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Menger Algebras of Alternating Terms

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Abstract. Let $\tau_n = (n_i)_{i\in I}$ be a particular language (type) of algebras such that $n_i = n$ for all i in I; n is a positive integer. This paper aims to introduce n-ary alternating terms (alt-terms) of type τ_n, based on the alternating group $Alt(n)$ of degree n. We demonstrate that the set of all n-ary alternating terms of type τ_n forms a Menger algebra of rank n; such algebra is denoted by $\mathcal{W}^{Alt(n)}_{\tau_n}(\Omega_n)$. We prove that the algebra $\mathcal{W}^{Alt(n)}_{\tau_n}(\Omega_n)$ is free with respect to the variety V_{Menger} of Menger algebras of rank n, and it is freely generated by the set $\{\omega_{(i,\sigma)} : i \in I, \sigma \in Alt(n)\}$. We introduce alternating hypersubstitutions of type τ_n and prove that the extension of an alternating hypersubstitution of type τ_n acts as an endomorphism on the algebra $\mathcal{W}^{Alt(n)}_{\tau_n}(\Omega_n)$. Furthermore, we have that the set of all alternating hypersubstitutions of type τ_n forms a monoid, denoted by $\mathcal{H}yp^{Alt(n)}(\tau_n)$. Finally, we establish that the set of all identities $s \approx t$ of a variety V of type τ_n, where s and t are n-ary alternating terms of type τ_n, constitutes a congruence on the algebra $\mathcal{W}^{Alt(n)}_{\tau_n}(\Omega_n)$. According to the monoid $\mathcal{H}yp^{Alt(n)}(\tau_n)$, we investigate alternating hyperidentities and alternating closed varieties.

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1. Introduction and preliminaries

Karl Menger (cf. [1], pp. 21-24) introduced the notion of Menger algebra as a generalization of the notion of semigroup; such algebras satisfy superassociative law, this is a generalization of associative law.

Throughout, n stands for a positive integer.

For a nonempty set M, an *n-ary operation* (*n-ary function*) on M is $f : M^n \to M$.

Definition 1. *A pair (M, f) consists of a nonempty set M and an n-ary operation f on M is a* Menger algebra *of rank n if*

$$f(f(\mu, \nu_1, \ldots, \nu_n), o_1, \ldots, o_n) = f(\mu, f(\nu_1, o_1, \ldots, o_n), \ldots, f(\nu_n, o_1, \ldots, o_n))$$

for any $\mu, \nu_1, \ldots, \nu_n, o_1, \ldots, o_n \in M$.

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Menger algebras of rank 1 are semigroups. Menger algebras of rank n are a significant extension of the classical Menger algebras, providing a powerful tool for modeling and analyzing multi-level and hierarchical systems. Their applications in fuzzy logic, probability, and artificial intelligence make them relevant in both theoretical and applied mathematics. However, their increasing complexity with higher ranks necessitates a careful balance between theoretical exploration and practical applicability. In theoretical mathematics, Dicker demonstrated in 1963 that every Menger algebra of rank n is isomorphic to a Menger algebra of n-ary operations defined on some set; the result is similar to Cayley's theorem for semigroups: any semigroup is isomorphic to a transformation semigroup.

Based on permutations and transformations (for example, full transformations, order-preserving transformations, order-decreasing transformations), several kinds of terms and of generalized terms (such as strongly full terms, full terms, order-preserving full terms, order-decreasing full terms, and generalized full terms) are introduced and studied. Menger algebras and unitary Menger algebras of particular terms have been extensively studied. According to the account provided in the reference, Denecke [2] studied Menger algebras and clone of terms of an arbitrary type. Since then several kinds of Menger algebras of particular terms have been investigated. Denecke and Freiberg [3] examined Menger algebras of strongly full terms defined using permutations. Denecke and Jampachon [4] introduced and studied Menger algebras of full terms, such terms are defined by full transformations. By order-decreasing transformations, Wattanatripop and Changphas [5, 6] introduced and investigated Menger algebras of order-decresing full terms. Puapong and Leeratanawalee [7] explored Menger algebras of generalized full terms. Denecke and Hounnon [8] (also Lekkoksung and Lekkoksung [9]) studied partial Menger algebras of linear terms and of r-terms; some algebraic properties such as generating systems, homomorphic images and freeness are investigated. Recently, Punigool, Phuapong and Chansuriya [10] introduced and studied Menger algebras of terms defined based on transformations with restricted range.

Definition 2. *A* type *(or language) of algebras is a nonempty indexed sequence $\tau = (n_i)_{i \in I}$ of nonnegative integers n_i such that for each n_i is assigned to a symbol f_i. This integer is called the* arity *(or rank) of f_i, and f_i is called an n_i-*ary operation (or function) symbol*.*

In particular, let $\tau_n = (n_i)_{i \in I}$ denote a type of algebras such that $n_i = n$ for all $i \in I$.

Drawing inspiration from research results mentioned above, this paper aims to introduce n-ary alternating terms of type τ_n, based on the alternating group $Alt(n)$ of degree n. Firstly, we demonstrate that the set of all n-ary alternating terms of type τ_n forms a Menger algebra of rank n; we denote this algebra by $\mathcal{W}_{\tau_n}^{Alt(n)}(\Omega_n)$. We prove that the algebra $\mathcal{W}_{\tau_n}^{Alt(n)}(\Omega_n)$ is free with respect to the variety V_{Menger} of Menger algebras of rank n, and it is freely generated by the set $\{\omega_{(i,\sigma)} : i \in I, \sigma \in Alt(n)\}$. Secondly, we introduce alternating hypersubstitutions of type τ_n and prove that the extension of an alternating hypersubstitution of type τ_n acts as an endomorphism on $\mathcal{W}_{\tau_n}^{Alt(n)}(\Omega_n)$. Furthermore, we have that the set of all alternating hypersubstitutions of type τ_n forms a monoid, denoted by $\mathcal{H}yp^{Alt(n)}(\tau_n)$. Finally, we establish that the set of all identities $s \approx t$ of a variety V

of type τ_n, where s and t are n-ary alternating terms of type τ_n, constitutes a congruence on $\mathcal{W}_{\tau_n}^{Alt(n)}(\Omega_n)$. According to the monoid $\mathcal{H}yp^{Alt(n)}(\tau_n)$, we investigate alternating hyperidentities and alternating closed vareities.

2. Alternating Terms

Using $Alt(n)$ the alternating group of degree n (see [11]), which is a subgroup of the symmetric group S_n on the set $\{1, 2, \ldots, n\}$, *n-ary alternating terms* (alt-terms) of type τ_n are defined as follows:

Definition 3. *Let $\Omega_n = \{\omega_1, \omega_2, \ldots, \omega_n\}$ denote a set of finite alphabet $\omega_1, \omega_2, \ldots, \omega_n$, called* variables. *Let $(f_i)_{i \in I}$ be an indexed sequence of n-ary operation symbols according to a type $\tau_n = (n_i)_{i\in I}$; this set is disjoint to the set Ω_n. For any $i \in I$ and $\sigma \in Alt(n)$,*

(1) $f_i(\omega_{\sigma(1)}, \omega_{\sigma(2)}, \ldots, \omega_{\sigma(n)})$ is an n-ary alt-term;

(2) If $\theta_1, \theta_2, \ldots, \theta_n$ are n-ary alt-terms, then $f_i(\theta_1, \theta_2, \ldots, \theta_n)$ is an n-ary alt-term.

Let $W_{\tau_n}^{Alt(n)}(\Omega_n)$ represent the smallest set of n-ary alt-terms of type τ_n which is closed under finite application of (2).

Example 1. *For the symmetric group $S_3 = \{(1), (12), (13), (23), (123), (132)\}$, the alternating group $Alt(3) = \{(1), (123), (132)\}$. Consider $\tau_3 = (3)$ with 3-ary operation symbol g. Then $g(\omega_1, \omega_2, \omega_3)$, $g(\omega_2, \omega_3, \omega_1)$, $g(\omega_3, \omega_1, \omega_2)$, $g(g(\omega_1, \omega_2, \omega_3), g(\omega_2, \omega_3, \omega_1), g(\omega_3, \omega_1, \omega_2))$ are 3-ary alt-terms, where as ω_1, ω_2, ω_3, $g(\omega_2, \omega_1, \omega_3)$, $g(\omega_3, \omega_2, \omega_1)$, $g(\omega_1, \omega_3, \omega_2)$, $g(\omega_1, \omega_2, g(\omega_3, \omega_1, \omega_2))$ are not 3-ary alt-terms.*

Example 2. *Consider the symmetric group S_4. For $\tau_4 = (4)$ with 4-ary operation symbol f, $f(\omega_1, \omega_2, \omega_3, \omega_4)$, $f(\omega_2, \omega_1, \omega_4, \omega_3)$, $f(\omega_3, \omega_4, \omega_1, \omega_2)$, $f(\omega_4, \omega_3, \omega_2, \omega_1)$, $f(\omega_2, \omega_3, \omega_1, \omega_4)$, $f(\omega_1, \omega_4, \omega_2, \omega_3)$, $f(\omega_1, \omega_1, \omega_3, \omega_2)$, $f(\omega_3, \omega_2, \omega_4, \omega_1)$, $f(\omega_3, \omega_1, \omega_2, \omega_4)$, $f(\omega_4, \omega_2, \omega_1, \omega_3)$, $f(\omega_1, \omega_3, \omega_4, \omega_2)$, $f(\omega_2, \omega_4, \omega_3, \omega_1) \in W_{\tau_4}^{Alt(4)}(\Omega_4)$. Clearly, $\omega_1, \omega_2, \omega_3, \omega_4 \notin W_{\tau_4}^{Alt(4)}(\Omega_4)$; hence $f(\omega_1, \omega_2, \omega_3, f(\omega_1, \omega_2, \omega_3, \omega_4))$, $f(\omega_1, f(\omega_2, \omega_4, \omega_3, \omega_1), \omega_3, f(\omega_1, \omega_2, \omega_3, \omega_4)) \notin W_{\tau_4}^{Alt(4)}(\Omega_4)$, as well.*

Example 3. *Let us consider the symmetric group S_4 again, but $\tau_4 = (4, 4, 4)$ with 4-ary operation symbols f, g and h. We have $f(\omega_1, \omega_2, \omega_3, \omega_4)$, $g(\omega_2, \omega_1, \omega_4, \omega_3)$, $f(\omega_3, \omega_4, \omega_1, \omega_2)$, $h(\omega_4, \omega_3, \omega_2, \omega_1)$ are 4-ary alternating terms of type τ_4; then*

$$g(f(\omega_1, \omega_2, \omega_3, \omega_4), g(\omega_2, \omega_1, \omega_4, \omega_3), f(\omega_3, \omega_4, \omega_1, \omega_2), h(\omega_4, \omega_3, \omega_2, \omega_1))$$

and

$$h(h(\omega_4, \omega_3, \omega_2, \omega_1), g(\omega_2, \omega_1, \omega_4, \omega_3), f(\omega_3, \omega_4, \omega_1, \omega_2), f(\omega_1, \omega_2, \omega_3, \omega_4))$$

are 4-ary alternating terms of type τ_4.

On $W_{\tau_n}^{Alt(n)}(\Omega_n)$, define

$$S^n : (W_{\tau_n}^{Alt(n)}(\Omega_n))^{n+1} \to W_{\tau_n}^{Alt(n)}(\Omega_n)$$

to be $(n+1)$-ary operation by:

(1) $S^n(f_i(\omega_{\sigma(1)}, \omega_{\sigma(2)}, \ldots, \omega_{\sigma(n)}), \vartheta_1, \ldots, \vartheta_n) = f_i(\vartheta_{\sigma(1)}, \vartheta_{\sigma(2)}, \ldots, \vartheta_{\sigma(n)})$;

(2) $S^n(f_i(\theta_1, \ldots, \theta_n), \vartheta_1, \ldots, \vartheta_n) = f_i(S^n(\theta_1, \vartheta_1, \ldots, \vartheta_n), \ldots, S^n(\theta_n, \vartheta_1, \ldots, \vartheta_n))$

for any f_i, $\sigma \in Alt(n)$, $\theta_1, \ldots, \theta_n, \vartheta_1, \ldots, \vartheta_n \in W_{\tau_n}^{Alt(n)}(\Omega_n)$.

Theorem 1.

$$\mathcal{W}_{\tau_n}^{Alt(n)}(\Omega_n) = (W_{\tau_n}^{Alt(n)}(\Omega_n), S^n)$$

is an algebra of type $(n+1)$.

Proof. We claim that for any $\theta, \theta_1, \theta_2, \ldots, \theta_n \in W_{\tau_n}^{Alt(n)}(\Omega_n)$, $S^n(\theta, \theta_1, \theta_2, \ldots, \theta_n) \in W_{\tau_n}^{Alt(n)}(\Omega_n)$, which can be proved by induction on the number of occurrence of operation symbols in θ. Assume $\theta = f_i(\omega_{\sigma(1)}, \omega_{\sigma(2)}, \ldots, \omega_{\sigma(n)})$ for some $\sigma \in Alt(n)$. By $\theta_1, \theta_2, \ldots, \theta_n \in W_{\tau_n}^{Alt(n)}(\Omega_n)$,

$$\begin{aligned} S^n(\theta, \theta_1, \theta_2, \ldots, \theta_n) &= S^n(f_i(\omega_{\sigma(1)}, \omega_{\sigma(2)}, \ldots, \omega_{\sigma(n)}), \theta_1, \theta_2, \ldots, \theta_n) \\ &= f_i(\theta_{\sigma(1)}, \theta_{\sigma(2)}, \ldots, \theta_{\sigma(n)}) \\ &\in W_{\tau_n}^{Alt(n)}(\Omega_n). \end{aligned}$$

Assume $\theta = f_i(\vartheta_1, \vartheta_2, \ldots, \vartheta_n)$ and

$$S^n(\vartheta_1, \theta_1, \theta_2, \ldots, \theta_n), S^n(\vartheta_2, \theta_1, \theta_2, \ldots, \theta_n), \ldots, S^n(\vartheta_n, \theta_1, \theta_2, \ldots, \theta_n) \in W_{\tau_n}^{Alt(n)}(\Omega_n).$$

Then

$$\begin{aligned} & S^n(\theta, \theta_1, \theta_2, \ldots, \theta_n) \\ &= S^n(f_i(\vartheta_1, \vartheta_2, \ldots, \vartheta_n), \theta_1, \theta_2, \ldots, \theta_n) \\ &= f_i(S^n(\vartheta_1, \theta_1, \theta_2, \ldots, \theta_n), S^n(\vartheta_2, \theta_1, \theta_2, \ldots, \theta_n), \ldots, S^n(\vartheta_n, \theta_1, \theta_2, \ldots, \theta_n)) \\ &\in W_{\tau_n}^{Alt(n)}(\Omega_n). \end{aligned}$$

So we have the claim.

Furthermore, we have

Theorem 2. *The algebra* $\mathcal{W}_{\tau_n}^{Alt(n)}(X_n)$ *is a Menger algebra of rank* n.

Proof. Let $\theta, \theta_1, \ldots, \theta_n, \vartheta_1, \ldots, \vartheta_n \in W_{\tau_n}^{Alt(n)}(\Omega_n)$. Suppose $\theta = f_i(\omega_{\sigma(1)}, \ldots, \omega_{\sigma(n)})$ for some $\sigma \in Alt(n)$. Then

$$S^n(S^n(\theta, \theta_1, \ldots, \theta_n), \vartheta_1, \ldots, \vartheta_n)$$

$$
\begin{aligned}
&= S^n(S^n(f_i(\omega_{\sigma(1)},\dots,\omega_{\sigma(n)}),\theta_1,\dots,\theta_n),\vartheta_1,\dots,\vartheta_n)\\
&= S^n(f_i(\theta_{\sigma(1)},\theta_{\sigma(2)},\dots,\theta_{\sigma(n)}),\vartheta_1,\dots,\vartheta_n)\\
&= f_i(S^n(\theta_{\sigma(1)},\vartheta_1,\dots,\vartheta_n),\dots,S^n(\theta_{\sigma(n)},\vartheta_1,\dots,\vartheta_n))\\
&= S^n(f_i(\omega_{\sigma(1)},\dots,\omega_{\sigma(n)}),S^n(\theta_1,\vartheta_1,\dots,\vartheta_n),\dots,S^n(\theta_n,\vartheta_1,\dots,\vartheta_n))\\
&= S^n(\theta,S^n(\theta_1,\vartheta_1,\dots,\vartheta_n),\dots,S^n(\theta_n,\vartheta_1,\dots,\vartheta_n)).
\end{aligned}
$$

Let $\theta = f_i(\theta'_1,\dots,\theta'_n)$ be such that

$$S^n(S^n(\theta'_j,\theta_1,\dots,\theta_n),\vartheta_1,\dots,\vartheta_n) = S^n(\theta'_j,S^n(\theta_1,\vartheta_1,\dots,\vartheta_n),\dots,S^n(\theta_n,\vartheta_1,\dots,\vartheta_n))$$

for any $1 \leq j \leq n$. Then, by induction hypothesis, we have

$$
\begin{aligned}
&S^n(S^n(\theta,\theta_1,\dots,\theta_n),\vartheta_1,\dots,\vartheta_n)\\
&= S^n(S^n(f_i(\theta'_1,\dots,\theta'_n),\theta_1,\dots,\theta_n),\vartheta_1,\dots,\vartheta_n)\\
&= S^n(f_i(S^n(\theta'_1,\theta_1,\dots,\theta_n),\dots,S^n(\theta'_n,\theta_1,\dots,\theta_n)),\vartheta_1,\dots,\vartheta_n)\\
&= f_i(S^n(S^n(\theta'_1,\theta_1,\dots,\theta_n),\vartheta_1,\dots,\vartheta_n),\dots,S^n(S^n(\theta'_n,\theta_1,\dots,\theta_n),\vartheta_1,\dots,\vartheta_n))\\
&= f_i(S^n(\theta'_1,S^n(\theta_1,\vartheta_1,\dots,\vartheta_n),\dots,S^n(\theta_n,\vartheta_1,\dots,\vartheta_n)),\dots,\\
&\quad S^n(\theta'_n,S^n(\theta_1,\vartheta_1,\dots,\vartheta_n),\dots,S^n(\theta_n,\vartheta_1,\dots,\vartheta_n)))\\
&= S^n(f_i(\theta'_1,\dots,\theta'_n),S^n(\theta_1,\vartheta_1,\dots,\vartheta_n),\dots,S^n(\theta_n,\vartheta_1,\dots,\vartheta_n))\\
&= S^n(\theta,S^n(\theta_1,\vartheta_1,\dots,\vartheta_n),\dots,S^n(\theta_n,\vartheta_1,\dots,\vartheta_n)).
\end{aligned}
$$

The proof is complete.

3. Freeness

Let V_{Menger} denote the variety of all Menger algebras of rank $n+1$, and let

$$\mathcal{F}_{V_{Menger}}(\Xi) = (F_{V_{Menger}}(\Xi), \widetilde{S}^n)$$

be the free algebra with respect to V_{Menger}, freely generated by an indexed set of alphabet of variables

$$\Xi = \{\omega_{(i,\sigma)} : (i,\sigma) \in I \times Alt(n)\}.$$

Theorem 3. *The Menger algebra $\mathcal{W}_{\tau_n}^{Alt(n)}(\Omega_n)$ is free with respect to the variety V_{Menger}, freely generated by Ξ.*

Proof. To prove the assertion we show that $\mathcal{W}_{\tau_n}^{Alt(n)}(\Omega_n)$ is isomorphic to the algebra $\mathcal{F}_{V_{Menger}}(\Xi)$. Define a mapping

$$\Phi : W_{\tau_n}^{Alt(n)}(\Omega_n) \to F_{V_{Menger}}(\Xi)$$

by :

(1) $\Phi(f_i(\omega_{\sigma(1)},\dots,\omega_{\sigma(n)}))=\omega_{(i,\sigma)}$;

(2) $\Phi(S^n(f_i(\omega_{\sigma(1)},\dots,\omega_{\sigma(n)}),\theta_1,\dots,\theta_n))=\widetilde{S}^n(\omega_{(i,\sigma)},\Phi(\theta_1),\dots,\Phi(\theta_n))$

for any $(i,\sigma)\in I\times Alt(n)$ and $\theta_1,\dots,\theta_n\in W_{\tau_n}^{Alt(n)}(\Omega_n)$.

The mapping Φ is a homomorphism, i.e.,

$$\Phi(S^n(\theta,\theta_1,\dots,\theta_n))=\widetilde{S}^n(\Phi(\theta),\Phi(\theta_1),\dots,\Phi(\theta_n))$$

for all $\theta,\theta_1,\dots,\theta_n\in W_{\tau_n}^{Alt(n)}(\Omega_n)$. Suppose $\theta=f_i(\omega_{\sigma(1)},\dots,\omega_{\sigma(n)})$ for some $i\in I$ and $\sigma\in Alt(n)$. Then

$$\begin{aligned}&\Phi(S^n(\theta,\theta_1,\dots,\theta_n))\\&=\Phi(S^n(f_i(\omega_{\sigma(1)},\dots,\omega_{\sigma(n)}),\theta_1,\dots,\theta_n))\\&=\widetilde{S}^n(\omega_{(i,\sigma)},\Phi(\theta_1),\dots,\Phi(\theta_n))\\&=\widetilde{S}^n(\Phi(f_i(\omega_{\sigma(1)},\dots,\omega_{\sigma(n)})),\Phi(\theta_1),\dots,\Phi(\theta_n))\\&=\widetilde{S}^n(\Phi(\theta),\Phi(\theta_1),\dots,\Phi(\theta_n)).\end{aligned}$$

Let $\theta=f_i(\theta_1',\dots,\theta_n')$ such that, for $1\le k\le n$,

$$\Phi(S^n(\theta_k',\theta_1,\dots,\theta_n))=S^n(\Phi(\theta_k'),\Phi(\theta_1),\dots,\Phi(\theta_n)).$$

From

$$\Phi(f_i(\theta_1',\dots,\theta_n'))=\widetilde{S}^n(\omega_{(i,id_n)},\Phi(\theta_1'),\dots,\Phi(\theta_n'))$$

for all $\theta_1',\dots,\theta_n'\in W_{\tau_n}^{Alt(n)}(\Omega_n)$ where id_n is the identity map in $Alt(n)$, we then have that

$$\begin{aligned}&\Phi(S^n(\theta,\theta_1,\dots,\theta_n))\\&=\Phi(S^n(f_i(\theta_1',\dots,\theta_n'),\theta_1,\dots,\theta_n))\\&=\Phi(f_i(S^n(\theta_1',\theta_1,\dots,\theta_n),\dots,S^n(\theta_n',\theta_1,\dots,\theta_n)))\\&=\widetilde{S}^n(\omega_{(i,id_n)},\Phi(S^n(\theta_1',\theta_1,\dots,\theta_n)),\dots,\Phi(S^n(\theta_n',\theta_1,\dots,\theta_n)))\\&=\widetilde{S}^n(\omega_{(i,id_n)},\widetilde{S}^n(\Phi(\theta_1'),\Phi(\theta_1),\dots,\Phi(\theta_n)),\dots,\widetilde{S}^n(\Phi(\theta_n'),\Phi(\theta_1),\dots,\Phi(\theta_n)))\\&=\widetilde{S}^n(\widetilde{S}^n(y_{(i,id_n)},\Phi(\theta_1'),\dots,\Phi(\theta_n')),\Phi(\theta_1),\dots,\Phi(\theta_n))\\&=\widetilde{S}^n(\Phi(\theta),\Phi(\theta_1),\dots,\Phi(\theta_n)).\end{aligned}$$

The mapping Φ is bijective: Assume

$$\Phi(f_i(\omega_{\sigma(1)},\dots,\omega_{\sigma(n)}))=\Phi(f_j(\omega_{\rho(1)},\dots,\omega_{\rho(n)}))$$

for some $\sigma,\rho\in Alt(n)$. Then

$$\omega_{(i,\sigma)}=\omega_{(j,\rho)}.$$

This means

$$(i,\sigma)=(j,\rho).$$

So

$$f_i(\omega_{\sigma(1)}, \ldots, \omega_{\sigma(n)}) = f_j(\omega_{\rho(1)}, \ldots, \omega_{\rho(n)}).$$

If $\omega_{(i,\sigma)} \in \Xi$, then

$$\Phi(f_i(\omega_{\sigma(1)}, \ldots, \omega_{\sigma(n)})) = \omega_{(i,\sigma)}.$$

Hence Φ is an isomorphism, and the proof is completed.

4. Alternating Hypersubstitutions

We begin this section with introducing alternating hypersubstitutions; hypersubstitutions are important notion for studying hyperidentities and solid vareities (see [12]).

Definition 4. *A mapping*

$$\alpha : \{f_i : i \in I\} \to W_{\tau_n}^{Alt(n)}(\Omega_n)$$

is called an alternating hypersubstitution *(or* alt-hypersubstitution*) of type* τ_n.

To define the extension of an alternating hypersubstitution of type τ_n to the set $W_{\tau_n}^{Alt(n)}(\Omega_n)$, for any $\theta \in W_{\tau_n}^{Alt(n)}(\Omega_n)$ and $\rho \in Alt(n)$, let

(1) $(\theta)_\rho = f_i(\omega_{\rho(\sigma(1))}, \omega_{\rho(\sigma(2))}, \ldots, \omega_{\rho(\sigma(n))})$ if $\theta = f_i(\omega_{\sigma(1)}, \omega_{\sigma(2)}, \ldots, \omega_{\sigma(n)})$ for some $i \in I, \sigma \in Alt(n)$;

(2) $(\theta)_\rho = f_i((\theta_1)_\rho, (\theta_2)_\rho, \ldots, (\theta_n)_\rho)$ if $\theta = f_i(\theta_1, \theta_2, \ldots, \theta_n)$ for some $i \in I, \theta_1, \theta_2, \ldots, \theta_n \in W_{\tau_n}^{Alt(n)}(\Omega_n)$.

It is observed that

$$(\theta)_\rho \in W_{\tau_n}^{Alt(n)}(\Omega_n)$$

for any $\rho \in Alt(n)$ and $\theta \in W_{\tau_n}^{Alt(n)}(\Omega_n)$. Moreover,

$$((\theta)_\rho)_\sigma = (\theta)_{\sigma\rho}$$

and

$$S^n(\theta, (\theta_1)_\sigma, \ldots, (\theta_n)_\sigma)) = (S^n(\theta, \theta_1, \ldots, \theta_n))_\sigma$$

for any $\theta, \theta_1, \ldots, \theta_n \in W_{\tau_n}^{Alt(n)}(\Omega_n)$ and $\sigma, \rho \in Alt(n)$.

Using notation introduced above, an alternating hypersubstitution $\alpha : \{f_i : i \in I\} \to W_{\tau_n}^{Alt(n)}(\Omega_n)$ of type τ_n can be extended to a mapping

$$\hat{\alpha} : W_{\tau_n}^{Alt(n)}(\Omega_n) \to W_{\tau_n}^{Alt(n)}(\Omega_n)$$

by:

(1) $\hat{\alpha}[f_i(\omega_{\sigma(1)}, \omega_{\sigma(2)}, \ldots, \omega_{\sigma(n)})] = (\alpha(f_i))_\sigma$;

(2) $\hat{\alpha}[f_i(\theta_1, \theta_2, \ldots, \theta_n)] = S^n(\alpha(f_i), \hat{\alpha}[\theta_1], \hat{\alpha}[\theta_2], \ldots, \hat{\alpha}[\theta_n])$.

Lemma 1. *Let α be an alt-hypersubstitution of type τ_n. For $\theta, \theta_1, \theta_2, \ldots, \theta_n \in W_{\tau_n}^{Alt(n)}(\Omega_n)$ and $\rho \in Alt(n)$,*

$$S^n(\theta, \hat{\alpha}[\theta_{\rho(1)}], \ldots, \hat{\alpha}[\theta_{\rho(n)}]) = S^n((\theta)_\rho, \hat{\alpha}[\theta_1], \ldots, \hat{\alpha}[\theta_n]).$$

Proof. Let $\theta, \theta_1, \ldots, \theta_n \in W_{\tau_n}^{Alt(n)}(X_n)$ and $\rho \in Alt(n)$. Suppose $\theta = f_i(\omega_{\sigma(1)}, \ldots, \omega_{\sigma(n)})$ for some $\sigma \in Alt(n)$. Then

$$\begin{aligned} S^n(\theta, \hat{\alpha}[\theta_{\rho(1)}], \ldots, \hat{\alpha}[\theta_{\rho(n)}]) &= S^n(f_i(\omega_{\sigma(1)}, \ldots, \omega_{\sigma(n)}), \hat{\alpha}[\theta_{\rho(1)}], \ldots, \hat{\alpha}[\theta_{\rho(n)}]) \\ &= f_i(\hat{\alpha}[\theta_{\rho(\sigma(1))}], \ldots, \hat{\alpha}[\theta_{\rho(\sigma(n))}]) \\ &= S^n(f_i(\omega_{\rho(\sigma(1))}, \ldots, \omega_{\rho(\sigma(n))}), \hat{\alpha}[\theta_1], \ldots, \hat{\alpha}[\theta_n]) \\ &= S^n((\theta)_\rho, \hat{\alpha}[\theta_1], \ldots, \hat{\alpha}[\theta_n]). \end{aligned}$$

Let $\theta = f_i(\theta'_1, \ldots, \theta'_n)$ and assume that, for $1 \leq k \leq n$,

$$S^n(\theta'_k, \hat{\alpha}[\theta_{\rho(1)}], \ldots, \hat{\alpha}[\theta_{\rho(n)}]) = S^n((\theta'_k)_\rho, \hat{\alpha}[\theta_1], \ldots, \hat{\alpha}[\theta_n]).$$

Then

$$\begin{aligned} & S^n(\theta, \hat{\alpha}[\theta_{\rho(1)}], \ldots, \hat{\alpha}[\theta_{\rho(n)}]) \\ &= S^n(f_i(\theta'_1, \ldots, \theta'_n), \hat{\alpha}[\theta_{\rho(1)}], \ldots, \hat{\alpha}[\theta_{\rho(n)}]) \\ &= f_i(S^n(\theta'_1, \hat{\alpha}[\theta_{\rho(1)}], \ldots, \hat{\alpha}[\theta_{\rho(n)}]), \ldots, S^n(\theta'_n, \hat{\alpha}[\theta_{\rho(1)}], \ldots, \hat{\alpha}[\theta_{\rho(n)}])) \\ &= f_i(S^n((\theta'_1)_\rho, \hat{\alpha}[\theta_1], \ldots, \hat{\alpha}[\theta_n]), \ldots, S^n((\theta'_n)_\rho, \hat{\alpha}[\theta_1], \ldots, \hat{\alpha}[\theta_n])) \\ &= S^n(f_i((\theta'_1)_\rho, \ldots, (\theta'_n)_\rho), \hat{\alpha}[\theta_1], \ldots, \hat{\alpha}[\theta_n]) \\ &= S^n((\theta)_\rho, \hat{\alpha}[\theta_1], \ldots, \hat{\alpha}[\theta_n]). \end{aligned}$$

Hence, the proof is complete.

Theorem 4. *For any $\alpha \in Hyp^{Alt(n)}(\tau_n)$, the extension $\hat{\alpha} : W_{\tau_n}^{Alt(n)}(\Omega_n) \to W_{\tau_n}^{Alt(n)}(\Omega_n)$ is an endomorphism on the Menger algebra $\mathcal{W}_{\tau_n}^{Alt(n)}(\Omega_n)$.*

Proof. Let $\alpha \in Hyp^{Alt(n)}(\tau_n)$. We have to show that, for any $\theta_0, \theta_1, \ldots, \theta_n \in W_{\tau_n}^{Alt(n)}(\Omega_n)$,

$$\hat{\alpha}[S^n(\theta_0, \theta_1, \ldots, \theta_n)] = S^n(\hat{\alpha}[\theta_0], \hat{\alpha}[\theta_1], \ldots, \hat{\alpha}[\theta_n]).$$

Let $\theta_0, \theta_1, \ldots, \theta_n \in W_{\tau_n}^{Alt(n)}(\Omega_n)$. Suppose $\theta_0 = f_i(\omega_{\sigma(1)}, \ldots, \omega_{\sigma(n)})$ for some $i \in I, \sigma \in Alt(n)$. By Lemma 1,

$$\begin{aligned} \hat{\alpha}[S^n(\theta_0, \theta_1, \ldots, \theta_n)] &= \hat{\alpha}[S^n(f_i(\omega_{\sigma(1)}, \ldots, \omega_{\sigma(n)}), \theta_1, \ldots, \theta_n)] \\ &= \hat{\alpha}[f_i(\theta_{\sigma(1)}, \ldots, \theta_{\sigma(n)})] \\ &= S^n(\alpha(f_i), \hat{\alpha}[\theta_{\sigma(1)}], \ldots, \hat{\alpha}[\theta_{\sigma(n)}]) \end{aligned}$$

$$
\begin{aligned}
&= S^n((\alpha(f_i))_\sigma, \hat{\alpha}[\theta_1], \ldots, \hat{\alpha}[\theta_n]) \\
&= S^n(\hat{\alpha}[f_i(\omega_{\sigma(1)}, \ldots, \omega_{\sigma(n)})], \hat{\alpha}[\theta_1], \ldots, \hat{\alpha}[\theta_n]) \\
&= S^n(\hat{\alpha}[\theta_0], \hat{\alpha}[\theta_1], \ldots, \hat{\alpha}[\theta_n]).
\end{aligned}
$$

Let $\theta_0 = f_i(\theta'_1, \ldots, \theta'_n)$, and assume, for $1 \leq k \leq n$, that

$$\hat{\alpha}[S^n(\theta'_k, \theta_1, \ldots, \theta_n)] = S^n(\hat{\alpha}[\theta'_k], \hat{\alpha}[\theta_1], \ldots, \hat{\alpha}[\theta_n]).$$

From Theorem 3,

$$
\begin{aligned}
& \hat{\alpha}[S^n(\theta_0, \theta_1, \ldots, \theta_n)] \\
= \;& \hat{\alpha}[S^n(f_i(\theta'_1, \ldots, \theta'_n), \theta_1, \ldots, \theta_n)] \\
= \;& \hat{\alpha}[f_i(S^n(\theta'_1, \theta_1, \ldots, \theta_n), \ldots, S^n(\theta'_n, \theta_1, \ldots, \theta_n))] \\
= \;& S^n(\alpha(f_i), \hat{\alpha}[S^n(\theta'_1, \theta_1, \ldots, \theta_n)], \ldots, \hat{\alpha}[S^n(\theta'_n, \theta_1, \ldots, \theta_n)]) \\
= \;& S^n(\alpha(f_i), S^n(\hat{\alpha}[\theta'_1], \hat{\alpha}[\theta_1], \ldots, \hat{\alpha}[\theta_n]), \ldots, S^n(\hat{\alpha}[\theta'_n], \hat{\alpha}[\theta_1], \ldots, \hat{\alpha}[\theta_n])) \\
= \;& S^n(S^n(\alpha(f_i), \hat{\alpha}[\theta'_1], \ldots, \hat{\alpha}[\theta'_n]), \hat{\alpha}[\theta_1], \ldots, \hat{\alpha}[\theta_n]) \\
= \;& S^n(\hat{\alpha}[f_i(\theta'_1, \ldots, \theta'_n)], \hat{\alpha}[\theta_1], \ldots, \hat{\alpha}[\theta_n]) \\
= \;& S^n(\hat{\alpha}[\theta_0], \hat{\alpha}[\theta_1], \ldots, \hat{\alpha}[\theta_n]).
\end{aligned}
$$

Hence

$$\hat{\alpha}[S^n(\theta_0, \theta_1, \ldots, \theta_n)] = S^n(\hat{\alpha}[\theta_0], \hat{\alpha}[\theta_1], \ldots, \hat{\alpha}[\theta_n]).$$

Define an operation $\circ_h$ on $Hyp^{Alt(n)}(\tau_n)$ by, for any $\alpha_1, \alpha_2 \in Hyp^{Alt(n)}(\tau_n)$,

$$(\alpha_1 \circ_h \alpha_2)(f_i) = \hat{\alpha}_1[\alpha_2(f_i)]$$

for all $i \in I$.

Lemma 2. *For any $\alpha_1, \alpha_2 \in Hyp^{Alt(n)}(\tau_n)$, $(\alpha_1 \circ_h \alpha_2)\hat{} = \hat{\alpha}_1 \circ \hat{\alpha}_2$.*

Proof. At first, we prove that for any $\alpha \in Hyp^{Alt(n)}(\tau_n)$, $\hat{\alpha}[(\theta)_\sigma] = (\hat{\alpha}[\theta])_\sigma$ for any $\theta \in W_{\tau_n}^{Alt(n)}(\Omega_n)$ and $\sigma \in Alt(n)$. Suppose $\theta = f_i(\omega_{\rho(1)}, \ldots, \omega_{\rho(n)})$ for some $i \in I, \rho \in Alt(n)$. Then $(\theta)_\sigma = f_i(\omega_{\sigma(\rho(1))}, \ldots, x_{\sigma(\rho(n))})$); so

$$
\begin{aligned}
\hat{\alpha}[(\theta)_\sigma] &= S^n(\alpha(f_i), \omega_{\sigma(\rho(1))}, \ldots, \omega_{\sigma(\rho(n))})) \\
&= S^n((\alpha(f_i))_{\sigma\rho}, \omega_1, \ldots, \omega_n)) \\
&= (\alpha(f_i))_{\sigma\rho}.
\end{aligned}
$$

Also,

$$(\hat{\alpha}[\theta])_\sigma = ((\alpha(f_i))_{rho})_\sigma = (\alpha(f_i))_{\sigma\rho}.$$

Let $\theta = f_i(\theta_1, \ldots, \theta_n)$ and assume $\hat{\alpha}[(\theta_k)_\sigma] = (\hat{\alpha}[\theta_k])_\sigma$ for any $1 \leq k \leq n$. So

$$(\theta)_\sigma = f_i((\theta_1)_\sigma, \ldots, (\theta_n)_\sigma).$$

Thus

$$\begin{aligned}\hat{\alpha}[(\theta)_\sigma] &= S^n(\alpha(f_i), \hat{\alpha}[(\theta_1)_\sigma], \dots, \hat{\alpha}[(\theta_n)_\sigma]) \\ &= S^n(\alpha(f_i), (\hat{\alpha}[t_1])_\sigma, \dots, (\hat{\alpha}[\theta_n])_\sigma)) \\ &= (S^n(\alpha(f_i), \hat{\alpha}[\theta_1], \dots, \hat{\alpha}[\theta_n]))_\sigma \\ &= (\hat{\alpha}[\theta])_\sigma.\end{aligned}$$

Now, let $\alpha_1, \alpha_2 \in Hyp^{Alt(n)}(\tau_n)$, and let $\theta \in W^{Alt(n)}_{\tau_n}(\Omega_n)$. If $\theta = f_i(\omega_{\sigma(1)}, \dots, \omega_{\sigma(n)})$ for some $i \in I, \sigma \in Alt(n)$, then

$$\widehat{(\alpha_1 \circ_h \alpha_2)}[\theta] = \widehat{(\alpha_1 \circ_h \alpha_2)}[f_i(\omega_{\sigma(1)}, \dots, \omega_{\sigma(n)})] = ((\alpha_1 \circ_h \alpha_2)(f_i))_\sigma = ((\hat{\alpha}_1[\alpha_2(f_i)])_\sigma$$

and

$$(\hat{\alpha}_1 \circ \hat{\alpha}_2)[\theta] = (\hat{\alpha}_1 \circ \hat{\alpha}_2)[f_i(\omega_{\sigma(1)}, \dots, \omega_{\sigma(n)})] = \hat{\alpha}_1(\hat{\alpha}_2[f_i(\omega_{\sigma(1)}, \dots, \omega_{\sigma(n)})]) = \hat{\alpha}_1((\alpha_2(f_i))_\sigma).$$

Thus $\widehat{(\alpha_1 \circ_h \alpha_2)}[\theta] = \hat{\alpha}_1 \circ \hat{\alpha}_2[\theta]$.

Let $\theta = f_i(\theta_1, \dots, \theta_n)$, and assume $\widehat{(\alpha_1 \circ_h \alpha_2)}[\theta_k] = (\hat{\alpha}_1 \circ \hat{\alpha}_2)[\theta_k]$ for any $1 \leq k \leq n$. Then

$$\begin{aligned}\widehat{(\alpha_1 \circ_h \alpha_2)}[\theta] &= \widehat{(\alpha_1 \circ_h \alpha_2)}[f_i(\theta_1, \dots, \theta_n)] \\ &= S^n((\alpha_1 \circ_h \alpha_2)(f_i), \widehat{(\alpha_1 \circ_h \alpha_2)}[\theta_1], \dots, \widehat{(\alpha_1 \circ_h \alpha_2)}[\theta_n]) \\ &= S^n((\hat{\alpha}_1[\alpha_2(f_i)], \hat{\alpha}_1[\hat{\alpha}_2[\theta_1]], \dots, \hat{\alpha}_1[\hat{\alpha}_2[\theta_n]]) \\ &= \hat{\alpha}_1(S^n(\alpha_2(f_i), \hat{\alpha}_2[\theta_1], \dots, \hat{\alpha}_2[\theta_n])) \\ &= \hat{\alpha}_1[\hat{\alpha}_2[f_i(\theta_1, \dots, \theta_n)]] \\ &= (\hat{\alpha}_1 \circ \hat{\alpha}_2)[\theta].\end{aligned}$$

Therefore the assertion holds.

The mapping $\alpha_{id} : \{f_i : i \in I\} \to W^{Alt(n)}_{\tau_n}(\Omega_n)$ which is defined by, for all $i \in I$,

$$\alpha_{id}(f_i) = f_i(\omega_1, \dots, \omega_n)$$

is an alt-hypersubstitution of type τ_n.

Theorem 5. *$\mathcal{H}yp^{Alt(n)}(\tau_n) = (Hyp^{Alt(n)}(\tau_n), \circ_h, \alpha_{id})$ forms a monoid.*

Proof. Let $\alpha_1, \alpha_2 \in Hyp^{Alt(n)}(\tau_n)$. Then, for any f_i, we have $(\alpha_1 \circ_h \alpha_2)(f_i) = \hat{\alpha_1}[\alpha_2(f_i)]$. Since $\alpha_2 \in Hyp^{Alt(n)}(\tau_n)$, $\alpha_2(f_i) \in W^{Alt(n)}_{\tau_n}(\Omega_n)$. If $\alpha_2(f_i) = f_i(\omega_{\sigma(1)}, \dots, \omega_{\sigma(n)})$ for some $\sigma \in Alt(n)$, then by $\alpha_1(f_i) \in W^{Alt(n)}_{\tau_n}(\Omega_n)$, we have

$$(\alpha_1 \circ_h \alpha_2)(f_i) = \hat{\alpha_1}[\alpha_2(f_i)] = (\alpha_1(f_i))_\sigma \in W^{Alt(n)}_{\tau_n}(\Omega_n).$$

Let $\alpha_2(f_i) = f_i(\theta_1, \ldots, \theta_n)$ and assume $\hat{\alpha_1}[\theta_1], \ldots, \hat{\alpha_1}[\theta_n] \in W_{\tau_n}^{Alt(n)}(\Omega_n)$. By

$$\hat{\alpha_1}[f_i(\theta_1, \ldots, \theta_n)] = S^n(\alpha_1(f_i), \hat{\alpha_1}[\theta_1], \ldots, \hat{\alpha_1}[\theta_n])$$

and $\alpha_1(f_i) \in W_{\tau_n}^{Alt(n)}(\Omega_n)$, it follows that $\hat{\alpha_1}[\alpha_2(f_i)] \in W_{\tau_n}^{Alt(n)}(\Omega_n)$. The associativity of $\circ_h$ follows by Lemma 2.

Lemma 3. *The Menger algebra $\mathcal{W}_{\tau_n}^{Alt(n)}(\Omega_n)$ of rank n is generated by*

$$F_{W_{\tau_n}^{Alt(n)}(\Omega_n)} = \{f_i(\omega_{\sigma(1)}, \ldots, \omega_{\sigma(n)}) : i \in I, \sigma \in Alt(n)\}.$$

Proof. Clearly, $f_i(\omega_{\sigma(1)}, \ldots, \omega_{\sigma(n)}) \in W_{\tau_n}^{Alt(n)}(\Omega_n)$ for any $i \in I, \sigma \in Alt(n)$. Let $\theta = f_i(\theta_1, \theta_2, \ldots, \theta_n) \in W_{\tau_n}^{Alt(n)}(\Omega_n)$ such that $F_{W_{\tau_n}^{Alt(n)}(\Omega_n)}$ generates $\theta_1, \theta_2, \ldots, \theta_n$. We have

$$S^n(f_i(\omega_1, \ldots, \omega_n), \theta_1, \ldots, \theta_n) = f_i(\theta_1, \ldots, \theta_n) = \theta.$$

Since the Menger algebra $\mathcal{W}_{\tau_n}^{Alt(n)}(\Omega_n)$ is generated by the set

$$F_{W_{\tau_n}^{Alt(n)}(\Omega_n)} = \{f_i(\omega_{\sigma(1)}, \ldots, \omega_{\sigma(n)}) : i \in I, \sigma \in Alt(n)\}$$

so any mapping

$$\eta : F_{W_{\tau_n}^{Alt(n)}(\Omega_n)} \to W_{\tau_n}^{Alt(n)}(\Omega_n)$$

can be uniquely extended to an endomorphism

$$\bar{\eta} : W_{\tau_n}^{Alt(n)}(\Omega_n) \to W_{\tau_n}^{Alt(n)}(\Omega_n).$$

Such mappings are called *alternating substitutions* (alt-substitutions). The set of all alt-substitutions will be denoted by $Subst_{Alt(n)}(\tau_n)$.

For $\eta_1, \eta_2 \in Subst_{Alt(n)}(\tau_n)$, define

$$\eta_1 \odot \eta_2 = \bar{\eta}_1 \circ \eta_2$$

where $\circ$ is the usual composition. The identity map $id_{F_{W_{\tau_n}^{Alt(n)}(\Omega_n)}}$ is an identity element with respect to $\odot$. It turns out that

$$\mathcal{S}ubst_{Alt(n)}(\tau_n) = (Subst_{Alt(n)}(\tau_n), \odot, id_{F_{W_{\tau_n}^{Alt(n)}(\Omega_n)}})$$

forms a monoid. Let $\alpha \in Hyp^{Alt(n)}(\tau_n)$. Since $\hat{\alpha} : W_{\tau_n}^{Alt(n)}(\Omega_n) \to W_{\tau_n}^{Alt(n)}(\Omega_n)$ is an endomorphism and $F_{W_{\tau_n}^{Alt(n)}(\Omega_n)}$ is a generating system of $\mathcal{W}_{\tau_n}^{Alt(n)}(\Omega_n)$, so $\hat{\alpha}|_{F_{W_{\tau_n}^{Alt(n)}(\Omega_n)}}$ is an alt-substitution such that

$$\overline{\hat{\sigma}|_{F_{W_{\tau_n}^{Alt(n)}(\Omega_n)}}} = \hat{\alpha}.$$

Define $\psi : Hyp^{Alt(n)}(\tau_n) \to Subst_{Alt(n)}(\tau_n)$ by

$$\psi(\alpha) = \hat{\alpha}|_{F_{W_{\tau_n}^{Alt(n)}(\Omega_n)}}$$

for any $\alpha \in Hyp^{Alt(n)}(\tau_n)$. Let $\alpha_1, \alpha_2 \in Hyp^{Alt(n)}(\tau_n)$. Then

$$\begin{aligned}\psi(\alpha_1 \circ_h \alpha_2) &= (\alpha_1 \circ_h \alpha_2)\hat{}|_{F_{W_{\tau_n}^{Alt(n)}(\Omega_n)}} \\ &= (\hat{\alpha}_1 \circ \hat{\alpha}_2)|_{F_{W_{\tau_n}^{Alt(n)}(\Omega_n)}} \\ &= \overline{\hat{\alpha}_1|_{F_{W_{\tau_n}^{Alt(n)}(\Omega_n)}}} \circ \hat{\alpha}_2|_{F_{W_{\tau_n}^{Alt(n)}(\Omega_n)}} \\ &= \overline{\psi(\alpha_1)} \circ \psi(\alpha_2) \\ &= \psi(\alpha_1) \odot \psi(\alpha_2).\end{aligned}$$

We have ψ is a homomorphism. Clearly, ψ is injective. We conclude the result as follows.

Theorem 6. *The monoid $\mathcal{H}yp^{Alt(n)}(\tau_n)$ can be embedded into the monoid $\mathcal{S}ubst_{Alt(n)}(\tau_n)$.*

5. Alternating Hyperidentities

In this section let V be a variety of type τ_n. The set of all identities of V will be represented by $Id(V)$; this is a congruence on the free algebra $\mathcal{F}_{\tau_n}(\Omega_n)$ (see [13]). Let $Id^{Alt(n)}(V)$ denote the set of all identities $\theta \approx \vartheta$ of V such that $\theta, \vartheta \in W_{\tau_n}^{Alt(n)}(\Omega_n)$, i.e.,

$$Id^{Alt(n)}(V) = (W_{\tau_n}^{Alt(n)}(\Omega_n))^2 \cap Id(V).$$

Lemma 4. *$Id^{Alt(n)}(V)$ is a congruence on the Menger algebra $\mathcal{W}_{\tau_n}^{Alt(n)}(\Omega_n)$.*

Proof. Let $\theta \approx \vartheta, \theta_1 \approx \vartheta_1, \ldots, \theta_n \approx \vartheta_n \in Id^{Alt(n)}(V)$. At first, we have

$$S^n(\theta, \theta_1, \ldots, \theta_n) \approx S^n(\theta, \vartheta_1, \ldots, \vartheta_n) \in Id^{Alt(n)}(V).$$

To see this, suppose $\theta = f_i(\omega_{\sigma(1)}, \ldots, \omega_{\sigma(n)})$ for some $i \in I, \sigma \in Alt(n)$. By the compatibility of $Id(V)$ with the operations $\bar{f}_i$ of the absolutely free algebra $\mathcal{F}_{\tau_n}(\Omega_n)$,

$$f_i(\theta_{\sigma(1)}, \ldots, \theta_{\sigma(n)}) \approx f_i(\vartheta_{\sigma(1)}, \ldots, \vartheta_{\sigma(n)}) \in Id^{Alt(n)}(V).$$

Thus

$$S^n(f_i(\omega_{\sigma(1)}, \ldots, \omega_{\sigma(n)}), \theta_1, \ldots, \theta_n) \approx S^n(f_i(\omega_{\sigma(1)}, \ldots, \omega_{\sigma(n)}), \vartheta_1, \ldots, \vartheta_n) \in Id^{Alt(n)}(V).$$

Hence

$$S^n(\theta, \theta_1, \ldots, \theta_n) \approx S^n(\theta, \vartheta_1, \ldots, \vartheta_n) \in Id^{Alt(n)}(V).$$

Let $\theta = f_i(\nu_1, \ldots, \nu_n)$ and assume, for all $1 \leq k \leq n$,

$$S^n(\nu_k, \theta_1, \ldots, \theta_n) \approx S^n(\nu_k, \vartheta_1, \ldots, \vartheta_n) \in Id^{Alt(n)}(V).$$

Then

$$f_i(S^n(\nu_1,\theta_1,\ldots,\theta_n),\ldots,S^n(\nu_n,\theta_1,\ldots,\theta_n))$$
$$\approx f_i(S^n(\nu_1,\vartheta_1,\ldots,\vartheta_n),\ldots,S^n(\nu_n,\vartheta_1,\ldots,\vartheta_n)) \in Id^{Alt(n)}(V).$$

So

$$S^n(f_i(\nu_1,\ldots,\nu_n),\theta_1,\ldots,\theta_n) \approx S^n(f_i(\nu_1,\ldots,\nu_n),\vartheta_1,\ldots,\vartheta_n) \in Id^{Alt(n)}(V).$$

Hence

$$S^n(\theta,\theta_1,\ldots,\theta_n) \approx S^n(\theta,\vartheta_1,\ldots,\vartheta_n) \in Id^{Alt(n)}(V).$$

Now, from

$$S^n(\theta,\vartheta_1,\ldots,\vartheta_n) \approx S^n(\vartheta,\vartheta_1,\ldots,\vartheta_n) \in Id^{Alt(n)}(V),$$

it follows that

$$S^n(\theta,\theta_1,\ldots,\theta_n) \approx S^n(\theta,\vartheta_1,\ldots,\vartheta_n) \approx S^n(\vartheta,\vartheta_1,\ldots,\vartheta_n) \in Id^{Alt(n)}(V).$$

Therefore the assertion holds.

Definition 5. *V is said to be* alternating closed *(alt-closed) if*

$$\hat{\alpha}[\theta] \approx \hat{\alpha}[\vartheta] \in Id^{Alt(n)}(V)$$

for all $\theta \approx \vartheta \in Id^{Alt(n)}(V)$ and for all $\alpha \in Hyp^{Alt(n)}(\tau_n)$.

Definition 6. *A congruence $Id^{Alt(n)}(V)$ on the Menger algebra $\mathcal{W}_{\tau_n}^{Alt(n)}(\Omega_n)$ is said to be* alternative fully invariant *(alt-fully invariant) if $Id^{Alt(n)}(V)$ is compatible with all endomorphisms $\hat{\alpha}$ on $\mathcal{W}_{\tau_n}^{Alt(n)}(\Omega_n)$.*

Theorem 7. *If $Id^{Alt(n)}(V)$ is alt-fully invariant, then V is alt-closed.*

Proof. Assume $Id^{Alt(n)}(V)$ is alt-fully invariant. Let $\theta \approx \vartheta \in Id^{Alt(n)}(V)$ and $\alpha \in Hyp^{Alt(n)}(\tau_n)$. By Theorem 4, $\hat{\alpha}$ is an endomorphism on $\mathcal{W}_{\tau_n}^{Alt(n)}(\Omega_n)$. Hence $\hat{\alpha}[\theta] \approx \hat{\alpha}[\vartheta] \in Id^{Alt(n)}(\tau_n)$, so V is alt-closed.

We have shown $Id^{Alt(n)}(V)$ is a congruence on $\mathcal{W}_{\tau_n}^{Alt(n)}(\Omega_n)$. Consequently, quotient algebra

$$W_{\tau_n}^{Alt(n)}(\Omega_n)/Id^{Alt(n)}(V)$$

belongs to V_{Menger}. Note that

$$nat_{Id^{Alt(n)}(V)} : W_{\tau_n}^{Alt(n)}(\Omega_n) \to W_{\tau_n}^{Alt(n)}(\Omega_n)/Id^{Alt(n)}(V)$$

is a homomorphism with

$$nat_{Id^{Alt(n)}(V)}(\theta) = [\theta]_{Id^{Alt(n)}(V)}.$$

Definition 7. *An identity $\theta \approx \vartheta \in Id^{Alt(n)}(V)$ is an alternative hyperidentity (alt-hyperidentity) if*

$$\hat{\alpha}[\theta] \approx \hat{\alpha}[\vartheta] \in Id^{Alt(n)}(V)$$

for all $\alpha \in Hyp^{Alt(n)}(\tau_n)$.

Theorem 8. *Let* $\theta \approx \vartheta \in Id^{Alt(n)}(V)$. *If* $\theta \approx \vartheta$ *is an identity in* $W_{\tau_n}^{Alt(n)}(\Omega_n)/Id^{Alt(n)}(V)$, *then* $\theta \approx \vartheta$ *is an alt-hyperidentity of* V.

Proof. Assume $\theta \approx \vartheta$ is an identity in $W_{\tau_n}^{Alt(n)}(\Omega_n)/Id^{Alt(n)}(V)$, and let $\alpha \in Hyp^{Alt(n)}(\tau_n)$. By

$$\hat{\alpha} : W_{\tau_n}^{Alt(n)}(\Omega_n) \to W_{\tau_n}^{Alt(n)}(\Omega_n)$$

is an endomorphism, we have

$$\kappa : W_{\tau_n}^{Alt(n)}(\Omega_n)/Id^{Alt(n)}(V) \to W_{\tau_n}^{Alt(n)}(\Omega_n)/Id^{Alt(n)}(V)$$

defined by

$$\kappa([\theta]_{Id^{Alt(n)}(V)}) = [\hat{\alpha}[\theta]]_{Id^{Alt(n)}(V)}$$

for all $[\theta]_{Id^{Alt(n)}(V)} \in W_{\tau_n}^{Alt(n)}(\Omega_n)/Id^{Alt(n)}(V)$ is an endomorphism. By assumption,

$$[\theta]_{Id^{Alt(n)}(V)} = [\vartheta]_{Id^{Alt(n)}(V)}.$$

Then

$$\kappa([\theta]_{Id^{Alt(n)}(V)}) = \kappa([\vartheta]_{Id^{Alt(n)}(V)}).$$

Thus

$$[\hat{\alpha}[\theta]]_{Id^{Alt(n)}(V)} = [\hat{\alpha}[\vartheta]]_{Id^{Alt(n)}(V)}.$$

Therefore

$$\hat{\alpha}[\theta] \approx \hat{\alpha}[\vartheta] \in Id^{Alt(n)}(V).$$

Hence $\theta \approx \vartheta$ is an alt-hyperidentity of V.

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