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# On Fuzzy Jacobsthal and Jacobsthal-Lucas Numbers

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**Abstract.** This study examines the generalization of the classical Jacobsthal and Jacobsthal-Lucas numbers to fuzzy numbers. Using the triangular membership function and the $\alpha$-cut method, a recurrence relation is given for newly defined fuzzy numbers, and formulas for the general terms of fuzzy sequences are derived. A detailed analysis of some fundamental equations satisfied by classical number sequences is also performed for newly defined sequences using fuzzy logic. An application in decision making is presented to compare classical and fuzzy sequences.

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**Key Words and Phrases**: Fuzzy numbers, fuzzy set, recurrence relations, Jacobsthal numbers

## 1. Introduction and Preliminaries

The first step in the field of fuzzy logic was presented in the paper 'Fuzzy sets' by Prof. Dr. Lotfi Asker Zadeh from the University of California, Berkeley in 1965[1]. Zadeh laid the foundations of Fuzzy Logic by stating that, unlike classical logic, uncertainty can belong to sets with no sharp boundaries and different memberships. Among the advantages of fuzzy logic are that it is simple and easy to understand, can respond to complex problems and solve uncertainty problems in various fields, and can provide solutions for poorly defined problems while its disadvantages include the need for extensive tests to verify the system, a capacity problem compared to machine learning, lack of self-learning capabilities, and lack of a specific method in the selection of membership functions. Membership functions are curves that describe how each point in the input space is mapped to a membership value between 0 and 1. It is a graphical representation of the magnitude of each input. Depending on the behavior of the input values, one-dimensional membership functions such as triangle, trapezoid, Gaussian, sigmoid, and bell-shaped can be used. Triangular and trapezoidal membership functions have a linear form, making them simple to design and represent. Membership functions such as the Gaussian and the sigmoid are more complex to design.

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Recently, many researchers have developed methods to compare and order fuzzy numbers. In 1977, Bass and Kwakernaak proposed a different method to extend the natural ordering of real numbers to fuzzy numbers[2]. In 1978, Dubois and Prade used set maximization to classify fuzzy numbers[3]. In 1979, Baldwin and Guild noted that these two methods have some drawbacks[4]. In 1980, Adamo[5] used the concept of a set of levels $\alpha$ to introduce the preference rule $\alpha$. In 1981, Chang [6] introduced the concept of a preference function of an alternative. Yager [7, 8] proposed four indices that can be used to classify fuzzy quantities in $[0, 1]$. Bortolan and Degani compared and reviewed some of these ranking methods[9]. Chen and Hwang[10] comprehensively reviewed existing approaches and pointed out some counterintuitive conditions that arise among them. Fuzzy decision-making methods have gained importance due to their ability to model uncertainties in real-world choices. One of the most widely used techniques in this context is the fuzzy TOPSIS method introduced by Chen. This method is particularly effective in multi-criteria decision-making problems under uncertainty and offers flexibility in weighting criteria and handling linguistic considerations[10–16]. Generalization of integer sequences using fuzzy logic has been discussed by some authors(see [17–21]). Irmak and Demirtaş defined fuzzy Fibonacci and Lucas numbers and presented important fundamental equations. Duman also obtained some new identities for fuzzy Fibonacci numbers. Catarino and his colleagues defined fuzzy Leonardo numbers in [19] and studied these numbers in detail. Erduvan, on the other hand, introduced fuzzy Gaussian Fibonacci numbers in [20]. A fuzzy real set is a function $\tilde{A} : \mathbb{R} \to [0, 1]$, and a fuzzy real set $\tilde{A}$ will be a fuzzy real number iff

1. $\tilde{A}$ is normal:$\tilde{A}(x) = 1$, $x \in \mathbb{R}$.
2. For all $\alpha \in (0, 1]$, the set $\tilde{A}[\alpha] = \{x \in \mathbb{R} : \tilde{A}(x) \geq \alpha\}$ is limited set.

The set of all fuzzy real numbers is denoted $\mathbb{R}_{\mathcal{F}}$. Notice that $\mathbb{R} \subset \mathbb{R}_{\mathcal{F}}$, since every $a \in \mathbb{R}$ can be written as $a : \mathbb{R} \to [0, 1]$, where $a(x) = 1$ if $x = a$ and $a(x) = 0$ if $x \neq a$.

One of the basics of calculating with fuzzy numbers is interval analysis. Interval analysis can be thought of as a type of tolerance or confidence interval with fuzzy numbers. Basic algebraic operations for numbers expressed as intervals can be listed as follows: $I(\mathbb{R}) = \{[a, b] : a, b \in \mathbb{R}\}$ endowed with the following arithmetic[22]

$$
\begin{aligned}
[a, b] + [c, d] &= [a + c, b + d],\\
[a, b] - [c, d] &= [a - d, b - c] = -[c, d] = [-d, -c],\\
[a, b] \cdot [c, d] &= [\min(a.c, a.d, b.c, b.d), \max(a.c, a.d, b.c, b.d)],\\
[c, d]^{-1} &= \frac{1}{[c, d]} = \left[\frac{1}{d}, \frac{1}{c}\right], \quad \text{if } 0 \notin [c, d],\\
\frac{[a, b]}{[c, d]} &= [a, b] \cdot [c, d]^{-1}, \quad 0 \notin [c, d].
\end{aligned}
$$

To perform arithmetic operations on a set of fuzzy numbers, different membership functions or $\alpha$-cut functions are usually used. For operations with fuzzy numbers, the alpha-cut method and the expansion rule are widely used in the literature. The alpha-cut sets of fuzzy numbers defined in a continuous set can be expressed parametrically. When calculating the alpha-cut set of a fuzzy number, the membership function of the fuzzy number is assumed to be equal to the specified alpha value. A triangular fuzzy number defined on the real number line is expressed parametrically by the following membership function [22]:

$$\mu_{\tilde{A}}(x) = \mu_{\tilde{A}}(x; a_1, a_2, a_3) = \begin{cases} \frac{x-a_1}{a_2-a_1}, & a_1 \leq x \leq a_2, \\ \frac{a_3-x}{a_3-a_2}, & a_2 \leq x \leq a_3, \\ 0, & x > a_3 \text{ or } x < a_1. \end{cases} \tag{1}$$

For the fuzzy number $\tilde{A} = (a_1, a_2, a_3)$, $\alpha-$ cut representation [3] is

$$A^\alpha = (a_1, a_2, a_3) = [(a_2 - a_1)\alpha + a_1,\ a_3 - \alpha(a_3 - a_2)],\ \ \alpha \in (0, 1]. \tag{2}$$

Operations applied to fuzzy numbers $\tilde{A}$ and $\tilde{B}$ result in a new fuzzy number. The following relations can be established between the alpha-cuts of these numbers. For $A^\alpha = [a_1^\alpha, a_3^\alpha]$ and $B^\alpha = [b_1^\alpha, b_3^\alpha]$, the arithmetic operations are as follows:

$$A^\alpha + B^\alpha = [a_1^\alpha + b_1^\alpha,\ a_3^\alpha + b_3^\alpha]. \tag{3}$$

$$A^\alpha - B^\alpha = [a_1^\alpha - b_3^\alpha,\ a_3^\alpha - b_1^\alpha]. \tag{4}$$

$$A^\alpha \times B^\alpha = [\min(a_1^\alpha.b_1^\alpha,\ a_3^\alpha.b_3^\alpha,\ a_3^\alpha.b_1^\alpha,\ a_1^\alpha.b_3^\alpha), \max(a_1^\alpha.b_1^\alpha,\ a_3^\alpha.b_3^\alpha,\ a_3^\alpha.b_1^\alpha,\ a_1^\alpha.b_3^\alpha)]\,. \tag{5}$$

$$A^\alpha \times k = [a_1^\alpha k,\ a_3^\alpha k], \quad k \in \mathbb{R}^+. \tag{6}$$

In [17], the authors defined the Fuzzy Fibonacci numbers as follows:

$$F_{rn+l}^\alpha = \left[F_{r(n-1)+l} + \alpha\left(F_{rn+l} - F_{r(n-1)+l}\right),\ F_{r(n+1)+l} - \alpha\left(F_{r(n+1)+l} - F_{rn+l}\right)\right]. \tag{7}$$

Where $r, n, l$ are positive integers and $\alpha \in (0, 1]$. Using this definition, the following equation can be written: The $n^{\text{th}}$ fuzzy Fibonacci number, $\tilde{F}n = (F_{n-1}, F_n, F_{n+1})$, $n \geq 1$, satisfies the recurrence relation:

$$\tilde{F}_{n+1} = \tilde{F}_n + \tilde{F}_{n-1}. \tag{8}$$

$F_0^\alpha = [1 - \alpha,\ 1 - \alpha]$ and $F_1^\alpha = [\alpha,\ 1]$ are the initial values of the recurrence relation (8). Similarly, the $n^{\text{th}}$ fuzzy Lucas number $\tilde{L}_n = (L_{n-1}, L_n, L_{n+1})$, satisfies the recurrence relation.

$$\tilde{L}_{n+1} = \tilde{L}_n + \tilde{L}_{n-1}, \tag{9}$$

where $L_0^\alpha = [-1 + 3\alpha,\ 1 + \alpha]$, $L_1^\alpha = [2 - \alpha,\ 3 - 2\alpha]$. In [17, 18], the authors defined the membership functions of these newly defined numbers as follows and investigated the fundamental properties of them.

$$\mu_{\tilde{F}_n}(x) = \begin{cases} 0, & x < F_{n-1} \\ \dfrac{x - F_{n-1}}{F_n - F_{n-1}}, & F_{n-1} \leq x \leq F_n \\ \dfrac{F_{n+1} - x}{F_{n+1} - F_n}, & F_n \leq x \leq F_{n+1} \\ 0, & x > F_{n+1} \end{cases}, \tag{10}$$

and

$$\mu_{\tilde{L}_n}(x) = \begin{cases} 0, & x < L_{n-1} \\ \dfrac{x - L_{n-1}}{L_n - L_{n-1}}, & L_{n-1} \le x \le L_n \\ \dfrac{L_{n+1} - x}{L_{n+1} - L_n}, & L_n \le x \le L_{n+1} \\ 0, & x > L_{n+1} \end{cases}. \tag{11}$$

As is well-known, the Jacobsthal and the Jacobsthal-Lucas sequences are

$$\{J_n\}_{n\ge 0} = \{0, 1, 1, 3, 5, 11, \dots\}, \quad \{j_n\}_{n\ge 0} = \{2, 1, 5, 7, 17, 31, \dots\}, \tag{12}$$

respectively. And the recurrence relations of these are as follows.

$$J_n = J_{n-1} + 2J_{n-2}, \ \ j_n = j_{n-1} + 2j_{n-2} \tag{13}$$

where

$$J_0 = 0, \ \ J_1 = 1, \ \ j_0 = 2, \ \ j_1 = 1.$$

In this study, we discussed and examined the triangular membership functions of Jacobsthal and Jacobsthal-Lucas sequences, as defined above, and the new fuzzy numbers obtained by using them.

## 2. Fuzzy Jacobsthal and Fuzzy Jacobsthal-Lucas Numbers

In this section, membership functions for well-known and recently frequently studied Jacobsthal numbers are defined, and basic identities related to their fuzzy forms are obtained using the $\alpha$-cut method. Binet's formula for these fuzzy Jacobsthal numbers is presented, and some additional algebraic identities are derived using this formula. Various summation formulas supporting the structure of the fuzzy generalization are also given. Triangular fuzzy numbers were chosen due to their simple structure and closure under addition and scalar multiplication, thereby preserving the algebraic structure. The $\alpha$-cut method allows for a consistent interpretation of operations at each level $\alpha$. These structures are considered a natural extension of the classical definitions in the fuzzy environment.

**Definition 1.** *The fuzzy Jacobsthal $\tilde{J}_n$ can be defined by the membership function as follows.*

$$\mu_{\tilde{J}_n}(x) = \begin{cases} 0, & x < J_{n-1} \\ \dfrac{x - J_{n-1}}{J_n - J_{n-1}}, & J_{n-1} \le x \le J_n \\ \dfrac{J_{n+1} - x}{J_{n+1} - J_n}, & J_n \le x \le J_{n+1} \\ 0, & x > J_{n+1} \end{cases}, \tag{14}$$

With the help of the membership function definition, the following equations are satisfied.

$$J_n^\alpha = \left[(J_n - J_{n-1})\alpha + J_{n-1},\ J_{n+1} - \alpha(J_{n+1} - J_n)\right], \tag{15}$$

$$j_n^\alpha = \left[(j_n - j_{n-1})\alpha + j_{n-1},\ j_{n+1} - \alpha(j_{n+1} - j_n)\right]. \tag{16}$$

Then, the two new sequences created with these numbers can be owned, respectively.

$$\{J_n^\alpha\}_{n\geq 0} = \left\{\left[\frac{1-\alpha}{2}, 1-\alpha\right], [\alpha, 1], [1, 3-2\alpha], [1+2\alpha, 5-2\alpha], [3+2\alpha, 11-6\alpha], \dots\right\}. \tag{17}$$

$$\{j_n^\alpha\}_{n\geq 0} = \left\{\left[\frac{1+5\alpha}{2}, 1+\alpha\right],\ [2-\alpha, 5-4\alpha],\ [4\alpha+1, 7-2\alpha],\ [5+2\alpha, 17-10\alpha], \dots\right\}. \tag{18}$$

The following equalities are true for the fuzzy numbers $J_n^\alpha$ and $j_n^\alpha$.

$$J_{n+2}^\alpha = J_{n+1}^\alpha + 2J_n^\alpha, \quad j_{n+2}^\alpha = j_{n+1}^\alpha + 2j_n^\alpha. \tag{19}$$

Indeed, the truth of these equations can be seen directly. From the equalities (15) (16), we get

$$J_{n+1}^\alpha + 2J_n^\alpha = [J_n + (J_{n+1} - J_n)\alpha, J_{n+2} - \alpha(J_{n+2} - J_{n+1})] + 2[J_{n-1} + (J_n - J_{n-1})\alpha, J_{n+1} - \alpha(J_{n+1} - J_n)],$$

$$J_{n+1}^\alpha + 2J_n^\alpha = [J_n + 2\alpha J_{n-1}, J_{n+2} - 2\alpha J_n] + 2[J_{n-1} + 2\alpha J_{n-2}, J_{n+1} - 2\alpha J_{n-1}],$$

$$J_{n+1}^\alpha + 2J_n^\alpha = [(J_n + 2J_{n-1}) + \alpha(2J_{n-1} - 4J_{n-2}),\ (J_{n+2} + 2J_{n+1}) - \alpha(2J_n - 4J_{n-1})],$$

$$J_{n+1}^\alpha + 2J_n^\alpha = [J_{n+1} + 2\alpha J_n, J_{n+3} - 2\alpha J_{n+1}],$$

$$J_{n+1}^\alpha + 2J_n^\alpha = J_{n+2}^\alpha.$$

For $\alpha = \frac{1}{2}, \alpha = \frac{1}{3}$ and $\alpha = \frac{1}{4}$, Fuzzy Jacobsthal and Jacobsthal-Lucas numbers are listed in the following tables, respectively.

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| $J_n^\alpha$ | $[1/4, 1/2]$ | $[1/2, 1]$ | $[1, 2]$ | $[2, 4]$ | $[4, 8]$ | $[8, 16]$ | $[16, 32]$ | $[32, 64]$ |
| $j_n^\alpha$ | $[3/4, 3/2]$ | $[3/2, 3]$ | $[3, 6]$ | $[6, 12]$ | $[12, 24]$ | $[24, 48]$ | $[48, 96]$ | $[96, 192]$ |

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $J_n^\alpha$ | $[1/3, 2/3]$ | $[1/3, 1]$ | $[1, 7/3]$ | $[5/3, 13/3]$ | $[11/3, 9]$ | $[7, 53/3]$ | $[43/3, 107/3]$ |
| $j_n^\alpha$ | $[1/3, 4/3]$ | $[5/3, 11/3]$ | $[7/3, 19/3]$ | $[17/3, 41/3]$ | $[31/3, 25]$ | $[65/3, 161/3]$ | $[127/3, 319/3]$ |

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $J_n^\alpha$ | $[3/8, 3/4]$ | $[1/4, 1]$ | $[1, 5/2]$ | $[3/2, 9/2]$ | $[7/2, 19/2]$ | $[13/2, 37/2]$ | $[27/2, 75/2]$ |
| $j_n^\alpha$ | $[1/8, 5/4]$ | $[7/4, 4]$ | $[2, 13/2]$ | $[11/2, 29/2]$ | $[19/2, 55/2]$ | $[41/2, 111/2]$ | $[79/2, 223/2]$ |

According to the above three tables, we can say that the range of fuzzy numbers narrows as $\alpha$ it increases and widens as it decreases, increasing uncertainty. This allows the flexible analysis of fuzzy numbers at different $\alpha$-cut values. For $\alpha = \frac{1}{2}$ and $\alpha = \frac{1}{3}$, we get

$$2J_0^{1/2} + J_1^{1/2} = J_2^{1/2} = 2[1/4, 1/2] + [1/2, 1] = [1, 2],$$

$$2j_0^{1/3} + j_1^{1/3} = j_2^{1/3} = 2[1/3, 4/3] + [5/3, 11/3] = [7/3, 19/3].$$

Notice that

$$J_n^\alpha = \left[(J_n - J_{n-1})\alpha + J_{n-1},\ J_{n+1} - \alpha(J_{n+1} - J_n)\right],$$

$$j_n^\alpha = \left[(j_n - j_{n-1})\alpha + j_{n-1},\ j_{n+1} - \alpha(j_{n+1} - j_n)\right].$$

Taking $\alpha = 1$ in the last two equations, the classical Jacobsthal and Jacobsthal-Lucas numbers are obtained.

In the following Proposition, we give the fundamental algebraic operations for the newly defined sequence elements.

**Proposition 1.** *For the numbers $J_n^\alpha$ and $j_n^\alpha$ the following equations are true.*

*1.*

$$J_n^\alpha \pm J_m^\alpha = [J_{n-1} + J_{m-1} + 2\alpha(J_{n-2} + J_{m-2}),\ J_{n+1} + J_{m+1} - 2\alpha(J_{n-1} - J_{m-1})]. \quad (20)$$

*2.*

$$J_n^\alpha \cdot J_m^\alpha = [A, B] : \quad (21)$$

$$A = J_{n-1}J_{m-1} + 2\alpha(J_{n-1}J_{m-2} + J_{n-2}J_{m-1}) + 4\alpha^2 J_{n-2}J_{m-2},$$

$$B = J_{n+1}J_{m+1} - 2\alpha(J_{n-1}J_{m+1} + J_{n+1}J_{m-1}) + 4\alpha^2 J_{n-1}J_{m-1}.$$

*3.*

$$j_n^\alpha \pm j_m^\alpha = [j_{n-1} + j_{m-1} + 2\alpha(j_{n-2} + j_{m-2}),\ j_{n+1} + j_{m+1} - 2\alpha(j_{n-1} - j_{m-1}) \quad (22)$$

*4.*

$$J_n^\alpha \cdot J_m^\alpha = [C, D]: \tag{23}$$

$$C = j_{n-1}j_{m-1} + 2\alpha(j_{n-1}j_{m-2} + j_{n-2}j_{m-1}) + 4\alpha^2 j_{n-2}j_{m-2},$$

$$D = j_{n+1}j_{m+1} - 2\alpha(j_{n-1}j_{m+1} + j_{n+1}j_{m-1}) + 4\alpha^2 j_{n-1}j_{m-1}.$$

*Proof.* The proof of the Proposition can be easily seen using definitions (15), (16).

**Theorem 1.** *The closed formulas giving the numbers $J_n^\alpha, j_n^\alpha$ are as follows.*

*1.*

$$J_n^\alpha = \frac{2^{n-1}}{3}[1+\alpha,\ 4-2\alpha] + \frac{(-1)^n}{3}(1-2\alpha). \tag{24}$$

*2.*

$$j_n^\alpha = 2^{n-1}[1+\alpha,\ 4-2\alpha] + (-1)^{n-1}(1-2\alpha). \tag{25}$$

*Proof.* From the definition of alpha-cut and the Binet formula of classical Jacobsthal numbers, $J_n = \frac{1}{3}(2^n - (-1)^n)$, we write

$$J_n^\alpha = [(J_n - J_{n-1})\alpha + J_{n-1},\ J_{n+1} - \alpha(J_{n+1} - J_n)] = [E, F].$$

Where,

$$E = \frac{2^{n-2}}{3}[1+\alpha,\ 4-2\alpha] - \frac{(-1)^{n-2}}{3}(1-2\alpha) + 2\alpha\left(\frac{2^{n-3}}{3}[1+\alpha,\ 4-2\alpha] - \frac{(-1)^{n-3}}{3}(1-2\alpha)\right),$$

$$F = \frac{2^{n}}{3}[1+\alpha,\ 4-2\alpha] - \frac{(-1)^{n}}{3}(1-2\alpha) - 2\alpha\left(\frac{2^{n-2}}{3}[1+\alpha,\ 4-2\alpha] - \frac{(-1)^{n-2}}{3}(1-2\alpha)\right).$$

If the necessary simplifications are made in the last two equations, then we get

$$J_n^\alpha = \frac{2^{n-1}}{3}[1+\alpha,\ 4-2\alpha] + \frac{(-1)^n}{3}(1-2\alpha)$$

which is the desired result.
It should be noted immediately that there are some remarkable relations between the fuzzy numbers $\tilde{J}_n$ and $\tilde{j}_n$. We give some of them in the following Theorem.

**Theorem 2.** *For the numbers $J_n^\alpha$ and $j_n^\alpha$, we have*

*1.*

$$j_n^\alpha = J_{n+1}^\alpha + 2J_{n-1}^\alpha. \tag{26}$$

*2.*

$$9J_n^\alpha = j_{n+1}^\alpha + 2j_{n-1}^\alpha. \tag{27}$$

*Proof.* 1. For the proof, both the recursive relations of the classical Jacobsthal numbers and the alpha-intercept representations of these numbers can be used. Accordingly, the following equations can be written.

$$\begin{aligned}
J_{n+1}^{\alpha}+2J_{n-1}^{\alpha} &= [J_n+(J_{n+1}-J_n)\alpha, J_{n+2}-\alpha(J_{n+2}-J_{n+1})],\\
&\quad +2[J_{n-2}+(J_{n-1}-J_{n-2})\alpha, J_n-\alpha(J_n-J_{n-1})],\\
&= [J_n+2\alpha J_{n-1}, J_{n+2}-2\alpha J_n]+2[J_{n-2}+2\alpha J_{n-3}, J_{n-4}-2\alpha J_{n-2}],\\
&= [(J_n+2J_{n-2})+2\alpha(J_{n-1}-2J_{n-3}), (J_{n+2}+2J_n)-2\alpha(J_n+2J_{n-2})],\\
&= [j_{n-1}+2\alpha j_{n-2}, j_{n+1}-2\alpha j_{n-1}]=j_n^{\alpha}.
\end{aligned}$$

2. Using the recursive relations of the classical Jacobsthal numbers, the alpha-intercept representations of these numbers can be used. Accordingly, the following equations can be written.

$$\begin{aligned}
j_{n+1}^{\alpha}+2j_{n-1}^{\alpha} &= [j_n+(j_{n+1}-j_n)\alpha, j_{n+2}-\alpha(j_{n+2}-j_{n+1})],\\
&\quad +2[j_{n-2}+(j_{n-1}-j_{n-2})\alpha, j_n-\alpha(j_n-j_{n-1})],\\
&= [(j_n+2j_{n-2})+2\alpha(j_{n-1}-2j_{n-3}), (j_{n+2}+2j_n)-2\alpha(j_n+2j_{n-2})],\\
&= [9J_{n-1}+18\alpha J_{n-2}, 9J_{n+1}-18\alpha J_{n-1}]=9J_n^{\alpha}.
\end{aligned}$$

We have given the important relation between the fuzzy numbers $\tilde{J}_n$, $\tilde{j}_n$ in the following corollary.

**Corollary 1.** *For the numbers $\tilde{J}_n$, $\tilde{j}_n$, we have*

$$j_n^{\alpha}J_n^{\alpha}=J_{2n}^{\alpha}. \tag{28}$$

*Proof.* From the algebraic operations performed between $\tilde{J}_n$ and $\tilde{j}_n$, then we get

$$j_n^{\alpha}J_n^{\alpha}=[j_{n-1}+2\alpha j_{n-2}, j_{n+1}-2\alpha j_{n-1}]\cdot[J_{n-1}+2\alpha J_{n-2}, J_{n+1}-2\alpha J_{n-1}]=[X,Y],$$

$$\begin{aligned}
X &= j_{n-1}J_{n-1}+\alpha(2j_{n-1}J_{n-2}+2j_{n-2}J_{n-1})+\alpha^2 4j_{n-2}J_{n-2},\\
Y &= J_{n+1}j_{n+1}-\alpha(2J_{n+1}j_{n+1}-2j_{n-1}J_{n+1})+\alpha^2 4j_{n-1}J_{n-1}.
\end{aligned}$$

$$j_n^{\alpha}J_n^{\alpha}=[J_{2n-2}+2\alpha(j_{n-1}J_{n-2}+j_{n-2}J_{n-1})+\alpha^2 4J_{2n-4}, J_{2n+2}-2\alpha(J_{n-1}j_{n+1}-j_{n-1}J_{n+1})+\alpha^2 4J_{2n-2}].$$

If the following equations are used,

$$J_m j_n+J_n j_m=2J_{m+n}, \quad \text{and} \quad 2j_{n-1}J_{n-2}+2j_{n-2}J_{n-1}=4J_{2n-3}.$$

Then, we obtain

$$j_n^{\alpha}J_n^{\alpha}=[J_{2n-2}+4\alpha J_{2n-3}+4\alpha^2 J_{2n-4}, J_{2n+2}-4\alpha J_{2n}+4\alpha^2 J_{2n-2}]=J_{2n}^{\alpha},$$

which is the desired result.

**Theorem 3.** *The sums of the first n fuzzy Jacobsthal and Jacobsthal-Lucas numbers are as follows.*

*1.*
$$\sum_{i=2}^{n} J_i^{\alpha} = \frac{1}{2}(J_{n+2}^{\alpha} - 3). \tag{29}$$

*2.*
$$\sum_{i=1}^{n} j_i^{\alpha} = \frac{1}{2}(j_{n+2}^{\alpha} - 5). \tag{30}$$

*Proof.* 2. From the mathematical induction method, for $n = \alpha = 1$,

$$j_1^{\alpha} = \frac{1}{2}(j_3^{\alpha} - 5).$$

Assume that for $n = k$,

$$\sum_{i=1}^{k} j_i^{\alpha} = \frac{1}{2}(j_{k+2}^{\alpha} - 5).$$

For $n = k+1$,

$$\sum_{i=1}^{k+1} j_i^{\alpha} = \left(\sum_{i=1}^{k} j_i^{\alpha}\right) + j_{k+1}^{\alpha},$$

$$= \frac{1}{2}(j_{k+2}^{\alpha} - 5) + [j_k + 2\alpha j_{k-1}, j_{k+2} - 2\alpha j_k],$$

$$= \frac{1}{2}[j_{k+1} + \alpha(j_{k+2} - j_{k+1}) - 5,\ j_{k+3} - \alpha(j_{k+3} - j_{k+2}) - 5] + [j_k + 2\alpha j_{k-1}, j_{k+2} - 2\alpha j_k],$$

$$= \frac{1}{2}[j_{k+2} + 2\alpha(j_k + 2j_{k-1}) - 5,\ j_{k+3} + 2j_{k+2} - 2\alpha(j_{k+1} + 2j_k) - 5],$$

$$= \frac{1}{2}[j_{k+2} + 2\alpha j_{k+1} - 5,\ j_{k+4} - 2\alpha j_{k+2} - 5] = \frac{1}{2}(j_{k+3}^{\alpha} - 5).$$

Thus, the proof is completed.

**Theorem 4.** *The odd and even indexed sums of the fuzzy Jacobsthal and Jacobsthal-Lucas numbers are as follows.*

*1.*
$$\sum_{i=0}^{n} J_{2i}^{\alpha} = \frac{1}{3}(J_{2n+2}^{\alpha} - n - 1). \tag{31}$$

*2.*
$$\sum_{i=1}^{n} j_{2i}^{\alpha} = \frac{1}{3}(J_{2n+2}^{\alpha} + n + 1). \tag{32}$$

*3.*

$$\sum_{i=0}^{n} J_{2i+1}^{\alpha} = \frac{1}{3}(2J_{2n+2}^{\alpha} + n + 1). \tag{33}$$

*4.*

$$\sum_{i=1}^{n} j_{2i+1}^{\alpha} = \frac{1}{3}(2J_{2n+2}^{\alpha} - n - 1). \tag{34}$$

*Proof.* Since the proof is done by a similar method, let us show only equations 1 and 3.

1. From the Binet formula, we write

$$\sum_{i=0}^{n} J_{2i}^{\alpha} = \sum_{i=0}^{n} \left( \frac{2^{2i+1}}{3}[1+\alpha, 4-2\alpha] - \frac{(-1)^{2i+1}}{3}(1-2\alpha) \right),$$

$$\sum_{i=0}^{n} J_{2i}^{\alpha} = \frac{1}{3} \left( \frac{2^{2n+1}}{3}[1+\alpha, 4-2\alpha] - \frac{(-1)^{2n+1}}{3}(1-2\alpha) - n - 1 \right),$$

$$\sum_{i=0}^{n} J_{2i}^{\alpha} = \frac{1}{3}(J_{2n+2}^{\alpha} - n - 1).$$

So, the claim is true.

3.

$$\sum_{i=0}^{n} J_{2i+1}^{\alpha} = \sum_{i=0}^{n} \left( \frac{2^{2i}}{3}[1+\alpha, 4-2\alpha] - \frac{(-1)^{2i}}{3}(1-2\alpha) \right),$$

$$\sum_{i=0}^{n} J_{2i+1}^{\alpha} = \frac{1}{3} \left\{ 2(\frac{2^{2n+1}}{3}[1+\alpha, 4-2\alpha] - \frac{1}{3}(1-2\alpha)) - (-1)(n+1) \right\},$$

$$\sum_{i=0}^{n} J_{2i+1}^{\alpha} = \frac{1}{3} \left( 2J_{2n+2}^{\alpha} + n + 1 \right),$$

which is the desired result.

**Theorem 5.** *For all positive integers n and m, we have*

$$J_{n+m}^{\alpha} = J_{m}^{\alpha} J_{n+1}^{\alpha} + 2J_{m-1}^{\alpha} J_{n}^{\alpha}. \tag{35}$$

*Proof.* Let's calculate the following equation

$$A = [J_{m-1} + 2\alpha J_{m-2}, J_{m+1} - 2\alpha J_{m-1}] \, [J_n + 2\alpha J_{n-1}, J_{n+2} - 2\alpha J_n, ] \, .$$

$$2B = 2 \, [J_{m-2} + 2\alpha J_{m-3}, J_m - 2\alpha J_{m-2}] \, [J_{n-1} + 2\alpha J_{n-2}, J_{n+1} - 2\alpha J_{n-1}] \, .$$

$A + 2B = [J_{m-1}J_n + 2J_{m-2}J_{n-1} + 2\alpha(J_{m-2}J_{n-1} + J_{m-2}J_n + 2J_{m-2}J_{n-2} + 2J_{m-3}J_{n-1})$
$+ 4\alpha^2(J_{m-2}J_{n-1} + 2J_{m-3}J_{n-2}), J_{m+1}J_{n+2} + 2J_mJ_{n+1}$
$-4\alpha(J_{m+1}J_n + J_{m-1}J_{n+2} + 2J_mJ_{n-1} + 2J_{m-2}J_{n+1}) + 4\alpha^2(J_{m-1}J_n + 2J_{m-2}J_{n-1})]$,

$$A+2B = \left[J_{n+m-2} + 2\alpha(J_{m-2}J_n + J_{n+m-3}) + 4\alpha^2 J_{n+m-4},\ J_{n+m+2} - 4\alpha J_{n+m} + 4\alpha^2 J_{n+m-2}\right],$$

Then, we have

$$A + 2B = J^{\alpha}_{n+m}.$$

Thus, the proof is complete.

**Theorem 6.** *For all positive integers $n$, the following equalities are true.*

1.
$$J^{\alpha}_n J^{\alpha}_{n-2} = \frac{1}{3}\left\{J^{\alpha}_{2n-2} + (-1)^{n+1}J^{\alpha}_{n-2} + (-1)^{n-1}J^{\alpha}_n\right\}. \tag{36}$$

2.
$$J^{\alpha}_n J^{\alpha}_n = \frac{1}{3}\left\{J^{\alpha}_{2n} + (-1)^{n+1}J^{\alpha}_n + (-1)^{n+1}J^{\alpha}_n\right\}. \tag{37}$$

3.
$$J^{\alpha}_{n+1} J^{\alpha}_{n-1} = \frac{1}{3}\left\{J^{\alpha}_{2n} + (-1)^{n+2}J^{\alpha}_{n-1} + (-1)^{n}J^{\alpha}_{n+1}\right\}. \tag{38}$$

*Proof.* 1. From the formulas (20) (21), we can write

$$J^{\alpha}_n J^{\alpha}_{n-2} = \frac{2^{n-1}}{3}[1+\alpha, 4-2\alpha] - \frac{(-1)^{n-1}}{3}(1-2\alpha) \times \frac{2^{n-3}}{3}[1+\alpha, 4-2\alpha] - \frac{(-1)^{n-3}}{3}(1-2\alpha),$$

$$= \frac{2^{2n-3}}{3}[1+\alpha, 4-2\alpha] - \frac{(-1)^{2n-3}}{3}(1-2\alpha) + (-1)^{n+1}\left(\frac{2^{n-3}}{3}[1+\alpha, 4-2\alpha] - \frac{(-1)^{n-3}}{3}(1-2\alpha)\right)$$
$$+(-1)^{n-1}\left(\frac{2^{n-1}}{3}[1+\alpha, 4-2\alpha] - \frac{(-1)^{n-1}}{3}(1-2\alpha)\right),$$
$$= \frac{1}{3}\left\{J^{\alpha}_{2n-2} + (-1)^{n+1}J^{\alpha}_{n-2} + (-1)^{n-1}J^{\alpha}_n\right\}.$$

The other equations can be easily seen.

**Theorem 7.** *For all positive integers $n$, the following equality is true.*

$$J^{\alpha}_{n+1}J^{\alpha}_{n-1} - (J^{\alpha}_n)^2 = \frac{1}{3}(-1)^n(J^{\alpha}_{n-1} + J^{\alpha}_{n+2}). \tag{39}$$

*Proof.* Using the following equalities

$$J^{\alpha}_{n+1}J^{\alpha}_{n-1} = \frac{1}{3}\left(J^{\alpha}_{2n} + (-1)^{n+2}J^{\alpha}_{n-1} + (-1)^{n}J^{\alpha}_{n+1}\right),$$

and

$$J_n^{\alpha} J_n^{\alpha} = \frac{1}{3}\left\{J_{2n}^{\alpha} + (-1)^{n+1} J_n^{\alpha} + (-1)^{n+1} J_n^{\alpha}\right\}$$

we get

$$\begin{aligned} J_{n+1}^{\alpha} J_{n-1}^{\alpha} - (J_n^{\alpha})^2 &= \frac{1}{3}\left(J_{2n}^{\alpha} + (-1)^{n+2} J_{n-1}^{\alpha} + (-1)^n J_{n+1}^{\alpha} - J_{2n}^{\alpha} - 2(-1)^{n+1} J_n^{\alpha}\right), \\ &= \frac{1}{3}(-1)^n \left(J_{n-1}^{\alpha} + J_{n+1}^{\alpha} + J_{n+2}^{\alpha}\right), \\ &= \frac{1}{3}(-1)^n \left(J_{n-1}^{\alpha} + J_{n+2}^{\alpha}\right). \end{aligned}$$

And so, the proof is completed.

## 3. A Fuzzy TOPSIS Application

In this section, an application example is presented to demonstrate how the proposed structure can be used in decision-making processes involving uncertainty. Fuzzy numbers are widely used in decision-making problems involving uncertainty. Triangular fuzzy numbers are preferred due to their ease of computation and their suitability for algebraic operations using the $\alpha$-cut method. The Jacobsthal number sequence, with its structure reflecting past uncertainty, can provide a meaningful modeling opportunity in dynamic decision-making processes.

Our aim in this study is to demonstrate the applicability of the proposed fuzzy structure to multi-criteria decision-making problems. For this purpose, the supplier selection problem is considered as an application, and the fuzzy TOPSIS method is used. This method uses linguistic variables based on various criteria to evaluate alternatives and examine their closeness to the ideal solution. The method produces consistent and understandable results through the steps of creating the decision matrix, normalization, weighted normalization, and calculating distances to ideal solutions. The vertex method is frequently used in calculating distances[11]. The distance between two triangular fuzzy numbers $\tilde{m} = (m_1, m_2, m_3)$ and $\tilde{n} = (n_1, n_2, n_3)$ is calculated using the vertex method proposed as follows.

$$d(\tilde{m}, \tilde{n}) = \sqrt{\frac{1}{3}\left[(m_1 - n_1)^2 + (m_2 - n_2)^2 + (m_3 - n_3)^2\right]}. \tag{40}$$

The algorithm of Fuzzy TOPSIS method can be summarized as:
Firstly, identify decision makers and define the evaluation criteria. Then, evaluate the alternatives using linguistic variables and convert linguistic terms into triangular fuzzy numbers. Then, construct the fuzzy decision and weight matrices normalize the fuzzy decision matrix. And then, calculate the weighted normalized matrix and determine the Fuzzy Positive Ideal Solution (FPIS) and Fuzzy Negative Ideal Solution (FNIS). Calculate the distances of each alternative from FPIS and FNIS. Finally, compute closeness coefficients and rank the alternatives accordingly.

Table 1: Linguistic Variables and Triangular Fuzzy Jacobsthal Numbers for $\alpha = \frac{1}{2}$

| **Linguistic Variable** | **Triangular Fuzzy Jacobsthal Number** |
|---|---|
| Very Low | $([1/4, 1/2], [1/2, 1], [1, 2])$ |
| Low | $([1/2, 1], [1, 2], [2, 4])$ |
| Medium | $([1, 2], [2, 4], [4, 8])$ |
| High | $([2, 4], [4, 8], [8, 16])$ |
| Very High | $([4, 8], [8, 16], [16, 32])$ |

The Topsis method is frequently used in solving decision-making problems. The evaluation criteria for supplier selection in decision-making problems can be listed as follows.

1. **Quality (C1):** The degree to which the product or service meets expectations.
2. **Cost (C2):** Procurement cost (lower is better).
3. **Delivery Performance (C3):** The rate of on-time and complete delivery.
4. **Flexibility (C4):** The supplier's ability to adapt to changing demands.
5. **Communication and Support (C5):** The quality of supplier's customer relations, information sharing, and technical support.

The linguistic variables used in the steps of the fuzzy TOPSIS algorithm are converted into triangular fuzzy numbers using the fuzzy Jacobsthal number definition for the necessary operations in the decision-making model. The $\alpha$cut method is used to manage the fuzziness of the numbers. In Table 2 below, we used the sequence elements in equation (17) according to the recursive relation for $\alpha = \frac{1}{2}$, and a $3 \times 5$ fuzzy decision matrix was created by converting the linguistic variables into triangular fuzzy numbers.

Table 2: Fuzzy Decision Matrix

| **Altern.** | **Quality (C1)** | **Cost (C2)** | **Delivery (C3)** | **Flexibility (C4)** | **Commun. (C5)** |
|---|---|---|---|---|---|
| A1 | $([4, 8], [8, 16], [16, 32])$ | $([1, 2], [2, 4], [4, 8])$ | $([4, 8], [8, 16], [16, 32])$ | $([1, 2], [2, 4], [4, 8])$ | $([2, 4], [4, 8], [8, 16])$ |
| A2 | $([1, 2], [2, 4], [4, 8])$ | $([1/2, 1], [1, 2], [2, 4])$ | $([1, 2], [2, 4], [4, 8])$ | $([4, 8], [8, 16], [16, 32])$ | $([1, 2], [2, 4], [4, 8])$ |
| A3 | $([2, 4], [4, 8], [8, 16])$ | $([1, 2], [2, 4], [4, 8])$ | $([2, 4], [4, 8], [8, 16])$ | $([2, 4], [4, 8], [8, 16])$ | $([4, 8], [8, 16], [16, 32])$ |

A normalized fuzzy decision matrix is created with the help of Table 2. It is sufficient to calculate the first row element of this matrix. The other elements of the matrix can be calculated similarly:

$$a_{11} = ([1/8, 1/2], [1/4, 1], [1/2, 2]), a_{12} = [1/8, 1/2], [1/4, 1], [1/2, 2]),$$
$$a_{13} = ([1/8, 1/2], [1/4, 1], [1/2, 2]), a_{14} = ([1/32, 1/8], [1/16, 1/4], [1/8, 1/2]),$$
$$a_{15} = ([1/16, 1/4], [1/8, 1/2], [1/4, 1]).$$

Each criterion is assigned a weight between 0 and 1, such that the sum of the weights is 1. These weights are multiplied by the corresponding column elements of the decision matrix. As a result of these operations, a weighted normal fuzzy decision matrix can be

Table 3: Weights of Decision Criteria

| Criterion | Weight |
|---|---|
| Quality ($C_1$) | 0.20 |
| Cost ($C_2$) | 0.30 |
| Delivery Performance ($C_3$) | 0.15 |
| Flexibility ($C_4$) | 0.20 |
| Communication and Support ($C_5$) | 0.15 |

created.
The first column elements of the weighted normalized fuzzy decision matrix can be calculated as follows.
$a_{11} = ([0.025, 0.1], [0.05, 0.2], [0.1, 0.4])$,
$a_{21} = ([0.00625, 0.025], [0.0125, 0.05], [0.025, 0.1])$,
$a_{31} = ([0.0125, 0.05], [0.025, 0.1], [0.05, 0.2])$.

After these processes, the decision-maker's optimal and unoptimal solutions, called fuzzy positive ideal solutions (FPIS) and fuzzy negative ideal solutions (FNIS), are determined, respectively. These solutions are obtained from the weighted normalized fuzzy decision matrix. FPIS and FNIS values are given in the following two tables.

Table 4: FPIS Values

| Criterion | FPIS |
|---|---|
| C1 | $(0.025, 0.1), (0.05, 0.2), (0.1, 0.4)$ |
| C2 | $(0.0375, 0.15), (0.075, 0.3), (0.15, 0.6)$ |
| C3 | $(0.01875, 0.075), (0.0375, 0.15), (0.075, 0.3)$ |
| C4 | $(0.025, 0.1), (0.05, 0.2), (0.1, 0.4)$ |
| C5 | $(0.01875, 0.075), (0.0375, 0.15), (0.075, 0.3)$ |

Table 5: FNIS Values

| Criterion | FNIS |
|---|---|
| C1 | $(0.00625, 0.025), (0.0125, 0.05), (0.025, 0.1)$ |
| C2 | $(0.01875, 0.075), (0.0375, 0.15), (0.075, 0.3)$ |
| C3 | $(0.0047, 0.01875), (0.0094, 0.0375), (0.01875, 0.075)$ |
| C4 | $(0.00625, 0.025), (0.0125, 0.05), (0.025, 0.1)$ |
| C5 | $(0.01875, 0.075), (0.0375, 0.15), (0.075, 0.3)$ |

For $\tilde{A}_i = (a_1, a_2, a_3)$ and $\tilde{B}_i = (b_1, b_2, b_3)$ triangular fuzzy numbers, we write using by (40):

$$d_i(\tilde{A}_i, \tilde{B}_i) = \sqrt{\frac{1}{3}\left[(a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2\right]}. \tag{41}$$

From here, we write

$$d_i^+ = \sum_{j=1}^{5}(d_i, C_j), \qquad d_i^- = \sum_{j=1}^{5}(d_i, C_j). \tag{42}$$

$$d_1^+ = (d_1, C_1) + (d_1, C_2) + (d_1, C_3) + (d_1, C_4) + (d_1, C_5) = 1.669.$$
$$d_2^+ = (d_2, C_1) + (d_2, C_2) + (d_2, C_3) + (d_2, C_4) + (d_2, C_5) = 1.441.$$
$$d_3^+ = (d_3, C_1) + (d_3, C_2) + (d_3, C_3) + (d_3, C_4) + (d_3, C_5) = 1.407.$$

Similar to the values $d_i^+$ calculated above, the other values $d_i^-$ can also be calculated easily. The decision is made based on the coefficient $CC_i$, the closeness coefficient, and so the closer this coefficient to 1, the better the solution.

$$CC_i = \frac{d_i^-}{d_i^- + d_i^+} \tag{43}$$

The first three closeness coefficients are follows.

$$CC_1 = \frac{0.589}{0.589 + 1.669} = \frac{0.589}{2.258} \approx 0.261.$$
$$CC_2 = \frac{0.846}{0.846 + 1.441} = \frac{0.846}{2.287} \approx 0.370.$$
$$CC_3 = \frac{1.015}{1.015 + 1.407} = \frac{1.015}{2.422} \approx 0.419.$$

Using the closeness coefficients, we deduce that $A_3$ is identified as the most preferable alternative with the highest value, followed by $A_2$ and then $A_1$. This ranking, with the help of the fuzzy TOPSIS method, provides clarity to the decision-making process by choosing among the given alternatives.
Now let us consider the problem solved above using classical Jacobsthal numbers, with the decision criteria being the same. Similar tables are given as follows.

Table 6: Jacobsthal Triples

| **Linguistic Variable** | **Jacobsthal Triples** |
|---|---|
| Very Low | $(0, 1, 1)$ |
| Low | $(1, 1, 3)$ |
| Medium | $(1, 3, 5)$ |
| High | $(3, 5, 11)$ |
| Very High | $(5, 11, 21)$ |

Table 7: Decision Matrix and Jacobsthal Numbers

| **Altern.** | **Quality(C1)** | **Cost(C2)** | **Delivery(C3)** | **Flexibility(C4)** | **Communication(C5)** |
|---|---|---|---|---|---|
| A1 | $(5, 11, 21)$ | $(1, 3, 5)$ | $(5, 11, 21)$ | $(1, 3, 5)$ | $(3, 5, 11)$ |
| A2 | $(1, 3, 5)$ | $(1, 1, 3)$ | $(1, 3, 5)$ | $(5, 11, 21)$ | $(1, 3, 5)$ |
| A3 | $(3, 5, 11)$ | $(1, 3, 5)$ | $(3, 5, 11)$ | $(3, 5, 11)$ | $(5, 11, 21)$ |

Table 8: Normalized Decision Matrix

| Altern. | C1 | C2 | C3 | C4 | C5 |
|---|---|---|---|---|---|
| A1 | $(5/21, 11/21, 1)$ | $(1/5, 3/5, 1)$ | $(5/21, 11/21, 1)$ | $(1/21, 3/21, 5/21)$ | $(3/21, 5/21, 11/21)$ |
| A2 | $(1/21, 3/21, 5/21)$ | $(1/5, 1/5, 3/5)$ | $(1/21, 3/21, 5/21)$ | $(5/21, 11/21, 1)$ | $(1/21, 3/21, 5/21)$ |
| A3 | $(3/21, 5/21, 11/21)$ | $(1/5, 3/5, 1)$ | $(3/21, 5/21, 11/21)$ | $(1/21, 3/21, 5/21)$ | $(5/21, 11/21, 1)$ |

Table 9: FPIS Values

| Crit. | FPIS |
|---|---|
| C1 | (0.04762, 0.10476, 0.2) |
| C2 | (0.06, 0.18, 0.3) |
| C3 | (0.03571, 0.07857, 0.15) |
| C4 | (0.04762, 0.10476, 0.2) |
| C5 | (0.03571, 0.07857, 0.15) |

Table 10: FNIS Values

| Crit. | FNIS |
|---|---|
| C1 | (0.00952, 0.02857, 0.04762) |
| C2 | (0.06, 0.18, 0.18) |
| C3 | (0.00714, 0.02143, 0.03571) |
| C4 | (0.00952, 0.02857, 0.04762) |
| C5 | (0.00714, 0.02143, 0.03571) |

The closeness coefficients obtained using the distances to the solutions are given below, and the alternatives are ranked accordingly.

$$\mathrm{CC}_1 = \frac{0.379}{0.379 + 0.176} = \frac{0.379}{0.555} \approx 0.683$$

$$\mathrm{CC}_2 = \frac{0.119}{0.119 + 0.450} = \frac{0.119}{0.569} \approx 0.209$$

$$\mathrm{CC}_3 = \frac{0.375}{0.375 + 0.136} = \frac{0.375}{0.511} \approx 0.734$$

So, taking into account the closeness coefficients, $A_3$ was determined as the most preferred option, and then $A_1$ and $A_2$ became the preferred options, respectively.

In this application, the supplier selection problem is solved using the TOPSIS method by classical and fuzzy Jacobsthal numbers. The results show that the fuzzy model provides a more discriminative and stable ranking among the alternatives under uncertainty. The values $CC_i$ close to the extreme values of the $[0, 1]$ range indicate that the model used distinguishes between the options quite sharply. Closer the values $CC_i$ indicate that the model used reflects uncertainty in the decision-making process more flexibly and solves the problem. Therefore, using fuzzy numbers and their properties in the problem to be solved is very advantageous in terms of decision making.

## 4. Conclusion

In this study, fuzzy extensions of Jacobsthal and Jacobsthal-Lucas number sequences are introduced using the $\alpha$-cut method. For these newly defined sequences, recurrence relations, Binet formulas, and some additive identities are investigated and obtained. A detailed analysis of some fundamental equations provided by classical number sequences is given in the literature for fuzzy sequences using fuzzy logic. A basic application is given to compare classical and fuzzy sequences. As a result of this application, it is observed that using fuzzy numbers produces more flexible solutions to the problem. Future studies will be advantageous in developing membership functions for the properties of integer sequences while examining the flexibility of a given problem using fuzzy logic.

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