EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 3, Article Number 6668
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

A Comprehensive Study of Generalized Bivariate q-Laguerre Polynomials: Structural Properties and Applications

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Abstract. In this paper, utilizing zeroth-order q-Bessel Tricomi functions, we introduce the generalized bivariate q-Laguerre polynomials. Then, we establish the generalized bivariate q-Laguerre polynomials from the context of quasi-monomiality. We examine some of their properties, such as q-multiplicative operator property, q-derivative operator property and two q-integro-differential equations. Additionally, we derive operational representations and three q-partial differential equations for the generalized bivariate q-Laguerre polynomials. Moreover, we draw the zeros of the new polynomials, forming 2D and 3D structures, and provide a table including approximate zeros of the generalized bivariate q-Laguerre polynomials.

2020 Mathematics Subject Classifications: 05A30; 11B83; 11B68, 33C45

Key Words and Phrases: Quantum calculus, q-Laguerre polynomials, generalized 2V q-Laguerre polynomials, Quasi monomiality, Extension of monomiality priciple, q-Dilatation operator, Partial differential equations, Differential equations

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DOI: https://doi.org/10.29020/nybg.ejpam.v18i3.6668

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1. Introduction

A set of orthogonal polynomials known as Laguerre polynomials is essential to many fields of applied mathematics and mathematical physics. The Laguerre polynomials stand out due to their applications in harmonic oscillator theory, coding theory, and quantum group theory [1]. These polynomials are instrumental in formulating covariant oscillator algebra [2, 3]. For additional information regarding the applications of Laguerre polynomials, refer to [4, 5]. Dattoli and Torre [6, 7] demonstrated that the hypothesis of bivariate Laguerre polynomials may be applied to ordinary Laguerre polynomials within the framework of quasi-monomials. Bivariate Laguerre polynomials are of great mathematical importance. Laguerre polynomials are used in overcoming radiation physics problems, including quantum beam lifetime in a storage ring and electromagnetic wave propagation [8–11]. They also appear as natural solutions to partial differential equations, such as the heat diffusion equation. To see more detail for the theory of two-variable Laguerre polynomials, one can look at the references [5, 12–16].
Dattoli and Torre [12] introduced the generalized bivariate Laguerre type polynomials, denoted by (2VgLtP) ${}_{[m]}L_{\omega}(\xi,\eta)$, are considered as follows:

$$\exp(\eta\psi^{m})C_0(\xi\psi)=\sum_{\omega=0}^{\infty}{}_{[m]}L_{\omega}(\xi,\eta)\frac{\psi^{\omega}}{\omega!}, \tag{1}$$

where the function $C_0(\xi)$ means the 0^{th} order Bessel Tricomi function:

$${}_{[m]}L_{\omega}(\xi,\eta)=\omega!\sum_{\theta=0}^{[\frac{\omega}{m}]}\frac{\eta^{\theta}(-\xi)^{\omega-m\theta}}{\theta!((\omega-m\theta)!)^{2}}. \tag{2}$$

The origins of quantum calculus can be traced back to the 18th century. This mathematical framework, commonly denoted as q-calculus, constitutes a substantial expansion of traditional calculus. Its significance lies in its intricate relationships with various scientific domains, including quantum mechanics, mathematical physics, mathematical analysis, combinatorics, and the theory of orthogonal polynomials. Subsequent to its inception, the q-calculus paradigm underwent extensive investigation and refinement by a multitude of scholars. This mathematical construct enables the scrutiny and examination of q-analogs, which serve as counterparts to fundamental and special functions under q-transformations. Contemporary academic pursuits have predominantly centered on exploring particular families of q-special polynomials. These investigations employ generating functions and their corresponding functional equations to illuminate and broaden the properties and applications of these polynomials across diverse scientific disciplines. The ongoing nature of this research emphasizes the profound influence and enduring relevance of q-calculus in the evolution of modern mathematical theory and its multidisciplinary applications. In a recent study, Fadel and colleagues [17] introduced and examined bivariate q-Hermite polynomials. Furthermore, certain q-special functions were analyzed and studied in the context of q-algebra representations [8, 18, 19].

In this study, we assume that $0 < q < 1$ and we adhere to the terminology and notions given in [17, 20]. The q-shift factorial $(a;q)_\omega$ is defined as [21]

$$(a;q)_\omega = \prod_{\phi=1}^{\omega-1}(1-q^\phi a) \text{ for } \omega \in \mathbb{N} \tag{3}$$

with $(a;q)_0 := 1$. Let $\omega \in \mathbb{C}$ with $\omega \geq 1$. The q-numbers and q-factorial are provided as follows

$$[\omega]_q = \frac{1-q^w}{1-q}, \qquad . \tag{4}$$

and

$$[\omega]_q! = \prod_{\phi=1}^{\omega}[\phi]_q \text{ for } \omega > 0, \tag{5}$$

with $[0]_q! = 1$.

The q-extension of $(\xi \pm a)^\omega$ is provided as follows

$$(\xi \pm a)_q^\omega = \sum_{\phi=0}^{\omega}\binom{\omega}{\phi}_q \xi^\phi(\pm a)^{\omega-\phi}q^{\binom{\omega-\phi}{2}}. \tag{6}$$

The two types of q-exponential functions are considered by [22–25]

$$e_q(\xi) = \sum_{\omega=0}^{\infty}\frac{\xi^\omega}{[\omega]_q!}, \tag{7}$$

and

$$E_q(\xi) = \sum_{\omega=0}^{\infty}\frac{q^{\binom{\omega}{2}}\xi^\omega}{[\omega]_q!}. \tag{8}$$

The following relations are valid:

$$e_q(\xi)E_q(\eta) = \sum_{\omega=0}^{\infty}\frac{(\xi \oplus \eta)_q^\omega}{[\omega]_q!}, \tag{9}$$

and

$$e_q(\xi)E_q(-\xi) = 1. \tag{10}$$

For $\xi \neq 0$, the q-derivative operator is provided as follows

$$\widehat{D}_{q,\xi}f(\xi) = \frac{f(q\xi)-f(\xi)}{q\xi-\xi}, \tag{11}$$

which satisfies the following operator rules

$$\widehat{D}_{q,\xi}\xi^\omega = [\omega]_q\xi^{\omega-1}, \tag{12}$$

$$\widehat{D}_{q,\xi}e_q(\alpha\xi) = \alpha e_q(\alpha\xi), \qquad \alpha \in \mathbb{C}, \tag{13}$$

and

$$\widehat{D}^{\phi}_{q,\xi}e_q(\alpha\xi) = \alpha^{\phi} e_q(\alpha\xi), \qquad \phi \in \mathbb{N},\ \alpha \in \mathbb{C}. \tag{14}$$

Here $\widehat{D}^{\phi}_{q,\xi}$ denotes the ϕ^{th} order q-derivative operator with respect to ξ. The product rule is provided as follows [25]

$$\widehat{D}_{q,\xi}(f(\xi)g(\xi)) = f(\xi)\widehat{D}_{q,\xi}g(\xi) + g(q\xi)\widehat{D}_{q,\xi}f(\xi). \tag{15}$$

In recent years, q-Gould-Hopper polynomials $qGHPH^{(m)}_{\omega,q}(\xi,\eta)$ are considered by Khan *et al.* [23] as follows

$$e_q(\xi\psi)e_q(\eta\psi^m) = \sum_{\omega=0}^{\infty} H^{(m)}_{\omega,q}(\xi,\eta)\frac{\psi^{\omega}}{[\omega]_q!}, \tag{16}$$

and also

$$H^{(m)}_{\omega,q}(\xi,\eta) = [\omega]_q! \sum_{\phi=0}^{[\frac{\omega}{m}]} \frac{\eta^{\phi}\xi^{\omega-m\phi}}{[\phi]_q![\omega-m\phi]_q!}. \tag{17}$$

The operational identity of q-Gould-Hopper polynomials $H^{(m)}_{\omega,q}(\xi,\eta)$ is given by (see [23]):

$$H^{(m)}_{\omega,q}(\xi,\eta) = e_q\left(\eta D^m_{q,\xi}\right)\{\xi^{\omega}\}. \tag{18}$$

Cao *et al.* [26] introduced the q-Laguerre polynomials $L_{\omega,q}(\xi,\eta)$ are provided as follows

$$C_{0,q}(\xi\psi)e_q(\eta\psi) = \sum_{\omega=0}^{\infty} L_{\omega,q}(\xi,\eta)\frac{\psi^{\omega}}{[\omega]_q!}, \tag{19}$$

the series definition, we have

$$L_{\omega,q}(\xi,\eta) = [\omega]_q! \sum_{\phi=0}^{\omega} \frac{(-1)^{\phi}\xi^{\phi}\eta^{\omega-\phi}}{([\phi]_q!)^2[\omega-\phi]_q!}. \tag{20}$$

where the 0^{th} order q-Bessel Tricomi functions $C_{0,q}(\xi)$ are defined by [26]:

$$C_{0,q}(\xi\psi) = e_q(-D^{-1}_{q,\xi}\psi)\{1\} \tag{21}$$

and also

$$C_{0,q}(\xi) = \sum_{\phi=0}^{\infty} \frac{(-1)^{\phi}\xi^{\phi}}{([\phi]_q!)^2}, \tag{22}$$

which converges absolutely for all ξ.
In view of the following notions [26]:

$$\widehat{D}^{-1}_{q,\xi}f(\xi) := \int_0^{\xi} f(\xi)d_q\xi, \tag{23}$$

Particularly, for $r \in \mathbb{N}$ and choosing $\widehat{D}_{q,\xi}^{-1}\{1\} = \xi$, it is seen that

$$\left(\widehat{D}_{q,\xi}^{-1}\right)^r \{1\} = \frac{\xi^r}{[r]_q!}. \tag{24}$$

From (5) and (21), equation (19) can be written as

$$e_q(\widehat{D}_{q,\xi}^{-1}\psi^m)e_q(\eta\psi)\{1\} = \sum_{\omega=0}^{\infty} {}_{[m]}L_{\omega,q}(\xi,\eta)\frac{\psi^\omega}{[\omega]_q!}, \tag{25}$$

For u being a complex variable, the q-dilation operator T_u is introduced as follows [27]:

$$\mathrm{T}_u^\phi f(u) = f(q^\phi u), \qquad \phi \in \mathbb{R}, \tag{26}$$

satisfies the property

$$\mathrm{T}_u^{-1}\mathrm{T}_u^1 f(u) = f(u). \tag{27}$$

The following operator rule is valid [26]:

$$\widehat{D}_{q,\psi}e_q(u\psi^m) = u\psi^{m-1}\mathrm{T}_{(u;m)}e_q(u\psi^m). \tag{28}$$

where

$$\mathrm{T}_{(u;m)} = \frac{1-q^m\mathrm{T}_u^m}{1-qT_u} = 1 + qT_u + \cdots + q^{m-1}\mathrm{T}_u^{m-1}. \tag{29}$$

Since, in view of equations (20) and (28), we have

$$T^r_{\widehat{D}_{q,\xi}^{-1}}C_{0,q}(\xi) = T_\xi^r C_{0,q}(\xi). \tag{30}$$

Therefore, the following operator rule is valid [17]:

$$\widehat{D}_{q,\psi}C_{0,q}(-\xi\psi^m) = \widehat{D}_{q,\xi}^{-1}\psi^{m-1}\mathrm{T}_{(\xi;m)}C_{0,q}(-\xi\psi^m), \tag{31}$$

where

$$\mathrm{T}_{(\xi;m)} = \frac{1-q^m\mathrm{T}_\xi^m}{1-q\mathrm{T}_\xi} = 1 + q\mathrm{T}_\xi + \cdots + q^{m-1}\mathrm{T}_\xi^{m-1}. \tag{32}$$

Also they proved the q-derivative and q-multiplicative operators of bivariate q-Laguerre polynomials as follows [26]

$$\widehat{D}_{q,\xi}\xi\widehat{D}_{q,\xi}C_{0,q}(\alpha\xi) = \frac{\partial_q}{\partial_q\widehat{D}_{q,\xi}^{-1}}C_{0,q}(\alpha\xi) = -\alpha C_{0,q}(\alpha\xi). \tag{33}$$

Taking into account equation (15), we have

$$\widehat{D}_{q,\xi}\xi\widehat{D}_{q,\xi}f(\xi) = (q\widehat{D}_{q,\xi} + \xi\widehat{D}_{q,\xi}^2)f(\xi). \tag{34}$$

The monomiality concept is a valuable technique for understanding certain special polynomials and functions in conjunction with relations and properties. This concept has

the potential to generate novel sets of the family of q-special polynomials and show the quasi-monomial nature of some previously established q-extension of special polynomials. For further information, one may look at the references [13, 14, 26, 27].
Implementing the monomiality principle to quantum calculus establishes a basis for comprehending q-special polynomials as specific solutions to expanded versions of q-integro differential equations and q-partial differential equations. Cao and colleagues [26] have recently expanded the idea of the monomiality principle to the field of quantum calculus. Let $p_{\omega,q}(\xi)$ be a q-polynomial set for $\omega \in \mathbb{N}$ and $\xi \in \mathbb{C}$. The q-multiplicative operator $\widehat{M_q}$ and q-derivative operator $\widehat{P_q}$ are provided as follows [26]

$$\widehat{M_q}\{p_{\omega,q}(\xi)\} = p_{\omega+1,q}(\xi), \tag{35}$$

and

$$\widehat{P_q}\{p_{\omega,q}(\xi)\} = \omega p_{\omega-1,q}(\xi), \tag{36}$$

which fulfill the following relation:

$$[\widehat{M_q}, \widehat{P_q}] = \widehat{P_q}\widehat{M_q} - \widehat{M_q}\widehat{P_q}. \tag{37}$$

The characteristics of the polynomials $p_{\omega,q}(\xi)$ can be deduced from the features of the $\widehat{M_q}$ and $\widehat{P_q}$ operators. If the angles $\widehat{M_q}$ and $\widehat{P_q}$ have a q-differential realization, then the polynomials $p_{\omega,q}(\xi)$ must satisfy the q-differential equations:

$$\widehat{M_q}\widehat{P_q}\{p_{\omega,q}(\xi)\} = [\omega]_q p_{\omega,q}(\xi), \tag{38}$$

and

$$\widehat{P_q}\widehat{M_q}\{p_{\omega,q}(\xi)\} = [\omega+1]_q p_{\omega,q}(\xi). \tag{39}$$

In view of (35) and (36), we have

$$[\widehat{M_q}, \widehat{P_q}] = [\omega+1]_q - [\omega]_q. \tag{40}$$

From (35), we have

$$\widehat{M_q}^{\,r}\{p_{n,q}\} = p_{n+r,q}(x). \tag{41}$$

In particular, we have

$$p_{n,q}(x) = \widehat{M_q}^{\,n}\{p_{0,q}\} = \widehat{M_q}^{\,n}\{1\}, \tag{42}$$

where $p_{0,q}(\xi) = 1$ is the q-sequel of polynomial $p_{\omega,q}(\xi)$ provided by

$$e_q(\widehat{M_q}\psi)\{1\} = \sum_{\omega=0}^{\infty} p_{\omega,q}(\xi)\frac{\psi^{\omega}}{[\omega]_q!}. \tag{43}$$

Using the idea of the q-monomiality principle, in the present paper, we describe and investigate the unique characteristics of generalized bivariate m^{th} order q-Laguerre polynomials, motivated by the possible applications of q-special functions in mathematics and science. Furthermore, we give applications of this recently certain members of generalized m^{th} order q-Laguerre polynomials family to show their graphs. We conclude this research article by computing the zeros of certain members of generalized m^{th} order q-Laguerre polynomials family numerically as well as presenting them graphically.

2. Generalized Bivariate q-Laguerre Polynomials

his part defines the generalized bivariate q-Laguerre polynomials ${}_{[m]}L_{\omega,q}(\xi,\eta)$ utilizing the function $C_{0,q}(\xi\psi)$ in (22) and derives explicit formulas, operational identities, q-quasi-monomiality characteristic and q-integro-differential equations for these polynomials.
We perform to define q-extension of the polynomials in (1) and we introduce the generalized bivariate q-Laguerre polynomials ${}_{[m]}L_{\omega,q}(\xi,\eta)$ as follows:

$$C_{0,q}(\xi\psi)e_q(\eta\psi^m)=\sum_{\omega=0}^{\infty}{}_{[m]}L_{\omega,q}(\xi,\eta)\frac{\psi^\omega}{[\omega]_q!}. \tag{44}$$

We readily derive the following explicit formula for the new polynomials in (44):

$${}_{[m]}L_{\omega,q}(\xi,\eta)=[\omega]_q!\sum_{\theta=0}^{[\frac{\omega}{m}]}\frac{(-1)^\omega\xi^{\omega-m\theta}\eta^\theta}{([\omega-m\theta]_q!)^2[\theta]_q!}. \tag{45}$$

We then observe from (24), (25), and (44) that

$$e_q(-D_{q,\xi}^{-1}\psi)e_q(\eta\psi^m)\{1\}=\sum_{\omega=0}^{\infty}{}_{[m]}L_{\omega,q}(\xi,\eta)\frac{\psi^\omega}{[\omega]_q!}. \tag{46}$$

Also we obtain using (18), (45) and (46) that

$${}_{[m]}L_{\omega,q}(\xi,\eta)=H_{\omega,q}^{(m)}(D_{q,\xi}^{-1},\eta)\{1\}. \tag{47}$$

Now, we provide the following theorem, including q-multiplicative operator and q-derivative operator of the new polynomials ${}_{[m]}L_{\omega,q}(\xi,\eta)$.

Theorem 1. *The polynomials ${}_{[m]}L_{\omega,q}(\xi,\eta)$ are quasi-monomials under the following q-multiplicative operator and q-derivative operator:*

$$\widehat{M}_{G2VqLP}=\eta\mathrm{T}_{(\eta;m)}\left(-\frac{\partial_q}{\partial_q\widehat{D}_{q,\xi}^{-1}}\right)^{m-1}-\widehat{D}_{q,\xi}^{-1}\mathrm{T}_{q^m,\eta}, \tag{48}$$

or, equally

$$\widehat{M}_{G2VqLP}=\eta\mathrm{T}_{(\eta;m)}\left(-\frac{\partial_q}{\partial_q\widehat{D}_{q,\xi}^{-1}}\right)^{m-1}\mathrm{T}_{q,\xi}-\widehat{D}_{q,\xi}^{-1}, \tag{49}$$

and

$$\widehat{P}_{G2VqLP}=-\widehat{D}_{q,\xi}\xi\widehat{D}_{q,\xi}=-\frac{\partial_q}{\partial_q\widehat{D}_{q,\xi}^{-1}}, \tag{50}$$

respectively.

Proof. Utilizing (15) and applying q-derivative operator to the both sides of (44) with respect to t, we derive that

$$\sum_{\omega=1}^{\infty} {}_{[m]}L_{\omega,q}(\xi,\eta)\widehat{D}_{q,\psi}\frac{\psi^{\omega}}{[\omega]_q!} = e_q(-\widehat{D}_{q,\xi}^{-1}\psi)\widehat{D}_{q,\psi}e_q(\eta\psi^m) + e_q(\eta q^m\psi^m)\widehat{D}_{q,\psi}e_q(-\widehat{D}_{q,\xi}^{-1}\psi).$$

Thus, we see from (23), (29) and (44) that

$$\sum_{\omega=1}^{\infty} {}_{[m]}L_{\omega,q}(\xi,\eta)\frac{\psi^{\omega-1}}{[\omega-1]_q!} = \eta\mathrm{T}_{(\eta;m)}\psi^{m-1}e_q(-\widehat{D}_{q,\xi}^{-1}\psi)e_q(\eta\psi^m) - \widehat{D}_{q,\xi}^{-1}T_{q^m,\eta}e_q(-\widehat{D}_{q,\xi}^{-1}\psi)e_q(\eta\psi^m). \tag{51}$$

Using equations (33) and (44) in the equation (51), we get

$$\sum_{\omega=1}^{\infty} {}_{[m]}L_{\omega,q}(\xi,\eta)\frac{\psi^{\omega-1}}{[\omega-1]_q!} = \sum_{\omega=0}^{\infty}\left(\eta\mathrm{T}_{(\eta;m)}\widehat{D}_{q,\eta}^{m-1} - \widehat{D}_{q,\xi}^{-1}\mathrm{T}_{q^m,\eta}\right) {}_{[m]}L_{\omega,q}(\xi,\eta)\frac{\psi^{\omega}}{[\omega]_q!}, \tag{52}$$

which means

$${}_{[m]}L_{n+1,q}(x,y) = \left(yT_{(y;m)}\left(-\frac{\partial_q}{\partial_q\widehat{D}_{q,x}^{-1}}\right)^{m-1} - \widehat{D}_{q,x}^{-1}T_{q^m,y}\right) {}_{[m]}L_{n,q}(x,y), \tag{53}$$

which is the first claimed result (48).
In the same way, utilizing (33) and (52) for $f_q(\psi) = e_q(\eta\psi^m)$ and $g_q(\psi) = e_q(-\widehat{D}_{q,\xi}^{-1}\psi)$, and applying q-derivative operator to the both sides of (44) with respect to ψ, we acquire

$$\sum_{\omega=1}^{\infty} {}_{[m]}L_{\omega,q}(\xi,\eta)\widehat{D}_{q,\psi}\frac{\psi^{\omega}}{[\omega]_q!} = \sum_{\omega=0}^{\infty}\left(\eta\left(-\frac{\partial_q}{\partial_q\widehat{D}_{q,\xi}^{-1}}\right)^{m-1}\mathrm{T}_{(\eta;m)}\mathrm{T}_{q,\xi} - \widehat{D}_{q,\xi}^{-1}\right) {}_{[m]}L_{\omega,q}(\xi,\eta)\frac{\psi^{\omega}}{[\omega]_q!}, \tag{54}$$

which yields

$${}_{[m]}L_{\omega+1,q}(\xi,\eta) = \left(\eta\left(-\frac{\partial_q}{\partial_q\widehat{D}_{q,\xi}^{-1}}\right)^{m-1}\mathrm{T}_{(\eta;m)}\mathrm{T}_{q,\xi} - \widehat{D}_{q,\xi}^{-1}\right) {}_{[m]}L_{\omega,q}(\xi,\eta), \tag{55}$$

which is the second asserted result (49).

If we apply the operator $\widehat{D}_{q,\xi}\xi\widehat{D}_{q,\xi}$ to the both sides of (44) and utilizing (33), we then obtain

$$\widehat{D}_{q,\xi}\xi C_{0,q}(\xi\psi)e_q(\eta\psi^m) = -\frac{\partial_q}{\partial_q D_{q,\xi}^{-1}}e_q(-\widehat{D}_{q,\xi}^{-1}t)e_q(\eta\psi) = \sum_{\omega=0}^{\infty}\widehat{D}_{q,\xi}\xi\widehat{D}_{q,\xi\,[m]}L_{\omega,q}(\xi,\eta)\frac{\psi^{\omega}}{[\omega]_q!}. \tag{56}$$

Using equation (33), we get

$$\psi C_{0,q}(\xi\psi)e_q(\eta\psi^m) = \sum_{\omega=0}^{\infty} -\frac{\partial_q}{\partial_q D_{q,\xi}^{-1}} {}_{[m]}L_{\omega,q}(\xi,\eta)\frac{\psi^\omega}{[\omega]_q!} = -\sum_{\omega=0}^{\infty} \widehat{D}_{q,\xi}\xi\widehat{D}_{q,\xi}{}_{[m]}L_{\omega,q}(\xi,\eta)\frac{\psi^\omega}{[\omega]_q!}. \tag{57}$$

By (57), we observe that

$$-\widehat{D}_{q,\xi}\xi\widehat{D}_{q,\xi}{}_{[m]}L_{\omega,q}(\xi,\eta) = -\frac{\partial_q}{\partial_q D_{q,\xi}^{-1}}{}_{[m]}L_{\omega,q}(\xi,\eta) = [\omega]_q{}_{[m]}L_{\omega-1,q}(\xi,\eta), \tag{58}$$

which means the third asserted operator formula (50).

Remark 1. *In view of* (33) *and Theorem 2.1, the multiplicative operators can also be represented as*

$$\widehat{M}_{G2VqLP} = \eta \mathrm{T}_{(\eta;m)}\left(-\widehat{D}_{q,\xi}\xi\widehat{D}_{q,\xi}\right)^{m-1} - \widehat{D}_{q,\xi}^{-1}\mathrm{T}_{q^m,\eta}, \tag{59}$$

or, equivalently

$$\widehat{M}_{G2VqLP} = \eta \mathrm{T}_{(\eta;m)}\left(-\widehat{D}_{q,\xi}\xi\widehat{D}_{q,\xi}\right)^{m-1}\mathrm{T}_{q,\xi} - \widehat{D}_{q,\xi}^{-1}. \tag{60}$$

Here, we provide the following theorem.

Theorem 2. *The following q-partial differential equations for* ${}_{[m]}L_{\omega,q}(\xi,\eta)$ *hold true:*

$$\left(-\widehat{D}_{q,\xi}\xi\widehat{D}_{q,\xi}\right)^m {}_{[m]}L_{\omega,q}(\xi,\eta) = \widehat{D}_{q,\eta}{}_{[m]}L_{\omega,q}(\xi,\eta), \tag{61}$$

or, equivalently

$$\left(-\left(q\widehat{D}_{q,\xi} + \xi\widehat{D}_{q,\xi}^2\right)\right)^m {}_{[m]}L_{\omega,q}(\xi,\eta) = \widehat{D}_{q,\eta}{}_{[m]}L_{\omega,q}(\xi,\eta), \tag{62}$$

$$\left(-\frac{\partial_q}{\partial_q D_{q,\xi}^{-1}}\right)^m {}_{[m]}L_{\omega,q}(\xi,\eta) = \widehat{D}_{q,\eta}{}_{[m]}L_{\omega,q}(\xi,\eta). \tag{63}$$

Proof. From equation (33) and (44), we have

$$\left(-\widehat{D}_{q,\xi}\xi\widehat{D}_{q,\xi}\right)C_{0,q}(\xi\psi)e_q(\eta\psi^m) = \psi^m C_{0,q}(\xi\psi)e_q(\eta\psi^m), \tag{64}$$

and

$$\widehat{D}_{q,\eta}C_{0,q}(\xi\psi)e_q(\eta\psi^m) = \psi^m C_{0,q}(\xi\psi)e_q(\eta\psi^m). \tag{65}$$

We readily get the asserted equation (61) utilizing (44) and some series manipulation methods, and also we obtain the claimed equation (62) just by using (61) and (34), equation (44) gives the assertion (62). We promptly acquire the argued equation (63) utilizing (44) and some series manipulation methods.

We now provide the q-integro-differential equations for $G2VqLP$ ${}_{[m]}L_{\omega,q}(\xi,\eta)$.

Theorem 3. *We have*

$$q\int_0^\xi \mathrm{T}_{q^m,\eta}\widehat{D}_{q,u[m]}L_{\omega,q}(u,\eta)d_qu+\int_0^\xi u\mathrm{T}_{q^m,\eta}\widehat{D}^2_{q,u[m]}L_{\omega,q}(u,\eta)d_qu$$
$$=\Big([\omega]_q-\eta(-\widehat{D}_{q,\xi}\xi\widehat{D}_{q,\xi})^m\mathrm{T}_{(\eta;m)}\Big)\,_{[m]}L_{\omega,q}(\xi,\eta), \quad (66)$$

and

$$\int_0^\xi (\widehat{D}_{q,u}u\widehat{D}_{q,u})_{[m]}L_{\omega,q}(u,\eta)d_qu=\Big([\omega]_q-\eta\mathrm{T}_{(\eta;m)}(-\widehat{D}_{q,\xi}\xi\widehat{D}_{q,\xi})^m\Big)\,_{[m]}L_{\omega,q}(\xi,\eta). \quad (67)$$

Proof. Taking into account (38) and (50) with equations (59) and (60), we get the assertions (66) and (67), respectively.

Remark 2. *In the special case* $m=2$ *in* (44) *and* (45), *we get*

$$C_{0,q}(\xi\psi)e_q(\eta\psi^2)=\sum_{\omega=0}^{\infty}\,_{[2]}L_{\omega,q}(\xi,\eta)\frac{\psi^\omega}{[\omega]_q!}, \quad (68)$$

and

$$_{[2]}L_{\omega,q}(\xi,\eta)=[\omega]_q!\sum_{\theta=0}^{[\frac{\omega}{2}]}\frac{(-1)^\omega\xi^{\omega-2\theta}\eta^\theta}{([\omega-2\theta]_q!)^2[\eta]_q!}. \quad (69)$$

Thus, we acquire from (21), (25), *and* (44) *that*

$$e_q(-D^{-1}_{q,\xi}\psi)e_q(\eta\psi^2)\{1\}=\sum_{\omega=0}^{\infty}\,_{[2]}L_{\omega,q}(\xi,\eta)\frac{\psi^\omega}{[\omega]_q!}. \quad (70)$$

We derive from (7) *and* (46) *that*

$$_{[2]}L_{\omega,q}(\xi,\eta)=H^{(2)}_{\omega,q}(D^{-1}_{q,\xi},\eta)\{1\}. \quad (71)$$

We acquire the following corollary in the special case $m=2$ in Theorem 1.

Corollary 1. *The following operator formulas are valid:*

$$\widehat{M}_{G2VqLP}=\eta\mathrm{T}_{(\eta;2)}\left(-\frac{\partial_q}{\partial_q\widehat{D}^{-1}_{q,\xi}}\right)-\widehat{D}^{-1}_{q,\xi}\mathrm{T}_{q^2,\eta}, \quad (72)$$

or, equally

$$\widehat{M}_{G2VqLP}=\eta\mathrm{T}_{(\eta;2)}\left(-\frac{\partial_q}{\partial_q\widehat{D}^{-1}_{q,\xi}}\right)\mathrm{T}_{q,x}-\widehat{D}^{-1}_{q,\xi}. \quad (73)$$

We acquire the following corollary in the special case $m=2$ in Theorem 2.

Corollary 2. *We have*

$$\left(-\widehat{D}_{q,\xi}\xi\widehat{D}_{q,\xi}\right)^2 {}_{[2]}L_{\omega,q}(\xi,\eta) = \widehat{D}_{q,\eta\,[2]}L_{\omega,q}(\xi,\eta), \tag{74}$$

or, equivalently

$$\left(-\left(q\widehat{D}_{q,\xi} + \xi\widehat{D}^2_{q,\xi}\right)\right)^2 {}_{[2]}L_{\omega,q}(\xi,\eta) = \widehat{D}_{q,\eta\,[2]}L_{\omega,q}(\xi,\eta), \tag{75}$$

and

$$\left(-\frac{\partial_q}{\partial_q D^{-1}_{q,\xi}}\right)^2 {}_{[2]}L_{\omega,q}(\xi,\eta) = \widehat{D}_{q,\eta\,[2]}L_{\omega,q}(\xi,\eta). \tag{76}$$

We acquire the following corollary in the special case $m = 2$ in Theorem 3.

Corollary 3. *We have*

$$q\int_0^x T_{q^2,y}\widehat{D}_{q,u\,[2]}L_{\omega,q}(u,\eta)d_qu + \int_0^\xi u\mathrm{T}_{q^2,\eta}\widehat{D}^2_{q,u\,[2]}L_{\omega,q}(u,\eta)d_qu$$
$$= \left([\omega]_q - \eta(-\widehat{D}_{q,\xi}\xi\widehat{D}_{q,\xi})^2 T_{(\eta;2)}\right) {}_{[2]}L_{\omega,q}(\xi,\eta), \tag{77}$$

and

$$\int_0^\xi (\widehat{D}_{q,u}u\widehat{D}_{q,u})_{[2]}L_{\omega,q}(u,\eta)d_qu = \left([\omega]_q - \eta\mathrm{T}_{(\eta;2)}(-\widehat{D}_{q,\xi}\xi\widehat{D}_{q,\xi})^2\right) {}_{[2]}L_{\omega,q}(\xi,\eta). \tag{78}$$

3. Distribution of Zeros and Graphical Representation

This section demonstrates how numerical analysis can be employed to confirm theoretical predictions and uncover new and interesting patterns in the zeros of certain members of a recently introduced hybrid polynomial family. Specifically, this paper utilizes computational methods to explore the "scattering" of the zeros of the generalized two-variable q-Laguerre polynomials, denoted as ${}_{[m]}L_{\omega,q}(\xi,\eta)$, within the complex plane a fascinating phenomenon to observe.
From (44) and (45), we remember that

$$C_{0,q}(\xi\psi)e_q(\eta\psi^m) = \sum_{\omega=0}^{\infty} {}_{[m]}L_{\omega,q}(\xi,\eta)\frac{\psi^\omega}{[\omega]_q!}, \tag{79}$$

and

$$ {}_{[m]}L_{\omega,q}(\xi,\eta) = [\omega]_q! \sum_{\theta=0}^{[\frac{\omega}{m}]} \frac{(-1)^\omega \xi^{\omega-m\theta}\eta^\theta}{([\omega-m\theta]_q!)^2[\theta]_q!}. \tag{80}$$

The first few generalized bivariate q-Laguerre polynomials are listed as follows:

$$
\begin{aligned}
{}_{[2]}L_{1,q}(\xi,\eta) &= -\xi,\\
{}_{[2]}L_{2,q}(\xi,\eta) &= \xi^2+\eta[2]_q!,\\
{}_{[2]}L_{3,q}(\xi,\eta) &= -\xi^3-\eta\xi[3]_q!,\\
{}_{[2]}L_{4,q}(\xi,\eta) &= \xi^4+\frac{\eta^2[4]_q!}{[2]_q!}+\frac{\eta\xi^2[4]_q!}{[2]_q!},\\
{}_{[2]}L_{5,q}(\xi,\eta) &= -\xi^5-\frac{\eta^2\xi[5]_q!}{[2]_q!}-\frac{\eta\xi^3[5]_q!}{[3]_q!},\\
{}_{[2]}L_{6,q}(\xi,\eta) &= \xi^6+\frac{\eta^2\xi^2[6]_q!}{[2]_q!^2}+\frac{\eta^3[6]_q!}{[3]_q!}+\frac{\eta\xi^4[6]_q!}{[4]_q!},\\
{}_{[2]}L_{7,q}(\xi,\eta) &= -\xi^7-\frac{\eta^3\xi[7]_q!}{[3]_q!}-\frac{\eta^2\xi^3[7]_q!}{[2]_q![3]_q!}-\frac{\eta\xi^5[7]_q!}{[5]_q!},\\
{}_{[2]}L_{8,q}(\xi,\eta) &= \xi^8+\frac{\eta^3\xi^2[8]_q!}{[2]_q![3]_q!}+\frac{\eta^4[8]_q!}{[4]_q!}+\frac{\eta^2\xi^4[8]_q!}{[2]_q![4]_q!}+\frac{\eta]\xi^6[8]_q!}{[6]_q!},\\
{}_{[2]}L_{9,q}(\xi,\eta) &= -\xi^9-\frac{\eta^3\xi^3[9]_q!}{[3]_q!^2}-\frac{\eta^4\xi[9]_q!}{[4]_q!}-\frac{\eta^2\xi^5[9]_q!}{[2]_q![5]_q!}-\frac{\eta\xi^7[9]_q!}{[7]_q!},\\
{}_{[2]}L_{10,q}(\xi,\eta) &= -\xi^{10}+\frac{\eta^4\xi^2[10]_q!}{[2]_q![4]_q!}+\frac{\eta^3\xi^4[10]_q!}{[3]_q![4]_q!}+\frac{\eta^5[10]_q!}{[5]_q!}+\frac{\eta^2\xi^6[10]_q!}{[2]_q![6]_q!}+\frac{\eta\xi^8[10]_q!}{[8]_q!}.
\end{aligned}
$$

We research the solutions of the equality ${}_{[m]}L_{\omega,q}(\xi,\eta)=0$, utilizing a computer programme. So, we draw these solutions for $m=2,\omega=50$ and $\eta=5$ by the following Figure 1:

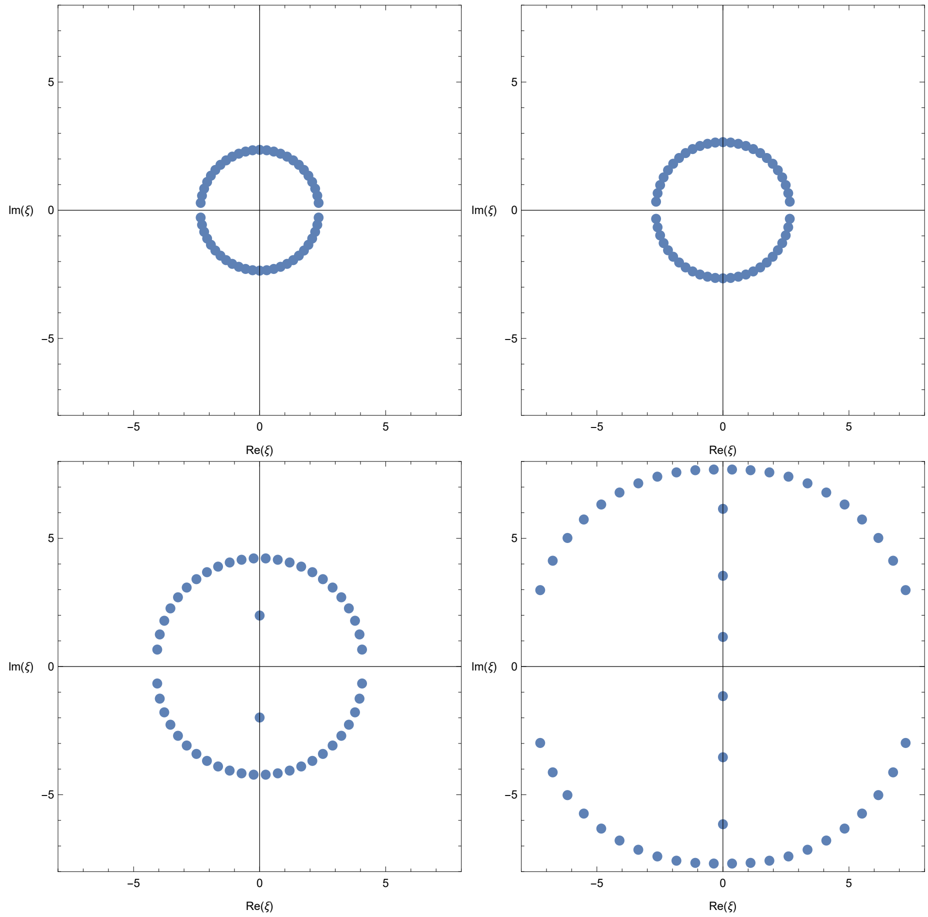

Figure 1: Zeros of ${}_{[m]}L_{\omega,q}(\xi,\eta)=0$

Especially, we take $q=\frac{1}{10}$ (top-left), $q=\frac{3}{10}$ (top-right), $q=\frac{7}{10}$ (bottom-left) and $q=\frac{9}{10}$ (bottom-right) in Figure 1.

We provide, forming a 3D structure, the stacks of zeros for the equality ${}_{[m]}L_{\omega,q}(\xi,\eta)=0$ for $m=2,\eta=5$, and $1\leq\omega\leq 50$ by the following Figure 2:

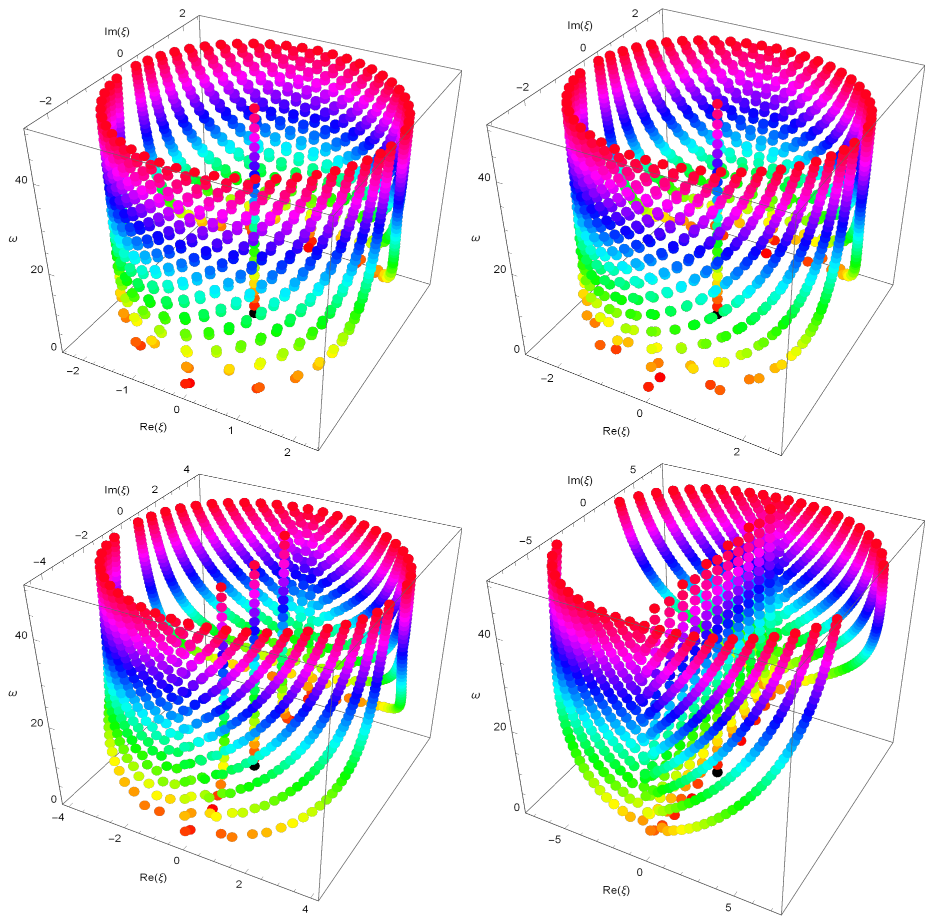

Figure 2: Zeros of ${}_{[m]}L_{\omega,q}(\xi,\eta)=0$

Here, we take $q=\frac{1}{10}$ (top-left), $q=\frac{3}{10}$ (top-right), $q=\frac{7}{10}$ (bottom-left) and $q=\frac{9}{10}$ (bottom-right) in Figure 2.

We give, forming a 2D structure, the stacks of zeros for the equality ${}_{[m]}L_{\omega,q}(\xi,\eta)=0$ for $q=\frac{9}{10}, \eta=5$, and $1\leq\omega\leq 50$ by the following Figure 3:

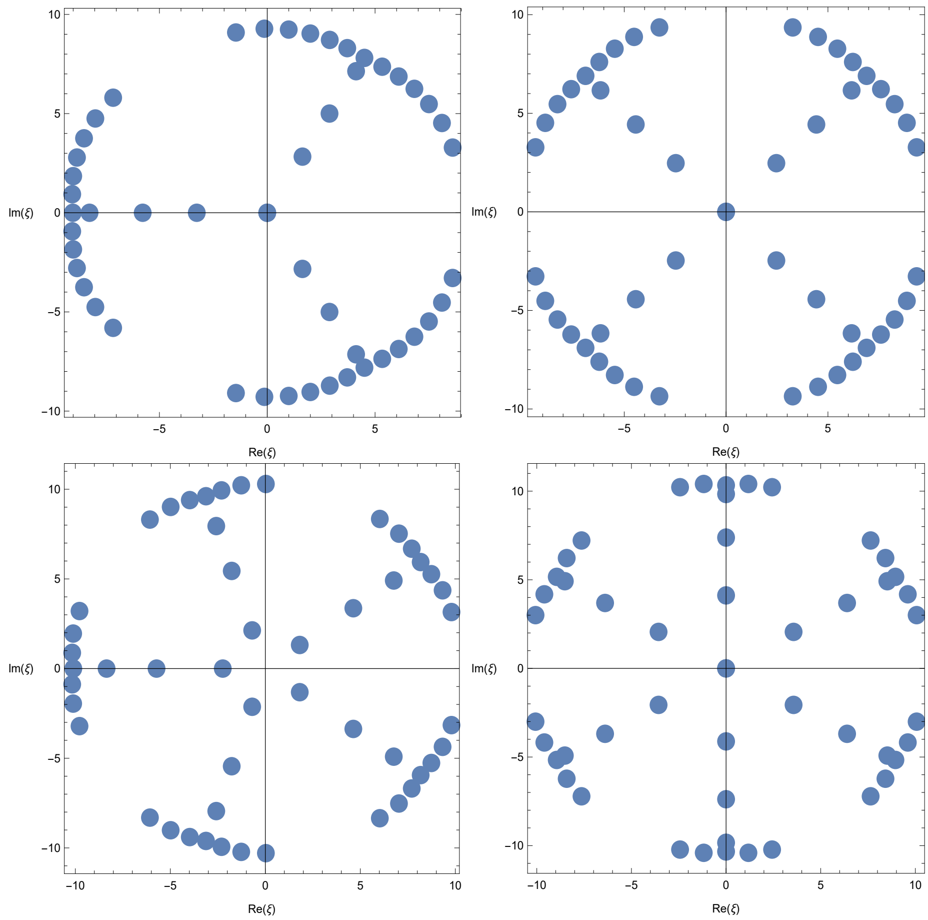

Figure 3: Zeros of ${}_{[m]}L_{\omega,q}(\xi,\eta)=0$

Here, we take $m=3$ (top-left), $m=4$ (top-right), $m=5$ (bottom-left) and $m=6$ (bottom-right) in Figure 3.

We show, forming a 3D structure, the stacks of zeros for the equality ${}_{[m]}L_{\omega,q}(\xi,\eta) = 0$ for $q = \frac{9}{10}, \eta = 5$, and $1 \leq \omega \leq 50$ by the following Figure 4:

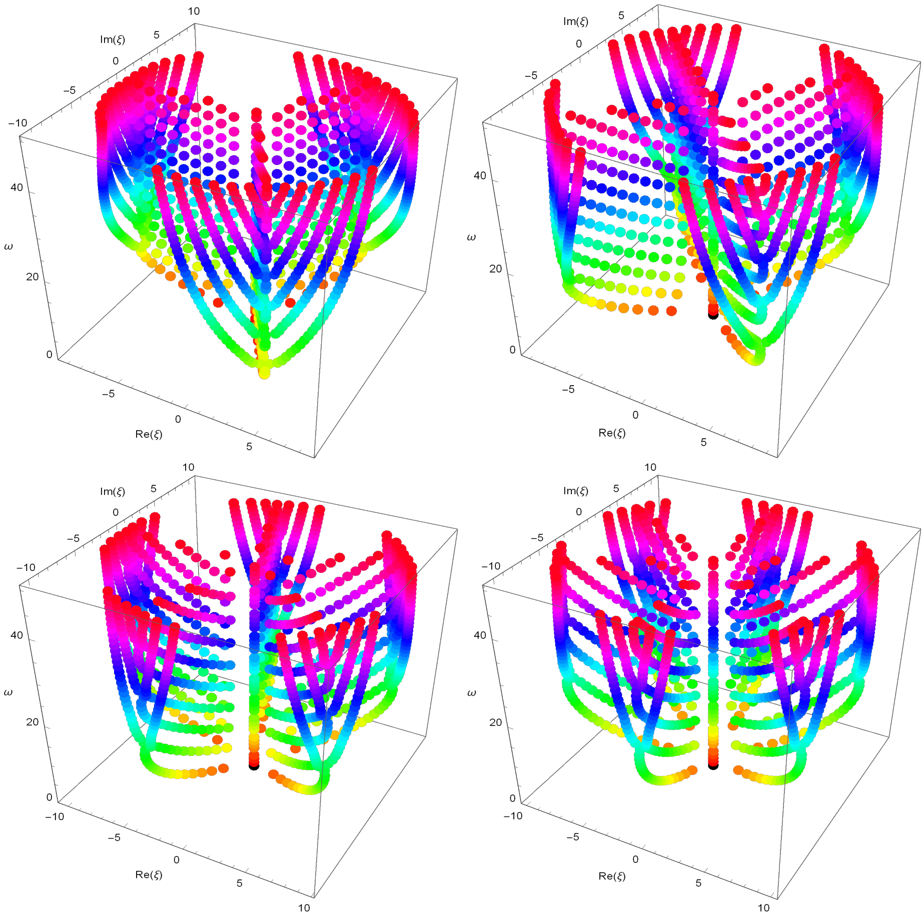

Figure 4: Zeros of ${}_{[m]}L_{\omega,q}(\xi,\eta) = 0$

Here, we take $m = 3$ (top-left), $m = 4$ (top-right), $m = 5$ (bottom-left) and $m = 6$ (bottom-right) in Figure 4.

We present the stacks of real zeros for the equality ${}_{[m]}L_{\omega,q}(\xi,\eta)=0$ for $q=\frac{9}{10},\eta=5$, and $1\leq\omega\leq 50$ by the following Figure 5:

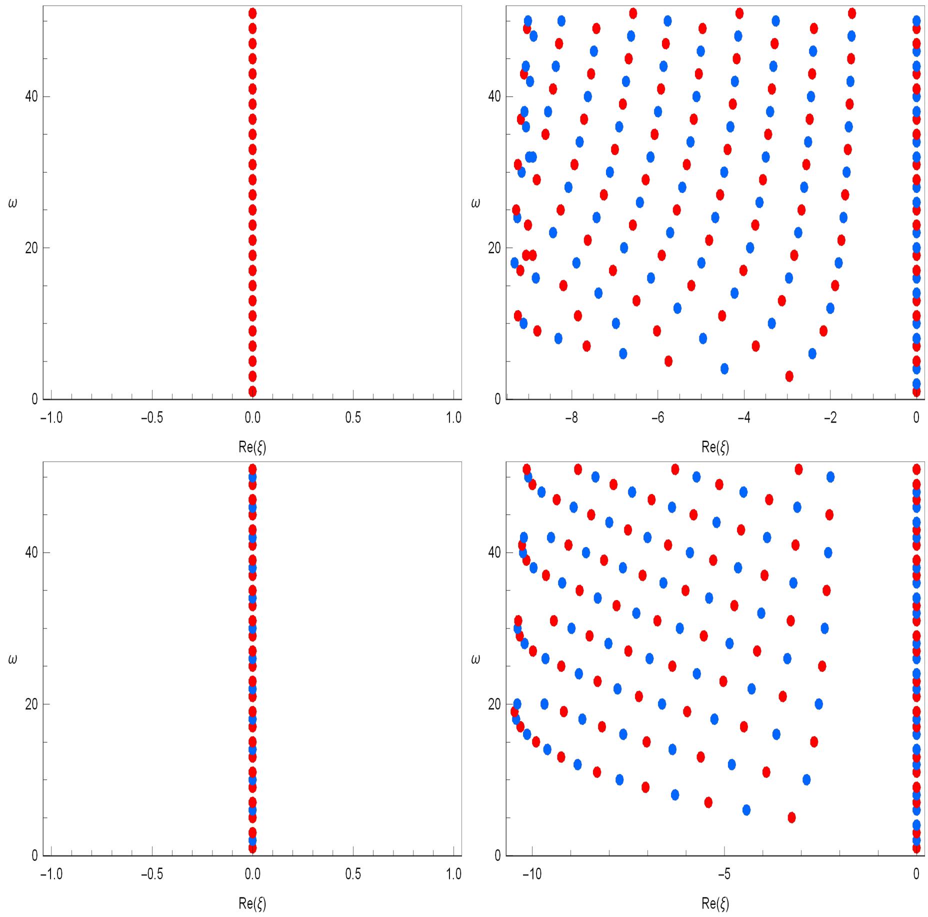

Figure 5: Real zros of ${}_{[m]}L_{\omega,q}(\xi,\eta)=0$

Here, we take $m=2$ (top-left), $m=3$ (top-right), $m=4$ (bottom-left) and $m=5$ (bottom-right) in Figure 5.

We provide the plots of the real zeros for the equality ${}_{[m]}L_{\omega,q}(\xi,\eta)=0$ for $m=\eta=5$, and $1\leq\omega\leq 50$ by the following Figure 6:

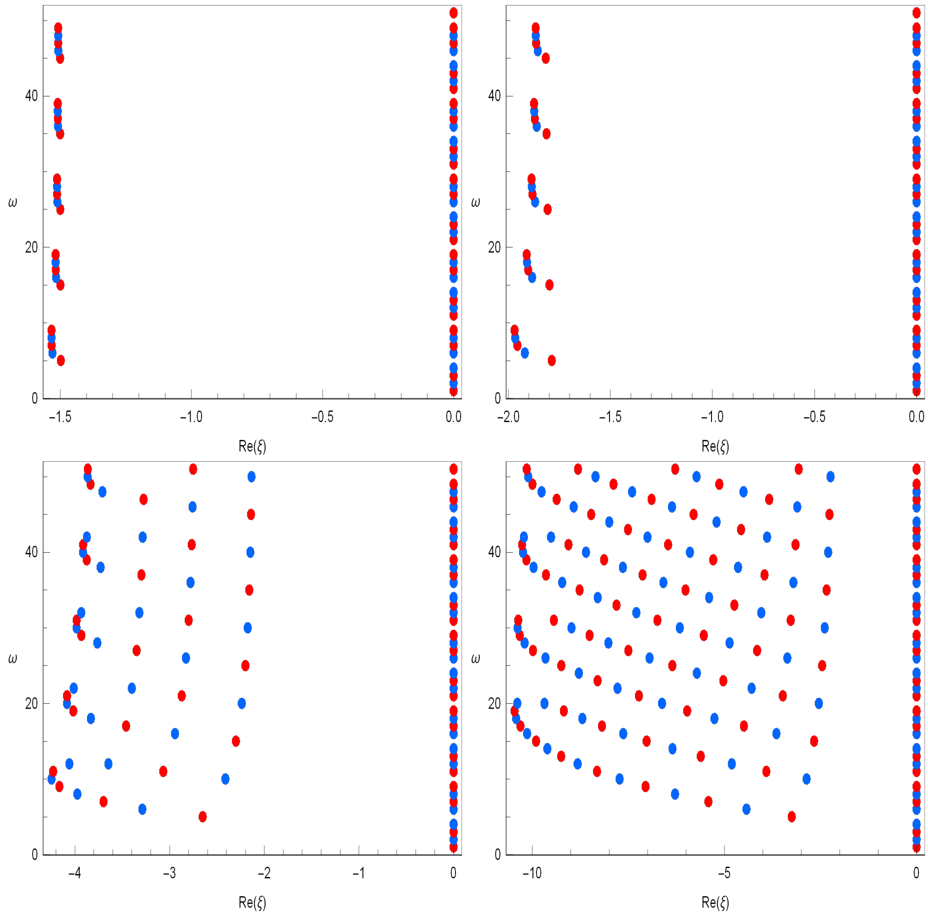

Figure 6: Real zros of ${}_{[m]}L_{\omega,q}(\xi,\eta)=0$

Here, we take $q=\frac{1}{10}$ (top-left), $q=\frac{3}{10}$ (top-right), $q=\frac{7}{10}$ (bottom-left) and $q=\frac{9}{10}$ (bottom-right) in Figure 6.

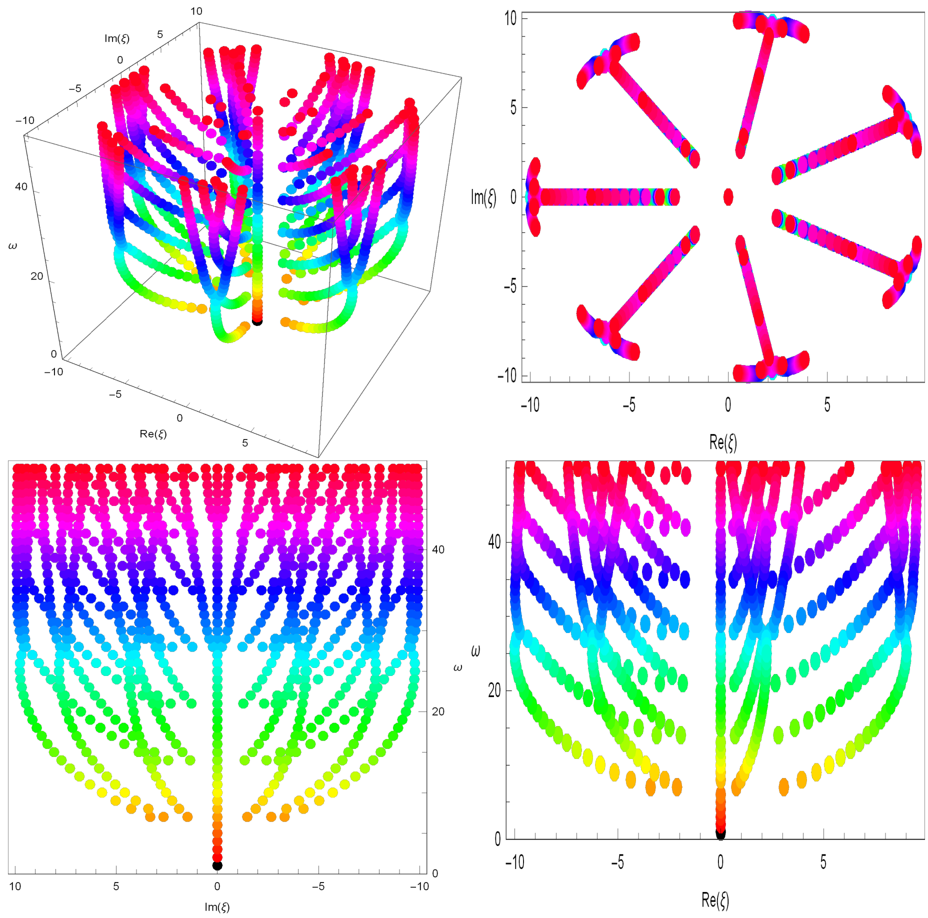

Figure 7: Real zeros of ${}_{[m]}L_{\omega,q}(\xi,\eta)=0$

In Figure 7, we take $m = 7, 1 \leq \omega \leq 50, \eta = 3$ and $q = \frac{9}{10}$. We plot, forming a 3D structure, the stacks of zeros for the equality ${}_{[m]}L_{\omega,q}(\xi,\eta) = 0$ in Figure 7 (top-left). We plot, forming a 3D structure, y and x axes but no z axis in the three dimensions in Figure 7 (top-right). We plot, forming a 3D structure, z and y axes but no x axis in Figure 7 (bottom-left). We plot, forming a 3D structure, z and x axes but no y axis in Figure 7 ((bottom-right).

Now, we computed an approximate solution fulfilling the equality ${}_{[m]}L_{\omega,q}(\xi,\eta)=0$ for $m=2, \eta=5$, and $q=\frac{9}{10}$ provided by the following Table 1.

Table 1. Approximate solutions of ${}_{[m]}L_{\omega,q}(\xi,\eta)=0$

degree ω	ξ
1	0
2	$-3.0822i,\quad 3.0822i$
3	$0,\quad -5.0740i,\quad 5.0740i$
4	$-2.3867i,\quad 2.3867i,\quad -6.3955i,\quad 6.3955i$
5	$0,\quad -4.2793i,\quad 4.2793i,$ $-7.2182i,\quad 7.2182i$
6	$-2.0470i,\quad 2.0470i,\quad -5.8493i,$ $5.8493i,\quad -7.5852i,\quad 7.5852i$
7	$-0.5543-7.3600i,\quad -0.5543+7.3600i,\quad 0,$ $-3.8079i,\quad 3.8079i,\quad 0.5543-7.3600i,\quad 0.5543+7.3600i$
8	$-1.0763-7.7067i,\quad -1.0763+7.7067i,\quad -1.8399i,$ $1.8399i,\quad -5.3580i,\quad 5.3580i,$ $1.0763-7.7067i,\quad 1.0763+7.7067i$
9	$-1.4703-7.7823i,\quad -1.4703+7.7823i,\quad 0,$ $-3.4948i,\quad 3.4948i,\quad -6.7398i,$ $6.7398i,\quad 1.4703-7.7823i,\quad 1.4703+7.7823i$
10	$-1.9450-7.8041i,\quad -1.9450+7.8041i,\quad -1.6989i,$ $1.6989i,\quad -5.0096i,\quad 5.0096i,\quad -7.5676i,$ $7.5676i,\quad 1.9450-7.8041i,\quad 1.9450+7.8041i$

4. Conclusion

In conclusion, this study has systematically investigated the unique characteristics and applications of generalized bivariate q-Laguerre polynomials. It has derived some of their properties, such as explicit formulas, operational identities, q-quasi-monomiality characteristics, and q-integro-differential equations for these polynomials. Moreover, we have provided the stacks of the zeros of the new polynomials, forming 2D and 3D structures, and provided a table including approximate zeros of the generalized bivariate q-Laguerre polynomials. Motivated by the potential applications of q-special functions in various scientific and mathematical fields, we have utilized the q-analog of the monomiality principle to describe the mentioned polynomials. The distribution of non-coherent or coherent radiation areas in quantum optics, electromagnetic radiation problems, and wave propagation phenomena are some practical applications that inspired this work. Furthermore, we have presented the graphical representations and numerical computations of the zeros of certain members of the m^{th}-order q-Laguerre polynomials family. The insights gained from these investigations provide a foundation for further exploration of q-special functions and their applications in more complex multidimensional systems. We anticipate that the results and methodologies discussed in this paper will stimulate future research in this promising area of mathematical analysis and its applications.

Availability of data and materials

Not applicable.

Competing interests

The authors declare no competing interests.

Acknowledgements

The authors acknowledge the financial support from Al-Zaytoonah University of Jordan, Amman 11733, Jordan.

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