EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2026, Vol. 19, Issue 1, Article Number 6675
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

On the Integration of the Dullin–Gottwald–Holm Equation with a Self-Consistent Source in the Class of Rapidly Decreasing Functions

Bazar Babajanov[1], Abbosbek Iskandarov[2,*]

[1] *Department of Applied Mathematics and Mathematical Physics, Urgench State University, Urgench, Uzbekistan*
[2] *Khorezm Mamun Academy, Khiva, Uzbekistan*

Abstract. In this paper, we investigate the Cauchy problem for the Dullin–Gottwald–Holm equation with a self-consistent source in the class of rapidly decreasing functions and present an algorithm for constructing a solution via the IST method. Physically, sources arise in solitary waves with variable speed and lead to a variety of dynamics in physical models. Such systems are commonly used to describe interactions between different solitary waves. We also present an efficient method to obtain the time evolution of scattering data. The advantage of this method lies in its reliability and applicability to other soliton equations with sources. The resulting equalities fully determine the scattering data at any time t, enabling the application of the IST method to solve the Cauchy problem for the Dullin–Gottwald–Holm equation with a self-consistent source. An illustrative example of a one-soliton solution is provided.
2020 Mathematics Subject Classifications: 35Q51, 37K15, 35P25
Key Words and Phrases: Dullin–Gottwald–Holm equation, self-consistent source, inverse scattering transform, soliton interaction, cauchy problem

1. Introduction

The Dullin-Gottwald-Holm (DGH) equation

$$u_t + u_{xxt} + 2\omega\, u_x + 3uu_x + \gamma u_{xxx} = \alpha^2(u_{xxt} + 2u_x u_{xx} + uu_{xxx}) \quad t > 0, \quad x \in \mathbb{R}, \qquad (1)$$

was derived in [1] describing the unidirectional propagation of surface waves in a shallow water regime. Where the constants α^2 and $\frac{\gamma}{2\omega}$ are squares of length scales and $2\omega > 0$ is the linear wave speed for undisturbed water at rest, at spatial infinity.It is important to note that the equation (1) is connected with two separately integrable equations: for $\alpha = 0$ and $\gamma \neq 0$ this equation becomes the Korteweg-de Vries equation

$$u_t + 2\omega\, u_x + 3uu_x = -\gamma\, u_{xxx},$$

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v19i1.6675

Email addresses: a.murod@mail.ru (B. Babajanov), iskandarovabbosbek@gmail.com (A. Iskandarov)

when $\alpha = 1$ and $\gamma = 0$, it reduces to the Camassa–Holm (CH) equation

$$u_t + 2\omega u_x + 3uu_x - u_{xxt} = 2u_x u_{xx} + uu_{xxx}.$$

Since [1] the DGH system has been the subject of various studies and several methods for solving it have been developed[2–15]. For instance, the Cauchy problem for the Eq.(1) has been studied in [2, 3]. It has been shown that this equation is locally well-posed for initial data $u_0 \in H^S(R)$, $s > \frac{3}{2}$. The scattering problem for equation (1) is investigated in [3] through the associated iso-spectral problem. The inverse scattering problem for the DGH equation is discussed in [4–6] where the reduction, by the Liouville transformation, of the iso-spectral problem to the the classical Sturm-Liouville problem is used. In [7, 8], the orbital stability of one single solitary waves and the sum of N peakons for Eq. [1] have been proved. In [9–15], some peakons, kinks, compactons, solitons, solitary wave solutions and exact solutions are obtained.

In the present paper, we study the Dullin-Gottwald-Holm equation with a source

$$\begin{cases} u_t - \alpha^2 u_{xxt} + 2\omega\, u_x + 3uu_x - \alpha^2 \left(2u_x u_{xx} + uu_{xxx}\right) + \gamma u_{xxx} = \sum\limits_{k=1}^{N} \left(m'_x g_k^2 + 2(m+\Omega)(g_k^2)'_x\right), \\ g''_{kxx} = \left(\dfrac{1}{4\alpha^2} + \eta_k (m+\Omega)\right) g_k, \quad k = 1, 2, \ldots, N, \quad x \in \mathbb{R},\ t > 0, \end{cases} \tag{2}$$

under the initial condition

$$u(x,t)|_{t=0} = u_0(x),\ x \in \mathbb{R} \tag{3}$$

and the normalizing conditions

$$\int_{-\infty}^{\infty} (m+\Omega) g_k^2 dx = A_k(t), \qquad k = 1, 2, ..., N, \tag{4}$$

where $A_k(t)$ are given arbitrary positive continuous functions for all $k \in \{1, 2, ..., N\}$, $m = u - \alpha^2 u_{xx}$, $\Omega = \omega + \frac{\gamma}{2\alpha^2} = const > 0$, $m + \Omega > 0$ and $g_k = g_k(x,t) = g(x, \eta_k, t)$ is an eigenfunction of the equation

$$\psi''_{xx} = \left(\frac{1}{4\alpha^2} + \eta\,(m+\Omega)\right)\psi \tag{5}$$

corresponding to the eigenvalue η_k. Moreover, the function $u_0(x)$ satisfy the following condition:

$$\int_{-\infty}^{\infty} (1+|x|)\,(|u_0(x)| + |u_{0xx}(x)|)dx < \infty. \tag{6}$$

Over the last few years, the interest has been growing in the soliton equations with a self-consistent source [16–40]. Physically, sources arise in solitary waves with a variable speed and lead to a variety of dynamics of physical models. With regard to their applications, these kinds of systems are usually used to describe interactions between different solitary waves.

The goal of this work is to obtain formulations for the solutions of the new system constructed within the framework of the inverse scattering theory for equation (5).

2. Main facts about the scattering problem

In this section, we give the basic information about the scattering theory for the problem (5)(see [4]). For convenience, we temporarily omit the variable t.

Consider the equation

$$\psi''_{xx} = \left(\frac{1}{4\alpha^2} + \eta\,(m+\Omega)\right)\psi, \tag{7}$$

where $m = u - \alpha^2 u_{xx}$, $\eta\,(k) = -\frac{1}{\Omega}\left(k^2 + \frac{1}{4\alpha^2}\right)$, $\Omega = \omega + \frac{\gamma}{2\alpha^2} = const > 0, m+\Omega > 0$, with the function $u\,(x)$ satisfying the condition

$$\int\limits_{-\infty}^{\infty} (1+|x|)\,(|u(x)| + |u_{xx}(x)|)dx < \infty. \tag{8}$$

Under assumption (8), equation (7) possesses the Jost solutions with the following asymptotics:

$$\begin{aligned} \psi_1(x,k) &= e^{-ikx} + o(1), \qquad x \to +\infty, \\ \psi_2(x,k) &= e^{ikx} + o(1), \qquad x \to +\infty, \end{aligned} \tag{9}$$

$$\begin{aligned} \varphi_1(x,k) &= e^{-ikx} + o(1), \qquad x \to -\infty, \\ \varphi_2(x,k) &= e^{ikx} + o(1), \qquad x \to -\infty. \end{aligned} \tag{10}$$

When k are real, the pairs $\{\varphi_1, \varphi_2\}$ and $\{\psi_1, \psi_2\}$ are pairs of linearly independent solutions for equation (7). Therefore, the following relation holds:

$$\varphi_1(x,k) = a(k)\psi_1(x,k) + b(k)\psi_2(x,k). \tag{11}$$

One can readily see that

$$a(k) = -\frac{1}{2ik} W\left\{\psi_2(x,k), \varphi_1(x,k)\right\}.$$

The function $a\,(k)$ admits an analytic continuation into the upper half-plane and has a finite number of zeros $k = ik_n,\;\; k_n > 0$ ([4]). Meanwhile,

$$\eta_n = -\frac{1}{\Omega}\left(-{k_n}^2 + \frac{1}{4\alpha^2}\right), \quad n = 1, 2, ..., N$$

is an eigenvalue of Equation (7) so that

$$\varphi_1(x, ik_n) = b_n \psi_2(x, ik_n), \qquad n = 1, 2, ..., N. \tag{12}$$

Moreover, the following expansion on the half-plane $\operatorname{Im} k > 0$ takes place for the coefficient $a(k)$:

$$\ln a(k) = -i\sigma k + \sum_{n=1}^{N} \ln \frac{k - ik_n}{k + ik_n} - \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\ln\left(1 - |R(k')|^2\right)}{k' - k} dk', \tag{13}$$

where $\sigma = \int\limits_{-\infty}^{\infty} \left(\sqrt{1 + \frac{m(x)}{\Omega}} - 1 \right) dx, \quad R(k) = \frac{b(k)}{a(k)}$.

The set $\left\{ R(k) = \frac{b(k)}{a(k)},\ k \in \mathbb{R},\ k_n,\ b_n,\ n = 1, 2, ..., N \right\}$ is called the scattering data for Equation (7). The inverse scattering problem consists in recovering the function $m(x)$, and consequently $u(x)$ of Equation (7) by the scattering data.

The inverse problem of recovering the function $u(x)$ by the scattering data is solved by means of the following equations [41]:

$$\overline{\psi_1}(x, k) = \left(\frac{\xi(x)}{\xi'(x)} \right)^{\frac{1}{2}} + \int_{-\infty}^{\infty} R(k') \overline{\psi_2}(x, k') \left[\xi(x)\right]^{2ik'} \frac{dk'}{k' - k} + \sum_{n=1}^{N} \frac{b_n \left[\xi(x)\right]^{-2k_n} \overline{\psi_1}(x, -ik_n)}{\dot{a}(ik_n)(ik_n - k)}, \tag{14}$$

p = 1, 2, . . . , N,

$$\overline{\psi_1}(x, -ik_p) = \left(\frac{\xi(x)}{\xi'(x)} \right)^{\frac{1}{2}} + \int_{-\infty}^{\infty} R(k') \overline{\psi_2}(x, k') \left[\xi(x)\right]^{2ik'} \frac{dk'}{k' + ik_p} + i \sum_{n=1}^{N} \frac{b_n \left[\xi(x)\right]^{-2k_n} \overline{\psi_1}(x, -ik_n)}{\dot{a}(ik_n)(k_p + k_n)}, \tag{15}$$

$$\begin{aligned} e^{-\frac{x}{2}} \left[\xi(x)\right]^{\frac{1}{2}} = \left(\frac{\xi(x)}{\xi'(x)} \right)^{\frac{1}{2}} + \int_{-\infty}^{\infty} R(k') \overline{\psi_2}(x, k') \left[\xi(x)\right]^{2ik'} \frac{dk'}{k' + i/2} + \\ + i \sum_{n=1}^{N} \frac{b_n \left[\xi(x)\right]^{-2k_n} \overline{\psi_1}(x, -ik_n)}{\dot{a}(ik_n)(k_n + \frac{1}{2})}. \end{aligned} \tag{16}$$

Here

$$\xi(x) = \exp\left\{ x + \int_{\infty}^{x} \left(\sqrt{\frac{m(y) + \Omega}{\Omega}} - 1 \right) dy \right\},$$

$$\overline{\psi_1}(x, k) \equiv \psi_1(x, k) \left[\xi(x)\right]^{ik},$$

$$\overline{\varphi_1}(x, k) \equiv \varphi_1(x, k) \exp\left\{ ik \left(x + \int_{-\infty}^{x} \left(\sqrt{\frac{m(y) + \Omega}{\Omega}} - 1 \right) dy \right) \right\},$$

$$\frac{\overline{\varphi_1}(x,k)}{e^{i\alpha k}a(k)} = \overline{\psi_1}(x,k) + R(k)\,\overline{\psi_2}(x,k)\,[\xi(x)]^{2ik}\,.$$

The function $m\,(x)$, and consequently $u\,(x)$ are defined as follows:

$$m(x) = \Omega\left[\left(\frac{\xi'(x}{\xi(x}\right)^2 - 1\right], \tag{17}$$

$$u(x) - \alpha^2 u''(x) = m(x). \tag{18}$$

We note that the functions

$$h_n(x) = \frac{\frac{d}{dk}\,(\varphi_1 - b_n\psi_2)\,|_{k=ik_n}}{\dot{a}(ik_n)}, \quad n = 1,2,...,N, \tag{19}$$

are solutions to the equation (7), and the asymptotics

$$\begin{aligned} &h_n(x) \to -b_n e^{-k_n x} \text{ when } x \to -\infty, \\ &h_n(x) \to e^{k_n x} \quad \text{when } \ x \to \infty \end{aligned} \tag{20}$$

are true. According to (12), (9), (20), the equalities

$$W\left\{\varphi_{1n}, h_n\right\} \equiv \varphi_{1n}{h'}_n - {\varphi'}_{1n}h_n = 2k_n b_n, \quad n = 1,2,...,N \tag{21}$$

hold, where $\varphi_{1n} = \varphi_1(x, ik_n),\ \ \psi_{2n} = \psi_2(x, ik_n),\ \ \ n = 1,2,...,N.$

Lemma 1. *If functions f and g are solutions to equations*

$$\begin{aligned} {f''}_{xx} &= \left(\frac{1}{4\alpha^2} + \tau(m+\Omega)\right)f, \\ {g''}_{xx} &= \left(\frac{1}{4\alpha^2} + \nu(m+\Omega)\right)g, \end{aligned}$$

the following equality holds for them:

$$(m+\Omega)fg = \frac{1}{\tau - \nu}\frac{d}{dx}W\left\{g,f\right\}.$$

The lemma is proved by direct verification.

3. Evolution of the Scattering Data

In this section, we derive the time evolution of the scattering data, which allows us to present an algorithm for solving problem (2)–(6). Let us consider the equation

$$u_t - \alpha^2 u_{xxt} + 2\omega\, u_x + 3uu_x - \alpha^2\left(2u_x u_{xx} + uu_{xxx}\right) + \gamma u_{xxx} = G\left(x,t\right), \tag{22}$$

where the function $G = G(x,t)$ is sufficiently smooth and $G(x,t) = o(1)$ when $x \to \pm\infty$, $t \geq 0$.

Lemma 2. *If the function $u(x,t)$ is a solution to Eq. (22), then the scattering data $\{R(k,t),\ k\in\mathbb{R},\ k_n(t),\ b_n(t),\ n=1,2,...,N\}$ of the problem*

$$\psi''_{xx}=\left(\frac{1}{4\alpha^2}+\eta\left(m(x,t)+\Omega\right)\right)\psi \tag{23}$$

depend on t as follows:

$$\frac{dR}{dt}=-\frac{ik}{\alpha^2}R\left(\frac{4\Omega}{4\alpha^2k^2+1}-2\gamma\right)-\frac{4\alpha^2k^2+1}{8ik\Omega a^2(k)}\int\limits_{-\infty}^{\infty}G\varphi_1^2dx\ \ (\operatorname{Im}k=0), \tag{24}$$

$$\frac{db_n(t)}{dt}=\frac{k_nb_n}{\alpha^2}\left(\frac{4\Omega}{1-4\alpha^2k_n^2}-2\gamma\right)+\frac{1-4\alpha^2k_n^2}{8\Omega k_n}\int\limits_{-\infty}^{\infty}G\varphi_{1n}h_ndx, \tag{25}$$

$$\frac{dk_n(t)}{dt}=i\frac{4\alpha^2k_n^2-1}{8\Omega k_nb_n\dot{a}(ik_n)}\int\limits_{-\infty}^{\infty}G\varphi_{1n}^2dx\,,\quad n=1,2,...,N\,. \tag{26}$$

Proof. We seek the Lax pair for Equation (22) in the form:

$$\varphi''_{1xx}=\left(\frac{1}{4\alpha^2}+\eta(m+\Omega)\right)\varphi_1, \tag{27}$$

$$\varphi'_{1t}=\left(\frac{1}{2\alpha^2}\left(\frac{1}{\eta}+2\gamma\right)-u\right)\varphi'_{1x}+\frac{1}{2}u'_x\varphi_1+\beta\varphi_1+F(x,k,t), \tag{28}$$

where $m(x,t)=u(x,t)-u''_{xx}(x,t)$, $\beta=\frac{ik}{2\alpha^2}\left(\frac{1}{\eta}+2\gamma\right)$, and $\varphi_1=\varphi_1(x,k,t)$ are the Jost solutions of the equation (27) with the asymptotics (10). Using the compatibility condition

$$\varphi'''_{1xxt}=\varphi'''_{1txx}, \tag{29}$$

and taking into account the equalities (22), (27) and (28), we obtain

$$F''_{xx}-\left(\frac{1}{4\alpha^2}+\eta(m+\Omega)\right)F=\eta G\varphi_1. \tag{30}$$

We seek solution of Eq. (30) in the form

$$F(x,k,t)=A(x,t)\varphi_1(x,k,t)+B(x,t)\varphi_2(x,k,t). \tag{31}$$

Substituting (31) into (30) we get

$$\begin{cases} A'_x\varphi_1+B'_x\varphi_2=0\\ A'_x\varphi'_{1x}+B'_x\varphi'_{2x}=\eta G\varphi_1. \end{cases} \tag{32}$$

Passing to the limit $x \to -\infty$ in the equality (28) we derive that $F(x,t) \to 0$. It follows that the solution of the system of equations (32) has the form:

$$A(x,t) = -\frac{\eta}{2ik}\int\limits_{-\infty}^{x} G\varphi_1\varphi_2\,dx, \tag{33}$$

$$B(x,t) = \frac{\eta}{2ik}\int\limits_{-\infty}^{x} G\varphi_1^2\,dx. \tag{34}$$

Putting (33) and (31) into (28) we have

$$\begin{aligned}\varphi_{1t}' = &\left(\frac{1}{2\alpha^2}\left(\frac{1}{\eta}+2\gamma\right)-u\right)\varphi_{1x}' + \left(\frac{u_x'}{2}+\beta\right)\varphi_1\\ &-\frac{\eta}{2ik}\left(\int\limits_{-\infty}^{x} G\varphi_1\varphi_2\,dx\right)\varphi_1 + \frac{\eta}{2ik}\left(\int\limits_{-\infty}^{x} G\varphi_1^2\,dx\right)\varphi_2\end{aligned} \tag{35}$$

Passing to the limit $x \to \infty$ in the equality (35) by virtue of (8), (9), (10) and (11), one obtains

$$\begin{aligned}a_t(k,t) = &-\frac{ik}{2\alpha^2}\left(\frac{1}{\eta}+2\gamma\right)a(k,t) + \beta a(k,t)\cdot\frac{\eta}{2ik}\int\limits_{-\infty}^{\infty} G\varphi_1\varphi_2\,dx\\ &+\frac{\eta}{2ik}\int\limits_{-\infty}^{\infty} G\varphi_1^2\,dx\cdot\overline{b}(k,t),\end{aligned} \tag{36}$$

$$\begin{aligned}b_t(k,t) = &\frac{ik}{2\alpha^2}\left(\frac{1}{\eta}+2\gamma\right)b(k,t) + \beta b(k,t) - \frac{\eta}{2ik}\int\limits_{-\infty}^{\infty} G\varphi_1\varphi_2\,dx\cdot b(k,t)\\ &+\frac{\eta}{2ik}\int\limits_{-\infty}^{\infty} G\varphi_1^2\,dx\cdot\overline{a}(k,t).\end{aligned} \tag{37}$$

Now, using the definition of the function $R\,(k,t)$ and substituting $\eta\,(k) = -\frac{1}{\Omega}\left(k^2+\frac{1}{4\alpha^2}\right)$, we get (24).

Further, we calculate the dependence of $b_n(t)$ and $k_n(t)$, $n = 1,2,...,N$, on time t.

Similarly to the case of the continuous spectrum, we seek the Lax pair in case of the discrete spectrum in the following form:

$$\varphi''_{1nxx} = \left(\frac{1}{4\alpha^2}+\eta_n(m+\Omega)\right)\varphi_{1n}, \tag{38}$$

$$\varphi'_{1nt} = \left(\frac{1}{2\alpha^2}\left(\frac{1}{\eta_n}+2\gamma\right)-u\right)\varphi'_{1nx}+\frac{1}{2}u'_x\varphi_{1n}+\beta_n\varphi_{1n}+F_n. \tag{39}$$

Then, using the compatibility condition (29) to the (38) and (39), we obtain the equation

$$F''_{nxx}-\left(\frac{1}{4\alpha^2}+\eta_n(m(x,t)+\Omega)\right)F_n=\eta_n G\varphi_{1n}. \tag{40}$$

Let us solve (40) in the form

$$F_n(x,t)=A_n(x,t)\varphi_{1n}+B_n(x,t)h_n. \tag{41}$$

Putting (41) into (40) we get

$$A_n(x,t)=-\frac{\eta_n}{2k_nb_n}\int\limits_{-\infty}^{x}G\varphi_{1n}h_n dx,$$

$$B_n(x,t)=\frac{\eta_n}{2k_nb_n}\int\limits_{-\infty}^{x}G\varphi_{1n}^2 dx.$$

Thus, the second equation (39) of the Lax pair in this case has the form :

$$\varphi'_{1nt} = \left(\frac{1}{2\alpha^2}\left(\frac{1}{\eta_n}+2\gamma\right)-u\right)\varphi'_{1nx}+\left(\frac{u'_x}{2}+\beta_n\right)\varphi_{1n},$$

$$-\frac{\eta_n}{2k_nb_n}\int\limits_{-\infty}^{x}G\varphi_{1n}h_n\,dx\cdot\varphi_{1n}+\frac{\eta_n}{2k_nb_n}\int\limits_{-\infty}^{x}G\varphi_{1n}^2\,dx\cdot h_n. \tag{42}$$

On the other hand, differentiating the equalities

$$\varphi_1(x,ik_n,t)=b_n(t)\,\psi_2(x,ik_n,t),\qquad n=1,2,...,N \tag{43}$$

with respect to t, we obtain

$$\frac{\partial\varphi_{1n}}{\partial t}+\frac{\partial\varphi_1}{\partial k}|_{k=ik_n}\frac{d(ik_n)}{dt}=\frac{db_n}{dt}\psi_{2n}+b_n\left(\frac{\partial\psi_{2n}}{\partial t}+\frac{\partial\psi_2}{\partial k}|_{k=ik_n}\frac{d(ik_n)}{dt}\right).$$

According to (19), the last equality can be written as follows

$$\frac{\partial\varphi_{1n}}{\partial t}=\frac{db_n}{dt}\psi_{2n}-\dot a(ik_n)h_n\frac{d(ik_n)}{dt}+b_n\frac{\partial\psi_{2n}}{\partial t}. \tag{44}$$

Passing to the limit in this equality (42) when $x\to\infty$, taking into account (44), and the using asymptotics (10), (20) and (43), we obtain

$$\frac{1}{2\alpha^2}\left(\frac{1}{\eta_n}+2\gamma\right)(-k_n)b_ne^{-k_nx}+\beta_nb_ne^{-k_nx}-\left(\frac{\eta_n}{2k_nb_n}\int\limits_{-\infty}^{\infty}G\varphi_{1n}h_n\,dx\right)b_ne^{-k_nx}$$

$$+\frac{\eta_n}{2k_nb_n}\int\limits_{-\infty}^{\infty} G\varphi_{1n}^2\,dx\cdot e^{k_nx}=\frac{db_n}{dt}e^{-k_nx}-\dot{a}(ik_n)\frac{d(ik_n)}{dt}e^{k_nx}.$$

Substituting $\eta_n=-\frac{1}{\Omega}\left(-k_n^2+\frac{1}{4\alpha^2}\right)$ and comparing the coefficients of the exponents, we derive (25) and (26). □

The following theorem contains the main results in the paper.

Theorem 1. *If the functions* $u(x,t)$, $g_k(x,t)$, $k=1,2,...,N$ *are the solution to the problem* (2)–(4) *then, the scattering data of the equation* (5) *changes with respect to* t *as follows:*

$$\frac{dR}{dt}=-\frac{ik}{\alpha^2}\left(\frac{4\Omega}{4\alpha^2k^2+1}-2\gamma\right)R,\qquad (\operatorname{Im}k=0),\tag{45}$$

$$\frac{db_n(t)}{dt}=\left(\frac{k_n}{\alpha^2}\left(\frac{4\Omega}{1-4\alpha^2k_n^2}-2\gamma\right)+\frac{4\alpha^2k_n^2-1}{4\Omega}A_n(t)\right)b_n,\tag{46}$$

$$\frac{dk_n(t)}{dt}=0,\qquad n=1,2,\dots,N.\tag{47}$$

Proof. Let us apply the result of Lemma 2 when

$$G=\sum_{k=1}^{N}({m'}_xg_k^2+2(m+\Omega)(g_k^2)'_x).$$

Using Lemma 1 for $Imk=0$, one obtains

$$\begin{aligned}
&\int\limits_{-\infty}^{\infty} G(x,t)\varphi_1^2(x,k,t)dx=\int_{-\infty}^{\infty}\left(2((m+\Omega)g_k^2)'_x-m'_xg_k^2\right)\varphi_1^2\,dx=\\
&=\int_{-\infty}^{\infty}\left(m'_xg_k^2+2(m+\Omega)g_kg'_{kx}-m'_xg_k^2\right)\varphi_1^2\,dx+(m+\Omega)g_k^2\varphi_1^2\Big|_{-\infty}^{\infty}-\\
&-\int_{-\infty}^{\infty}(m+\Omega)g_k^2(\varphi_1^2)'_x\,dx=2\int_{-\infty}^{\infty}(m+\Omega)\left(g_kg'_{kx}\varphi_1^2-g_k^2\varphi_1\varphi_1'\right)dx=\\
&=\frac{2}{\eta_k-\eta}\int_{-\infty}^{\infty}W\{\varphi_1,g_k\}\frac{d}{dx}W\{\varphi_1,g_k\}\,dx=\frac{1}{\eta_k-\eta}W^2\{\varphi_1,g_k\}\Big|_{-\infty}^{\infty}=0.
\end{aligned}\tag{48}$$

According to (24) we get (45).

To prove (46) we use the equality $g_n(x,t)=c_n\varphi_{1n}(x,t)$. Let $k\neq n$, then we have

$$\int\limits_{-\infty}^{\infty} G(x,t)\varphi_{1n}h_ndx=\int_{-\infty}^{\infty}\left(2\left((m+\Omega)g_k^2\right)'_x-m'_xg_k^2\right)\varphi_{1n}h_n\,dx=$$

$$=\int_{-\infty}^{\infty}\left((m+\Omega)g_k^2\varphi'_{1n,x}h_n-g_k^2\varphi_{1n}h'_n\right)\,dx+(m+\Omega)g_k^2\varphi_{1n}h_n\Big|_{-\infty}^{\infty}-\int_{-\infty}^{\infty}(m+\Omega)g_k^2(\varphi_{1n}h_n)'_x\,dx=$$

$$= \int_{-\infty}^{\infty} (m+\Omega)\left(g_k h_n \left(g'_{kx}\varphi_{1n} - g_k\varphi'_{1n,x}\right) + g_k\varphi_{1n}\left(g'_{kx}h_n - g_k h'_{n,x}\right)\right)dx =$$

$$= \frac{1}{\eta_k-\eta_n}\int_{-\infty}^{\infty}\left(\frac{d}{dx}W\{h_n,g_k\}W\{\varphi_{1n},g_k\} + \frac{d}{dx}W\{\varphi_{1n},g_k\}W\{h_n,g_k\}\right)dx =$$

$$= \frac{1}{\eta_k-\eta_n}W\{h_n,g_k\}W\{\varphi_{1n},g_k\}\Big|_{-\infty}^{\infty} = 0. \tag{49}$$

For $k=n$, according to (21) and (4) we get

$$\int_{-\infty}^{\infty} G(x,t)\varphi_{1n}h_n dx = \int_{-\infty}^{\infty}\left(2((m+\Omega)g_n^2)'_x - m'_x g_n^2\right)\varphi_{1n}h_n dx =$$

$$\int_{-\infty}^{\infty}\left(m'_x g_n^2 + 2(m+\Omega)g_n g'_{nx} - m'_x g_n^2\right)\varphi_{1n}h_n dx - \int_{-\infty}^{\infty}(m+\Omega)g_n^2(\varphi_{1n}h_n)'_x dx =$$

$$= \int_{-\infty}^{\infty}(m+\Omega)\left[g_n h_n(g'_{nx}\varphi_{1n} - g_n\varphi'_{1nx}) + g_n\varphi_{1n}(g'_{nx}h_n - g_n h'_{nx})\right]dx = \tag{50}$$

$$= \int_{-\infty}^{\infty}(m+\Omega)g_n^2 W\{h_n,\varphi_{1n}\}dx = -2k_n b_n A_n(t).$$

Putting (49) and (50) into (25) we obtain (46).

Now we prove (47). For $k \neq n$ we deduce

$$\int_{-\infty}^{\infty}\left(2((m+\Omega)g_k^2)'_x - m'_x g_k^2\right)\varphi_{1n}^2 dx = \int_{-\infty}^{\infty}\left(m'_x g_k^2 + 2(m+\Omega)g_k g'_{kx} - m'_x g_k^2\right)\varphi_{1n}^2 dx +$$

$$+(m+\Omega)g_k^2\varphi_{1n}^2\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}(m+\Omega)g_k^2(\varphi_{1n}^2)'_x dx = 2\int_{-\infty}^{\infty}(m+\Omega)(g_k g'_{kx}\varphi_{1n}^2 - g_k^2\varphi_{1n}\varphi'_{1n})dx =$$

$$= 2\int_{-\infty}^{\infty}(m+\Omega)g_k\varphi_{1n}(g'_{kx}\varphi_{1n} - g_k\varphi'_{1n})dx = \frac{2}{\eta_k-\eta_n}\int_{-\infty}^{\infty}W\{\varphi_{1n},g_k\}\frac{d}{dx}W\{\varphi_{1n},g_k\}dx \tag{51}$$

$$= \frac{1}{\eta_k-\eta_n}W^2\{\varphi_{1n},g_k\}\Big|_{-\infty}^{\infty} = 0.$$

For $k = n$, using the equality $g_n(x,t) = c_n\varphi_{1n}(x,t)$, one obtains

$$\int\limits_{-\infty}^{\infty} \left(m'_x g_n^2 + 2(m+\Omega)g_n g'_{nx} - m'_x g_n^2\right)\varphi_{1n}^2 dx + (m+\Omega)g_n^2\varphi_{1n}^2\Big|_{-\infty}^{\infty} - \int\limits_{-\infty}^{\infty}(m+\Omega)g_n^2(\varphi_{1n}^2)'_x dx =$$
$$= 2\int\limits_{-\infty}^{\infty}(m+\Omega)g_n\varphi_{1n}(g'_{nx}\varphi_{1n} - g_n\varphi'_{1n})dx = 2\int\limits_{-\infty}^{\infty}(m+\Omega)c_n\varphi_{1n}^2 W\{\varphi_{1n}, c_n\varphi_{1n}\}dx = 0. \tag{52}$$

It follows from (51) and (52) that

$$\int\limits_{-\infty}^{\infty} G(x,t)\varphi_{1n}^2(x,t)dx = 0. \tag{53}$$

Substituting (53) into (26) we obtain (47).

Theorem is proved.

4. An Illustrative Example

We illustrate the application of Theorem 1 to solving problem (2)–(4) for the initial condition

$$u_0(x) = \frac{1}{2}\int\limits_{x}^{\infty} e^{x-z}m_0(z)dz + \frac{1}{2}\int\limits_{-\infty}^{x} e^{-(x-z)}m_0(z)dz$$

where

$$m_0(x) = \Omega\left[\frac{4}{(3e^x+4)}\left[3e^x\left(\cos\left(\frac{1}{3}\left(\pi - \arccos\frac{3}{\sqrt{3e^x+4}}\right)\right) + \right.\right.\right.$$
$$\left. + \frac{1}{\sqrt{3e^x-5}}\sin\left(\frac{1}{3}\left(\pi - \arccos\frac{3}{\sqrt{3e^x+4}}\right)\right)\right)\Big/\Big(2\sqrt{3e^x+4}\,\times$$
$$\left.\left.\times\cos\left(\frac{1}{3}\left(\pi - \arccos\frac{3}{\sqrt{3e^x+4}}\right)\right) - 3\Big)\right]^2 - 1\right],$$

with $\alpha = 1$. Solving the direct problem for the equation

$$\psi''_{xx} = \left(\frac{1}{4\alpha^2} + \eta\left(m_0(x) + \Omega\right)\right)\psi,$$

we find the scattering data for $t = 0$:

$$b_1(0) = \sqrt{3}, \quad k_1(0) = \frac{1}{4}, \quad R(k,0) = 0.$$

By Theorem 1, we have

$$R(k,t) = 0,$$

$$k_1(t) = \frac{1}{4},$$

$$b_1(t) = \sqrt{3}e^{B(t)},$$

where $B(t) = (\frac{4\Omega}{3} - \frac{\gamma}{2})t + \frac{3}{16\Omega}\int\limits_0^t A_1(\tau)d\tau$.

Applying the inverse problem method, we find:

$$\sqrt{\xi(x,t)} = \frac{2}{3}\sqrt{3e^{x-2B(t)}+4}\cos\left(\frac{1}{3}\arccos(-\frac{3}{\sqrt{3e^{x-2B(t)}+4}})\right) - 1.$$

Therefore,

$$m(x,t) = \Omega\left[\left(\frac{\xi'(x,t}{\xi(x,t}\right)^2 - 1\right] =$$

$$= \Omega\left[\frac{4}{\left(3e^{x-2B(t)}+4\right)}\left[3e^{x-2B(t)}\left(\cos\left(\frac{1}{3}\left(\pi - \arccos\frac{3}{\sqrt{3e^{x-2B(t)}+4}}\right)\right) + \right.\right.\right.$$
$$+\frac{1}{\sqrt{3e^{x-2B(t)}-5}}\sin\left(\frac{1}{3}\left(\pi - \arccos\frac{3}{\sqrt{3e^{x-2B(t)}+4}}\right)\right)\Bigg)/\Bigg(2\sqrt{3e^{x-2B(t)}+4}\times$$
$$\left.\left.\left.\times\cos\left(\frac{1}{3}\left(\pi - \arccos\frac{3}{\sqrt{3e^{x-2B(t)}+4}}\right)\right) - 3\right)\right]^2 - 1\right].$$

Thus, we find

$$u(x,t) = \frac{1}{2}\int\limits_x^\infty e^{x-z}m(z,t)dz + \frac{1}{2}\int\limits_{-\infty}^x e^{-(x-z)}m(z,t)dz.$$

Therefore, using representation (15), we obtain

$$\psi_1(x, \frac{i}{4}, t) = \frac{[\xi(x,t)]^{\frac{5}{4}}}{\sqrt{\xi'_x(x,t)}}\frac{1}{\sqrt{\xi(x,t)}+e^{B(t)}},$$

and from the normalization (4), we obtain

$$c_1^2\int\limits_{-\infty}^\infty (m(x,t)+\Omega)\psi_1^2\left(x, \frac{i}{4}, t\right)dx = A_1(t),$$

$$c_1^2 = \frac{A_1(t)}{\int\limits_{-\infty}^\infty (m(x,t)+\Omega)\psi_1^2\left(x, \frac{i}{4}, t\right)dx},$$

$$c_1 = \frac{\sqrt{A_1(t)}}{\sqrt{\int\limits_{-\infty}^{\infty} (m(x,t)+\Omega)\psi_1^2\left(x,\frac{i}{4},t\right)dx}},$$

$$g_1(x,t) = \frac{\sqrt{A_1(t)}}{\sqrt{\int\limits_{-\infty}^{\infty} (m(x,t)+\Omega)\psi_1^2\left(x,\frac{i}{4},t\right)dx}} \frac{[\xi(x,t)]^{\frac{5}{4}}}{\sqrt{\xi'_x(x,t)}} \frac{1}{\sqrt{\xi(x,t)}+e^{B(t)}}.$$

5. Conclusion

We have solved the integrable Dullin–Gottwald–Holm equation with a self-consistent source via IST method. Physically, sources arise in solitary waves with a variable speed and lead to a variety of dynamics of physical models. With regard to their applications, these kinds of systems are usually used to describe interactions between different solitary waves. We also present an efficient method to obtain the time evolution of scattering data. The advantage of this method is its reliability and the possibility of using other soliton equations with a source to obtain the time evolution of scattering data. An example is given illustrating the application of the obtained results to one soliton solution. The results obtained will be useful in carrying out further analysis in the context of shallow water waves that arises in the context of oceanography.

References

[1] H. Dullin, G. Gottwald, and D. Holm. An integrable shallow water equation with linear and nonlinear dispersion. *Physical Review Letters*, 87:4501–4504, 2001.

[2] L. Tian, G. Gui, and Y. Liu. On the well-posedness problem and the scattering problem for the dullin–gottwald–holm equation. *Communications in Mathematical Physics*, 257:667–701, 2005.

[3] Z. Yin. Well-posedness, global existence and blowup phenomena for an integrable shallow water equation. *Discrete and Continuous Dynamical Systems*, 10:393–411, 2004.

[4] O. Christov and S. Hakkaev. On the inverse scattering approach and action-angle variables for the dullin–gottwald–holm equation. *Physica D: Nonlinear Phenomena*, 238:919–925, 2009.

[5] X. Ai and G. Gui. On the inverse scattering problem and the low regularity solutions for the dullin–gottwald–holm equation. *Nonlinear Analysis: Real World Applications*, 11:888–894, 2010.

[6] D. Shepelsky and L. Zielinski. The inverse scattering transform in the form of a riemann–hilbert problem for the dullin–gottwald–holm equation. *Opuscula Mathematica*, 37(1):167–187, 2017.

[7] S. Hakkaev. Stability of peakons for an integrable shallow water equation. *Physics Letters A*, 354:137–144, 2006.

[8] X. X. Liu and Z. Y. Yin. Orbital stability of the sum of n peakons for the dullin–gottwald–holm equation. *Nonlinear Analysis: Real World Applications*, 13:2414–2422, 2012.

[9] A. M. Wazwaz. Peakons, kinks, compactons and solitary pattern solutions for a family of camassa–holm equations by using new hyperbolic schemes. *Applied Mathematics and Computation*, 182:412–424, 2006.

[10] G. C. Xiao, D. Q. Xian, and X. Q. Liu. Application of exp-function method to dullin–gottwald–holm equation. *Applied Mathematics and Computation*, 210:536–541, 2009.

[11] A. Biswas and A. H. Kara. 1-soliton solution and conservation law of the generalized dullin–gottwald–holm equation. *Applied Mathematics and Computation*, 217:929–932, 2010.

[12] Q. Meng, B. He, Y. Long, and Z. Y. Li. New exact periodic wave solutions for the dullin–gottwald–holm equation. *Applied Mathematics and Computation*, 218:4533–4537, 2011.

[13] J. B. Zhou, L. X. Tian, W. B. Zhang, and S. Kumar. Peakon-antipeakon interaction in the dullin–gottwald–holm equation. *Physics Letters A*, 377:1233–1238, 2013.

[14] S. H. Zhang and Z. Y. Yin. Global weak solutions for the dullin–gottwald–holm equation. *Nonlinear Analysis: Theory, Methods & Applications*, 72:1690–1700, 2010.

[15] B. A. Babajanov and D. O. Atajonov. Integration of the generalized camassa–holm equation in the class of periodic functions. *Theoretical and Mathematical Physics*, 223(1):114–126, 2025.

[16] V. K. Melnikov. Integration of the korteweg–de vries equation with a source. *Inverse Problems*, 6(2):233–246, 1990.

[17] J. Leon and A. Latifi. Solution of an initial-boundary value problem for coupled nonlinear waves. *Journal of Physics A: Mathematical and General*, 23(8):1385–1403, 1990.

[18] C. Claude, A. Latifi, and J. Leon. Nonlinear resonant scattering and plasma instability: an integrable model. *Journal of Physics A: Mathematical and General*, 32(12):3321–3330, 1991.

[19] V. K. Melnikov. Integration of the nonlinear schrödinger equation with a self-consistent source. *Communications in Mathematical Physics*, 137:359–381, 1991.

[20] D. J. Zhang and D. Y. Chen. The n-soliton solutions of the sine-gordon equation with self-consistent sources. *Physica A*, 321:467–481, 2003.

[21] Y. Huang, Y. Zeng, and O. Ragnisco. The degasperis–procesi equation with self-consistent sources. *Journal of Physics A: Mathematical and Theoretical*, 41(35):1–10, 2008.

[22] D. Zhang. The n-soliton solutions of some soliton equations with self-consistent sources. *Chaos, Solitons & Fractals*, 18:31–43, 2003.

[23] S. F. Deng, D. Y. Chen, and D. J. Zhang. The multisoliton solutions of the kp equation with self-consistent sources. *Journal of the Physical Society of Japan*, 72:2184–2192, 2003.

[24] D. J. Zhang and D. Y. Chen. The n-soliton solutions of the sine-gordon equation with self-consistent sources. *Physica A*, 321:467–481, 2003.

[25] B. A. Babajanov and A. Sh. Azamatov. Integration of the kaup–boussinesq system with a self-consistent source via inverse scattering method. *Vestnik Udmurtskogo Universiteta*, 32(2):153–170, 2022.

[26] B. A. Babajanov, A. Sh. Azamatov, and R. B. Atajanova. Integration of the kaup–boussinesq system with time-dependent coefficients. *Theoretical and Mathematical Physics*, 216(1):961–972, 2023.

[27] B. A. Babajanov and A. B. Khasanov. Periodic toda chain with an integral source. *Theoretical and Mathematical Physics*, 184:1114–1128, 2015.

[28] I. I. Baltaeva and G. U. Urazboev. About the camassa–holm equation with a self-consistent source. *Ufa Mathematical Journal*, 3(2):10–18, 2011.

[29] G. U. Urazboev and I. I. Baltaeva. Integration of camassa–holm equation with a self-consistent source of integral type. *Ufa Mathematical Journal*, 14(1):84–94, 2022.

[30] B. A. Babajanov, A. K. Babadjanova, and A. Sh. Azamatov. Integration of the differential-difference sine-gordon equation with a self-consistent source. *Theoretical and Mathematical Physics*, 210(3):327–336, 2022.

[31] G. U. Urazboev and A. K. Babadjanova. On the integration of the matrix modified korteweg–de vries equation with a self-consistent source. *Tamkang Journal of Mathematics*, 50(3):281–291, 2019.

[32] A. B. Yakhshimuratov. The nonlinear schrödinger equation with a self-consistent source in the class of periodic functions. *Mathematical Physics, Analysis and Geometry*, 14:153–169, 2011.

[33] A. Yakhshimuratov, T. Kriecherbauer, and B. Babajanov. On the construction and integration of a hierarchy for the kaup system with a self-consistent source in the class of periodic functions. *Mathematical Physics, Analysis and Geometry*, 17(2):233–257, 2021.

[34] Y. Zeng, W. X. Ma, and R. Lin. Integration of the soliton hierarchy with self-consistent sources. *Journal of Mathematical Physics*, 41(8):5453–5488, 2000.

[35] Y. B. Zeng, W. X. Ma, and Y. J. Shao. Two binary darboux transformations for the kdv hierarchy with self-consistent sources. *Journal of Mathematical Physics*, 42:2113–2128, 2001.

[36] A. B. Yakhshimuratov and B. A. Babajanov. Integration of equation of kaup system kind with a self-consistent source in the class of periodic functions. *Ufa Mathematical Journal*, 12(1):104–114, 2020.

[37] B. A. Babajanov, M. Feckan, and G. U. Urazbaev. On the periodic toda lattice hierarchy with an integral source. *Communications in Nonlinear Science and Numerical Simulation*, 52:110–123, 2017.

[38] S. G. Vladimir, G. Grahovski, and R. Ivanov. On the integrability of kdv hierarchy with self-consistent sources. *Communications on Pure and Applied Analysis*, 11(4):1439–1452, 2012.

[39] A. B. Khasanov and U. A. Xaitmetov. Integration of the general loaded korteweg–de vries equation with an integral type source in the class of rapidly decreasing complex-valued functions. *Russian Mathematics*, 65(7):43–57, 2021.

[40] B. Babajanov and F. Abdikarimov. The application of the functional variable method

for solving the loaded non-linear evaluation equations. *Frontiers in Applied Mathematics and Statistics*, pages 1–9, 2022.

[41] A. Constantin, V. Gerdjikov, and R. I. Ivanov. Inverse scattering transform for the camassa–holm equation. *Inverse Problems*, 22:2197, 2006.