

Figure 3: Real zeros of ${}_{Bel}\mathbb{A}_{\varepsilon}^{(\alpha)}(\sigma,\rho;\mu)$

Figure 3 (top-left) shows the real zeros of ${}_{Bel}\mathbb{A}_{\varepsilon}^{(5)}(\sigma,5;3)$ for $1 \leq \varepsilon \leq 20$. Figure 3 (top-right) shows the real zeros of ${}_{Bel}\mathbb{A}_{\varepsilon}^{(5)}(\sigma,-5;3)$ for $1 \leq \varepsilon \leq 20$. Figure 3 (bottom-left) shows the real zeros of ${}_{Bel}\mathbb{A}_{\varepsilon}^{(5)}(\sigma,5;-3)$ for $1 \leq \varepsilon \leq 20$. Figure 3 (bottom-right) shows the real zeros of ${}_{Bel}\mathbb{A}_{\varepsilon}^{(5)}(\sigma,-5;-3)$ for $1 \leq \varepsilon \leq 20$.

Lastly, we compute approximate solutions of ${}_{Bel}\mathbb{A}_{\varepsilon}^{(5)}(\sigma, 5; 3) = 0$ in Table 1:

Table 1. Approximate solutions of ${}_{Bel}\mathbb{A}_{\varepsilon}^{(5)}(\sigma, 5; 3) = 0$

degree ε	σ
1	-10.000
2	$-10.0000 - 4.4721i, \quad -10.0000 + 4.4721i$
3	$-11.063, \quad -9.4684 - 7.8005i, \quad -9.4684 + 7.8005i$
4	$-11.3040 - 3.4430i, \quad -11.3040 + 3.4430i,$ $-8.6960 - 10.5616i, \quad -8.6960 + 10.5616i$
5	$-12.134, \quad -11.1517 - 6.3553i, \quad -11.1517 + 6.3553i,$ $-7.781 - 12.967i, \quad -7.781 + 12.967i$
6	$-12.4670 - 2.9563i, \quad -12.4670 + 2.9563i, \quad -10.7625 - 8.9281i,$ $-10.7625 + 8.9281i, \quad -6.770 - 15.121i, \quad -6.770 + 15.121i$
7	$-13.208, \quad -12.4943 - 5.5954i, \quad -12.4943 + 5.5954i, \quad -10.213 - 11.259i,$ $-10.213 + 11.259i, \quad -5.689 - 17.086i, \quad -5.689 + 17.086i$
8	$-13.5902 - 2.6625i, \quad -13.5902 + 2.6625i, \quad -12.312 - 8.006i,$ $-12.312 + 8.006i, \quad -9.547 - 13.404i, \quad -9.547 + 13.404i,$ $-4.552 - 18.901i, \quad -4.552 + 18.901i$
9	$-14.284, \quad -13.723 - 5.110i, \quad -13.723 + 5.110i,$ $-11.975 - 10.240i, \quad -11.975 + 10.240i, \quad -8.791 - 15.403i,$ $-8.791 + 15.403i, \quad -3.369 - 20.595i, \quad -3.369 + 20.595i$

6. Conclusion

Hermite-based special polynomials have been studied for a long time, and many mathematicians and scientists have improved their properties and applications; [2, 14] and the references cited therein.

In recent years, by the similar motivation mentioned above, Bell-based special polynomials such as Bell-based Bernoulli polynomials in [8], Bell-based Bernoulli polynomials of the first kind in [7], and Bell-based Frobenius-type sine-(and cosine-)Eulerian polynomials in [15] have been defined, and diverse properties, applications, and relations have been derived. As a link in this working chain, we have worked on Bell-based Frobenius-type Eulerian polynomials of order α, and we then have acquired some relations, formulas, and derivative properties. Also, we have examined some implicit summation formulas and symmetric identities for Bell-based Frobenius-type Eulerian polynomials of order α. Lastly, we have shown the stack of zeros and surface representations of Bell-based Frobenius-type Eulerian polynomials for several specific parameters with specific values.

Future research can explore several directions, including the development of q-analogues and degenerate forms of the proposed polynomials to study associated q-difference equations and limiting behaviors in [22, 23]. Investigating orthogonality conditions and suitable weight functions will help identify inner product spaces where these polynomials are orthogonal. Their application in interpolation, approximation theory, and spectral methods also warrants attention, particularly in solving differential or integral equations. Potential uses in mathematical physics and engineering such as quantum systems and signal analysis highlight their applied significance. Additionally, a detailed study of asymptotic properties and zero distributions using analytic and numerical tools could offer deeper insights into their structural behavior.

Availability of data and materials

Not applicable.

Competing interests

The authors declare no competing interests.

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