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Some New Weighted Results for the Hardy Operator on Variable-Exponent λ-central Morrey Space

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Abstract. The main purpose of this study is to demonstrate that the fractional Hardy operators are bounded on the weighted variable central Morrey space. When the symbol functions belong to the λ-central BMO space with a variable exponent, the estimates for their commutators are similar.
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1. Introduction

Let $L^1_{loc}(\mathbb{R}^n)$ denote the set consisting of all complex-valued integrable functions on $\mathbb{R}^n$, In [1], the Hardy operator is defined as follows:

$$Hf(x) = \frac{1}{x}\int_0^x f(t)dt, \qquad x > 0,$$

which was generalized by Faris [2] to n-dimensional Euclidean space and defined as:

$$Hf(x) = \frac{1}{v_n|x|^n}\int_{|y|<|x|} f(y)dy,$$

where $x \in \mathbb{R}^n \setminus \{0\}$, $f \in L^1_{loc}(\mathbb{R}^n)$, and v_n represents the volume of the unit ball. The n-dimensional fractional Hardy operator H_α and its dual operator are defined as follows:

$$H_\alpha f(x) = |x|^{\alpha-n}\int_{|y|<|x|} f(y)dy, \qquad H^*_\alpha f(x) = \int_{|y|>|x|} |y|^{\alpha-n} f(y)dy.$$

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Let b be a locally integrable function, and define the BMO norm as follows:

$$\|b\|_{BMO} = \sup_{B} \frac{1}{|B|} \int_{B} |b(x) - b_B| dx,$$

where the supremum is taken over all balls B in $\mathbb{R}^n$, and b_B is the average of b on B. A function is called bounded mean oscillation if $\|b\|_{BMO} < +\infty$. If $b \in BMO(\mathbb{R}^n)$, then the commutator generated by the Hardy operator is defined as:

$$H_b f(x) = b(x) H f(x) - H(bf)(x).$$

The commutator for the n-dimensional fractional Hardy operator H_α and their dual operator H^*_α are defined in [3] as follows:

$$H_{\alpha,b} f(x) = b(x) H_\alpha f(x) - H_\alpha(bf)(x)$$

$$H^*_{\alpha,b} f(x) = b(x) H^*_\alpha f(x) - H^*_\alpha(bf)(x).$$

In real analysis, variable exponents function spaces are being observed with keen interest. The theory of variable exponents has great applications in partial differential equations and applied mathematics because they are used in image restoration and the modeling of electrorheological fluids. Some basic properties for variable exponent function spaces were defined by Kovacik and Rakosnik [4]. The boundedness of Hardy operators on some function spaces is one of the main problems in this theory. The boundedness of the Hardy operator on Lebesgue spaces and Sobolev spaces with variable exponents can be verified in [5–7]. The boundedness of the commutator of the Hardy operator on λ-central Morrey space and the variable exponent Herz space $\dot{K}^{\alpha,p}_{q(\cdot)}$ is discussed in [8–13]. More generalized results, incorporating weights and variable exponents, can be found in [14–29]. Some of these results include duality, boundedness of the Littlewood maximal Hardy operator and sublinear operator, wavelet characterization, commutator of fractional and singular integral, and more.

In this paper, our main focus is on the weighted λ-central Morrey space, which holds importance in the theory of Harmonic analysis. The idea of λ-central Morrey space and λ-central BMO space for variable exponents was initially discussed in [30]. Central Morrey space, central BMO, and their corresponding function spaces have delightful applications in exploring results for operators, including singular integral operators, as explained in [31–38]. In this article, we utilize the Reisz type potential operator to establish the boundedness of the Hardy operator, defined as follows:

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy,$$

f is locally integrable function, $0 < \alpha < n$ and $x \in \mathbb{R}^n \setminus \{0\}$.

Permit me to elucidate the structural framework of the present manuscript. In Section 3, a recapitulation shall be furnished of certain pivotal lemmas and propositions, articulated within the analytical milieu of weighted Lebesgue space endowed with variable exponents. Moving on to Section 4.1, we will establish the boundedness of fractional Hardy operators in the weighted central Morrey space concerning variable exponents. In Section 4.2, we will investigate the estimates of commutators induced by Hardy operators and weighted λ-central BMO functions on the central Morrey space.

Additionally, $|S|$ and χ_S represent the Lebesgue measure and characteristic function of a measurable set $S \subset \mathbb{R}^n$, respectively. When we write $g \approx h$, it indicates the existence of constants C_1 and C_2, both greater than zero, such that $C_1 g \leq h \leq C_2 g$. We define $S_j = S(0, 2^j) = \{x \in \mathbb{R}^n : |x| \leq 2^j\}$, and for $j \in \mathbb{Z}$, we set $\chi_j = \chi_{A_j}$.

2. Preliminaries

According to some basic books and papers [4, 39–41], we have established the Lebesgue space with a variable exponent. For a measurable function $p(\cdot) : \mathbb{R}^n \to [1, +\infty)$, the Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ with a variable exponent is a set containing complex valued function g such that

$$P_p(g) = \int_{\mathbb{R}^n} |g(x)|^{p(x)} dx < +\infty.$$

$L^{p(\cdot)}$ is a Banach function space with respect to the norm

$$\|g\|_{L^{P(\cdot)}} = \inf\{\omega > 0 : P_p(\frac{g}{\omega}) \leq 1\}.$$

The set $\mathcal{P}$ represents a set consisting of all measurable function $p(\cdot)$ such that

$$p^- = ess\inf p(x) > 1 \qquad x \in \mathbb{R}^n$$

$$+\infty > p^+ = ess\sup p(x) \qquad x \in \mathbb{R}^n.$$

A function $p(\cdot)$ is said to be globally log Holder continuous ($LH(\mathbb{R}^n)$) if it fulfills the following conditions:

$$|p(y) - p(x)| \lesssim \frac{1}{log(|y-x|)}, \qquad x, y \in \mathbb{R}^n : |y-x| \leq \frac{1}{2} \tag{1}$$

$$|p_\infty - p(x)| \lesssim \frac{1}{log(e + |x|)}, \tag{2}$$

for some real number p_∞.

The Hardy maximal Littlewood operator denoted by $\mathcal{M}$, is defined as:

$$\mathcal{M}g(x) = \sup \frac{1}{|B|} \int_B g(t)dt,$$

where B is a ball and $x \in B$. It is known that $\mathcal{M}$ is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$, for $p(\cdot) \in \mathcal{P}(\mathbb{R}^n) \bigcap LH(\mathbb{R}^n)$ [42–44].

Let w be a weight, and $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. Then $L^{p(\cdot)}(w)$ represents the weighted variable exponent Lebesgue spaces variable exponent, which contains all complex valued measurable function f such that $fw^{\frac{1}{p(\cdot)}} \in L^{p(\cdot)}(w)$. $L^{p(\cdot)}(w)$ forms a Banach function space with respect to the given norm

$$\|f\|_{L^{p(\cdot)}(w)} = \|fw^{\frac{1}{p(\cdot)}}\|_{L^{p(\cdot)}}.$$

Since, $p^{'}(\cdot)$ is the conjugate of $p(\cdot)$ and $\frac{1}{p(\cdot)} + \frac{1}{p^{'}(\cdot)} = 1$, we have the conjugate exponent relationship.

We introduce the concept of a A_1 Muckenhoupt weight, defined as follows:

Definition 1. *Weight w is called a A_1 Muckenhoupt weight if it fulfills the condition $\mathcal{M}w(x) \lesssim w(x)$, $x \in \mathbb{R}^n$.*

Definition 2. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. A weight is said to be $A_{p(\cdot)}$ if it satisfies the condition:*

$$\sup_B \frac{1}{|B|} \|w^{\frac{1}{p(\cdot)}}\chi_B\|_{L^{p(\cdot)}} \|w^{-\frac{1}{p(\cdot)}}\chi_B\|_{L^{p'(\cdot)}} < +\infty.$$

and a weight is called $\tilde{A_{p(\cdot)}}$ if the following condition holds:

$$\sup_B \frac{1}{|B|^{p_B}} \|w\chi_B\|_{L^1} \|w^{-1}\chi_B\|_{L^{\frac{p'(\cdot)}{p(\cdot)}}} < +\infty.$$

Where $p_B = \left(\frac{1}{|B|}\int_B \frac{1}{p(x)}dx\right)^{-1}$.

Based on this definition Izuki and Noi [16] have proved the following monotone property.

Definition 3. *Let $\alpha \in (0,n)$ and $p_2(\cdot), p_1(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, and $\frac{1}{p_1(\cdot)} = \frac{1}{p_2(\cdot)} + \frac{\alpha}{n}$. A weight w is known as $A(p_2(\cdot), p_1(\cdot))$ if it satisfies the following inequality for all balls $B \subset \mathbb{R}^n$*

$$\|w^{-1}\chi_B\|_{L^{p_1'(\cdot)}}^{\frac{1}{1-\frac{\alpha}{n}}} \|w\chi_B\|_{L^{p_2(\cdot)}}^{\frac{1}{1-\frac{\alpha}{n}}} \leq |B|.$$

Definition 4. *If $p(\cdot) \in \mathcal{P}$ and $\lambda \in \mathbb{R}$, the weighted Morrey space with a variable exponent $\dot{B}^{p(\cdot),\lambda}(w^{p(\cdot)})$ is defined as:*

$$\dot{B}^{p(\cdot),\lambda}(w) = \{f \in L^{p(\cdot)}_{loc}(w) : \ \|f\|_{\dot{B}^{p(\cdot),\lambda}(w)} < +\infty\},$$

where

$$\|f\|_{\dot{B}^{p(\cdot),\lambda}(w)} = \sup_{R>0} \frac{\|f\chi_{B(0,R)}\|_{L^{p(\cdot)}(w)}}{|B(0,R)|^\lambda \|\chi_{B(0,R)}\|_{L^{p(\cdot)}(w)}}.$$

Definition 5. *If $p(\cdot) \in \mathcal{P}$ and $\lambda < \frac{1}{n}$ the weighted $\lambda-$ BMO space with variable exponent $CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})$ is defined as:*

$$CBMO^{p(\cdot),\lambda}(w) = \{f \in L^{p(\cdot)}_{loc}(w) : \ \|f\|_{CBMO^{p(\cdot),\lambda}(w)} < +\infty\},$$

where

$$\|f\|_{CBMO^{p(\cdot),\lambda}(w)} = \sup_{R>0} \frac{\|(f - f_{B(0,R))}\chi_{B(0,R)}\|_{L^{p(\cdot)}(w)}}{|B(0,R)|^\lambda \|\chi_{B(0,R)}\|_{L^{p(\cdot)}(w)}}.$$

3. Important Lemmas

Lemma 1. *[16] If $p_1(\cdot), p_2(\cdot) \in \mathcal{P}(\mathbb{R}^n) \bigcap LH(\mathbb{R}^n)$ and $p_1(\cdot) < p_2(\cdot)$, then*

$$A_1 \subset A_{p_1(\cdot)} \subset A_{p_2(\cdot)}.$$

Lemma 2. *[16] Let $0 < \alpha < n$ and $p_2(\cdot), p_1(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ where $\frac{1}{p_1(\cdot)} = \frac{1}{p_2(\cdot)} + \frac{\alpha}{n}$. Then $w \in A(p_1(\cdot), p_2(\cdot)) \Leftrightarrow w^{p_2(\cdot)} \in A_{1+\frac{p_2(\cdot)}{p_1'(\cdot)}}$.*

Lemma 3. *[45] Consider Y is a Banach function space. Then*

1. Y' will also be a Banach function space. Specifically, $\|\cdot\|_{(Y')'}$ and $\|\cdot\|_Y$ are both equivalent norms and satisfy $(Y')' = Y$.

2. If $f \in Y$ and $g \in Y'$, then we have

$$\int_{\mathbb{R}^n} |f(x)g(x)|dx \leq \|f\|_Y \|g\|_{Y'},$$

which is known as Hölder inequality.

Lemma 4. *[45] Let Y be a Banach function space. Then*

$$1 \leq \frac{1}{|B|}\|\chi_B\|_Y \|\chi_B\|_{Y'}.$$

This holds for all balls B.

Lemma 5. *[46] If $\mathcal{M}$ is weakly bounded on a Banach function space Y, i.e,*

$$\|\chi_{\{Mf>\lambda\}}\|_Y \leq \lambda^{-1}\|f\|_Y, \tag{3}$$

where $\lambda > 0$ and $f \in Y$, then

$$\sup \frac{1}{|B|}\|\chi_B\|_Y \|\chi_B\|_{Y'} < +\infty. \tag{4}$$

Now we define weighted Banach function space and explain some of their properties. For a function $W(x) \in (0, +\infty)$, $W(x) \in Y_{loc}$ and $W^{-1}(x) \in Y'_{loc}$, where $Y_{loc}(\mathbb{R}^n)$ comprises of all measurable functions f such that $f\chi_B \in Y$ for any compact set B with $|B| < +\infty$. Then the weighted Banach function space is defined as

$$Y(\mathbb{R}^n, W) = \{f \in M : fW \in Y\}.$$

Then the accompanying lemma is true.

Lemma 6. *[47]*

1. $Y(\mathbb{R}^n, W)$ is a Banach function space with the given norm

$$\|f\|_{Y(\mathbb{R}^n,W)} = \|fW\|_Y.$$

2. The associated space of the weighted Banach function space $Y(\mathbb{R}^n, W)$ is also a Banach function space .

Remark 1. *Take $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. Now, by the definition of the weighted Banach function space $Y(\mathbb{R}^n, W)$, $L^{q(\cdot)}(w^{q(\cdot)})$ and $L^{q'(\cdot)}(w^{-q'(\cdot)})$, we observe that*

1. Let $Y = L^{q(\cdot)}(\mathbb{R}^n)$ and $W = w$, then we have $L^{q(\cdot)}(\mathbb{R}^n, w) = L^{q(\cdot)}(w^{q(\cdot)})$.

2. For $Y = L^{q'(\cdot)}(\mathbb{R}^n)$ and $W = w$, we obtain $L^{q'(\cdot)}(\mathbb{R}^n, w^{-1}) = L^{q'(\cdot)}(w^{-q'(\cdot)})$. Therefore, from lemma 6, we have

$$(L^{q(\cdot)}(w^{q(\cdot)}))' = (L^{q(\cdot)}(\mathbb{R}^n, w^{-1}))' = L^{q'(\cdot)}(w^{-q'(\cdot)}).$$

Furthermore, define $p_1(\cdot)$ so that $\frac{1}{p_1(\cdot)} = \frac{\alpha}{n} + \frac{1}{p_2(\cdot)}$. If $p_1(\cdot) \in \mathcal{P} \cap LH(\mathbb{R}^n)$ and $0 < \alpha < \frac{n}{p_1}$, now by applying the monotone property, we obtain $w^{p_1(\cdot)} \in A_1 \subset A_{1+\frac{p_2(\cdot)}{p_1'(\cdot)}}$. So, by lemma 2 we obtain $w \in A(p_1(\cdot), p_2(\cdot))$.

Lemma 7. *[48] Consider a weight w on $\mathbb{R}^n$. There exist $p \in [1, +\infty)$ such that $w \in A_p$, then for any measurable set E subset of B, we have*

$$\frac{w(B)}{w(E)} \leq C \left(\frac{|B|}{|E|} \right)^p$$

$$\frac{w(E)}{w(B)} \leq C \left(\frac{|E|}{|B|} \right)^\delta,$$

where $0 < \delta < 1$ represents a constant independent of E and B.

Lemma 8. *[48] Let us consider $p(\cdot) \in \mathcal{P} \cap LH(\mathbb{R}^n)$. If $w^{p_2(\cdot)} \in A_{p_2(\cdot)}$ and $w^{p_1(\cdot)} \in A_{p_1(\cdot)}$ imply $w^{-p_2'(\cdot)} \in A_{p_2'(\cdot)}$, $w^{-p_1'(\cdot)} \in A_{p_1'(\cdot)}$ respectively . Thus, $\mathcal{M}$ is bounded on $L^{p_2'(\cdot)}(w^{-p_2'(\cdot)})$. There exists constants δ_1, $\delta_2 \in (0, 1)$ and $E \subset B$ such that*

$$\frac{\|\chi_E\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}}{\|\chi_B\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}} = \frac{\|\chi_E\|_{(L^{p_2'(\cdot)} w^{-p_2'(\cdot)})'}}{\|\chi_B\|_{(L^{p_2'(\cdot)} w^{-p_2'(\cdot)})'}} \lesssim \left(\frac{|E|}{|B|} \right)^{\delta_1}, \tag{5}$$

$$\frac{\|\chi_E\|_{(L^{p_1(\cdot)} w^{p_1(\cdot)})'}}{\|\chi_B\|_{(L^{p_1(\cdot)} w^{p_1(\cdot)})'}} \lesssim \left(\frac{|E|}{|B|} \right)^{\delta_2}. \tag{6}$$

Lemma 9. *[16] Let $p_1(\cdot) \in \mathcal{P} \cap LH(\mathbb{R}^n)$ and $0 < \alpha < \frac{n}{p_1(\cdot)+}$, and $\frac{1}{p_2(\cdot)} = \frac{1}{p_1(\cdot)} - \frac{\alpha}{n}$. If $w \in A(p_1(\cdot), p_2(\cdot))$, then I_α is bounded from $L^{p_1(\cdot)}(w^{p_1(\cdot)})$ to $L^{p_2(\cdot)}(w^{p_2(\cdot)})$.*

Lemma 10. *[49] If the Hardy Littlewood maximal operator is bounded on the Banach function space Y, then for a measurable set $E \subset B$ we have the following result*

$$\frac{\|\chi_B\|_Y}{\|\chi_E\|_Y} \lesssim \frac{|B|}{|E|}.$$

4. Main Results and Their Proof

Lemma 11. *If $w^{q_1(\cdot)} \in A_1$, where $q_1(\cdot) \in \mathcal{P}(\mathbb{R}^n) \bigcap LH(\mathbb{R}^n)$, define the variable exponent $q_2(\cdot)$ by*

$$\frac{1}{q_2(x)} = \frac{1}{q_1(x)} - \frac{\alpha}{n},$$

then

$$\|\chi_{B_k}\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})} \leq C2^{k(n-\alpha)}\|\chi_{B_k}\|^{-1}_{(L^{q_1(\cdot)}(w^{q_1(\cdot)}))'}.$$

Proof. Based on lemmas 9 and 5, we have

$$I_\alpha(\chi_{B_k})(x) \geq C2^{k\alpha}\chi_{B_k}(x)$$

$$\chi_{B_k}(x) \leq C2^{-k\alpha}I_\alpha(\chi_{B_k})(x)$$

$$\begin{aligned} \|\chi_{B_k}\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})} &\leq C2^{-k\alpha}\|I_\alpha(\chi_{B_k})\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})} \\ &\leq C2^{-k\alpha}\|\chi_{B_k}\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})} \\ &\leq C2^{k(n-\alpha)}\|\chi_{B_k}\|^{-1}_{(L^{q_1(\cdot)}(w^{q_1(\cdot)}))'}. \end{aligned} \tag{7}$$

4.1. Boundedness of Fractional Hardy Operators

Theorem 1. *Let $q_1(\cdot) \in \mathcal{P}(\mathbb{R}^n) \bigcap LH(\mathbb{R}^n)$. Define the variable exponent $q_2(\cdot)$ by*

$$\frac{1}{q_2(x)} = \frac{1}{q_1(x)} - \frac{\alpha}{n}.$$

If $w^{q_1(\cdot)} \in A_1$, $\lambda_2 = \lambda_1 + \frac{\alpha}{n}$, and $\delta_2 + \delta\lambda_2 + \delta_1 > 0$, then

$$\|H_\alpha f\|_{\dot{B}^{q_2(\cdot),\lambda_2}(w^{q_2(\cdot)})} \leq C\|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}(w^{q_1(\cdot)})}.$$

Proof of Theorem 1

Using generalized Hölder inequality given in Lemma 3.

$$\begin{aligned} |H_\alpha f(x)\cdot\chi_k(x)| &\leq \frac{1}{|x|^{n-\alpha}}\int_{B_k}|f(t)|dt\cdot\chi_k(x) \\ &\leq C2^{-k(n-\alpha)}\sum_{j=-\infty}^{k}\|\chi_j\|_{(L^{q_1(\cdot)}(w^{q_1(\cdot)}))'}\|f_j\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\cdot\chi_k(x). \end{aligned}$$

$$\begin{aligned} &\|H_\alpha f(x)\cdot\chi_k\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})} \\ &\leq C2^{-k(n-\alpha)}\sum_{j=-\infty}^{k}\|f_j\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\|\chi_j\|_{(L^{q_1(\cdot)}(w^{q_1(\cdot)}))'}\|\chi_k\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}. \end{aligned}$$

By means of Lemma 5, we have

$$
\begin{aligned}
&\|H_\alpha f(x)\cdot\chi_k\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}\\
&\le C2^{k\alpha}\sum_{j=-\infty}^{k}\|\chi_j\|_{(L^{q_1(\cdot)}(w^{q_1(\cdot)}))'}\|f_j\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\|\chi_k\|^{-1}_{(L^{q_2(\cdot)})(w^{q_2(\cdot)}))'}\\
&\le C2^{k\alpha}\sum_{j=-\infty}^{k}\|f_j\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\frac{\|\chi_j\|_{(L^{q_1(\cdot)}(w^{q_1(\cdot)}))'}}{\|\chi_k\|_{(L^{q_1(\cdot)}(w^{q_1(\cdot)}))'}}\|\chi_k\|_{(L^{q_1(\cdot)}(w^{q_1(\cdot)}))'}\|\chi_k\|^{-1}_{(L^{q_2(\cdot)}(w^{q_2(\cdot)}))'}.
\end{aligned}
$$

By Lemma 8 and condition (6), we acquire

$$
\begin{aligned}
&\|H_\alpha f(x)\cdot\chi_k\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}\\
&\le C2^{k\alpha}\sum_{j=-\infty}^{k}2^{n\delta_2(j-k)}\|\chi_k\|_{(L^{q_1(\cdot)}(w^{q_1(\cdot)}))'}\|f_j\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\|\chi_k\|^{-1}_{(L^{q_2(\cdot)}(w^{q_2(\cdot)}))'}.
\end{aligned}
\tag{8}
$$

Using inequality (7) in (8) and by the result of Lemma 5

$$
\begin{aligned}
&\|H_\alpha f(x)\cdot\chi_k\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}\\
&\le C2^{k\alpha}\sum_{j=-\infty}^{k}2^{n\delta_2(j-k)}2^{k(n-\alpha)}\|f_j\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\|\chi_k\|^{-1}_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}\|\chi_k\|^{-1}_{(L^{q_2(\cdot)}(w^{q_2(\cdot)}))'}\\
&\le C\sum_{j=-\infty}^{k}2^{n\delta_2(j-k)}\|f_j\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\left(2^{-kn}\|\chi_k\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}\|\chi_k\|_{(L^{q_2(\cdot)}(w^{q_2(\cdot)}))'}\right)^{-1}\\
&\le C\sum_{j=-\infty}^{k}2^{n\delta_2(j-k)}\|f_j\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\\
&\le C\|f\|_{\dot B^{q_1(\cdot),\lambda_1}(w^{q_1(\cdot)})}\sum_{j=-\infty}^{k}2^{n\delta_2(j-k)}w(B_j)^{\lambda_1}\|\chi_j\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}.
\end{aligned}
$$

$$
\|\chi_j\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\approx w(B)^{\frac{1}{q_1(\cdot)}}\approx w(B)^{\frac{1}{q_2(\cdot)}+\frac{\alpha}{n}}\approx w(B)^{\frac{\alpha}{n}}\|\chi_j\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}.
$$

$$
\begin{aligned}
&\|H_\alpha f(x)\cdot\chi_k\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}\\
&\le C\|f\|_{\dot B^{q_1(\cdot),\lambda_1}(w^{q_1(\cdot)})}\sum_{j=-\infty}^{k}2^{n\delta_2(j-k)}w(B_j)^{\lambda_1+\frac{\alpha}{n}}\|\chi_j\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}\\
&= C\|f\|_{\dot B^{q_1(\cdot),\lambda_1}(w^{q_1(\cdot)})}\sum_{j=-\infty}^{k}2^{n\delta_2(j-k)}w(B_k)^{\lambda_2}\frac{w(B_j)^{\lambda_2}}{w(B_k)^{\lambda_2}}\|\chi_k\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}\frac{\|\chi_j\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}}{\|\chi_k\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}}.
\end{aligned}
$$

Applying Lemmas 8 and 7

$$\begin{aligned}
&\|H_\alpha f(x)\cdot\chi_k\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}\\
&\leq C\|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}(w^{q_1(\cdot)})}\sum_{j=-\infty}^{k}2^{n\delta_2(j-k)}|B_k|^{\lambda_2}\left(\frac{|B_j|}{|B_k|}\right)^{\delta\lambda_2}\|\chi_k\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}\frac{\|\chi_j\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}}{\|\chi_k\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}}\\
&\leq C\|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}(w^{q_1(\cdot)})}|B_k|^{\lambda_2}\|\chi_k\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}\sum_{j=-\infty}^{k}2^{(j-k)(n\delta_3+n\delta_1+n\delta\lambda_2)},
\end{aligned}$$

$$\|H_\alpha f(x)\cdot\chi_k\|_{\dot{B}^{q_2(\cdot),\lambda_2}(w^{q_2(\cdot)})}\leq C\|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}(w^{q_1(\cdot)})}\sum_{j=-\infty}^{k}2^{n(j-k)(\delta_2+\delta_1+\delta\lambda_2)}.$$

Since it is given that $\delta_2+\delta_1+\delta\lambda_2>0$, which gives the required result:

$$\|H_\alpha f(x)\cdot\chi_k\|_{\dot{B}^{q_2(\cdot),\lambda_2}(w^{q_2(\cdot)})}\leq C\|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}(w^{q_1(\cdot)})}.\square$$

Theorem 2. *Let $q_1(\cdot)$, $q_2(\cdot)$ and α be the same as in theorem 1 . If $\lambda_2=\lambda_1+\frac{\alpha}{n}$, $w^{q_1(\cdot)}\in A_1$ and $\alpha<-n(1+\lambda_2)$, then*

$$\|H^*_\alpha f\|_{\dot{B}^{q_2(\cdot),\lambda_2}(w^{q_2(\cdot)})}\leq C\|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}(w^{q_1(\cdot)})}.$$

Proof of Theorem 2

$$\begin{aligned}
|H^*_\alpha f(x)\cdot\chi_k(x)|&\leq\int_{\mathbb{R}^n\setminus B_k}|f(t)||t|^{\alpha-n}dt\cdot\chi_k(x)\\
&\leq C\sum_{j=k+1}^{\infty}2^{j(\alpha-n)}\|\chi_j\|_{(L^{q_1(\cdot)}(w^{q_1(\cdot)}))'}\|f_j\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\chi_k(x).
\end{aligned}$$

$$\begin{aligned}
&\|H^*_\alpha f(x)\cdot\chi_k\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}\\
&\leq C\sum_{j=k+1}^{\infty}2^{j(\alpha-n)}\|\chi_j\|_{(L^{q_1(\cdot)}(w^{q_1(\cdot)}))'}\|f_j\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\|\chi_k\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}.\\
&\leq C\sum_{j=k+1}^{\infty}2^{j(\alpha-n)}\frac{\|\chi_j\|_{(L^{q_1(\cdot)}(w^{q_1(\cdot)}))'}}{\|\chi_k\|_{(L^{q_1(\cdot)}(w^{q_1(\cdot)}))'}}\|\chi_k\|_{(L^{q_1(\cdot)}(w^{q_1(\cdot)}))'}\|\chi_k\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}\|f_j\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}.
\end{aligned}$$

By virtue of Lemmas 2, 10, 6 and by the definition of $A(q_2(\cdot),q_1(\cdot))$ we obtain the following inequalities:

$$\begin{aligned}
&\|H^*_\alpha f(x)\cdot\chi_k\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}\\
&\leq C\sum_{j=k+1}^{\infty}2^{j(\alpha-n)}2^{n(j-k)}\|\chi_k\|_{(L^{q_1(\cdot)}(w^{q_1(\cdot)}))'}\|f_j\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\|\chi_k\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}.\\
&\leq C\sum_{j=k+1}^{\infty}2^{(j-k)(\alpha-n)}2^{n(j-k)}\|f_j\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}.
\end{aligned}$$

$$\|H_{\alpha}^{*}f(x)\cdot\chi_k\|_{\dot{B}^{q_2(\cdot),\lambda_2}(w^{q_2(\cdot)})}\leq C\|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}(w^{q_1(\cdot)})}\sum_{j=k+1}^{\infty}2^{(j-k)(\alpha+n+n\lambda_2)}.$$

By using $\alpha<-n(1+\lambda_2)$, we get the final result:

$$\|H_{\alpha}^{*}f\|_{\dot{B}^{q_2(\cdot),\lambda_2}(w^{q_2(\cdot)})}\leq C\|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}(w^{q_1(\cdot)})}.\square$$

4.2. Commutators of Fractional Hardy Operators

Theorem 3. *Let $0<\alpha<n$, and $p_1(\cdot),p(\cdot)\in\mathcal{P}(\mathbb{R}^n)\bigcap LH(\mathbb{R}^n)$. Define the variable exponent $p_2(\cdot)$ by*

$$\frac{1}{p_2(x)}=\frac{1}{p(x)}+\frac{1}{p_1(x)}-\frac{\alpha}{n}.$$

If $w^{P_1(\cdot)}\in A_1$, $b\in\|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})}$, $\mu=\lambda_1+\frac{\alpha}{n}$ and $\lambda_2=\lambda+\lambda_1+\frac{\alpha}{n}$, then

$$\|[b,H_{\alpha}]f\|_{\dot{B}^{p_2(\cdot),\lambda_2}(w^{p_2(\cdot)})}\leq C\|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})}\|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})}.$$

Proof of Theorem 3

$$\begin{aligned}|[b,H_{\alpha}]f(x)\cdot\chi_B(x)|&\leq\frac{1}{|x|^{n-\alpha}}\int_{B(0,|x|)}|(b(x)-b(v))f(v)|dv\cdot\chi_B(x)\\&\leq\frac{1}{|x|^{n-\alpha}}\int_{B(0,|x|)}|(b(x)-b_B)f(v)|dv\cdot\chi_B(x)\\&+\frac{1}{|x|^{n-\alpha}}\int_{B(0,|x|)}|(b(v)-b_B)f(v)|dv\cdot\chi_B(x)\\&=A_1+A_2.\end{aligned}$$

First, we estimate A_1. Denote $\frac{1}{S(x)}=\frac{1}{p_1(x)}-\frac{\alpha}{n}$, then $\frac{1}{p_2(x)}=\frac{1}{S(x)}+\frac{1}{p(x)}$.

$$\begin{aligned}A_1&=\frac{1}{|x|^{n-\alpha}}\int_{B(0,|x|)}|(b(x)-b_B)f(v)|dv\cdot\chi_B(x)\\&=|(b(x)-b_B)\chi_B(x)||H_{\alpha}f(x)|,\end{aligned}$$

$$\|A_1\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}=\|(b(x)-b_B)\chi_B(x)H_{\alpha}f(x)\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}.$$

Using Hölder inequality ($\frac{1}{p_2(\cdot)}=\frac{1}{S(\cdot)}+\frac{1}{p(\cdot)}$)

$$\begin{aligned}\|A_1\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}&\leq\|(b(x)-b_B)\chi_B(x)\|_{L^{p(\cdot)}(w^{p(\cdot)})}\|H_{\alpha}f(x)\chi_B\|_{L^{S(\cdot)}(w^{S(\cdot)})}\\&=C\|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})}|B|^{\lambda}\|\chi_B\|_{L^{p(\cdot)}(w^{p(\cdot)})}|B|^{\mu}\|\chi_B\|_{L^{S(\cdot)}(w^{S(\cdot)})}\|H_{\alpha}f\|_{\dot{B}^{\mu,S(\cdot)}(w^{S(\cdot)})},\end{aligned}$$

$$\|A_1\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}\leq C\|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})}|B|^{\lambda}\|\chi_B\|_{L^{p(\cdot)}(w^{p(\cdot)})}|B|^{\mu}\|\chi_B\|_{L^{S(\cdot)}(w^{S(\cdot)})}\|H_{\alpha}f\|_{\dot{B}^{\mu,S(\cdot)}(w^{S(\cdot)})}.$$

$$\|\chi_B\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} \approx w(B)^{\frac{1}{p_2(\cdot)}} \approx w(B)^{\frac{1}{S(\cdot)}+\frac{1}{p(\cdot)}} \approx \|\chi_B\|_{L^{p(\cdot)}(w^{p(\cdot)})}\|\chi_B\|_{L^{S(\cdot)}(w^{S(\cdot)})}$$

Given that $\mu = \lambda_1 + \frac{\alpha}{n}$, using the result of theorem 1

$$\|A_1\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} \leq C\|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})}|B|^{\lambda_2}\|\chi_B\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}\|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})},$$

$$A_2 = \frac{1}{|x|^{n-\alpha}}\int_{B(0,|x|)}|(b(v)-b_B)f(v)|dv \cdot \chi_B(x).$$

$$\begin{aligned}
A_2 &= \sum_{k=-\infty}^{0}\frac{1}{|x|^{n-\alpha}}\sum_{l=-\infty}^{k}\int_{2^lB\backslash 2^{l-1}B}|(b(v)-b_B)f(v)|dv \cdot \chi_{2^kB\backslash 2^{k-1}B}(x)\\
&\leq \sum_{k=-\infty}^{0}\frac{1}{|x|^{n-\alpha}}\sum_{l=-\infty}^{k}\int_{2^lB\backslash 2^{l-1}B}|(b(v)-b_{2^lB})f(v)|dv \cdot \chi_{2^kB\backslash 2^{k-1}B}(x)\\
&+ \sum_{k=-\infty}^{0}\frac{1}{|x|^{n-\alpha}}\sum_{l=-\infty}^{k}\int_{2^lB\backslash 2^{l-1}B}|(b_B-b_{2^lB})f(v)|dv \cdot \chi_{2^kB\backslash 2^{k-1}B}(x)\\
&= A_{21} + A_{22}
\end{aligned}$$

$$A_{21} = \sum_{k=-\infty}^{0}|2^kB|^{\frac{\alpha}{n}-1}\sum_{l=-\infty}^{k}\int_{2^lB\backslash 2^{l-1}B}|(b(v)-b_{2^lB})f(v)|dv \cdot \chi_{2^kB\backslash 2^{k-1}B}(x).$$

Using Hölder inequality ($\frac{1}{p_1(\cdot)} + \frac{1}{t(\cdot)} + \frac{1}{p(\cdot)} = 1$).

$$\begin{aligned}
A_{21} &\leq C\sum_{k=-\infty}^{0}|2^kB|^{\frac{\alpha}{n}-1}\chi_{2^kB\backslash 2^{k-1}B}(x)\sum_{l=-\infty}^{k}\|(b(v)-b_{2^lB})\chi_{2^lB}\|_{L^{p(\cdot)}(w^{p(\cdot)})}\|f\chi_{2^lB}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\|\chi_{2^lB}\|_{L^{t(\cdot)}(w^{t(\cdot)})}\\
&= C\sum_{k=-\infty}^{0}|2^kB|^{\frac{\alpha}{n}-1}\chi_{2^kB\backslash 2^{k-1}B}(x)\sum_{l=-\infty}^{k}\|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})}|2^lB|^{\lambda}\|\chi_{2^lB}\|_{L^{p(\cdot)}(w^{p(\cdot)})}\\
&\quad \|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})}|2^lB|^{\lambda_1}\|\chi_{2^lB}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\|\chi_{2^lB}\|_{L^{t(\cdot)}(w^{t(\cdot)})}
\end{aligned}$$

$$\|\chi_{2^lB}\|_{(L^{p_1'(\cdot)}(w^{p_1(\cdot)}))'} \approx w(2^lB)^{\frac{1}{p_1'(\cdot)}} \approx w(2^lB)^{\frac{1}{p(\cdot)}+\frac{1}{t(\cdot)}} \approx \|\chi_{2^lB}\|_{L^{p(\cdot)}(w^{p(\cdot)})}\|\chi_{2^lB}\|_{L^{t(\cdot)}(w^{t(\cdot)})}$$

$$\begin{aligned}
A_{21} &= C\sum_{k=-\infty}^{0}|2^kB|^{\frac{\alpha}{n}-1}\chi_{2^kB\backslash 2^{k-1}B}(x)\|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})}\|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})}\\
&\quad \sum_{l=-\infty}^{k}|2^lB|^{\lambda+\lambda_1}\|\chi_{2^lB}\|_{(L^{p_1'(\cdot)}(w^{p_1(\cdot)}))'}\|\chi_{2^lB}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}.
\end{aligned}$$

By using the result of lemma 5, we will have

$$A_{21} = C\sum_{k=-\infty}^{0}|2^kB|^{\frac{\alpha}{n}-1}\chi_{2^kB\backslash 2^{k-1}B}(x)\|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})}\|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})}\sum_{l=-\infty}^{k}|2^l|^{\lambda+\lambda_1+1}|B|^{\lambda+\lambda_1+1}$$

$$= C\|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})}\|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})}\sum_{k=-\infty}^{0}|2^k|^{\frac{\alpha}{n}+\lambda+\lambda_1}\chi_{2^kB\backslash 2^{k-1}B}(x)|B|^{\lambda+\lambda_1+\frac{\alpha}{n}}$$

$$\begin{aligned}\|A_{21}\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} &\leq C\|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})}\|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})}\sum_{k=-\infty}^{0}|2^k|^{\frac{\alpha}{n}+\lambda+\lambda_1}\|\chi_{2^kB}\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}|B|^{\lambda+\lambda_1+\frac{\alpha}{n}} \\
&= C\|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})}\|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})}\sum_{k=-\infty}^{0}|2^k|^{\frac{\alpha}{n}+\lambda+\lambda_1}w(2^kB)^{\frac{1}{p_2(\cdot)}}|B|^{\lambda+\lambda_1+\frac{\alpha}{n}} \\
&= C\|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})}\|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})}w(B)^{\frac{1}{p_2(\cdot)}}|B|^{\lambda_2}\sum_{k=-\infty}^{0}|2|^{k(\lambda_2+\frac{1}{p_2(\cdot)})}\end{aligned}$$

$$\|A_{21}\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} \leq C\|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})}\|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})}\|\chi_B\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}|B|^{\lambda_2}$$

$$A_{22} = \sum_{k=-\infty}^{0}|2^kB|^{\frac{\alpha}{n}-1}\sum_{l=-\infty}^{k}\int_{2^lB\backslash 2^{l-1}B}|(b_B - b_{2^lB})f(y)|dy\cdot\chi_{2^kB\backslash 2^{k-1}B}(x)$$

$$\begin{aligned}|(b_B - b_{2^lB})| &= \sum_{i=l}^{-1}|(b_{2^{i+1}B} - b_{2^iB}) \\
&= \sum_{i=l}^{-1}\frac{1}{|2^iB|}\int_{2^iB}|b(y) - b_{2^{i+1}B}|dy \\
&\leq C\sum_{i=l}^{-1}\frac{1}{|2^iB|}\|(b - b_{2^{i+1}b})\chi_{2^{i+1}B}\|_{L^{p(\cdot)}(w^{p(\cdot)})}\|\chi_{2^{i+1}B}\|_{(L^{p(\cdot)}(w^{p_1(\cdot)}))'}\end{aligned}$$

By virtue of Lemma 5, we have

$$\begin{aligned}|(b_B - b_{2^lB})| &\leq C\sum_{i=l}^{-1}\frac{1}{|2^iB|}\|(b - b_{2^{i+1}b})\chi_{2^{i+1}B}\|_{L^{p(\cdot)}(w^{p(\cdot)})}\frac{|2^{i+1}B|}{\|\chi_{2^{i+1}B}\|_{L^{p(\cdot)}(w^{p(\cdot)})}} \\
&\leq C\sum_{i=l}^{-1}\|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})}|2^{i+1}B|^{\lambda} \\
&\leq C\|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})}\sum_{i=l}^{-1}|2^{i+1}B|^{\lambda} \\
&\leq C\|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})}|2^{l+1}B|^{\lambda}|l| \qquad (9)\end{aligned}$$

$$A_{22} \leq C \sum_{k=-\infty}^{0} \chi_{2^kB\backslash 2^{k-1}B}(x)|2^kB|^{\frac{\alpha}{n}-1} \sum_{l=-\infty}^{k} \|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})}|2^{l+1}B|^{\lambda}|l|$$
$$\|f\chi_{2^lB}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\|\chi_{2^lB}\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}$$
$$\leq C\|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})} \sum_{k=-\infty}^{0} \chi_{2^kB\backslash 2^{k-1}B}(x)|2^kB|^{\frac{\alpha}{n}-1} \sum_{l=-\infty}^{k} |2^{l+1}B|^{\lambda}|l|$$
$$\|f\chi_{2^lB}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\frac{|2^lB|}{\|\chi_{2^lB}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}}$$
$$\leq C\|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})} \sum_{k=-\infty}^{0} \chi_{2^kB\backslash 2^{k-1}B}(x)|2^kB|^{\frac{\alpha}{n}-1} \sum_{l=-\infty}^{k} |2^lB|^{\lambda+1}|l|\|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})}|2^lB|^{\lambda_1}$$
$$\leq C\|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})}\|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})} \sum_{k=-\infty}^{0} \chi_{2^kB\backslash 2^{k-1}B}(x)|2^kB|^{\frac{\alpha}{n}-1} \sum_{l=-\infty}^{k} |2^lB|^{\lambda+\lambda_1+1}|l|$$
$$\leq C\|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})}\|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})} \sum_{k=-\infty}^{0} \chi_{2^kB\backslash 2^{k-1}B}(x)|2^kB|^{\frac{\alpha}{n}-1}|2^kB|^{\lambda+\lambda_1+1}|k|$$
$$\leq C\|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})}\|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})} \sum_{k=-\infty}^{0} |k||2^kB|^{\lambda+\lambda_1+\frac{\alpha}{n}}\chi_{2^kB\backslash 2^{k-1}B}(x)$$

$$\|A_{22}\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} \leq C\|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})}\|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})} \sum_{k=-\infty}^{0} |k||2^kB|^{\lambda+\lambda_1+\frac{\alpha}{n}}\|\chi_{2^kB}\|_{L^{p_2(\cdot)}}$$
$$\leq C\|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})}\|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})} \sum_{k=-\infty}^{0} |k||2^kB|^{\lambda+\lambda_1+\frac{\alpha}{n}}w(2^kB)^{\frac{1}{p_2(\cdot)}}$$
$$\leq C\|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})}\|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})} \sum_{k=-\infty}^{0} |k||2^k|^{\lambda_2+\frac{1}{p_2(\cdot)}}|B|^{\lambda_2}w(B)^{\frac{1}{p_2(\cdot)}}$$
$$\leq C\|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})}\|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})}|B|^{\lambda_2}\|\chi_B\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}$$

Combine all the results of A_1, A_2, A_{21}, A_{22}, we obtain the required result

$$\|[b,H_\alpha]f\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} \leq C\|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})}\|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})}|B|^{\lambda_2}\|\chi_B\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}$$

$$\|[b,H_\alpha]f\|_{\dot{B}^{p_2(\cdot),\lambda_2}(w^{p_2(\cdot)})} \leq C\|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})}\|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})}.\square$$

Theorem 4. *Let $p_1(\cdot)$, $p_2(\cdot)$, $p(\cdot)$, and α be defined the same way as in Theorem 3. If $b \in \|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})}$, $\mu = \lambda_1 + \frac{\alpha}{n}$ and $\lambda_2 = \lambda + \lambda_1 + \frac{\alpha}{n}$, then*

$$\|[b,H^*_\alpha]f\|_{\dot{B}^{p_2(\cdot),\lambda_2}(w^{p_2(\cdot)})} \leq C\|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})}\|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})}.$$

Proof of Theorem 4

$$\begin{aligned}|[b,H_\alpha^*]f(x)\cdot\chi_B(x)| &\le \int_{B(0,|x|)^c}\frac{|(b(x)-b(v))f(v)|}{|v|^{n-\alpha}}dv\cdot\chi_B(x)\\ &\le \int_{B(0,|x|)^c}\frac{|(b(x)-b_B)f(v)|}{|v|^{n-\alpha}}dv\cdot\chi_B(x)\\ &+\int_{B(0,|x|)^c}\frac{|(b(v)-b_B)f(v)|}{|v|^{n-\alpha}}dv\cdot\chi_B(x)\\ &= D_1+D_2.\end{aligned}$$

$$\begin{aligned}D_1 &= \int_{B(0,|x|)^c}\frac{|(b(x)-b_B)f(v)|}{|v|^{n-\alpha}}dv\cdot\chi_B(x)\\ &= |(b(x)-b_B)\chi_B(x)||H_\alpha^*f(x)|,\\ \|D_1\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} &= \|(b(x)-b_B)\chi_B H_\alpha^* f\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}.\end{aligned}$$

Using Hölder inequality $(\frac{1}{p_2(\cdot)}=\frac{1}{S(\cdot)}+\frac{1}{p(\cdot)})$

$$\begin{aligned}\|D_1\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} &\le C\|(b(x)-b_B)\chi_B\|_{L^{p(\cdot)}(w^{p(\cdot)})}\|H_\alpha^* f\chi_B\|_{L^{S(\cdot)}(w^{S(\cdot)})}\\ &= C\|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})}|B|^{\lambda}\|\chi_B\|_{L^{p(\cdot)}(w^{p(\cdot)})}|B|^{\mu}\|\chi_B\|_{L^{S(\cdot)}(w^{S(\cdot)})}\|H_\alpha^* f\|_{\dot{B}^{\mu,S(\cdot)}(w^{S(\cdot)})},\end{aligned}$$

Given that $\mu=\lambda_1+\frac{\alpha}{n}$*, using the result of Theorem 2*

$$\begin{aligned}\|D_1\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} &\le C\|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})}|B|^{\lambda_2}\|\chi_B\|_{L^{p(\cdot)}(w^{p(\cdot)})}\|\chi_B\|_{L^{S(\cdot)}(w^{S(\cdot)})}\|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})},\\ &\le C\|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})}|B|^{\lambda_2}\|\chi_B\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}\|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})},\end{aligned}$$

$$D_2=\int_{B(0,|x|)^c}\frac{|(b(v)-b_B)f(v)|}{|v|^{n-\alpha}}dv\cdot\chi_B(x).$$

$$\begin{aligned}D_2 &= \sum_{k=-\infty}^{0}\int_{2^lB\backslash 2^{l-1}B}\frac{|(b(v)-b_B)f(v)|}{|v|^{n-\alpha}}dv\cdot\chi_{2^kB\backslash 2^{k-1}B}(x)\\ &\le \sum_{k=-\infty}^{0}\chi_{2^kB\backslash 2^{k-1}B}(x)\sum_{l=k+1}^{\infty}|2^lB|^{\frac{\alpha}{n}-1}\int_{2^lB\backslash 2^{l-1}B}|(b(v)-b_{2^lB})f(v)|dv\\ &+\sum_{k=-\infty}^{0}\chi_{2^kB\backslash 2^{k-1}B}(x)\sum_{l=k+1}^{\infty}|2^lB|^{\frac{\alpha}{n}-1}\int_{2^lB\backslash 2^{l-1}B}|(b_B-b_{2^lB})f(v)|dv\\ &= D_{21}+D_{22}\end{aligned}$$

$$D_{21}=\sum_{k=-\infty}^{0}\chi_{2^kB\backslash 2^{k-1}B}(x)\sum_{l=k+1}^{\infty}|2^lB|^{\frac{\alpha}{n}-1}\int_{2^lB\backslash 2^{l-1}B}|(b(v)-b_{2^lB})f(v)|dv.$$

Using Hölder inequality $(\frac{1}{t(\cdot)}+\frac{1}{p_1(\cdot)}+\frac{1}{p(\cdot)}=1)$.

$$\begin{aligned}
D_{21} &\leq C\sum_{k=-\infty}^{0}\chi_{2^kB\backslash 2^{k-1}B}(x)\sum_{l=k+1}^{\infty}|2^lB|^{\frac{\alpha}{n}-1}\|(b(v)-b_{2^lB})\chi_{2^lB}\|_{L^{p(\cdot)}(w^{p(\cdot)})}\|f\chi_{2^lB}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\|\chi_{2^lB}\|_{L^{t(\cdot)}(w^{t(\cdot)})}\\
&= C\sum_{k=-\infty}^{0}\chi_{2^kB\backslash 2^{k-1}B}(x)\sum_{l=k+1}^{\infty}|2^lB|^{\frac{\alpha}{n}-1}\|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})}|2^lB|^{\lambda}\|\chi_{2^lB}\|_{L^{p(\cdot)}(w^{p(\cdot)})}\\
&\quad \|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})}|2^lB|^{\lambda_1}\|\chi_{2^lB}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\|\chi_{2^lB}\|_{L^{t(\cdot)}(w^{t(\cdot)})}\\
&\leq C\sum_{k=-\infty}^{0}\chi_{2^kB\backslash 2^{k-1}B}(x)\sum_{l=k+1}^{\infty}|2^lB|^{\frac{\alpha}{n}-1}\|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})}|2^lB|^{\lambda}\\
&\quad \|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})}|2^lB|^{\lambda_1}\|\chi_{2^lB}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\|\chi_{2^lB}\|_{(L^{p(\cdot)}(w^{p(\cdot)}))'}\\
&= C\sum_{k=-\infty}^{0}\chi_{2^kB\backslash 2^{k-1}B}(x)\|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})}\|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})}\sum_{l=k+1}^{\infty}|2^lB|^{\frac{\alpha}{n}-1}|2^lB|^{\lambda+\lambda_1+1}\\
&= C\sum_{k=-\infty}^{0}\chi_{2^kB\backslash 2^{k-1}B}(x)\|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})}\|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})}\sum_{l=k+1}^{\infty}|2^lB|^{\lambda_2}\\
&= C\|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})}\|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})}\sum_{k=-\infty}^{0}|2^{(k+1)}B|^{\lambda_2}\chi_{2^kB\backslash 2^{k-1}B}(x)
\end{aligned}$$

$$\begin{aligned}
\|D_{21}\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} &\leq C\|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})}\|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})}\sum_{k=-\infty}^{0}|2^{(k+1)}B|^{\lambda_2}\|\chi_{2^kB}\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}\\
&\leq C\|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})}\|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})}\sum_{k=-\infty}^{0}|2^{(k+1)}B|^{\lambda_2}w(2^kB)^{\frac{1}{p_2(\cdot)}}\\
&\leq C\|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})}\|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})}|B|^{\lambda_2}w(B)^{\frac{1}{p_2(\cdot)}}\sum_{k=-\infty}^{0}|2^{(k+1)}|^{\lambda_2+\frac{1}{p_2(\cdot)}}\\
&\leq C\|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})}\|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})}|B|^{\lambda_2}\|\chi_B\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}
\end{aligned}$$

$$D_{22}=\sum_{k=-\infty}^{0}\chi_{2^kB\backslash 2^{k-1}B}(x)\sum_{l=-\infty}^{k}|2^lB|^{\frac{\alpha}{n}-1}\int_{2^lB\backslash 2^{l-1}B}|(b_B-b_{2^lB})f(v)|dv.$$

Here we use inequality (9)

$$
\begin{aligned}
D_{22} \leq C \sum_{k=-\infty}^{0} \chi_{2^kB\backslash 2^{k-1}B}(x) \sum_{l=k+1}^{\infty} &|2^lB|^{\frac{\alpha}{n}-1} \|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})} |2^{l+1}B|^{\lambda}|l| \\
&\|f\chi_{2^lB}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \|\chi_{2^lB}\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'} \\
\leq C\|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})} &\sum_{k=-\infty}^{0} \chi_{2^kB\backslash 2^{k-1}B}(x) \sum_{l=k+1}^{\infty} |2^lB|^{\frac{\alpha}{n}-1} |2^{l+1}B|^{\lambda}|l| \\
&\|f\chi_{2^lB}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \frac{|2^lB|}{\|\chi_{2^lB}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}} \\
\leq C\|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})} &\sum_{k=-\infty}^{0} \chi_{2^kB\backslash 2^{k-1}B}(x) \sum_{l=k+1}^{\infty} |2^lB|^{\frac{\alpha}{n}-1} |2^lB|^{\lambda+1}|l| \|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})} |2^lB|^{\lambda_1} \\
\leq C\|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})} &\|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})} \sum_{k=-\infty}^{0} \chi_{2^kB\backslash 2^{k-1}B}(x) \sum_{l=k+1}^{\infty} |2^lB|^{\lambda+\lambda_1+\frac{\alpha}{n}}|l| \\
\leq C\|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})} &\|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})} \sum_{k=-\infty}^{0} \chi_{2^kB\backslash 2^{k-1}B}(x) |2^{k+1}B|^{\lambda_2}|k+1| \\
\leq C\|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})} &\|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})} \sum_{k=-\infty}^{0} |k+1||2^{k+1}B|^{\lambda_2} \chi_{2^kB\backslash 2^{k-1}B}(x)
\end{aligned}
$$

$$
\begin{aligned}
\|D_{22}\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} &\leq C\|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})} \|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})} \sum_{k=-\infty}^{0} |k+1||2^{k+1}B|^{\lambda_2} \|\chi_{2^kB}\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} \\
&\leq C\|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})} \|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})} \sum_{k=-\infty}^{0} |k+1||2^{k+1}B|^{\lambda_2} w(2^kB)^{\frac{1}{p_2(\cdot)}} \\
&\leq C\|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})} \|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})} \sum_{k=-\infty}^{0} |k+1||2^{k+1}|^{\lambda_2+\frac{1}{p_2(\cdot)}} |B|^{\lambda_2} w(B)^{\frac{1}{p_2(\cdot)}} \\
&\leq C\|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})} \|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})} |B|^{\lambda_2} \|\chi_B\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}
\end{aligned}
$$

Combine all results of D_1, D_2, D_{21}, D_{22}, we obtain the required result

$$
\|[b, H_{\alpha}^*]f\chi_B\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} \leq C\|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})} \|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})} |B|^{\lambda_2} \|\chi_B\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}
$$

$$
\|[b, H_{\alpha}^*]f\|_{\dot{B}^{p_2(\cdot),\lambda_2}(w^{p_2(\cdot)})} \leq C\|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})} \|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})} . \square
$$

5. Conclusion

This scholarly treatise vouchsafes momentous progressions in the analytical dissection of fractional Hardy operators, meticulously ensconced within the structural fabric of *weighted central Morrey spaces imbued with variable exponents. The present intellectual enterprise is poised to inaugurate uncharted pathways for erudition in the mathematical and physical sciences—most conspicuously within the abstruse territories of Quantum Mechanics and Mathematical Modeling—thereby engendering novel prospects for exploratory inquiry and theoretical augmentation.*

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Authors' contributions

M.A., S.H. and N.M. wrote the main manuscript text. All authors reviewed the manuscript.

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